Ideal Closures and Sheaf Stability

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Introduction

"It is impossible to be a mathematician
without being a poet in soul."
Sofia Kovalevskaya

One of the main goals of mathematics is to structure and classify the objects and concepts arising in mathematical discourse. To make progress towards this goal two methods are of special importance to lay bare the hidden structure. One of those methods is to strip the descriptions of the objects of as much unnecessary data as possible, to focus on the important aspects. The other method is to characterize subclasses of objects that give rise to better classification structures than would be possible for the whole class.

Ideal closure operations are an example for the former method applied to ideals in commutative rings and how they relate to different aspects of algebraic geometry. Semistability on the other hand is a property that enables the latter method by allowing to algebraically describe the moduli space of vector bundles that exhibit it. The two main components of this thesis are a theorem showing containment of one ideal closure operation in another, namely that continuous closure often contains tight closure, and a theorem that allows to algorithmically determine semistability of a vector bundle. This algorithm was implemented by the author as a computer program.

We will now introduce the relevant concepts properly. The most basic example of an ideal closure operation is the radical of an ideal. The radical of an ideal $I \subseteq R$ consists of the elements in the ring R for which a power lies in I. The radical condenses the information of I as far as it relates to elementary algebraic geometry, specifically the zero set $V(I)$: While there are many ideals with the same zero set, they all have the same radical.

The central ideal closures in this thesis are tight closure and continuous closure. Tight closure was introduced by Craig Huneke and Melvin Hochster

[28] and quickly showed some interesting results with the most prominent being a new and simpler proof of the theorem of Briançon-Skoda. Formally the *tight closure* I^* of an ideal I in a commutative ring R of characteristic p is the ideal consisting of the elements $f \in R$ such that there exists a $c \in R$ not contained in any minimal prime of R (for example a nonzerodivisor) for which cf^q is an element of the Frobenius power $I^{[q]}$ of the ideal for almost all $q = p^e$.

Continuous closure was introduced by Holger Brenner [8] and naturally comes up when thinking of ideal closures in terms of forcing equations, i.e. equations that admit a solution if and only if an element is contained in an ideal. An ideal $I = (f_1, \ldots, f_n) \subseteq R$ consists precisely of the elements f that admit a solution $(g_1, \ldots, g_n) \in R^n$ to the forcing equation $f = g_1 f_1 + \ldots + g_n f_n$. Various ideal closures can be constructed by instead allowing solutions in ring extensions of R. The *continuous closure* in a finitely generated $\mathbb{C}$-algebra R is the ideal of elements that have a continuous solution to the forcing equation, i.e. where the g_i are continuous maps $\operatorname{Spec} R(\mathbb{C}) \longrightarrow \mathbb{C}$. Tight closure containment can also be checked via forcing equations, but in a more subtle way, which opens up an avenue for computation of tight closure which we will get back to later.

Now let's describe the first part of the thesis, the relation between continuous closure and tight closure. As continuous closure is a strictly characteristic 0 concept and tight closure is originally a characteristic p concept, one first has to find the right common ground to compare these notions. This is found via valuative criteria for the containment of an element in a closure. The continuous closure is very close to another closure, the axes closure, which can be defined via rings that geometrically correspond to a scheme that consists of normal curves which intersect transversally in one point, i.e. a cross. For each of the axes of the ring we can define a valuation because of the fact that a one-dimensional normal ring is a discrete valuation ring. These valuations give a numerical criterion to check whether an element is in the axes closure or not.

The axes closure has been introduced by Brenner in [8] as an attempt to characterize algebraically the continuous closure, though it has failed to exactly do that, since the two closures are different as Neil Epstein and Melvin Hochster have shown [19, Example 9.2]. Still they are close: We always have $I^{\mathrm{cont}} \subseteq I^{\mathrm{ax}}$ and they coincide for primary ideals.

We prove that the tight closure of an ideal in an excellent, normal ring with perfect residue fields at the maximal ideals is contained in its axes closure. An important step of the proof uses special tight closure introduced by Adela Vraciu and Craig Huneke [34]. Their theory allows – as long as the ring is normal – the splitting of tight closure into two summands, the ideal itself and a 'deeper' part, the special tight closure. We show the containment of special tight closure in the axes closure in Theorem 2.4.1 for rings of characteristic p as well as in Theorem 2.5.1 for rings of equal characteristic 0. The characteristic 0 case is done by reduction to positive characteristic, a method that connects properties that hold modulo the primes on a dense subset of the spectrum of $\mathbb{Z}$ to their characteristic 0 property counterparts.

An application of semistability of vector bundles led us to the development of the second main component of this thesis. A vector bundle is a topological space that locally looks like the product of a base space and a vector space and where on the overlapping of the local environments there is a homeomorphism induced by linear maps on the vector spaces. For example, if we have a circle as base space, there are – up to isomorphism – only two one-dimensional vector bundles, a cylinder and a Möbius strip. There is an equivalence of categories between isomorphism classes of vector bundles and isomorphism classes of locally free sheaves, and sometimes we may use these terms interchangeably.

For the computation of tight closure Melvin Hochster considered the closely related solid closure instead, for which the question whether an element belongs to the closure is equivalent to the question whether the complement of a vector bundle is an affine scheme. The complement in question is by construction isomorphic to the projective sprectrum of the respective forcing algebra. Even with Hochsters insights, computing the tight closure can be quite tricky. But Brenner developed some helpful methods for it [6]. However, those methods themselves rely on the semistability of vector bundles or at least the ability to decide whether a vector bundle is semistable. This is what the algorithm developed in the second part of this thesis wants to address.

What is semistability? There exist several definitions depending on context. All definitions of semistability in some sense express that the vector bundle is at least as ample as all its subbundles, in the sense that subbundles don't have 'more' sections. We will restrict ourselves to curves in projective space, i.e. projective varieties corresponding to two-dimensional rings. In that context we get a very ample sheaf $\mathcal{O}_X(1)$ from the embedding of the curve X in projective space. With respect to this very ample sheaf we can define a slope $\mu(V)$ of a vector bundle as the fraction of degree and rank. A sheaf is semistable if its slope is at least as big as the slopes of all subbundles. In some contexts this is also called μ-stability or Mumford-Takemoto-stability. See also the book by Daniel Huybrechts and Manfred Lehn [35] and the original work by David Mumford [39].

Originally, semistable vector bundles were introduced in the context of moduli spaces. As it turned out, there is no moduli space for the class of all vector bundles, which can be shown using Castelnuovo-Mumford-regularity. In general, sheaves can have any Castelnuovo-Mumford-regularity. As an example, sheaves of the form $\mathcal{O}(l) \oplus \mathcal{O}(-l)$ are not semistable and have Castelnuovo-Mumford regularity l. For semistable locally free sheaves, however, the Castelnuovo-Mumford-regularity is bounded by a number that only depends on the underlying scheme. This makes it possible to construct a moduli space.

Our method to compute semistability is to use Riemann-Roch-theory and methods of multilinear algebra and twists to make destabilizing subsheaves visible. Previous work on this has been done by Brenner in [12, Section 2](only in positive characteristic), and by Almar Kaid and Ralf Kasprowitz in [36](only for sheaves over $\mathbb{P}^n$). We will also give a concrete implementation using linear algebra methods to compute destabilizing sections.

Since our method relies heavily on linear and multilinear algebra, we need to provide the vector bundle in a computationally accessible way, which for us means as a kernel sheaf that can be described globally as the kernel of a matrix. Consequently we will consider sequences of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus \mathcal{O}_X(-d_i) \xrightarrow{A} \mathcal{O}_X.$$

We want to decide semistability of $\mathcal{F}$ computationally. This leads us to two problems. First we need to be aware of all subbundles $\mathcal{E} \subseteq \mathcal{F}$ and check whether there is one with slope $\mu(\mathcal{E}) > \mu(\mathcal{F})$. In addition we will only be able to find global sections, so we need to make all subbundles visible as global sections.

The solution we present is to use multilinear algebra methods, namely symmetric and exterior powers in addition to twisting in order to transform the situation in such a way that any destabilizing subbundle $\mathcal{E}$ has sufficiently high slope so that it will have global sections, while $\mathcal{F}$ has negative slope so that $\mathcal{F}$ can't have global sections if it is semistable. The main theorems are Theorem 4.4.1 and Theorem 4.4.2.

The implementation of the semistability algorithm requires very efficient linear algebra implementations which were built for this purpose by the author. This can be found here: `https://github.com/JonathanSteinbuch/sheafstability`.

Chapters

Chapter 1: Chapter 1 is a background chapter and recapitulates some of the central concepts used in the subsequent chapter. It deals with ideal closure operations, particularly tight closure, continuous closure and axes closure. Also the chapter includes a short description of the process of reduction modulo a prime number which is the central concept used to relate characteristic p notions like tight closure with characteristic 0 notions like continuous closure.

Some of the content in Chapter 1 was also part of the author's Master Thesis on János Kollár's algebraic description of continuous closure [45].

Chapter 2: This chapter deals with the connection between tight closure and continuous closure and axes closure respectively. The main theorem is that for primary ideals over excellent, normal rings with perfect residue field the tight closure is contained in the axes closure. This is proved using special tight closure [34] and valuative criteria for axes closure. This is then used to also relate tight closure to continuous closure for primary ideals in affine $\mathbb{C}$-algebras as in that context continuous closure and axes closure coincide.

The main contents of Chapter 2 were published in the paper [10] that the author wrote together with Holger Brenner.

Chapter 3: In this second background chapter locally free sheaves and their relation to vector bundles are introduced. Next introductions to some tools needed to deal with locally free sheaves like divisors, degree, twist and stability follow. Lastly the chapter deals with curves and their embedding into projective space and what it means for a curve to be smooth.

Chapter 4: The algorithm for deciding semistability of kernel bundles on curves is the central part of this chapter. Before that, however, we compute how rank, degree and slope of a vector bundle on a curve change with symmetric and exterior powers. We also show how to describe the symmetric and exterior powers of a kernel sheaf as kernel sheaves again. This is necessary to give an algorithm that can actually be implemented in a computer. To relate the way the algorithm works, we give several examples.

There is also a section describing how to alter the algorithm to work in positive characteristic using the Frobenius pullback.

The contents of Chapter 4 are going to be published in an upcoming paper together with Holger Brenner [11].

Chapter 5: Here some applications of the semistability algorithm are described. Semistable sheaves were first introduced to give a class of sheaves where moduli functors can be corepresented, i.e. their moduli spaces have a scheme structure. In this chapter we recall what this means.

We also describe the concept of Harder-Narasimhan-filtration, which gives the power to apply some methods that require semistability to non-semistable sheaves. We explain how this is used in [6] to compute the tight closure of an ideal and how the semistability algorithm from this thesis can be employed here.

Chapter 6: In the last chapter we shine a light on several aspects of the implementation of the semistability algorithm from chapter 4. In particular we describe our use of monomial orderings, Gröbner bases, monomial bases, Hilbert polynomials and how to compute them.

Additionally we describe the data structures and methods used to implement efficient integer value sparse matrix echelonization, which is the work horse behind the algorithm. We also analyze some performance characteristics of the implementation. Lastly we describe exactly how to use the implementation to do your own semistability computations.

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Last - but with regards to this thesis most - I thank my advisor Holger Brenner not only for in countless sessions helping me understand a tiny wave in the vast ocean of maths but also for encouragement and being a friend.

Chapter 1

Ideal closure operations

This chapter recapitulates ideal closure operations.

The treatment of the topics in this chapter is relatively brief as giving each the attention it deserves would make this thesis very long. For more in-depth information we refer the reader to the relevant literature. A good first source on ideal closure operations is the book of Irena Swanson and Craig Huneke [33]. For tight closure we give several citations in the chapter to the works of Melvin Hochster, Craig Huneke, Holger Brenner and others.

1.1 Ideal closure operations

The general idea behind ideal closure operations is to enlarge an ideal with the goal to get rid of peculiarities of the ideal which are irrelevant to the task at hand. As a result, how similar or different the closures of two ideals are, often tells us more than the relation of the ideals themselves. Which properties are highlighted depend on the specific closure, but often those properties are of a geometrical nature.

For this work tight closure, continuous closure and axes closure are of special importance.

Definition 1.1.1. A *closure operation* cl on a commutative ring R is an unary operation on the set of ideals in R, satisfying the following properties:

extension $I \subseteq I^{\mathrm{cl}}$.

idempotence $\left(I^{\mathrm{cl}}\right)^{\mathrm{cl}} = I^{\mathrm{cl}}$.

order preservation $J^{\mathrm{cl}} \subseteq I^{\mathrm{cl}}$ for all ideals $J \subseteq I$.

If we have $I = I^{\mathrm{cl}}$ then the ideal is considered cl-closed.

An elementary example of a closure operation is the radical of an ideal.

Definition 1.1.2. Let R be a commutative ring. The *radical* of an ideal $I \subseteq R$ is

$$\sqrt{I} := \bigcap \{\mathfrak{p} \in \operatorname{Spec} R | I \subseteq \mathfrak{p}\} .$$

In particular this means that $\sqrt{I}$ contains all roots of elements of I in R.

It's obvious from the definition that all ideals with the same radical define the same zero set in $\operatorname{Spec} R$. In this sense the radical is the correct geometrical essence of an ideal.

Related to the radical is the following notion, which will become important later. A *primary* ideal I is an ideal where $xy \in I$ implies that either $x \in I$ or $y \in I$ or $x, y \in \sqrt{I}$.

Lemma 1.1.3. *If the radical $\sqrt{I}$ of an ideal I is a maximal ideal then I is primary.*

Proof. Otherwise for $xy \in I$, $x \notin \sqrt{I}$ the ideal $(x) + \sqrt{I}$ would properly contain $\sqrt{I}$. □

For a primary ideal I we often write that I is $\sqrt{I}$-primary to emphasize the radical.

All closure operations cl have the basic properties that the finite intersection of cl-closed ideals is cl-closed and that I^{cl} is the intersection of all cl-closed ideals containing I. Also closure operations are well-behaved regarding sums, i.e. $\left(\sum_\alpha I_\alpha^{\mathrm{cl}}\right)^{\mathrm{cl}} = \left(\sum_\alpha I_\alpha\right)^{\mathrm{cl}}$.

Another important classical closure operation is integral closure.

Definition 1.1.4. Let R be a commutative ring. The *integral closure* of an Ideal $I \subseteq R$ is

$$\overline{I} := \left\{ r \in R \middle| \exists n \in \mathbb{N}\ \forall i \in \{1, \ldots, n\}\ \exists a_i \in I^i : r^n + \sum_{i=1}^{n} a_i r^{n-i} = 0 \right\} .$$

For further reading on integral closure we recommend [33].

An important property that some closure operations have is persistence.

Definition 1.1.5. A closure operation cl is persistent if $\varphi(I^{\mathrm{cl}}) \subseteq (\varphi(I)S)^{\mathrm{cl}}$ for all ring homomorphisms $\varphi : R \longrightarrow S$ and all ideals $I \subseteq R$.

Lemma 1.1.6. *Integral closure is persistent.*

Proof. If we have an equation of integral dependence for $r \in R$ over $I \subseteq R$, we can map the equation via $\varphi : R \longrightarrow S$ to get an equation of integral dependence for $\varphi(r)$ over $\varphi(I)S$. □

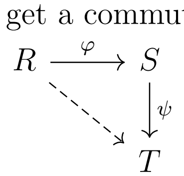
Persistence is closely related to having test rings for a closure operation.

A *class of test rings* to a closure operation cl is a subclass $\mathcal{T}$ of the category of rings such that $f \in I^{\mathrm{cl}} \subseteq R$ if and only if $\varphi(f) \in \varphi(I)T$ for all algebra homomorphisms $\varphi : R \longrightarrow T$ with $T \in \mathcal{T}$.

Lemma 1.1.7. *An ideal closure with a class of test rings is persistent.*

Proof. Let I be an ideal in a ring R and let $f \in I^{\mathrm{cl}}$ for a closure cl for which $\mathcal{T}$ is a class of test rings.

Let $\varphi : R \longrightarrow S$ be a homomorphism. We have to check if $\varphi(f) \in (\varphi(I)S)^{\mathrm{cl}}$. For this we take a test ring $T \in \mathcal{T}$ that allows a ring homomorphism $\psi : S \longrightarrow T$. In total we get a commutative diagram as follows.



The concatenation $\psi \circ \varphi$ makes T a ring for which $f \in I^{\mathrm{cl}}$ implies $\psi(\varphi(f)) \in \psi(\varphi(I))T$. Thus $\psi(\varphi(f)) \in \psi(\varphi(I)S)T$ and cl is persistent. □

Lemma 1.1.8. *Let R be a commutative ring. A class of test rings for the radical of an ideal $I \subseteq R$ is the class of all fields.*

Proof. For any map $\varphi : R \longrightarrow K$ into a field the ideal $\varphi(I)K$ is either 0 or K. In either case an element in the radical of I is mapped to the ideal generated by $\varphi(I)$.

On the other hand, because the radical is an intersection of prime ideals, for any element $f \notin \sqrt{I}$ we always have a prime ideal $\mathfrak{p}$ which contains $\sqrt{I}$ but does not contain f. By construction we have a map $R \longrightarrow R/\mathfrak{p}$ where f is not in the image of I, and $R/\mathfrak{p}$ is a field. □

Lemma 1.1.9. *Let R be a noetherian integral domain. Then a test class for integral closure is the class of discrete valuation domains.*

An even smaller test class for integral closure consists of just all valuation rings between $R/\mathfrak{p}$ and $\kappa(\mathfrak{p})$ for every minimal prime $\mathfrak{p}$.

Proof. [33, Proposition 6.8.2 and Proposition 6.8.3]. □

1.2 Tight closure

In the introduction to [28], Melvin Hochster and Craig Huneke mention that the notion of tight closure, which was introduced by them, facilitated some remarkable algebraic results. For example they were able to give a simpler proof to the theorem of Briançon-Skoda, which states that the integral closure of the n-th power of an ideal with n generators of a regular ring is contained in the ideal. In a survey article Winfried Bruns likens tight closure to be synonymous with characteristic p methods in commutative algebra [13]. In this section we recapitulate the important definitions.

To a ring R we denote with R^o the multiplicative system containing all elements which are not in any minimal prime ideal.

Definition 1.2.1. Let $N \subseteq M$ be modules over a noetherian ring R of characteristic $p > 0$. We define the *tight closure* of N in M as

$$N_M^* := \{x \in M | \exists c \in R^o, q \in \mathbb{N} : \forall p^e > q : cx^{p^e} \in N_M^{[p^e]}\}.$$

Here $N_M^{[p^e]} := \operatorname{im}(\mathrm{F}^{e*}(N) \longrightarrow \mathrm{F}^{e*}(M))$, where $\mathrm{F}^{e*}(M)$ is the e-th iteration of the Frobenius functor. Consequently x^{p^e} stands for the image of $x \in M$ under the e-th Frobenius pullback of M.

We defined tight closure for modules and not just for ideals, to keep the subsequent definition of test elements consistent with the exposition in [28]. However, the important case for us is when $N = I$ is an ideal of R and $M = R$ itself. In this case $I^{[p^e]}$ is the ideal generated by the p^e-th powers of a set of generators of I.

Lemma 1.2.2. *The tight closure of an ideal I in a noetherian ring of positive characteristic is contained in the integral closure of I.*

Proof. [28, Theorem 5.2] □

Definition 1.2.3. Let q be a power of p. An element $c \in R^o$ is called a *q-weak test element* if for every set of finitely generated modules $N \subseteq M$ we have that $x \in N_M^*$ if and only if $cx^{p^e} \in N_M^{[p^e]}$ for all $p^e \geq q$.

An element $c \in R^o$ is called a *completely stable q-weak test element* if its image in the completion of every local ring of R is a q-weak test element.

If $q = 1$ we just call c a *test element*, or *completely stable test element* respectively.

Theorem 1.2.4. *Let R be a noetherian ring of characteristic $p > 0$. If R is an algebra essentially of finite type over an excellent local ring or if the Frobenius endomorphism is a finite morphism then R has a completely stable test element.*

Proof. [29, Theorem 5.10 and Theorem 6.1]. □

The following notion was introduced by Craig Huneke and Adela Vraciu in [34].

Definition 1.2.5. Let $I \subseteq R$ be an ideal in a local noetherian ring of prime characteristic p. The *special tight closure* $I^{*\mathrm{sp}}$ is the ideal of all elements f for which there exists a $q_0 > 0$ such that $f^{q_0} \in \left(\mathfrak{m}I^{[q_0]}\right)^*$.

The most important point about special tight closure is that for any ideal I in a local excellent normal ring with perfect residue field we have $I^* = I + I^{*\mathrm{sp}}$. Note that despite its name special tight closure is not a closure operation and in general doesn't contain the ideal itself. Special tight closure will play a crucial role in our proof that tight closure is contained in the axes closure (Theorem 2.4.1).

Next we introduce solid closure, which is closely related to tight closure and was introduced by Melvin Hochster [27]. We recall the definition here because solid closure is susceptible to computations for which our algorithm to decide semistability can be put to use.

Originally solid closure was defined via solid modules which we will not do. Our definition follows the exposition in [3] and is equivalent to the original because of [27, Corollary 2.4 and Proposition 5.3].

Definition 1.2.6. Let R be a noetherian ring and $I := (f_1, \ldots, f_n) \subseteq R$. Then we define the *solid closure* $I^\star$ of I as follows: For an element $f_0 \in R$ we have $f_0 \in I^\star$ if and only if for every maximal ideal $\mathfrak{m} \subseteq R$ and every minimal prime $\mathfrak{q} \subseteq \hat{R_\mathfrak{m}}$ we have (with $R' := \hat{R_\mathfrak{m}}/\mathfrak{q}$)

$$H^{\dim R'}_{\mathfrak{m}R'}\left(R'[X_1, \ldots, X_n]/(f_1X_1 + \ldots + f_nX_n + f_0)\right) = 0.$$

Here $H^{\dim R'}_{\mathfrak{m}R'}$ denotes local cohomology, see [32].

The definition does not depend on the characteristic and in many cases in positive characteristic it coincides with tight closure.

Theorem 1.2.7. *Let I be an ideal in a noetherian ring R of characteristic $p > 0$. Then $I^* \subseteq I^\star$. If R contains a completely stable weak test element we have $I^* = I^\star$. In particular this is the case for rings essentially of finite type over an excellent local ring or rings such that the Frobenius endomorphism is finite.*

Proof. This is [27, Theorem 8.6]. □

For the computations connections a characterization of solid closure with fiber bundles is important. This is the topic of Section 3.7.

1.3 Continuous closure and axes closure

Definition 1.3.1. Let $R = \mathbb{C}[X_1, \ldots, X_n]/\mathfrak{a}$ be an algebra and let $X = \operatorname{Spec} R(\mathbb{C})$ be the space of $\mathbb{C}$-valued points of $\operatorname{Spec} R$. We can embed R into the ring of continuous functions $C^0(X)$. The *continuous closure* I^{cont} is the ideal consisting of all elements $f \in R$, that we can write as a linear combination $f = c_1 r_1 + \ldots + c_m r_m$ with continuous functions $c_i \in C^0(X)$ and $r_i \in I$.

In other words $I^{\text{cont}} := i^{-1}(i(I)C^0(X))$, if $i : R \hookrightarrow C^0(X)$.

It's important to note, that the continuous functions mentioned here refer to continuity in the euclidean topology, i.e. the topology with open sets the sets that contain for each point an open $\mathbb{C}$-ball. In other contexts in this thesis, when we talk about open sets we mostly refer to Zariski-topology, in which the open sets are the complements of (closed) algebraic sets.

The continuous closure is persistent, because for a ring homomorphism $R \longrightarrow S$, the induced map $(\operatorname{Spec} S)(\mathbb{C}) \longrightarrow (\operatorname{Spec} R)(\mathbb{C})$ is continuous. This induces naturally a ring homomorphism of the rings of continuous functions.

A more algebraic closure operation that is somewhat close to continuous closure is axes closure. There are multiple possible classes of test rings to define it, all of which have in common that their geometric spectrum consists of smooth irreducible curves (read *axes*) intersecting in one point.

Definition 1.3.2. Let K be an algebraically closed field. A *complete axes ring* is a ring of the form

$$K[[x_1, \ldots, x_n]]/(x_i x_j, i \neq j).$$

An *axes ring* is a reduced, one-dimensional, finitely generated ring R with $X = \operatorname{Spec} R$ the union of smooth irreducible curves that meet in one singular point $P \in X$ and for which the completion at P is isomorphic to a complete axis ring. An example for an axes ring is the *polynomial axes ring*, which is a ring of the form

$$K[x_1, \ldots, x_n]/(x_i x_j, i \neq j).$$

Definition 1.3.3. Let K be an algebraically closed field of characteristic 0 and R a finitely generated, noetherian K-algebra. The axes closure I^{ax} of an ideal $I \subseteq R$ is defined by either of the following classes of test rings.

1. Axes rings.
2. Complete axes rings.
3. Complete, excellent, seminormal, one-dimensional, local R-algebras.

For proofs of the equivalence and additional equivalent definitions of axes closure see [19, Theorem 4.1 and Corollary 4.2].

Lemma 1.3.4. *All ideals in axes rings over $\mathbb{C}$ are continuously closed.*

Proof. This is [8, Lemma 3.6]. □

Together with persistence of continuous closure this means that for an ideal $I \subseteq R$, R a finite type $\mathbb{C}$-algebra and a morphism $\varphi : R \longrightarrow T$ to an axes ring T we have that $f \in I^{\text{cont}}$ implies $f \in \varphi(I)T$. But axes rings are not test rings for continuous closure, because the converse is not true. An explicit example is $I = (u^2, v^2, uvx^2) \subset \mathbb{C}[u, v, x]$, where we have $uvx \in I^{\text{ax}}$, but $uvx \notin I^{\text{cont}}$ [19, Example 9.2].

Nonetheless we have the following relations.

Proposition 1.3.5. *In finite type $\mathbb{C}$-algebras we have:*

$$I \subseteq I^{cont} \subseteq I^{ax} \subseteq \overline{I} \subseteq \sqrt{I}.$$

Proof.

$I \subseteq I^{\mathbf{cont}}$ This is the extension property of closure operations.

$I^{\mathbf{cont}} \subseteq I^{\mathbf{ax}}$ Let $I \subseteq R$ be an ideal in a $\mathbb{C}$-algebra of finite type and let $\varphi : R \longrightarrow T$ be a homomorphism for T an axes ring. Because of persistence we have $\varphi(I^{\text{cont}}) \subseteq (\varphi(I)T)^{\text{cont}}$. Lemma 1.3.4 tells us $\varphi(I)T = \varphi(I)T^{\text{cont}}$ for axes rings T. This shows $I^{\text{cont}} \subseteq I^{\text{ax}}$.

$I^{\mathbf{ax}} \subseteq \overline{I}$ Discrete valuation domains, the test rings for integral closure are axes rings with one axis (see also [8, Remark 4.2]).

$\overline{I} \subseteq \sqrt{I}$ An integral relation $r^n + \sum_{i=1}^{n} a_i r^{n-i} = 0$ with $a_i \in I$ implies $r^n \in (a_1, \ldots, a_n) \subseteq I$.

□

This proposition will be extended in Chapter 2 of this thesis, where we show that $I^* \subseteq I^{\text{ax}}$ to give a connection between tight closure and continuous closure. This is a meaningful connection between tight closure and continuous closure, because I^{cont} and I^{ax} are so close together, they are even the same in some cases.

1.4 Monomial ideals

An ideal $I \subseteq R$ in a polynomial ring $R = K[x_1, \ldots, x_m]$ is called monomial if it has a monomial set of generators $f_1, \ldots, f_n$.

Monomial ideals make for good examples because they have some nice properties. They appear naturally for example as the initial ideals in Gröbner basis theory and also allow to compute some things for more general ideals, for example when it comes to the Hilbert polynomial.

Square-free monomial ideals are combinatorially relevant, as they correspond to simplicial sets. In this correspondence the variables of the ring correspond to the nodes of the simplicial set. The simplicial set contains a face if the product of the variables corresponding to its nodes is not in the ideal.

In this section we will recall some properties of monomial ideals with regards to ideal closure operations. One of the nice properties of monomial ideals is that we can visualize them easily as subsets of $\mathbb{N}^n$. The set $\Gamma_I = \{\gamma \in \mathbb{N}^m : z^\gamma \in I\}$ consists of all exponents of I.

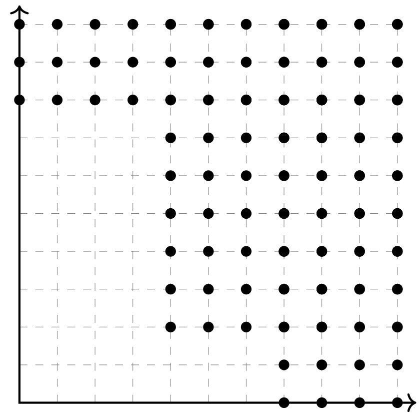


Figure 1.1: This figure shows the set Γ_I for $I = (x^7, x^4y^2, y^8)$.

Lemma 1.4.1. *The integral closure of a monomial ideal I in a polynomial ring over an algebraically closed field is generated by the monomials with exponents in the convex hull of Γ_I.*

Proof. Let the polynomial ring be $K[x_1, \ldots, x_m]$ and let I be generated by monomials $f_1, \ldots, f_n$. First of all the integral closure of a monomial ideal is again a monomial ideal [33, Proposition 1.4.2].

Any monomial r in the integral closure $\overline{I}$ has by definition an equation of integral dependence $r^k - \sum_{i=0}^{k-1} a_i r^i = 0$ with $a_i \in I$. We can assume that all the $a_i r^i$ are associated to r^k, because the equation especially has to hold for that monomial. Furthermore we can assume that r is nonzero and thus there is an a_i that is nonzero. Because the vector space generated by r^k is one-dimensional and all $a_i r^i$ are in it, we can also assume $r^k - a_i r^i = 0$, and after division by r^i we have $r^{k-i} = a_i$, where a_i is a product of monomials in i. Thus for a monomial to be in the integral closure is the same as finding natural numbers $a, b_1, \ldots, b_n$ such that $r^a = f_1^{b_1} \cdots f_n^{b_n}$. On exponent vectors ($e_r$ of r and e_{f_i} for the f_i) this is the same as finding positive rational numbers such that $e_r = b_1 e_{f_1} + \ldots + b_n e_{f_n}$. The exponents of this form are exactly the grid points in the convex hull of Γ_I. $\square$

A more detailed discussion of this can be found in the rest of [33, Section 1.4].

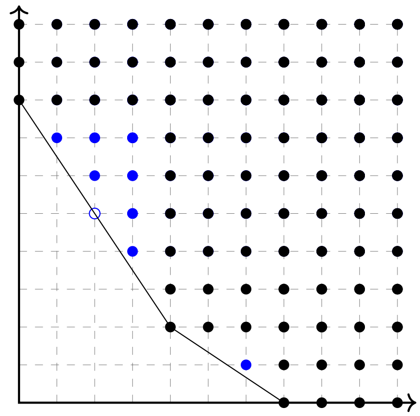


Figure 1.2: Here we see the convex hull (all marked points) and the interior of the convex hull (all marked points minus the hollow point) of Γ_I for $I = (x^7, x^4y^2, y^8)$.

The following lemma is built upon a lemma which we prove in Chapter 2.

Lemma 1.4.2. *The axes closure of an R_+-primary, monomial ideal I in a polynomial ring R over an algebraically closed field is generated by the union of I and the monomials with exponents in the interior of the convex hull of Γ_I.*

Proof. Let (V, ν) be a discrete valuation ring with homomorphism $\varphi : R \longrightarrow V$ and let $w \in \bar{\Gamma}^0$ be an element in the interior of the convex hull of Γ. Let $w = r_1e_1 + \ldots + r_ne_n$ be a convex combination for w, with the e_i the exponents of a generating family of I and $r_i \in \mathbb{Q}_{\geq 0}$ such that $\sum_{i=0}^{n} r_i > 1$. Let k be the least common multiple of the r_i. We have

$$\begin{aligned} k\nu(\varphi(z^w)) &= \nu(\varphi(z^{kw})) \\ &= \nu(\varphi(z^{\sum_{i=1}^{n} kr_ie_i})) \\ &= \sum_{i=1}^{n} kr_i\nu(\varphi(z^{e_i})) \\ &= k\sum_{i=1}^{n} r_i\nu(\varphi(z^{e_i})). \end{aligned}$$

Thus $\nu(\varphi(z^w)) > \min \nu(\varphi(z^{e_i}))$ and this implies because of Lemma 2.1.1, that $z^w \in I^{\text{ax}}$.

An element on the boundary of the convex hull, which is not in Γ, lies in a hyperplane which because of the dimension m is spanned by exactly m exponents $e_1, \ldots, e_m$ of the generating family of I. We can apply a module-finite transformation $z_i \mapsto z_i^{\delta_i}$ such that all monomials of the hyperplane have the same degree d. All ideal generators have degree at least d after this transformation, because otherwise the hyperplane would intersect the interior of the convex hull. [8, Lemma 4.6] deals with the fact that such a module-finite transformation does not affect the axes closure.

For any polynomial we can assume that all but one monomial is in I by enlarging the ideal if necessary. This means we can assume that we have a monomial on the boundary but not in I, which after a module-finite transformation is of the same degree as the generators of I. Now Lemma 2.1.2 tells us that $f \notin I^{\text{ax}}$.

Because the axes closure is contained in the integral closure, monomials outside of the convex hull can't be in the axes closure as of Lemma 1.4.1.

This shows the assertion by handling the three possible cases: monomials in the interior, monomials on the boundary and monomials outside the convex hull. □

1.5 Descent and reduction modulo p

Reduction modulo p is the main ingredient in applying characteristic p methods to characteristic 0 cases. In order to perform reduction modulo p, we need to fix descent data.

We will use descent in the sense of [30, 2.1]. We start with a finitely generated noetherian algebra R over a field K and some finite number of finitely generated algebraic structures (let's denote one exemplarily as M here, since most of these structures will be modules). The general idea is to construct a finitely generated $\mathbb{Z}$-algebra A and an A-free A-algebra R_A such that

- There is a prime ideal $\nu \subset A$ such that $A_\nu/(\nu A_\nu)$ is isomorphic to a subfield of K.
- $R_A \otimes_A K \cong R$.

- Important properties of R are also true for R_A. For example we may want to preserve normality of R in R_A.
- To every structure M we have an M_A such that $M_A \otimes_A K \cong M$.
- Important properties of M continue to hold for M_A. In particular we generally want to represent a free module M by a free module M_A.

For every prime ideal μ in A we get a field $\kappa(\mu) = A_\mu/(\mu A_\mu)$. If we tensor R_A with one of these fields $\kappa := \kappa(\mu)$ we get $R_\kappa := R_A \otimes_A \kappa$.

Example 1.5.1. In many cases K will be a field of characteristic 0. This is the setting for reduction modulo p, which is used to relate rings in equal characteristic 0 to rings in positive characteristic. Many properties hold for R if and only if they hold for $R_{\kappa(\mu)}$ for all μ in a Zariski-dense open subset of $\operatorname{MaxSpec} A$.

We will construct A from $\mathbb{Z}$ by adjoining the necessary elements of K. This will be a finite number of elements since all the structures are finitely generated.

Since R is finitely generated and noetherian, it is of the form $R = K[x_1, \dots, x_n]/(f_1, \dots, f_m)$. We start by adjoining the coefficients of the f_i to get A_0. Now with $R_{A_0} = \mathbb{Z}[x_1, \dots, x_n]/(f_1, \dots, f_m)$ we are already in a situation, where $R_{A_0} \otimes_{A_0} K \cong R$. The same holds for all subrings of K containing A_0. We adjoin to A_0 all coefficients in M and all the finite number of inverses to get a larger $\mathbb{Z}$-algebra $A \supseteq A_0$ which makes M_A free.

We represent an ideal $I \subseteq R$ by further enlarging the $\mathbb{Z}$-algebra $A \subseteq R$, such that generators for I are in R_A. If we want to check containment of a specific element $f \in R$ we also add an element $f_A \in R_A$ with $f_A \otimes 1 = f$.

This allows us to define tight closure for rings in characteristic 0.

Definition 1.5.2. In the setting as above we define $u \in I^{K*}$ if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq R$ such that $u_\kappa \in I_\kappa^*$ in the fibers R_κ for κ in a dense open subset of $\operatorname{Spec} A$.

It is worth noting that the tight closure I^{K*} does not depend on the choice of A. That there exists such a subalgebra of R is equivalent to the same being true for all sufficiently large $\mathbb{Z}$-subalgebras of R.

Chapter 2

Continuous and tight closure

We show that for excellent, normal equicharacteristic rings with perfect residue fields the tight closure of an ideal is contained in its axes closure. First we prove this for rings in characteristic p. This is achieved by using the notion of special tight closure established by Huneke and Vraciu.

By reduction to positive characteristic we show that the containment of tight closure in axes closure also holds in characteristic 0. From this we deduce that for a normal ring of finite type over $\mathbb{C}$ the tight closure of a primary ideal is inside its continuous closure.

The containment of tight closure inside the continuous closure is strict. A basic result of tight closure theory tells us that tight closure of an ideal in a regular ring is the ideal itself, but in the polynomial ring $\mathbb{C}[X,Y]$ we have $X^2Y^2 \in (X^3,Y^3)^{\mathrm{cont}}$.

The inclusion result doesn't hold for nonnormal domains. In the ring $R = K[X,Y,Z]/(X^2 - YZ^2)$ we have $\left(\frac{X}{Z}\right)^2 = Y$ and its normalization is $K[U,Z]$ with $U = \frac{X}{Z}$. Therefore $X \in (Z)^*$ in R. The map

$$R \longrightarrow K[X,Z]/(X^2 - Z^2) = K[X,Z]/(X+Z)(X-Z), \qquad Y \mapsto 1, X \mapsto X, Z \mapsto Z$$

shows however that X does not belong to the axes closure of Z and hence for $K = \mathbb{C}$ not to the continuous closure.

2.1 Axes closure and valuations

We will test axes closure with complete, excellent, local, one-dimensional, seminormal rings as Definition 1.3.3 says we can do. These are very similar to complete axes rings.

In fact, according to [19, Theorem 3.3], these rings can be constructed as follows: Let $\mathfrak{m}$ be the unique maximal ideal of R and $k = R/\mathfrak{m}$ the residue field. Then R is isomorphic to a subring of the product ring $\prod_{i=1}^{n} V_i$, where $(V_i, \mathfrak{m}_i, L_i)$ are discrete valuation rings whose residue fields are finite extension fields of k.

In this product, R is the subring of all elements $(v_1, \dots, v_n) \in \prod_{i=1}^{n} V_i$ such that the $v_i \mod \mathfrak{m}_i$ are congruent to the same element α in k. The units of R are exactly the elements congruent to a non-zero α.

We define functions $\mathrm{val}_i : R \longrightarrow \mathbb{N} \cup \{\infty\}, (v_1, \dots, v_n) \mapsto \mathrm{val}_{V_i}(v_i)$, where val_{V_i} is the valuation of V_i. Let $\mathrm{val}_i(I) := \min\{\mathrm{val}_i(f) : f \in I\}$.

Over an algebraically closed field of equal characteristic every complete, local, one-dimensional seminormal ring is isomorphic to a complete axes ring and vice versa, a result for which in [19, Proposition 3.4] Bombieri [2] is credited. For complete axes rings, [8, Corollary 3.4] gives a valuative criterion for ideal membership which can be extended with the same methods as follows.

Lemma 2.1.1. *Let R be a complete, excellent, local, one-dimensional, seminormal ring, $I \subseteq R$ an ideal in R and $f \in R$. If $\mathrm{val}_i(f) > \mathrm{val}_i(I)$ for all $1 \leq i \leq n$, then $f \in I$.*

Proof. For every $i \in \{1, \dots, n\}$ there is, by definition, a $g_i \in I$ with $\mathrm{val}_i(g_i) = \mathrm{val}_i(I)$. Let x_i be the generator of the maximal ideal $\mathfrak{m}_i$. We can interpret it as an element $(0, \dots, 0, x_i, 0, \dots, 0)$ of R, where x_i is in the i-th position. The product $g_i \cdot x_i$ has order $\mathrm{val}_i(I) + 1$ in V_i. All elements of higher or equal order are divisible by $g_i \cdot x_i$. Hence there exists $h_i \in V_i$ such that $f_i = g_i x_i h_i$, where f_i is the ith component of f. The element $x_i h_i$ has positive order, so its value modulo $\mathfrak{m}_i$ is 0 and thus there exists the global element

$$y_i = (0, \dots, 0, x_i h_i, 0 \dots, 0) \in R\,.$$

So we can write

$$f = (f_1, \dots, f_n) = \sum_{i=1}^{n} g_i y_i$$

and hence $f \in I$. □

Note that this is also true if $\text{val}_i(f) = \text{val}_i(I) = \infty$ for some i, we thus consider $\infty > \infty$ for the purpose of this lemma.

Lemma 2.1.2. *Let $R = K[x_1, \ldots, x_m]$ be a polynomial ring over a field and let $I = (f_1, \ldots, f_n) \subset R$ be an ideal, with all generators having the same positive degree d. For all other polynomials f of degree d we have $f \in I^{ax} \Leftrightarrow f \in I$.*

Proof. We always have $f \in I \Rightarrow f \in I^{\text{ax}}$ because of the extension property of closures, so we only have to prove the other direction.

Recall the identity theorem for polynomials [42, Satz 54.7] saying that there is a $k \in \mathbb{N}$ and points $\alpha_j = (\alpha_{j1}, \ldots, \alpha_{jm}) \in K^m$ for all $j \in \{1, \ldots, k\}$ such that any pair of polynomials $f, g \in R$ with degree $\leq d$ are the same if and only if $f(\alpha_j) = g(\alpha_j)$ for all $j \in \{1, \ldots, k\}$.

For any $\alpha_{1l}, \ldots, \alpha_{kl} \in K$ we define a homomorphism

$$\varphi : K[z_1, \ldots, z_m] \longrightarrow K[x_1, \ldots, x_k]/(x_i x_j, i \neq j), z_l \mapsto \alpha_{1l} x_1 + \ldots + \alpha_{kl} x_k.$$

We apply φ to a monomial $g \in R$ of degree d and get

$$\begin{aligned} \varphi(g) &= \varphi\left(\prod_{i=1}^{m} z_i^{r_i}\right) \\ &= \sum_{j=1}^{k} \left(\prod_{i=1}^{m} \alpha_{ji}^{r_i}\right) x_j^d \\ &= \sum_{j=1}^{k} g(\alpha_j) x_j^d. \end{aligned}$$

Because φ is a ring homomorphism the the equation $\varphi(g) = \sum_{j=1}^{k} g(\alpha_j) x_j^d$ holds for any polynomial g of degree d.

Take an $f \in I^{\text{ax}}$. Because $S = K[x_1, \ldots, x_k]/(x_i x_j, i \neq j)$ is a polynomial ring of axes this implies that $\varphi(f) \in \varphi(I)$, i.e. that $\varphi(f)$ is a S-linear combination of the $\varphi(f_i)$. Because both $\varphi(f)$ and the $\varphi(f_i)$ have degree d they are of the form $\sum_{j=1}^{k} f(\alpha_j) x_j^d$ and $\sum_{j=1}^{k} f_i(\alpha_j) x_j^d$. Because we have the same degree on the left and right side we get $\varphi(f) = \sum_{i=1}^{n} c_i \varphi(f_i)$ with $c_i \in K$. With a coefficient comparison we get $f(\alpha_j) = \sum_{i=1}^{n} c_i f_i(\alpha_j)$ for all $j \in \{1, \ldots, k\}$. We can apply the identity theorem and choose the α_j such that $f = \sum_{i=1}^{n} c_i f_i$. □

2.2 Smooth and étale

In this thesis both smoothness of schemes and smoothness of morphisms play an important role.

Definition 2.2.1. A morphism of schemes $f : Y \longrightarrow X$ of finite type over a field K is called *smooth (of relative dimension n)* if

1. f is flat,
2. For irreducible components X' and Y' with $f(Y') \subseteq X'$ we have
$$\dim Y' = \dim X' + n,$$
3. For all points $a \in Y$ we have $\dim_{K(a)}(\Omega_{Y/X} \otimes K(a)) = n$.

We also say that Y is smooth over X.

Of special importance to us is the case when $X = \operatorname{Spec} K$ and Y is integral. In this case the first condition is always true and the other two become that $\Omega_{Y/K}$ is locally free of rank equal to $\dim Y$. In this case we also just call Y smooth.

$\Omega_{Y/X}$ is the sheaf of Kähler differentials, which over an affine subset $\operatorname{Spec} S \subset Y$ (with $f(\operatorname{Spec} S) \subseteq \operatorname{Spec} R \subseteq X$) is isomorphic to a module $\Omega_{S/R}$ generated by forms $\{ds | s \in S\}$ modulo the relations given by derivation: $d(s + t) = ds + dt$, $d(st) = sdt + tds$ for $s, t \in S$ and $dr = 0$ for $r \in R$. Further treatment of Kähler differentials can be found in [26, Section 2.8].

Note that in the case $X = \operatorname{Spec} K, Y = K[x_1, \ldots, x_n]/\mathfrak{a}$ we always have $\dim_{K(a)}(\Omega_{Y/K} \otimes K(a)) \geq \dim Y$ for every point $a \in Y$. This can be shown with Noether normalization and transcendence degrees. Consider also [26, Theorem II.8.6A].

Definition 2.2.2. The *Jacobian* to a family $f_1, \ldots, f_m \in K[x_1, \ldots, x_n]$ is
$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

Here $\frac{\partial f_i}{\partial x_j}$ denotes the formal partial derivative of f_i with respect to x_j.

The Jacobian can be used to determine smoothness for affine algebras. If $S = R[x_1, \ldots, x_n]$ then $\Omega_{S/R} = \bigoplus_{i=1}^n S dx_i$. If $T = S/I$, where $I = (f_1, \ldots, f_m)$ then we have $\Omega_{T/R} = \Omega_{S/R}/M$, the submodule

$$M \subseteq \bigoplus_{i=1}^{n} S dx_i$$

being generated by the entries in $(dx_1, \ldots, dx_n) \cdot J^t$.

Thus for closed points a we have $\Omega_{T/R} \otimes K(a) \cong \operatorname{coker} J(a)^t$, where $J(a)$ stands for the matrix over $K(a)$ where we input a in every entry of J.

Let $Y = \operatorname{Proj}\left(K[x_0, \ldots, x_n]/(f_1, \ldots, f_m)\right)$, with the $f_1, \ldots, f_m$ homogeneous and let C_Y^* be the affine cone over Y without the origin, with the projection $p : C_Y^* \longrightarrow Y$. Then there is an exact sequence

$$0 \longrightarrow \Omega_{Y/K} \longrightarrow \Omega_{\widetilde{C_Y/K}(-1)} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

induced by the Euler derivation. Thus $\operatorname{rank}(\Omega_{C_Y^*/K}) = \operatorname{rank}(\Omega_{Y/K}) + 1$. This allows us to check smoothness on the affine cone.

The following Lemma is known as the Jacobian Criterion for smoothness.

Lemma 2.2.3. *Let $X = \operatorname{Proj}\left(K[x_0, \ldots, x_n]/(f_1, \ldots, f_m)\right)$ be a normal projective scheme of finite type over an algebraically closed field K. Then X is smooth if and only if at every closed point $a \in C_X^*$ we have*

$$\operatorname{rank}\begin{pmatrix} \frac{\partial f_1}{\partial x_0}(a) & \ldots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_0}(a) & \ldots & \frac{\partial f_m}{\partial x_n}(a) \end{pmatrix} \geq \operatorname{codim}_{\mathbb{P}^n} X.$$

Proof. The point a projects to a closed point $a' \in X$.

We compute that $\operatorname{rank} J(a) \geq \operatorname{codim}_{\mathbb{P}^n} X$ if and only if

$$\begin{aligned} \dim_{K(a')} \Omega_{X/K} \otimes K(a') &= \dim \Omega_{C_X^*/K} \otimes K(a) - 1 \\ &= \operatorname{rank} \operatorname{coker} J(a)^t - 1 \\ &= \operatorname{rank} \ker J(a) - 1 \\ &= n + 1 - \operatorname{rank} J(a) - 1 \\ &= n - \operatorname{rank} J(a) \\ &\leq n - \operatorname{codim}_{\mathbb{P}^n} X \\ &= \dim X. \end{aligned}$$

Because we always have $\dim \Omega_{X/K} \otimes K(a') \geq \dim X$ this means that X is smooth at a'. Because K is algebraically closed, the behaviour on closed points already determines the behaviour on all points.

Note that we would technically only have to check the criterion on one point of the cone above each closed point of X. $\square$

In the setting of the previous Lemma let $t = \operatorname{codim}_{\mathbb{P}^n} X$. We call the ideal $I_t(J)$ generated by the t-minors of J the *Jacobian ideal* of X.

Lemma 2.2.4. *Let* $X = \operatorname{Proj}(K[x_0, \ldots, x_n]/(f_1, \ldots, f_m))$ *be a normal projective scheme of finite type over an algebraically closed field* K*. Then* X *is smooth if and only if* $V_+(I_t(J)) = \emptyset$*.*

Proof. By construction a point $a \in V_+(I_t(J))$ is a point for which $\operatorname{rank} J(a) < t = \operatorname{codim}_{\mathbb{P}^n} X$, thus there is such a point if and only if X is not smooth. $\square$

2.3 Étaleness descends

The lemma in this section states the fact that for morphisms being smooth and being étale descend. (For descent see Section 1.5) Remember that being étale for schemes of finite type over a field means being smooth of relative dimension 0.

Standard smoothness will play a role, which is defined as follows.

Definition 2.3.1. Let $S = R[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be an R-algebra, R a ring and $n \geq m$. Then S is called *standard smooth* if and only if

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_m} \end{pmatrix}$$

is invertible in S.

Lemma 2.3.2. *Let* L *be a field of characteristic* 0*. Consider an* L*-algebra homomorphism* $f : S \longrightarrow T$ *of finitely generated* L*-algebras.*

As described in Example 1.5.1 we can descend this, i.e.: We can find a finitely generated $\mathbb{Z}$*-subalgebra* A *of* L*, finitely generated* A*-algebras* S_A *and* T_A*. Also we find an* A*-algebra homomorphism* $f_A : S_A \longrightarrow T_A$ *such that* $S = S_A \otimes_A L$*,* $T = T_A \otimes_A L$ *and* $f = f_A \otimes_A L$*.*

Furthermore we get:

1. *If f is smooth, we can choose A such that f_A is smooth.*

2. *If f is étale, we can choose A such that f_A is étale.*

Proof. The process of descending diagrams of ring homomorphisms is described in [30, 2.1.18]. So we only have to show that we can descend the smoothness and étaleness.

1. There are finitely many elements $g_i \in T, i \in I = \{1, \dots, r\}$ such that $D(g_i)$ cover $\operatorname{Spec} T$ and that T_{g_i} is standard smooth over S [44, Tag 00TA]. We can assume that T_A contains the g_i and also the coefficients with which they generate the unit ideal so we can also assume that the $D(g_i)$ cover $\operatorname{Spec} T_A$. If we show that $(T_A)_{g_i}$ is standard smooth over S_A for all $i \in I$ then T_A will be smooth over S_A. In the following we will just assume that T is standard smooth.

   This means that there exist $f_1, \dots, f_c$ in $S[Y_1, \dots, Y_n]$ such that $T = S[Y_1, \dots, Y_n]/(f_1, \dots, f_c)$ and that the determinant g of the Jacobian of the f_i is invertible in T. We will add the finitely many coefficients of the f_i and the inverse of g to A. Then we have the same representation for T_A which makes it standard smooth over S_A.

2. We will use the characterization that a ring homomorphism $S \longrightarrow T$ is étale iff it is smooth and $\Omega_{T/S} = 0$. Smoothness descends by (1).

   The sequence $L \longrightarrow S \longrightarrow T$ induces the exact sequence of T-modules

   $$\Omega_{S/L} \otimes_S T \longrightarrow \Omega_{T/L} \longrightarrow \Omega_{T/S} \longrightarrow 0.$$

   Here this means that we have a surjective map $\Omega_{S/L} \otimes_S T \twoheadrightarrow \Omega_{T/L}$.

   We have a commutative diagram as follows.

   $$\begin{array}{ccc} T_A & \longrightarrow & T \\ \uparrow & & \uparrow \\ S_A & \longrightarrow & S \\ \uparrow & & \uparrow \\ A & \longrightarrow & L \end{array}$$

According to [23, Proposition 0.20.5.5] there is a canonical isomorphism $\Omega_{T/L} \cong \Omega_{T_A/A} \otimes_A L$. By the same proposition we have

$$\Omega_{S/L} \otimes_S T \cong \left(\Omega_{S_A/A} \otimes_A L\right) \otimes_S T \cong \left(\Omega_{S_A/A} \otimes_{S_A} T_A\right) \otimes_A L.$$

The second isomorphism is just a standard property of base change.

After choosing a basis each we can write the surjective map

$$\left(\Omega_{S_A/A} \otimes_{S_A} T_A\right) \otimes_A L \twoheadrightarrow \Omega_{T_A/A} \otimes_A L$$

as a matrix multiplication. We can, by further enlargement of A, assume that the finitely many coefficients occuring in the matrix are contained in A. Then this matrix also describes the induced homomorphism $\Omega_{S_A/A} \otimes_{S_A} T_A \twoheadrightarrow \Omega_{T_A/A}$ which thus is surjective.

If we put that into the respective exact sequence of T_A-modules

$$\Omega_{S_A/A} \otimes_{S_A} T_A \longrightarrow \Omega_{T_A/A} \longrightarrow \Omega_{T_A/S_A} \longrightarrow 0$$

the rightmost module Ω_{T_A/S_A} is zero, thus $S_A \longrightarrow T_A$ is étale.

□

The lemma will be important for the characteristic 0 tight closure inclusion theorem, Theorem 2.5.1.

2.4 Positive characteristic

We prove the main result in positive characteristic using a result of Huneke and Vraciu about the decomposition of tight closure in normal rings. Namely, in [34, Theorem 2.1] they show that for a local, excellent, normal ring of characteristic p the tight closure of any ideal can be written as

$$I^* = I + I^{*\mathrm{sp}}.$$

Here $I^{*\mathrm{sp}}$ denotes the *special tight closure*, which is the ideal of all elements f for which there exists a $q_0 > 0$ such that $f^{q_0} \in \left(\mathfrak{m} I^{[q_0]}\right)^*$.

Theorem 2.4.1. *Let R be an excellent, normal ring of characteristic p such that the residue field at every maximal ideal is perfect. Let I be an ideal which is primary to a maximal ideal. Then $I^* \subseteq I^{ax}$.*

Proof. Let I be primary to the maximal ideal $\mathfrak{m}$. [19, Theorem 4.1(3)] says that we can test the containment in the axes closure on complete, excellent, local, one-dimensional, seminormal rings S with a ring homomorphism $\varphi : R \longrightarrow S$. Thus we take such a ring and test whether the image of an element $f \in I^*$ is in $IS := \varphi(I)S$.

Let $\mathfrak{n}$ be the unique maximal ideal of S. If $\mathfrak{m} \neq \varphi^{-1}(\mathfrak{n})$ then $\mathfrak{m} \not\subseteq \varphi^{-1}(\mathfrak{n})$ and so I extends to the unit ideal in S and then $f \in IS$. So we can assume that $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$. Then φ factors through $R_\mathfrak{m}$. We also have $f \in I^* R_\mathfrak{m} \subseteq (IR_\mathfrak{m})^*$ by the persistence of tight closure. This means it suffices to show $f \in IS$ in the case that R is local (in effect we exchange R with $R_\mathfrak{m}$) and $R \longrightarrow S$ is a local homomorphism. This together with the other conditions on R gives us a decomposition $I^* = I + I^{*\mathrm{sp}}$ [34, Theorem 2.1]. Thus we write $f = g + h$, where $g \in I$ and $h \in I^{*\mathrm{sp}}$.

By the definition of special tight closure there exists a $q_0 > 0$ such that $h^{q_0} \in \left(\mathfrak{m} I^{[q_0]}\right)^*$.

As above we can write S as the subring of a product $\prod_{i=1}^n V_i$. Let $p_i : S \longrightarrow V_i$ be the i-th projection. Let us denote $I_i := IV_i$ and $h_i := p_i(\varphi(h))$ and $\mathfrak{a}_i = \mathfrak{a} V_i$ with $\mathfrak{a} = \mathfrak{m} S$. Note that $\mathrm{val}_{V_i}(\mathfrak{a}_i) > 0$ as $\mathfrak{a}_i$ contains no units.

In a discrete valuation domain we have $J = \overline{J} = J^*$ for all ideals J. Thus,

$$h^{q_0} \in \left(\mathfrak{m} I^{[q_0]}\right)^* \Rightarrow h_i^{q_0} \in \left(\mathfrak{a}_i I_i^{[q_0]}\right)^* = \mathfrak{a}_i I_i^{[q_0]},$$

by persistence of tight closure [30, Theorem 1.4.13], as S and V_i have completely stable weak test elements. For the valuation $\mathrm{val} := \mathrm{val}_{V_i}$ on V_i this gives the following inequalities.

$$\begin{aligned} \mathrm{val}(h_i^{q_0}) &\geq \mathrm{val}(\mathfrak{a}_i I_i^{[q_0]}) \\ \Rightarrow q_0 \,\mathrm{val}(h_i) &\geq \mathrm{val}(\mathfrak{a}_i) + q_0 \,\mathrm{val}(I_i) \\ \Rightarrow \mathrm{val}(h_i) &\geq \frac{1}{q_0}\mathrm{val}(\mathfrak{a}_i) + \mathrm{val}(I_i) \\ \Rightarrow \mathrm{val}(h_i) &> \mathrm{val}(I_i). \end{aligned}$$

The last inequality (which might be $\infty > \infty$) gives

$$\mathrm{val}_i(\varphi(h)) > \mathrm{val}_i(IS).$$

By Lemma 2.1.1 the membership $\varphi(h) \in IS$ follows. Thus $h \in I^{\mathrm{ax}}$ and $f = g + h \in I^{\mathrm{ax}}$. □

Theorem 2.4.2. *Let I be an ideal in a normal domain R of finite type over a perfect field K of characteristic p. Then $I^* \subseteq I^{ax}$.*

Proof. Let $f \in I^*$. We work with ring of axes of finite type over K. So let A be a seminormal onedimensional ring of finite type over K with only one meeting point corresponding to a maximal ideal $\mathfrak{n}$ of A and let $\varphi : R \longrightarrow A$ be a ring homomorphism. Then $\mathfrak{m} := \varphi^{-1}(\mathfrak{n})$ is a maximal ideal of R. The residue field at $\mathfrak{m}$ is perfect and by the persistence of tight closure we also have $f \in IR_{\mathfrak{m}}$. So we work with the factorization $R_{\mathfrak{m}} \longrightarrow A$ and since all conditions for special tight closure hold true in $R_{\mathfrak{m}}$ we can proceed as in the previous proof. □

Example 2.4.3. In the proof of the Theorem we use that the special tight closure gives us elements which are in some way deeper inside the tight closure. The inner integral closure $I_{>1}$ also measures "deeper" elements in a similar way and because $I_{>1} \subseteq I^{\text{ax}}$ it seems one might think to prove the statement also for inner integral closure instead of axes closure. For $\mathfrak{m}$-primary ideals this is possible but in general it doesn't work. We have the following simple example which was given to us by Neil Epstein that shows $I^{*\text{sp}} \nsubseteq I_{>1}$: Let $R = K[[X, Y]], I = (X), \mathfrak{m} = (X, Y)$. Then $XY \in I^{*\text{sp}}$ (we can put $c = q_o = 1$ in the definition of special tight closure), but $XY \notin I_{>1}$.

2.5 Equal characteristic 0

In the following we show a result similar to Theorem 2.4.2 for a ring R of finite type over a field K of characteristic 0. For this we use the notion of tight closure in characteristic 0 developed in several variants in [28] and in more detail in [30].

The general idea is that of reduction modulo p as we elaborated on in Example 1.5.1.

We want to check whether an element is in the axes closure and for this we need a suitable version of axes ring. The finitely generated version for axes rings in [19, Theorem 4.1(6)] is that of finitely generated étale extensions S of polynomial axes rings

$$T = L[X_1, \ldots, X_n]/(X_iX_j, i \neq j)$$

over the algebraic closure L of K.

Theorem 2.5.1. *Let I be an ideal in a geometrically normal ring R of finite type over a field K of characteristic* 0. *Then $I^{K*} \subseteq I^{ax}$.*

Proof. To test whether an element $f \in I^{K*}$ is in the axes closure, we take a K-algebra homomorphism $\varphi : R \longrightarrow T$ to a finitely generated étale extension T of a polynomial axes ring

$$S = L[X_1, \ldots, X_n]/(X_i X_j, i \neq j)$$

over the algebraic closure L of K (these rings characterize axes closure due to [19, Theorem 4.1(6)]). We may assume that T is defined and étale over a polynomial ring of axes over a finite extension field K' of K. Moreover, we may assume that φ is defined over K'. We replace K by K' and denote it K again.

We perform the reduction described above such that we will get a finitely generated $\mathbb{Z}$-Algebra A and the following diagram of finitely generated A-algebras.

$$\begin{array}{ccc} R_A & \xrightarrow{\varphi_A} & T_A \\ & & \uparrow{\scriptstyle \psi_A} \\ & & S_A \end{array}$$

By further shrinking as described in Lemma 2.3.2, using the characterization with Kähler differentials, we make sure that ψ_A is étale. We also choose A so that R_κ will be normal for a dense open subset of fibers κ ([30, Propositon 2.3.17]).

We take a closed fiber R_κ of $A \longrightarrow R_A$ with characteristic p for which $f_\kappa \in I_\kappa^*$. By Theorem 2.4.1 we have $f_\kappa \in I_\kappa^{\mathrm{ax}}$.

Going from R_A to R_κ is a base change, i.e. done by tensoring with κ. Thus the induced morphism ψ_κ is étale. The inclusion $f_\kappa \in I_\kappa T_\kappa$ follows.

Now assume that $f \notin IT$. Then $(f, IT)/IT$ is nonzero and free when localized at a single element. Thus for a dense open subset of fibers it will still be nonzero. But we have shown $f_\kappa \in I_\kappa T_\kappa$ for the fibers over a dense open subset, so we have a contradiction. It follows that $f \in IT$. □

Theorem 2.5.2. *For an ideal I primary to a maximal ideal in a normal, affine $\mathbb{C}$-algebra R we have $I^* \subseteq I^{cont} = I^{ax}$.*

Proof. This follows immediately from [19, Corollary 7.14] and Theorem 2.5.1. □

Example 2.5.3. Let $R = K[X,Y,Z]/(X^m + Y^m + Z^m)$, $m \geq 2$, and $I = (X,Y)$. Assume that K is algebraically closed and that the characteristic does not divide m. Then $Z^2 \in (X,Y)^*$ but $Z \notin (X,Y)^*$. These are well known results with several proofs. We revover the second part by showing that Z is not in the axes closure of (X,Y). We write

$$X^m + Z^m = (X - \xi_1 Z) \cdots (X - \xi_m Z)$$

with some roots of unity $\xi_1, \ldots, \xi_m$. We have a map

$$R \longrightarrow S := K[U,V]/(UV), X \longrightarrow \frac{\xi_2 U - \xi_1 V}{\xi_2 - \xi_1}, Z \longrightarrow \frac{U - V}{\xi_2 - \xi_1}, Y \longrightarrow 0\,.$$

This is well defined as we have $U = X - \xi_1 Z$ and $V = X - \xi_2 Z$ and the map sends $X^m + Y^m + Z^m = (X - \xi_1 Z)(X - \xi_2 Z) \cdot Q + Y^m$ to a multiple of UV.

Modulo $IS = \left(\frac{\xi_2 U - \xi_1 V}{\xi_2 - \xi_1}\right)$, the ring S becomes $S' \cong K[V]/(V^2)$ but the image of Z in S' is not 0, thus $ZS \notin IS$.

This means, that there is an axes ring for which Z is not in the image of the ideal, thus Z can't be in the axes closure of I. Thus, by Theorem 2.5.1, it can also not be in the tight closure, since R is normal.

Example 2.5.4. The containment of tight closure can not be extended to one-dimensional test rings with regular components, we can not drop the condition that the curves meet transversally. To see this, let $R = K[X,Y,Z]/(X^3 + Y^3 - Z^3)$ with a field K of characteristic $\neq 3$ containing a primitive third root of unity ζ and let again $I = (X,Y)$. Then $Z^2 \in (X,Y)^*$. However, if we go modulo Y we get

$$A = K[X,Z]/(X^3 - Z^3) = K[X,Z](X - Z)(X - \zeta Z)(X - \zeta^2 Z)$$

and its spectrum consists of three lines lying in a plane meeting in one point. In this ring we have $Z^2 \notin IA = (X)$.

We conclude with some remarks and questions.

Remark 2.5.5. Our main results Theorem 2.4.2 and Theorem 2.5.1 are also true under the weaker condition that R is a domain of finite type over a field with the property that its seminormalization is already normal. This rests upon the fact that any ringhomomorphism to an axes ring factors through the seminormalization. This property means basically that the ring is unibranched in the sense that the completion is a domain.

Remark 2.5.6. One might ask whether the used result of Huneke and Vraciu is also true under weaker conditions. Is it true for a complete domain? Is it true for an excellent local analytically irreducible domain? Is it true without the assumption that the residue field is perfect? The example $K[X,Y,Z]/(X^2-YZ^2)$ localized at $(X,Y-1,Z)$ mentioned in the introduction shows that it can not be true for local domains essentially of finite type without any further assumption. This follows from the proof of Theorem 2.4.1, but can also be seen directly. We write $W = Y - 1$ and work in the ring $S = K[X,W,Z]/(X^2 - WZ^2 - Z^2)$ localized at (X,W,Z). We have $X \in (Z)^*$. We claim that it is not possible to write $X = ZH + X - ZH$ with $H \in S$ and $X - ZH$ inside the special tight closure of (Z).

To understand the special tight closure of (Z) one has to look at the tight closure $(ZZ^{q_0}, WZ^{q_0}, XZ^{q_0})^*$ for various $q_0 = p^{e_0}$. The tight closure can be computed in the normalization $K[U,Z]$, the extended ideal is

$$(Z^{q_0+1}, (U^2-1)Z^{q_0}, UZ^{q_0+1}) = (Z^{q_0+1}, (U^2-1)Z^{q_0}) \,.$$

Now assume that $X - ZH \in (Z)^{sp*}$, so

$$X^q - Z^{q_0}H^{q_0} \in (Z^{q_0+1}, (U^2-1)Z^{q_0})$$

for some q_0. Then using $X = ZU$ and cancelling with Z^{q_0} gives

$$U^{q_0} - H^{q_0} \in (Z, U^2-1) \,.$$

Writing $H = ZA + WB + P(X)$ (with $A, B \in R$ and $P(X)$ a polynomial in X) shows that the containment is equivalent to $U^{q_0} - P_0^{q_0} \in (Z, U^2-1)$, which is for odd characteristic a contradiction.

Remark 2.5.7. Is tight closure for normal affine $\mathbb{C}$-algebras always inside the continuous closure? Is tight closure for normal noetherian rings always inside the axes closure?

Remark 2.5.8. As mentioned in the introduction, J. Kollár has given in [37] an algebraic characterization of continuous closure, meaning an algebraically defined closure operation for varieties which coincides with the continuous closure if the base field is $\mathbb{C}$. A natural question is whether tight closure of a normal domain, in particular in positive characteristic, fulfills this algebraic characterization.

Remark 2.5.9. Is solid closure for normal noetherian rings inside the axes closure? Solid closure agrees with tight closure in positive characteristic, but can be strictly larger than tight closure in characteristic zero. An example of P. Roberts shows in [41] that in the polynomial ring $K[X,Y,Z]$ over a field K of characteristic zero the inclusion $X^2Y^2Z^2 \in (X^3,Y^3,Z^3)^{\mathrm{sc}}$ holds. In this example we have indeed the containment in the axes closure and in the continuous closure.

Remark 2.5.10. The completion of a ring of axes is the completion of a (local version of a) one-dimensional Stanley-Reisner ring. Can the inclusion of the tight closure (in the normal case) inside the axes closure be strengthend to an inclusion inside the Stanley-Reisner closure I^{SR} of an ideal? This closure is defined by taking up to completion the Stanley-Reisner rings as a test category. Note that $I \subseteq I^{\mathrm{SR}} \subseteq I^{\mathrm{ax}} \cap I^{\mathrm{reg}}$ and $I^* \subseteq I^{\mathrm{ax}} \cap I^{\mathrm{reg}}$. If we consider Stanley-Reisner rings where the sheets always meet in one point, then the arguments used in this chapter go through. The main problem is to find a replacement for Lemma 2.1.1 in this setting.

Remark 2.5.11. Is there a reasonable subclass of continuous functions which defines a tight closure type theory for normal $\mathbb{C}$-algebras of finite type? In particular, for a smooth variety it should not change the ideals.

Chapter 3

Sheaves and vector bundles

This chapter recapitulates some background on vector bundles and locally free sheaves.

3.1 Locally free sheaves and vector bundles

A fiber bundle is a topological space that locally looks like the product of a base space and a fiber space. Precisely it is defined as follows.

Definition 3.1.1. A *fiber bundle* consists of topological spaces V, F, X and a continuous surjective projection map $f : V \longrightarrow X$ that satisfies the following condition: There is a covering of X by open sets such that for each open set U in the covering there is a homeomorphism $\psi : f^{-1}(U) \longrightarrow U \times F$ such that the following diagram commutes.

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\psi} & U \times F \\ \downarrow f & \swarrow & \\ U & & \end{array}$$

V is called the total space, X the base space and F the fiber.

A special case of a fiber bundle are geometric vector bundles, where V and X are schemes and the fiber is a vector space and the transition mappings are linear.

Definition 3.1.2. A *geometric vector bundle* of rank n over a scheme X is a scheme V together with the following data:

1. A projection morphism $f : V \longrightarrow X$.

2. An affine open covering $\{U_i\}$ of X.

3. Isomorphisms $\psi_i : f^{-1}(U_i) \longrightarrow \mathbb{A}^n_{U_i}$ such that for all i, j and all open affine subsets $U = \operatorname{Spec} R \subseteq U_i \cap U_j$ we have that $\psi_i \circ \psi_j^{-1} : \mathbb{A}^n_U \longrightarrow \mathbb{A}^n_U$ is induced by an R-linear algebra automorphism on $R[x_1, \ldots, x_n]$.

Remark 3.1.3. Note that if X is a scheme over a field k we will have $\mathbb{A}^n_{U_i} \cong U_i \times \mathbb{A}^n_k$. This means that over each closed point a vector bundle is isomorphic to a vector space k^n. Also, in this case the R-algebra automorphisms are automorphisms on $U \times \mathbb{A}^n_k$ which are given by a linear automorphism on $\mathbb{A}^n_k$.

A *section* of a vector bundle is a scheme morphism $s : U \longrightarrow V$ from an open subset $U \subseteq X$ such that $f \circ s = \mathrm{id}_U$. To every vector bundle there is a *trivial (or canonical) global section* which is given by mapping to zero everywhere.

Proposition 3.1.4. *Let X be a scheme. There is a contravariant equivalence of categories between isomorphism classes of vector bundles of rank n on X and isomorphism classes of locally free sheaves of rank n on X. To a vector bundle $f : V \longrightarrow X$ we associate the sheaf of sections $S(V/X)$ which is defined on an open set $U \subseteq X$ by*

$$S(V/X)(U) := \{s : U \longrightarrow V : f \circ s = \mathrm{id}_U\}.$$

To a locally free sheaf $\mathcal{F}$ we associate the vector bundle

$$V(\mathcal{F}) := \operatorname{Spec}(\operatorname{Sym} \mathcal{F}).$$

On an affine open subset U for which $\mathcal{F}(U) \cong \mathcal{O}_X(U)^n$ the symmetric algebra takes the form $\operatorname{Sym}(\mathcal{F})(U) \cong \mathcal{O}_X(U)[x_1, \ldots, x_n]$. This isomorphism of sheaves of algebras induces the following isomorphism of schemes

$$\psi : \operatorname{Spec}(\operatorname{Sym} \mathcal{F})(U) \longrightarrow \mathbb{A}^n_U = \operatorname{Spec} \mathcal{O}_X(U)[x_1, \ldots, x_n].$$

The functors V and S are related by the following correspondence:

$$S(V(\mathcal{F})/X)^\vee \cong \mathcal{F}.$$

Proof. [24, Proposition 11.7]. □

Because of this equivalence of categories we will sometimes use the terms vector bundle and locally free sheaf interchangeably.

An example that will be important are syzygy sheaves. Let $X = \operatorname{Proj} R$, the ring R a standard graded domain. A *syzygy sheaf* is the kernel sheaf of a map $\bigoplus_{i=1}^{n} \mathcal{O}_X(-d_i) \longrightarrow \mathcal{O}_X$, given by a one row matrix $(f_1, \ldots, f_n)$, where $f_i \in R$ has degree d_i. We call this kernel $\operatorname{Syz}(f_1, \ldots, f_n)$.

On the other hand consider

$$R' = R[T_1, \ldots, T_n]/(\sum_{i=1}^{n} f_i T_i),$$

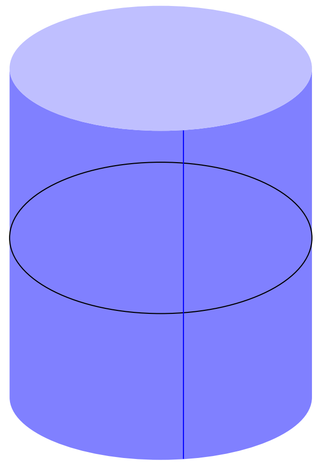
where the T_i have degree $-d_i$. Then the open set $V := D_+(R_+) \in \operatorname{Proj} R$ is a vector bundle over X. The sheaf of sections of V is $\operatorname{Syz}(f_1, \ldots, f_n)$. In particular a global section corresponds to a homomorphism $R' \longrightarrow R$, which is given by the images s_i of the T_i. By construction these images form a global syzygy $(s_1, \ldots, s_n)$ of $f_1, \ldots, f_n$.

Of particular importance are vector bundles of rank 1, also called line bundles. The corresponding locally free sheaves are called invertible sheaves. Invertible sheaves form a group $\operatorname{Pic} X$ with tensor multiplication as the group operation. The neutral element is $\mathcal{O}_X$. To a rank 1 sheaf $\mathcal{L}$ the inverse is $\mathcal{L}^{-1} = \mathcal{L}^{\vee} = \operatorname{Hom}(\mathcal{L}, \mathcal{O}_X)$.

In the following we will restrict ourselves to the case of smooth projective curves.

Example 3.1.5. Take as an example the smooth projective curve $X = \operatorname{Proj} R$, where $R = \mathbb{R}[x, y, z]/(x^2 + y^2 - z^2)$. On the affine chart given by $D_+(z)$, this is the unit circle. Since all real points of the circle are already in this chart, the curve is essentially the same as $\operatorname{Spec} \mathbb{R}[x, y](x^2 + y^2 - 1)$.

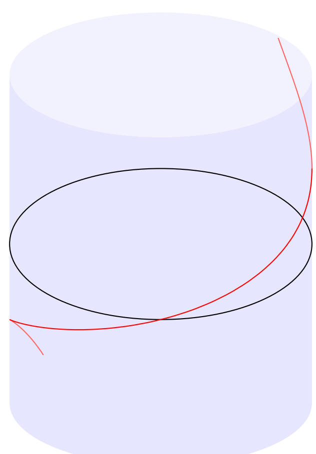
A vector bundle of rank 1 consists as a set of a line above each point. The simplest vector bundle of rank one over the circle is a cylinder. The cylinder can be described as $V = D_+(R_+) \subseteq \operatorname{Proj} R[T]$, where T has degree 0.



The nonisomorphic alternative to the cylinder bundle is the Möbius bundle, in which the lines turn around halfway along the circle.

A section of the cylinder bundle is a map $X \longrightarrow V$. On any open set $U \subseteq X$ it is defined by a map $\mathcal{O}_U[T] \longrightarrow \mathcal{O}_U$, mapping T to a degree 0 rational function without poles on U. If we use the affine construction of the curve the degree condition is removed. A rational function associates to every point $x \in X$ on which it is defined a value on the line over x.

For example the section defined by $f(x) = \frac{x}{z-y}$ on the cylinder can be viewed as the red line in the following picture.



3.2 Divisors on curves

For the Riemann-Roch theory and to define the degree of a sheaf we will need the concept of divisors. For a more in-depth treatment of the topic we refer to [26, Section II.6], nonetheless we give an introduction.

There are Weil divisors and Cartier divisors, which will be defined below. On a noetherian, integral, separated and locally factorial scheme there is a 1-to-1 correspondence. Also in this case the divisor class group denoted $\operatorname{Cl} X$ is isomorphic to the Picard group $\operatorname{Pic} X$. In particular smooth projective curves have the property to be noetherian, integral, separated and locally factorial, and we will restrict ourselves to that case.

A *Weil divisor* on a curve X is an element D of the free abelian group $\operatorname{Div} X$ generated by closed points on X. The closed points in this group are also called prime divisors.

The degree of a Weil divisor $D = \sum_{i=1}^{n} a_i P_i$ on a scheme over an algebraically closed field is $\deg D = \sum_{i=1}^{n} a_i$.

Let K be the function field of X. To a closed point P the valuation $v_P : K^* \longrightarrow \mathbb{Z}$, maps a nonzero rational function $f \in K^*$ to the multiplicity of a zero or pole of f in P (poles are counted negative). If f has neither a zero nor a pole in P then $v_P(f) = 0$.

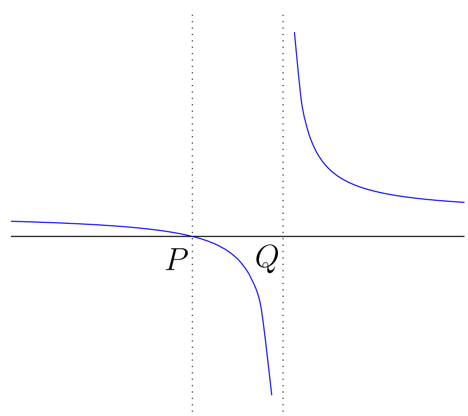


Figure 3.1: Rational function with zero and pole. $\nu(P) = 1$, $\nu(Q) = -1$.

The divisor associated to a nonzero rational function f is the linear combination of all points with their valuations $(f) = \sum_{P \in X} v_P(f) \cdot P$ and called a *principal divisor*. Note that if the function has no pole or zero at a point the valuation is 0. Two divisors D, E are called *linearly equivalent* $D \sim E$ if their difference $D - E$ is a principal divisor. We call the quotient group to this equivalence relation the *divisor class group* $\mathrm{Cl}(X) := \mathrm{Div}\, X/\sim$.

Next we define Cartier divisors for which we need the sheaves $\mathcal{K}^*$ and $\mathcal{O}_X^*$. Because we assume X to be integral the field of fractions defines a constant sheaf $\mathcal{K}$ of which $\mathcal{O}_X$ is a subsheaf. By $\mathcal{O}_X^*$ and $\mathcal{K}^*$ we denote the subsheaves of invertible elements of the respective sheaves.

A *Cartier divisor* is a global section of the sheaf $\mathcal{K}^*/\mathcal{O}_X^*$. This means that a Cartier divisor can be described by giving a rational function f_i for each open set of an open cover U_i such that on an intersection $U_i \cap U_j$ the quotient $\frac{f_i}{f_j}$ of the rational functions is a section of $\mathcal{O}_X^*$.

It's clear that a Cartier divisor defines a Weil divisor with the coefficient $v_Y(f_i)$ on every point $Y \in U_i$. On the other hand we can express any Weil divisor locally as a principal divisor and glue the defining functions.

Every divisor D defines an *associated invertible sheaf* $\mathcal{L}(D)$. If D is described as a Cartier divisor $\{(U_i, f_i)\}$ the associated sheaf is the subsheaf of the sheaf of total quotient rings $\mathcal{K}$ generated as an $\mathcal{O}_X$-module on U_i by f_i^{-1}. This gives a 1-1 correspondence between invertible subsheaves of $\mathcal{K}$ and Cartier divisors, because to an invertible subsheaf $\mathcal{L} \subseteq \mathcal{K}$ we can associate the Cartier divisor given by inverses of local generators of $\mathcal{L}$.

On any integral scheme we have for any invertible sheaf $\mathcal{L}$ that $\mathcal{L} \otimes \mathcal{K} \cong \mathcal{K}$, because $\mathcal{K}$ is a constant sheaf. The mapping $\mathcal{L} \longrightarrow \mathcal{L} \otimes \mathcal{K}$ thus describes $\mathcal{L}$ as a subsheaf of $\mathcal{K}$.

In addition for two divisors D and E we have that

$$\mathcal{L}(D - E) \cong \mathcal{L}(D) \otimes \mathcal{L}(E)^{-1}.$$

and

$$D \sim E \Longleftrightarrow \mathcal{L}(D) \cong \mathcal{L}(E).$$

In total this gives us a natural isomorphism $\mathrm{Cl}\, X \cong \mathrm{Pic}\, X$.

In the more general case of a noetherian normal scheme, which is not necessarily a curve, the prime divisors are closed integral subschemes of codimension one, but the rest of the theory stays the same.

Example 3.2.1. On $\mathbb{P}_K^n$ any divisor D is linearly equivalent to dH, where $H = \{(x_0 : \ldots : x_n) \in \mathbb{P}_K^n | x_0 = 0\}$ is a hyperplane and $d = \deg D$. In this situation the divisor can be written as $D = V_+(f)$, where f is a homogeneous polynomial of degree d. This also determines the degree of D.

Because the divisors are equivalent to those hyperplanes, we have

$$\operatorname{Cl}\mathbb{P}^n_K \cong \operatorname{Pic}\mathbb{P}^n_K \cong \mathbb{Z}.$$

On the other hand we look at the twisting sheaf $\mathcal{O}(l)$ for $l \in \mathbb{Z}$. The twisting is based on the grading of the underlying graded ring, so assume that we work on $X = \operatorname{Proj} R$, where R is a graded ring. For any graded R-module M we define $M(l)_d = M_{d+l}$. For $M = R$ and sheafification this can be used analogously to the construction of $\mathcal{O}$ to define the twisting sheaf $\mathcal{O}(l)$.

All the twisting sheaves $\mathcal{O}(l)$ are invertible (with inverse $\mathcal{O}(-l)$), so they have to correspond to divisors. We have $\mathcal{O}(l) \otimes \mathcal{K} \cong \mathcal{K}$. With regards to this isomorphism the subsheaf $\mathcal{O}(l) \subseteq \mathcal{K}$ is locally on $D(x_i)$ generated by x_j^{-l}, where $j \neq i$. Thus the x_j^l describe the Cartier divisor on U_i and $\mathcal{O}(l)$ corresponds to lH. In other words the $\mathcal{O}(l)$ for all $l \in \mathbb{Z}$ generate $\operatorname{Pic}\mathbb{P}^n_K$. Note that the invertible sheaf is generated by the inverse of the local function describing the Cartier divisor.

3.3 Degree and twist

In this section we describe how to use divisors and invertible sheaves in the case of projective curves.

Definition 3.3.1. If we take a smooth projective curve X it comes with an embedding $\varphi : X \hookrightarrow \mathbb{P}^n_K$. We call $\mathcal{O}_X(l) = \varphi^*(\mathcal{O}_{\mathbb{P}^n_K}(l))$ the *twisting sheaf* on X.

In particular the twisting sheaves depend on the embedding of X.

We use multiple notions of degree. We already encountered the degree of a divisor. Next we will define the degree of a projective variety and of a locally free sheaf on a curve.

The *degree* of a projective variety X of dimension d is $\deg X = r! \cdot a_d$, where a_d is the leading coefficient of the Hilbert polynomial $\operatorname{Hilb} X = a_d X^d + \ldots + a_0$. We will handle Hilbert polynomials in more detail in Section 6.3.

For a plane curve $X = \operatorname{Proj}[x, y, z]/(f)$, the degree is $\deg X = \deg f$.

Definition 3.3.2. For an invertible sheaf on a curve we define the degree as

$$\deg(\mathcal{L}(D)) := \deg(D).$$

To a locally free sheaf the determinant bundle $\bigwedge^{\operatorname{rank}\mathcal{F}} \mathcal{F}$ has rank 1 and is thus an invertible sheaf. Thus we can define the degree as

$$\deg(\mathcal{F}) = \deg(\bigwedge^{\operatorname{rank}\mathcal{F}} \mathcal{F}).$$

Remember that the degree of a divisor is the sum of the zeroes minus the sum of the poles, each counted with multiplicity. The twisting sheaves $\mathcal{O}(l)$ correspond to l-multiples of hyperplanes in the embedding space. Counted with multiplicities there are exactly $l \cdot \deg X$ intersection points with a curve X. Thus we have

$$\deg \mathcal{O}_X(l) = l \cdot \deg X.$$

With the definition of degree we can define semistability.

Definition 3.3.3. The *slope* of a locally free sheaf $\mathcal{F}$ on a smooth projective curve X is defined as

$$\mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}.$$

A locally free sheaf $\mathcal{F}$ is called *semistable* if for all proper subsheaves $\mathcal{E}$ we have $\mu(\mathcal{E}) \leq \mu(\mathcal{F})$. It is called *stable* if for all proper subsheaves $\mathcal{E}$ we have $\mu(\mathcal{E}) < \mu(\mathcal{F})$.

3.4 Embeddings and the affine cone

This chapter deals with the embedding and representation of curves in projective space, which will be important in Chapter 4. The best way to represent curves from a computational perspective is as $X = \operatorname{Proj} K[x_0, \dots, x_n]/\mathfrak{a}$, where $\mathfrak{a}$ is a homogeneous ideal of $K[x_0, \dots, x_n]$. The ring homomorphism $K[x_0, \dots, x_n] \longrightarrow K[x_0, \dots, x_n]/\mathfrak{a}$ induces a natural embedding $X \hookrightarrow \mathbb{P}^n$.

In this sense the embedding is equivalent to the ring over which we define the curve.

As we worked out in Section 3.3 every embedding into projective space comes with a twisting sheaf $\mathcal{O}_X(1)$. Any invertible sheaf which can be retrieved by the pullback via an embedding to projective space is called a *very ample invertible* sheaf. Because for the theoretical aspects to work only this very ample invertible sheaf is needed, many authors require the scheme to be equipped with a very ample invertible sheaf instead of the embedding, which is equivalent. For the computations we do, however, need the embedding in the standard projective space, so for us the embedding approach makes more sense.

In general let S be a graded affine ring and $S_+ = \bigoplus_{i>0} S_i$ the irrelevant ideal. $V(S_+) \subseteq \operatorname{Spec} S$ is the set of all prime ideals containing S_+ and corresponds to the origin. $X = \operatorname{Proj} S$ is defined as the set of all homogeneous prime ideals not containing S_+. We call $C_X := \operatorname{Spec} S$ the affine cone of $X = \operatorname{Proj} S$.

The affine cone is important because any actual computations, such as finding destabilizing global sections are actually done in graded rings and its spectrum instead of the projective schemes themselves. Thus we have to be sure that the important properties translate from the rings to the projective curve.

Over projective space $\mathbb{P}^n_K = \operatorname{Proj} R$ for $R = K[x_0, \ldots, x_n]$ the global sections of $\mathcal{O}(l)$ are homogeneous degree l elements of R. In general for a normal ring $S = K[x_0, \ldots, x_n]/\mathfrak{a}$ and $X = \operatorname{Spec} S$ we have an isomorphism $\Gamma_*(\mathcal{O}_X) := \bigoplus_{l \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(l)) \cong S$.

To any locally free sheaf $\mathcal{F}$ we define the sheaf *twisted* by $l \in \mathbb{Z}$ to be $\mathcal{F}(l) := \mathcal{F} \otimes \mathcal{O}_X(l)$. By twisting, we can make "hidden parts" of the sheaf visible in its global sections.

Remark 3.4.1. For a scheme of the form $X = \operatorname{Proj} S$ we always have a map $S_d \longrightarrow \Gamma(X, \mathcal{O}_X(d))$, which maps a homogeneous degree d element f to $\frac{f}{1}$. In computations we work in S_d so we have to make sure that this map is bijective. In general this is not the case, but it is if we assume S to be at least two-dimensional and normal, i.e. an integrally closed domain. The bijectivity can be shown in this case, because the elememts of $\Gamma(X, \mathcal{O}_X(d))$ can be represented by elements of the quotient field fulfilling integrality equations.

Smooth schemes are always normal, but S is not necessarily normal. If X is a normal complete intersection (in particular a normal plane curve), then S is normal however, which covers most examples in the next section.

If we start with an integrally closed domain, the affine cone will be a normal affine surface. The only possible nonregular point is the origin, thus the projective curve associated to the ring is smooth.

On the other hand if we start with a sheaf $\mathcal{F}$ over a smooth scheme $X = \operatorname{Proj} S$, for which S is not normal, we take the integral closure $S \longrightarrow S'$. The induced morphism on schemes $\nu : X' = \operatorname{Proj} S' \longrightarrow X$ is a finite morphism between normal schemes. Thus the inverse image $\nu^*\mathcal{F}$ is semistable if and only if $\mathcal{F}$ is [35, Lemma 3.2.2].

3.5 The moduli space of semistable sheaves

A very important aspect of semistable sheaves is that their moduli spaces exhibit a scheme structure.

There are various constructions of moduli spaces of sheaves, one of the first being Gieseker's construction [20]. The following description of the moduli space of sheaves follows the exposition in Huybrechts and Lehn [35].

Let X be a projective scheme with a fixed very ample line bundle $\mathcal{O}_X(1)$ (this is also called a polarized projective scheme) over an algebraically closed field k. The moduli functor to a fixed polynomial $P \in \mathbb{Q}[n]$ is a functor $\mathcal{M} = \mathcal{M}'/\sim$ constructed as a quotient functor:

The contravariant functor $\mathcal{M}'$ maps a scheme S over k to the set of isomorphism classes of coherent sheaves on $S \times_k X$ which have Hilbert polynomial P and that are flat over S. The morphisms $f : S' \longrightarrow S$ are mapped by pulling back via $f \times \mathrm{id}_X$. Let $p : S \times_k X \longrightarrow S$ be the projection. The equivalence relation $\sim$ is defined by declaring two S-flat sheaves $\mathcal{F}$ and $\mathcal{G}$ on X to be equivalent if there is a line bundle $\mathcal{L}$ on S such that $\mathcal{F} \cong \mathcal{G} \otimes p^*\mathcal{L}$ on $X \times S$.

A scheme corepresenting the functor $\mathcal{M}$ is called a *moduli space* $M_{\mathcal{O}_X(1)}(P)$ of semistable sheaves with polarization $\mathcal{O}_X(1)$ and Hilbert polynomial P. For given data there is only one moduli space up to unique isomorphism. Sometimes we write $M(P)$, making the polarization implicit.

Example 3.5.1. Let $X, \mathcal{O}_X(1)$ be a projective curve with $d = \deg X$ and g the genus of X. Then for any coherent sheaf $\mathcal{F}$ of rank r we have

$$\mathrm{Hilb}(\mathcal{F})(n) = \chi(\mathcal{F}(n)) = rd \cdot n + \deg \mathcal{F} + r(1-g).$$

Thus as the curve is fixed the only data that are determined by the Hilbert polynomial are the rank and the degree of the sheaf. This means that the moduli space $M_r(d) := M(P)$ parameterizes all semistable sheaves of fixed rank and degree.

There is an alternative construction characterizing different moduli spaces on X not by rank and degree but by rank r and isomorphism classes of determinant bundles $\mathcal{L} \cong \bigwedge^r \mathcal{F}$. Let's denote these moduli spaces as $M_r(\mathcal{L})$. This is a finer classification, since the determinant bundle determines the degree but not the other way round. In fact if we look at the morphism $M_r(d) \longrightarrow \mathrm{Pic}_d, \mathcal{F} \mapsto \bigwedge^r \mathcal{F}$ we can view $M(\mathcal{L})$ as the fiber over $\mathcal{L} \in \mathrm{Pic}_d$.

To be able to construct the moduli space it's important that the sheaves in question form a bounded family.

Definition 3.5.2. A family of isomorphism classes of coherent sheaves on a scheme X on k are called bounded if there is a k-scheme S of finite type and a sheaf $\mathcal{F}$ on $S \times X$ such that every member of the family is a fiber $\mathcal{F}|_{\operatorname{Spec} k(s) \times X}$ for a closed point $s \in S$.

In particular this implies that there is an $n_0 \in \mathbb{Z}$ such that $H^i(\mathcal{F}(n_0 - i)) = 0$ for all $i > 0$ and any $\mathcal{F}$ in the family, i.e. that the Castelnuovo-Mumford-regularity of the members of the family is bounded from above (see [35, Lemma 1.7.6]). For any sheaf there is always a Castelnuovo-Mumford-regularity but in general the regularity depends on the sheaf.

Example 3.5.3. Fix a smooth projective curve $(X, \mathcal{O}_X)$. The family of sheaves $\mathcal{F}_l = \mathcal{O}(-l) \oplus \mathcal{O}(l)$ has fixed degree and rank (0 and 2 respectively), thus fixed Hilbert polynomial. But the Castelnuovo-Mumford-regularity of $\mathcal{F}_l$ is l and thus not bounded. Of course these sheaves are not semistable either as for $l > 0$ the subsheaf $\mathcal{O}(l)$ has slope $l > \mu(\mathcal{F}_l) = 0$.

Semistable sheaves with fixed Hilbert polynomial do form a bounded family though [35, Theorem 3.3.7]. This is a vital ingredient for the construction of the moduli space of semistable sheaves.

3.6 Multilinear powers

So far in this exposition we have used the tensor product as the vehicle of ring change. In that function the tensor product allows us to see a module under the light of a different ring. In this section we take tensor products of a module with itself and the related operations of symmetric and exterior powers. This is a technique used to reveal some inner properties of modules and morphisms - for example it is central via diagonal morphisms to the notion of separated morphisms. We will use these multilinear powers to make semistability visible in the global sections of a sheaf.

We remind the reader that the tensor product $M \otimes_R N$ of two R-modules M and N is the module generated by pairs $a \otimes b, a \in M, b \in N$. Between these generators we have the relations that are necessary to make the tensor product multilinear, i.e.

$$\begin{aligned}(a_1 + a_2) \otimes b &= a_1 \otimes b + a_2 \otimes b,\\ a \otimes (b_1 + b_2) &= a \otimes b_1 + a \otimes b_2,\\ (r \cdot a) \otimes b &= r \cdot (a \otimes b) = a \otimes (r \cdot b).\end{aligned}$$

The tensor product is functorial, it preserves direct sums in each component and it is right exact.

The n-th tensor power $T^n M$ is $\bigotimes_{i=1}^n M$, i.e. the n-th iterated tensor product of M with itself. Based on this we define the symmetric and exterior powers.

Definition 3.6.1. The n-th symmetric power $\operatorname{Sym}^n M$ of an R-module M is the quotient of the n-th tensor power where all tensors which are permutations of each other are identified, i.e. $a \otimes b = b \otimes a$. We write the residue classes as $a_1 \cdots a_n := \overline{a_1 \otimes \ldots \otimes a_n}$.

Definition 3.6.2. Similarly the n-th exterior power $\bigwedge^n M$ is the n-th tensor power modulo the module generated by tensors which have the same factor occurring multiple times (like $a \otimes a$). We write the residue classes as $a_1 \wedge \ldots \wedge a_n$.

This means that in the symmetric powers the products are made commutative (hence it is called symmetric). Similarly in the exterior powers they are antisymmetric (as from $(a+b) \otimes (a+b) = 0$ and $a \otimes a = 0$, $b \otimes b = 0$ it follows that $a \otimes b = -b \otimes a$).

3.7 Open sets that are affine schemes and solid closure

In this section we talk about the characterization of solid closure via the affineness of open sets of certain bundles. Again let $I = (f_1, \ldots, f_n) \subseteq R$ and $f_0 \in R$ another element. Let $U = D(I)$ be the open set defined by the ideal I.

$V := R[T_1, \ldots, T_n] / \left(\sum_{i=1}^n f_i T_i\right)|_U$ is a vector bundle. The sheaf corresponding to V is $\mathcal{F} := \operatorname{Syz}(f_1, \ldots, f_n)$. We define another vector bundle

$V' := R[T_0, \ldots, T_n]/\left(\sum_{i=0}^{n} f_i T_i\right)|_U$ and a short exact sequence

$$0 \longrightarrow V \longrightarrow V' \xrightarrow{T_0} \mathbb{A}^1_U \longrightarrow 0.$$

Note that the complement $\mathbb{P}(V') \setminus \mathbb{P}(V)$ is isomorphic to the spectrum of the forcing algebra to the forcing equation $f_0 + f_1 T_1 + \ldots + f_n T_n$ restricted to the subset U[7, Remark 1.3].

If we want to work on projective schemes, we can use a graded version of this construction. In this case we start with a standard graded algebra R and a homogeneous R_+-primary ideal $I = (f_1, \ldots, f_n)$ and $f_0 \in R$ homogeneous. Then we fix an $m \in \mathbb{N}$ and introduce new variables T_i with $\deg T_i = m - \deg f_i$. With this we can similarly define vector bundles

$$V_m := D_+\left(R_+ R[T_1, \ldots, T_n]/\left(\sum_{i=1}^{n} f_i T_i\right)\right)$$

and

$$V'_m := D_+\left(R_+ R[T_0, \ldots, T_n]/\left(\sum_{i=0}^{n} f_i T_i\right)\right),$$

i.e. each time the open set defined by the irrelevant ideal $R_+ \subseteq R$. There is also again a short exact sequence

$$0 \longrightarrow V_m \longrightarrow V'_m \xrightarrow{T_0} \mathbb{A}^1_U(-e_0) \longrightarrow 0.$$

We state the following lemma for the graded construction, but it can essentially also be stated without the grading for any ideal primary to a maximal ideal (see [3, Propositon 1.3]).

Lemma 3.7.1. *Let R be a normal, standard graded ring of dimension $d \geq 2$. Let $I = (f_1, \ldots, f_n)$ be a homogeneous R_+-primary ideal and $f_0 \in R$ homogeneous. Then $f_0 \in I^\star$ if and only if the cohomological dimension of $\mathbb{P}(V'_m) \setminus \mathbb{P}(V_m)$ is $d - 1$. If $d = 2$ this is equivalent to $\mathbb{P}(V'_m) \setminus \mathbb{P}(V_m)$ not being an affine scheme.*

Proof. See [3, Proposition 3.9]. □

The proof of this depends on a theorem of Serre relating the property of being an affine scheme to cohomology[26, Theorem III.3.7].

Another way to look at the short exact sequence of V_m and V'_m is the corresponding short exact sequence of sheaves. To the short exact sequence

$$0 \longrightarrow \mathcal{F} := \mathrm{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \longrightarrow \mathcal{O}_X \longrightarrow 0$$

there is the connecting homomorphism $\delta : H^0(X, \mathcal{O}_X) \longrightarrow H^1(X, \mathcal{F})$. Thus for any homogeneous f_0 of degree d_0 there is a cohomology class $c := \delta(f_0) \in H^1(X, \mathcal{F})$. As $H^1(X, \mathcal{F}) \cong \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{F})$ the class c corresponds to an extension $0 \longrightarrow \mathcal{F}(d_0) \longrightarrow \mathcal{F}'(d_0) \longrightarrow \mathcal{O}_X \longrightarrow 0$. The sheaf $\mathcal{F}'$ corresponds to the vector bundle V'_m.

Chapter 4

Deciding stability of sheaves

In the following sections we will describe in detail how the method to determine semistability works, work out the machinery involved, give the algorithm and compute some simple examples.

All our bundles will be on a smooth curve $X \subseteq \mathbb{P}^r_K$ over an algebraically closed field K.

4.1 Degree and global sections

We recall basic notions for bundles on curves.

Any locally free sheaf is reflexive. All torsion-free sheaves on a curve are locally free. As such torsion-free rank 1 sheaves on a curve are invertible.

Lemma 4.1.1. *Let $\mathcal{F}$ be a locally free sheaf of rank r on a curve. There exist an invertible sheaf $\mathcal{L}$, a locally free sheaf $\mathcal{F}'$ of rank $r-1$ and an exact sequence:*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}' \longrightarrow 0.$$

Proof. See [46, Lemma 1.15]. □

For a noetherian, smooth scheme X, we have a natural isomorphism between the divisor class group and the Picard group: $\operatorname{Cl} X \cong \operatorname{Pic} X$. As such we have a divisor class for every isomorphism class of invertible sheaves on X and in particular if X is a smooth projective curve over an algebraically closed field we can define the degree of an invertible sheaf $\mathcal{L} = \mathcal{L}(D)$ by the degree of the corresponding divisor.

The *degree* of a locally free sheaf of rank r over a smooth projective curve X is defined as

$$\deg \mathcal{F} := \deg \bigwedge^r \mathcal{F}.$$

The definition reduces the degree of a bundle to the degree of an invertible sheaf $\mathcal{L}$.

The *slope* of a locally free sheaf $\mathcal{F}$ on X is $\mu(\mathcal{F}) := \frac{\deg \mathcal{F}}{\operatorname{rank} \mathcal{F}}$. A sheaf is called *stable* if its slope is larger than the slope of any proper subsheaf and *semistable* if its slope is at least as big as that of any subsheaf. We call a subsheaf *destabilizing* if it violates the stability condition.

The theorem of Riemann-Roch for sheaves relates the degree and rank to global sections and the genus g of the curve:

$$\deg \mathcal{F} = \chi(\mathcal{F}) + r(g-1).$$

$\chi(\mathcal{F})$ denotes the Euler-Poincaré characteristic, which on projective curves is

$$\chi(\mathcal{F}) = \dim H^0(X, \mathcal{F}) - \dim H^1(X, \mathcal{F}),$$

and g is the genus of the curve. Here $H^0(X, \mathcal{F})$ is the vector space of global sections of $\mathcal{F}$. With the theorem of Riemann-Roch we get the following inequality:

$$\dim H^0(X, \mathcal{F}) \geq \deg D - r(g-1).$$

From this follows immediately:

Lemma 4.1.2. *Let $\mathcal{F}$ be a sheaf on a smooth projective curve X. If $\mu(\mathcal{F}) > g-1$ then $\mathcal{F}$ must have a nontrivial global section.*

We will also use the following fact.

Lemma 4.1.3. *Let $\mathcal{F}$ be a sheaf on a smooth projective curve X. If $\mu(\mathcal{F}) < 0$ and $\mathcal{F}$ has a nontrivial global section, then $\mathcal{F}$ is not semistable.*

Proof. Every nontrivial global section defines a map $\mathcal{O}_X \longrightarrow \mathcal{F}$. We can factor it through a line bundle of nonnegative slope to get $\mathcal{O}_X \twoheadrightarrow \mathcal{L} \hookrightarrow \mathcal{F}$. Then $\mathcal{L}$ is a subsheaf of nonnegative slope, which can not exist in a semistable sheaf of negative slope. □

In the following sections we will use these two lemmas like follows: The definition of slope gives a numerical minimum on the difference of the slopes of a subsheaf and the sheaf itself. We will enlarge this minimal gap using multilinear algebra. This will allow us to make any potential destabilizing subsheaf visible in terms of global sections.

Remark 4.1.4. If we don't want to restrict ourselves to curves another similar notion to (μ-)stability is Gieseker stability. A locally free sheaf $\mathcal{F}$ is Gieseker stable if for every proper subsheaf $\mathcal{E} \subset \mathcal{F}$ we have $p(\mathcal{E}) < p(\mathcal{F})$, where $p(\mathcal{F}) = \frac{\mathrm{Hilb}(F)}{a_d}$ is the reduced Hilbert polynomial, a_d the leading coefficient of the Hilbert polynomial. We say for two polynomials $P, Q \in K[X]$ that $P \geq Q$ if and only if for $n \gg 0$ we have $P(n) \geq Q(n)$. On curves the two notions of stability agree, thus we will work with the somewhat more easily accesible notion of μ-stability.

4.2 Symmetric and exterior powers

In this section we describe the multilinear operations we use. We will use the sheaf versions of the tensor product, the symmetric power Sym^n and the exterior power $\bigwedge^n$. These are each the sheafifications of their respective module versions. We want to apply these operations to short exact sequences.

Lemma 4.2.1. *Let $0 \longrightarrow \mathcal{E} \xrightarrow{\psi} \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \longrightarrow 0$ be a short exact sequence of locally free sheaves of finite rank over a scheme over characteristic 0. This induces for every $n \in \mathbb{N}_{>0}$ exact sequences*

$$
\begin{aligned}
0 \longrightarrow \bigwedge^n \mathcal{E} \longrightarrow \bigwedge^n \mathcal{F} \longrightarrow & \left(\bigwedge^{n-1} \mathcal{F}\right) \otimes \mathrm{Sym}^1 \mathcal{G} \longrightarrow \\
& \left(\bigwedge^{n-2} \mathcal{F}\right) \otimes \mathrm{Sym}^2 \mathcal{G} \longrightarrow \\
& \cdots \longrightarrow \\
& \left(\bigwedge^{2} \mathcal{F}\right) \otimes \mathrm{Sym}^{n-2} \mathcal{G} \longrightarrow \\
& \left(\bigwedge^{1} \mathcal{F}\right) \otimes \mathrm{Sym}^{n-1} \mathcal{G} \longrightarrow \mathrm{Sym}^n \mathcal{G} \longrightarrow 0.
\end{aligned}
$$

The leftmost map in this sequence is $\bigwedge^n \psi$. *The maps in the middle of the sequence are*

$$\left(\bigwedge^{k} \mathcal{F}\right) \otimes \operatorname{Sym}^{n-k} \mathcal{G} \longrightarrow \left(\bigwedge^{k-1} \mathcal{F}\right) \otimes \operatorname{Sym}^{n-k+1} \mathcal{G},$$

which are given by

$$\begin{aligned} &f_1 \wedge \ldots \wedge f_k \otimes g_{k+1} \cdots g_n \mapsto \\ &\qquad \sum_{i=1}^{k} (-1)^{i-1} f_1 \wedge \ldots \wedge f_{i-1} \wedge f_{i+1} \wedge \ldots \wedge f_k \otimes \varphi(f_i) \cdot g_{k+1} \cdots g_n, \end{aligned}$$

and

$$\begin{aligned} 0 \longrightarrow \operatorname{Sym}^n \mathcal{E} \longrightarrow \operatorname{Sym}^n \mathcal{F} \longrightarrow &\left(\operatorname{Sym}^{n-1} \mathcal{F}\right) \otimes \bigwedge^{1} \mathcal{G} \longrightarrow \\ &\left(\operatorname{Sym}^{n-2} \mathcal{F}\right) \otimes \bigwedge^{2} \mathcal{G} \longrightarrow \\ &\cdots \longrightarrow \\ &\left(\operatorname{Sym}^{2} \mathcal{F}\right) \otimes \bigwedge^{n-2} \mathcal{G} \longrightarrow \\ &\left(\operatorname{Sym}^{1} \mathcal{F}\right) \otimes \bigwedge^{n-1} \mathcal{G} \longrightarrow \bigwedge^{n} \mathcal{G} \longrightarrow 0. \end{aligned}$$

The leftmost map in this sequence is $\operatorname{Sym}^n \psi$. *The maps in the middle of the sequence are*

$$\left(\operatorname{Sym}^{k} \mathcal{F}\right) \otimes \bigwedge^{n-k} \mathcal{G} \longrightarrow \left(\operatorname{Sym}^{k-1} \mathcal{F}\right) \otimes \bigwedge^{n-k+1} \mathcal{G},$$

which are given by

$$\begin{aligned} &f_1 \cdots f_k \otimes g_{k+1} \wedge \ldots \wedge g_n \mapsto \\ &\qquad \sum_{i=1}^{k} f_1 \cdots f_{i-1} \cdot f_{i+1} \cdots f_k \otimes \varphi(f_i) \wedge g_{k+1} \wedge \ldots \wedge g_n. \end{aligned}$$

Proof. Since exactness is a local property we can assume we are working with free modules over a ring and that the original sequence is splitting, i.e. $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{G}$.

We prove the exactness of the first sequence by induction over the rank r of $\mathcal{E}$. For $r = 0$ we have $\mathcal{F} \cong \mathcal{G}$ and the sequence is a well-known sequence of multilinear algebra (see for example [43, §86,Aufgabe 20]), where the maps are as stated in our Lemma.

For $r \geq 1$ we fix an element v in a basis of $\mathcal{E}$. Because $\mathcal{F} \cong \mathcal{E} \oplus \mathcal{G}$ it is also an element of a basis of $\mathcal{F}$. We write $\mathcal{E} = \langle v \rangle \oplus \mathcal{U}$, where U is a free module of rank $r - 1$. Similarly we get $\mathcal{F} = \langle v \rangle \oplus \mathcal{U}'$.

We have $\bigwedge^k \mathcal{F} \cong \bigwedge^k \mathcal{U}' \oplus \left(\bigwedge^{k-1} \mathcal{U}'\right) \otimes \langle v \rangle$ by the map which concentrates every contribution of v to the last component. Because taking the tensor product with a free rank 1 module is an isomorphism we even have $\bigwedge^k \mathcal{F} \cong \bigwedge^k \mathcal{U}' \oplus \bigwedge^{k-1} \mathcal{U}'$. Similarly for $\bigwedge^k \mathcal{E}$.

This allows us to write the sequence as the direct sum of two sequences we know are exact by induction:

$$
\begin{aligned}
0 \longrightarrow \bigwedge^{n} \mathcal{U} \longrightarrow \bigwedge^{n} \mathcal{U}' \longrightarrow \left(\bigwedge^{n-1} \mathcal{U}'\right) \otimes \operatorname{Sym}^1 \mathcal{G} \longrightarrow \\
\cdots \longrightarrow \left(\bigwedge^{1} \mathcal{U}'\right) \otimes \operatorname{Sym}^{n-1} \mathcal{G} \longrightarrow \operatorname{Sym}^n \mathcal{G} \longrightarrow 0
\end{aligned}
$$

and

$$
\begin{aligned}
0 \longrightarrow \bigwedge^{n-1} \mathcal{U} \longrightarrow \bigwedge^{n-1} \mathcal{U}' \longrightarrow \left(\bigwedge^{n-2} \mathcal{U}'\right) \otimes \operatorname{Sym}^1 \mathcal{G} \longrightarrow \\
\cdots \longrightarrow \left(\bigwedge^{1} \mathcal{U}'\right) \otimes \operatorname{Sym}^{n-2} \mathcal{G} \\
\longrightarrow \operatorname{Sym}^{n-1} \mathcal{G} \longrightarrow 0 \longrightarrow 0.
\end{aligned}
$$

Thus the sum sequence is exact as well. We want to prove that the map of the sum sequence is correct. Take a look at the diagram.

$$\begin{array}{ccc}
\left(\bigwedge^k \mathcal{F}\right) \otimes \mathrm{Sym}^{n-k} \mathcal{G} & \longrightarrow & \left(\bigwedge^{k-1} \mathcal{F}\right) \otimes \mathrm{Sym}^{n-k+1} \mathcal{G} \\
\uparrow & & \uparrow \\
\left(\bigwedge^k \mathcal{U}'\right) \otimes \mathrm{Sym}^{n-k} \mathcal{G} & \longrightarrow & \left(\bigwedge^{k-1} \mathcal{U}'\right) \otimes \mathrm{Sym}^{n-k+1} \mathcal{G} \\
\oplus & & \oplus \\
\left(\bigwedge^{k-1} \mathcal{U}'\right) \otimes \mathrm{Sym}^{n-k} \mathcal{G} & \longrightarrow & \left(\bigwedge^{k-1} \mathcal{U}'\right) \otimes \mathrm{Sym}^{n-k+1} \mathcal{G}
\end{array}$$

If we map an element $f_1 \wedge \ldots \wedge f_k \otimes g + f_1' \wedge \ldots \wedge f_{k-1}' \otimes g'$ via the lower right route we get

$$\begin{aligned}
&\sum_{i=1}^{k} (-1)^{i-1} f_1 \wedge \ldots \wedge f_{i-1} \wedge f_{i+1} \wedge \ldots \wedge f_k \otimes \varphi(f_i) \cdot g \\
+&\sum_{i=1}^{k-1} (-1)^{i-1} f_1' \wedge \ldots \wedge f_{i-1}' \wedge f_{i+1}' \wedge \ldots \wedge f_{k-1}' \wedge v \otimes \varphi(f_i') \cdot g'.
\end{aligned}$$

If we map via the upper left route we get

$$\begin{aligned}
&\sum_{i=1}^{k} (-1)^{i-1} f_1 \wedge \ldots \wedge f_{i-1} \wedge f_{i+1} \wedge \ldots \wedge f_k \otimes \varphi(f_i) \cdot g \\
+&\sum_{i=1}^{k-1} (-1)^{i-1} f_1' \wedge \ldots \wedge f_{i-1}' \wedge f_{i+1}' \wedge \ldots \wedge f_{k-1}' \wedge v \otimes \varphi(f_i') \cdot g' \\
+&(-1)^{k-1} f_1' \wedge \ldots \wedge f_{k-1}' \otimes \varphi(v) \cdot g',
\end{aligned}$$

but since $v \in \mathcal{E}$ we have $\varphi(v) = 0$, so the diagram commutes.

The second sequence works similarly. Again we start with a locally free sheaf $\mathcal{E}$ of rank 0, where the dual of the exterior power case gives us the sequence. For the dual sheaves we have the canonical isomorphisms $(\bigwedge^k \mathcal{F})^\vee \cong \bigwedge^k(\mathcal{F}^\vee)$ and as we work over a ring of characteristic 0 we also have $(\mathrm{Sym}^k \mathcal{F})^\vee \cong \mathrm{Sym}^k(\mathcal{F}^\vee)$ [43, Satz 83.7 and Satz 86.12].

We can also write the symmetric product of $\mathcal{F} \cong \mathcal{U}' \oplus \langle v \rangle$ as a direct sum as follows:

$$\mathrm{Sym}^k \mathcal{F} \cong \mathrm{Sym}^k \mathcal{U}' \oplus \mathrm{Sym}^{k-1} \mathcal{F} \otimes \langle v \rangle \cong \mathrm{Sym}^k \mathcal{U}' \oplus \mathrm{Sym}^{k-1} \mathcal{F}.$$

Notice how this time we have $\mathcal{F}$ itself in the second summand, so we have to do a double induction over the exponent k and the rank. $\square$

For us the most important part of these sequences are the maps $\bigwedge^n \mathcal{F} \longrightarrow \left(\bigwedge^{n-1} \mathcal{F}\right) \otimes \operatorname{Sym}^1 \mathcal{G}$ and $\operatorname{Sym}^n \mathcal{F} \longrightarrow \left(\operatorname{Sym}^{n-1} \mathcal{F}\right) \otimes \bigwedge^1 \mathcal{G}$, as they allow us to explicitly describe the exterior and symmetric powers of a kernel bundle as another kernel bundle.

Because we will first take the exterior power and afterwards the symmetric power we will need the following two lemmas which shows that we can in the same way describe the kernel of a sequence which is only left exact. The two lemmas generalize [36, Proposition 4.1].

Lemma 4.2.2. *Let* $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \ldots \longrightarrow 0$ *be an exact sequence of locally free sheaves of finite rank over a scheme over a field of characteristic* 0*, where* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ *is the second map. Then* $\bigwedge^n \mathcal{E}$ *is the kernel of the map*

$$\bigwedge^n \mathcal{F} \longrightarrow \left(\bigwedge^{n-1} \mathcal{F}\right) \otimes \mathcal{G},$$
$$f_1 \wedge \ldots \wedge f_n \mapsto \sum_{i=1}^{n} (-1)^{i-1} f_1 \wedge \ldots \wedge f_{i-1} \wedge f_{i+1} \wedge \ldots \wedge f_k \otimes \varphi(f_i).$$

Also, $\operatorname{Sym}^n \mathcal{E}$ *is the kernel of the map*

$$\operatorname{Sym}^n \mathcal{F} \longrightarrow \left(\operatorname{Sym}^{n-1} \mathcal{F}\right) \otimes \mathcal{G},$$
$$f_1 \cdots f_n \mapsto \sum_{i=1}^{n} f_1 \cdots f_{i-1} \cdot f_{i+1} \cdots f_k \otimes \varphi(f_i).$$

Proof. Because all sheaves in the sequence are locally free so are the kernels by induction starting from the right. We take the short exact sequence $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \operatorname{im} \varphi \longrightarrow 0$ and construct the sequences of Lemma 4.2.1.

Let's look at the sequence for the exterior power. The kernel of the map is $\bigwedge^n \mathcal{F} \longrightarrow \left(\bigwedge^{n-1} \mathcal{F}\right) \otimes \operatorname{im} \varphi$ is $\bigwedge^n \mathcal{E}$. Because of the local freeness of the involved modules the map $\left(\bigwedge^{n-1} \mathcal{F}\right) \otimes \operatorname{im} \varphi \longrightarrow \left(\bigwedge^{n-1} \mathcal{F}\right) \otimes \mathcal{G}$ is injective, so the kernel doesn't change if we concatenate with this map. The same is true for Sym^n. $\square$

Lemma 4.2.3. *Let* $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G}$ *be an exact sequence of locally free sheaves of finite rank over a smooth curve over a field of characteristic*

0, *where* $\varphi : \mathcal{F} \longrightarrow \mathcal{G}$ *is the second map. Then* $\bigwedge^n \mathcal{E}$ *and* $\operatorname{Sym}^n \mathcal{E}$ *have the same description as in Lemma 4.2.2.*

Proof. The image $\operatorname{im}\varphi \subseteq \mathcal{G}$ is torsion free as a subsheaf of $\mathcal{G}$. Thus $\operatorname{im}\varphi$ is locally free because over every point the structure sheaf is a principal ideal domain [40, Theorem II.1.1.6]. We take the short exact sequence $0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \operatorname{im}\varphi \longrightarrow 0$ and construct the sequences of Lemma 4.2.1.

The rest of the proof is the same as for Lemma 4.2.2. □

4.3 Rank, degree and slope

To work more easily with the degree and slope of the sheaves involved we present some rank and degree computations.

First note that degree and rank are additive. In particular for a short exact sequence, the degree and the rank of the middle sheaf is the sum of the degrees and ranks respectively of the outer sheaves.

Lemma 4.3.1. *Let* $\mathcal{E}$ *and* $\mathcal{F}$ *be locally free sheaves. We have* $\deg(\mathcal{F} \otimes \mathcal{E}) = \operatorname{rank}\mathcal{E} \cdot \deg\mathcal{F} + \operatorname{rank}\mathcal{F} \cdot \deg\mathcal{E}$ *and* $\operatorname{rank}\mathcal{F} \otimes \mathcal{E} = \operatorname{rank}\mathcal{F} \cdot \operatorname{rank}\mathcal{E}$.

Proof. [46, Lemma 1.16]. □

Lemma 4.3.2. *Let* $\mathcal{F}$ *be a locally free sheaf of finite* $\operatorname{rank}\mathcal{F} \geq 1$ *on a smooth projective curve over an algebraically closed field* K *and* $n \in \mathbb{N}_{>0}$. *We have* $\operatorname{rank}\operatorname{Sym}^n \mathcal{F} = \binom{n+\operatorname{rank}\mathcal{F}-1}{n}$ *and for the degree we have* $\deg\operatorname{Sym}^n \mathcal{F} = \binom{n+\operatorname{rank}\mathcal{F}-1}{n-1}\deg\mathcal{F}$.

Also $\operatorname{rank}\bigwedge^n \mathcal{F} = \binom{\operatorname{rank}\mathcal{F}}{n}$ *and* $\deg\bigwedge^n \mathcal{F} = \binom{\operatorname{rank}\mathcal{F}-1}{n-1}\deg\mathcal{F}$.

Proof. We want to compute this via induction over $n + \operatorname{rank}\mathcal{F}$. For $n = 1$ the assertions are clear. For Sym^n of a line bundle $\mathcal{L}$ use that $\operatorname{Sym}^n \mathcal{L} = \mathcal{L}^{\otimes n}$. So the assertions are also true for every sheaf of rank 1.

We apply Lemma 4.1.1 to the dual of $\mathcal{F}$ and dualize again to get a short exact sequence $0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow 0$, where $\mathcal{U}$ has one rank less than $\mathcal{F}$ and $\mathcal{L}$ is a line bundle. Note that $\deg\mathcal{L} = \deg\mathcal{F} - \deg\mathcal{U}$.

We apply Lemma 4.2.1 to the sequence $0 \longrightarrow \mathcal{U} \longrightarrow \mathcal{F} \longrightarrow \mathcal{L} \longrightarrow 0$. Because of $\bigwedge^2 \mathcal{L} = 0$ we get the short exact sequence

$$0 \longrightarrow \operatorname{Sym}^n \mathcal{U} \longrightarrow \operatorname{Sym}^n \mathcal{F} \longrightarrow \left(\operatorname{Sym}^{n-1} \mathcal{F}\right) \otimes \mathcal{L} \longrightarrow 0.$$

First we compute the rank of the symmetric powers by induction over $n + \operatorname{rank} F$.

$$\begin{aligned}\operatorname{rank} \operatorname{Sym}^n \mathcal{F} &= \operatorname{rank} \operatorname{Sym}^n \mathcal{U} + \operatorname{rank}\left(\mathcal{L} \otimes \operatorname{Sym}^{n-1} \mathcal{F}\right)\\ &= \operatorname{rank} \operatorname{Sym}^n \mathcal{U} + \operatorname{rank} \operatorname{Sym}^{n-1} \mathcal{F}\\ &= \binom{n + \operatorname{rank} \mathcal{F} - 2}{n} + \binom{n + \operatorname{rank} \mathcal{F} - 2}{n-1}\\ &= \binom{n + \operatorname{rank} \mathcal{F} - 1}{n}.\end{aligned}$$

Now we do induction over $n + \operatorname{rank} F$ for the degree.

$$\begin{aligned}\deg \operatorname{Sym}^n \mathcal{F} &= \deg \operatorname{Sym}^n \mathcal{U} + \deg\left(\mathcal{L} \otimes \operatorname{Sym}^{n-1} \mathcal{F}\right)\\ &= \deg \operatorname{Sym}^n \mathcal{U} + \deg \operatorname{Sym}^{n-1} \mathcal{F} + \operatorname{rank}\left(\operatorname{Sym}^{n-1} \mathcal{F}\right) \cdot \deg \mathcal{L}\\ &= \binom{n + \operatorname{rank} \mathcal{F} - 2}{n-1} \deg(\mathcal{U}) + \binom{n + \operatorname{rank} \mathcal{F} - 2}{n-2} \deg \mathcal{F}\\ &+ \binom{n + \operatorname{rank} \mathcal{F} - 2}{n-1} \cdot (\deg \mathcal{F} - \deg \mathcal{U})\\ &= \binom{n + \operatorname{rank} \mathcal{F} - 2}{n-2} \deg \mathcal{F} + \binom{n + \operatorname{rank}(\mathcal{F}) - 2}{n-1} \cdot \deg \mathcal{F}\\ &= \binom{n + \operatorname{rank} \mathcal{F} - 1}{n-1} \deg \mathcal{F}.\end{aligned}$$

For $\bigwedge^n \mathcal{F}$ we consider the short exact sequence

$$0 \longrightarrow \bigwedge^n \mathcal{U} \longrightarrow \bigwedge^n \mathcal{F} \longrightarrow \left(\bigwedge^{n-1} \mathcal{U}^{n-1}\right) \otimes \mathcal{L} \longrightarrow 0.$$

The right map is on affine subsets given by concentration of the contributions of $\mathcal{L}$ on the last factor. For $n \geq 2$ we do induction over rank F.

$$\begin{aligned}
\operatorname{rank} \bigwedge^n \mathcal{F} &= \operatorname{rank} \bigwedge^n \mathcal{U} + \operatorname{rank} \left(\mathcal{L} \otimes \bigwedge^{n-1} \mathcal{U} \right) \\
&= \operatorname{rank} \bigwedge^n \mathcal{U} + \operatorname{rank} \bigwedge^{n-1} \mathcal{U} \\
&= \binom{\operatorname{rank} \mathcal{U}}{n} + \binom{\operatorname{rank} \mathcal{U}}{n-1} \\
&= \binom{\operatorname{rank} \mathcal{F}}{n}.
\end{aligned}$$

And similarly for the degree.

$$\begin{aligned}
\deg \bigwedge^n \mathcal{F} &= \deg \bigwedge^n \mathcal{U} + \deg \left(\mathcal{L} \otimes \bigwedge^{n-1} \mathcal{U} \right) \\
&= \deg \bigwedge^n \mathcal{U} + \deg \bigwedge^{n-1} \mathcal{U} + \operatorname{rank} \left(\bigwedge^{n-1} \mathcal{U} \right) \cdot \deg \mathcal{L} \\
&= \binom{\operatorname{rank} \mathcal{U} - 1}{n-1} \deg \mathcal{U} + \binom{\operatorname{rank}(\mathcal{U}) - 1}{n-2} \deg \mathcal{U} \\
&+ \binom{\operatorname{rank} \mathcal{U}}{n-1} \cdot (\deg \mathcal{F} - \deg \mathcal{U}) \\
&= \binom{\operatorname{rank} \mathcal{U}}{n-1} \deg \mathcal{U} + \binom{\operatorname{rank} \mathcal{U}}{n-1} \cdot (\deg \mathcal{F} - \deg \mathcal{U}) \\
&= \binom{\operatorname{rank} \mathcal{F} - 1}{n-1} \deg \mathcal{F}.
\end{aligned}$$

□

Note that the rank of $\operatorname{Sym}^n(\mathcal{F})$ of a rank 1 sheaf stays 1 as we have used in the proof.

Corollary 4.3.3. *Let $\mathcal{F}$ and $\mathcal{E}$ be locally free sheaves on X and $s \in \mathbb{N}_{>0}$. We have $\mu(\bigwedge^s \mathcal{F}) = \mu(\operatorname{Sym}^s \mathcal{F}) = s \cdot \mu(\mathcal{F})$ and $\mu(\mathcal{F} \otimes \mathcal{E}) = \mu(\mathcal{F}) + \mu(\mathcal{E})$.*

Proof. This follows from Lemma 4.3.1 and Lemma 4.3.2. □

Lemma 4.3.4. *Let $\mathcal{F}$ be a locally free sheaf over a normal projective variety over an algebraically closed field of characteristic* 0*. Fix $n \in \mathbb{N}_{>0}, k \in \mathbb{Z}$. The following are equivalent:*

1. *$\mathcal{F}$ is semistable.*
2. *$\operatorname{Sym}^n \mathcal{F}$ is semistable.*
3. *$\mathcal{F} \otimes \mathcal{O}(k)$ is semistable.*

Also the following is true: If $\mathcal{F}$ is semistable so is $\bigwedge^n \mathcal{F}$.

Proof. Let first $\mathcal{F}$ be not semistable. Then a destabilizing subsheaf $\mathcal{E} \subset \mathcal{F}$ gives a destabilizing subsheaf $\operatorname{Sym}^n \mathcal{E} \subset \operatorname{Sym}^n \mathcal{F}$ and a destabilizing subsheaf $\mathcal{E} \otimes \mathcal{O}(k) \subset \mathcal{F} \otimes \mathcal{O}(k)$ (as $\mathcal{O}(k)$ is flat).

If however $\mathcal{F}$ is semistable, then it shown (for bundles over characteristisc 0) in [35, Corollary 3.2.10] that the symmetric and exterior powers are also semistable. □

4.4 Destabilizing subbundles and destabilizing sections

We want to determine if $\mathcal{F}$ is semistable by only looking at global sections. By definition it is semistable exactly when there is no proper subbundle $\mathcal{E}$ with $\mu(\mathcal{E}) > \mu(\mathcal{F})$.

Theorem 4.4.1. *Let $\mathcal{F}$ be a locally free sheaf on a smooth projective curve X of genus g over an algebraically closed field of characteristic* 0 *and $r := \operatorname{rank} \mathcal{F}$. Then $\mathcal{F}$ is semistable if and only if there does not exist a nontrivial global section of $(\operatorname{Sym}^q \mathcal{F}) \otimes \mathcal{O}(k)$, where $q = (g - 1 + \deg X)n + 1$, $n = \frac{r(r-1)}{\gcd(r, \deg \mathcal{F})}$ and $k = \left\lceil \frac{-q\mu(\mathcal{F})}{\deg X} \right\rceil - 1$.*

Proof. First assume that $\mathcal{F}$ is not semistable. Take a destabilizing subbundle $\mathcal{E} \subseteq \mathcal{F}$ of rank $s < r$. Then

$$\frac{\deg \mathcal{E}}{s} = \mu(\mathcal{E}) > \mu(\mathcal{F}) = \frac{\deg \mathcal{F}}{r}$$

and thus

$$\mu(\mathcal{E}) - \mu(\mathcal{F}) = \left(\frac{r \cdot \deg \mathcal{E} - s \cdot \deg \mathcal{F}}{r \cdot s} \right) \geq \frac{\gcd(r, \deg \mathcal{F})}{r(r-1)} = \frac{1}{n} > 0.$$

With this we calculate

$$\begin{aligned}
\mu\left((\operatorname{Sym}^q \mathcal{E}) \otimes \mathcal{O}(k)\right) &= q\mu(\mathcal{E}) + k \deg X \\
&= q\mu(\mathcal{E}) + \left(\left\lceil \frac{-q\mu(\mathcal{F})}{\deg X} \right\rceil - 1\right) \deg X \\
&\geq q(\mu(\mathcal{E}) - \mu(\mathcal{F})) - \deg X \\
&\geq \frac{q}{n} - \deg X \\
&= \frac{(g - 1 + \deg X)n + 1}{n} - \deg X \\
&= g - 1 + \frac{1}{n} \\
&> g - 1.
\end{aligned}$$

Thus $(\operatorname{Sym}^q \mathcal{E}) \otimes \mathcal{O}(k)$ has a global section by Lemma 4.1.2 and this is also a global section of $(\operatorname{Sym}^q \mathcal{F}) \otimes \mathcal{O}(k)$.

Now let's assume $\mathcal{F}$ is semistable. Then so is $(\operatorname{Sym}^q \mathcal{F}) \otimes \mathcal{O}(k)$ by Lemma 4.3.4. We calculate its slope

$$\begin{aligned}
\mu\left((\operatorname{Sym}^q \mathcal{F}) \otimes \mathcal{O}(k)\right) &= q\mu(\mathcal{F}) + k \deg X \\
&= q\mu(\mathcal{F}) + \left(\left\lceil \frac{-q\mu(\mathcal{F})}{\deg X} \right\rceil - 1\right) \deg X \\
&< q\mu(\mathcal{F}) + \frac{-q\mu(\mathcal{F})}{\deg X} \deg X \\
&= 0.
\end{aligned}$$

Being semistable with negative slope it can't have any nontrivial global sections by Lemma 4.1.3, so we are done. □

We will also prove the following variant of this theorem, which allows for a smaller q (by at least a factor $r - 1$), but introduces the need to compute the global sections of several exterior powers. In general it is unclear which of these is easier to compute. While Theorem 4.4.1 seems simpler, there are cases where a (parallelized) implementation of Theorem 4.4.2 is much faster and easier (see Example 4.6.5). Note also that for rank $r = 2$ the theorems are identical.

Theorem 4.4.2. *Let $\mathcal{F}$ be a locally free sheaf on a smooth projective curve X of genus g over an algebraically closed field of characteristic* 0 *and* $r :=$

rank $\mathcal{F}$. *Then $\mathcal{F}$ is semistable if and only if for every $s < r$ there does not exist a nontrivial global section of* $\operatorname{Sym}^q(\bigwedge^s \mathcal{F}) \otimes \mathcal{O}(k)$, *where* $q = (g - 1 + \deg X)n + 1$, $n = \frac{r}{\gcd(r, s\cdot\deg\mathcal{F})}$ *and* $k = \left\lceil \frac{-qs\mu(\mathcal{F})}{\deg X} \right\rceil - 1$.

Proof. First assume that $\mathcal{F}$ is not semistable. Take a destabilizing subbundle $\mathcal{E} \subseteq \mathcal{F}$ of rank $s < r$. By taking the s-th exterior product we get $\bigwedge^s \mathcal{E} \subseteq \bigwedge^s \mathcal{F}$, where $\bigwedge^s \mathcal{E}$ is the determinant bundle of $\mathcal{E}$ and as such invertible. Then

$$\frac{\deg\mathcal{E}}{s} = \mu(\mathcal{E}) > \mu(\mathcal{F}) = \frac{\deg\mathcal{F}}{r}$$

and thus

$$s(\mu(\mathcal{E}) - \mu(\mathcal{F})) = \left(\deg\mathcal{E} - \frac{s\deg\mathcal{F}}{r}\right) \geq \frac{\gcd(r, s\deg\mathcal{F})}{r} = \frac{1}{n} > 0.$$

With this we calculate

$$\begin{aligned}
\mu\left(\operatorname{Sym}^q\left(\bigwedge^s \mathcal{E}\right) \otimes \mathcal{O}(k)\right) &= qs\mu(\mathcal{E}) + k\deg X \\
&= qs\mu(\mathcal{E}) + \left(\left\lceil \frac{-qs\mu(\mathcal{F})}{\deg X} \right\rceil - 1\right)\deg X \\
&\geq qs(\mu(\mathcal{E}) - \mu(\mathcal{F})) - \deg X \\
&\geq \frac{q}{n} - \deg X \\
&= \frac{(g - 1 + \deg X)n + 1}{n} - \deg X \\
&= g - 1 + \frac{1}{n} \\
&> g - 1.
\end{aligned}$$

Thus by Lemma 4.1.2 the invertible sheaf $\operatorname{Sym}^q(\bigwedge^s \mathcal{E}) \otimes \mathcal{O}(k)$ has a global section and this is also a global section of $\operatorname{Sym}^q(\bigwedge^s \mathcal{F}) \otimes \mathcal{O}(k)$.

Now let's assume $\mathcal{F}$ is semistable. Then so is $\operatorname{Sym}^q(\bigwedge^s \mathcal{F}) \otimes \mathcal{O}(k)$ by

Lemma 4.3.4. We calculate its slope

$$\begin{aligned}\mu\left(\operatorname{Sym}^q\left(\bigwedge^s \mathcal{F}\right)\otimes\mathcal{O}(k)\right) &= qs\mu(\mathcal{F}) + k\deg X\\ &= qs\mu(\mathcal{F}) + \left(\left\lceil\frac{-qs\mu(\mathcal{F})}{\deg X}\right\rceil - 1\right)\deg X\\ &< qs\mu(\mathcal{F}) + \left(\frac{-qs\mu(\mathcal{F})}{\deg X}\right)\deg X\\ &= 0.\end{aligned}$$

Being semistable with negative slope it can have no nontrivial global sections, so we are done. □

4.5 Syzygy sheaves

We want to describe the algorithm to decide semistability for kernel (or syzygy) sheaves.

Let $\mathcal{F}$ be a *kernel sheaf* over a smooth projective curve $X = \operatorname{Proj} S$, S a normal 2-dimensional standard graded domain, which means that it is embedded in an exact sequence as follows.

$$0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \xrightarrow{A} \bigoplus_{j=1}^{m} \mathcal{O}_X(-d_j).$$

As F is a subsheaf of a free sheaf it is torsion free and thus, because X is a smooth curve, it is already locally free [40, Theorem II.1.1.6].

Mainly we deal with syzygy sheaves, i.e. the case where $m = 1$ and A is a single row matrix $A = (f_1, \dots, f_n)$. Then we denote the kernel as $\operatorname{Syz}(f_1, \dots, f_n)$.

Remark 4.5.1. All vector bundles on a smooth projective curve $X = \operatorname{Proj} S$ are isomorphic to bundles described in this way, at least after twisting and change of coordinate ring. Concretely we have for a rank r sheaf $\mathcal{F}$ as in [5, Lemma 2.3] a presentation $\mathcal{O}_X^{r+1} \longrightarrow \mathcal{F} \longrightarrow 0$. The kernel is a line bundle.

The determinant sheaf of $\mathcal{F}$ is a line bundle. There is a twist n, where it becomes a very ample line bundle $\mathcal{L} := \det \mathcal{F} \otimes \mathcal{O}_X(l)$. To every very ample line bundle on X exists an embedding of X, we take the one for $\mathcal{L}$. In practice we take as the new coordinate ring the ring generated by the determinant bundle (The Proj of which is again the curve differently embedded). With regard to this embedding the twisting sheaf becomes $\mathcal{L}$. Thus without loss of generality we can assume that the determinant bundle of $\mathcal{F}$ is $\mathcal{O}_X(1)$. Then the kernel line bundle must also be a twist of the structure sheaf.

In this case we have a short exact sequence

$$0 \longrightarrow \mathcal{O}_X(d) \longrightarrow \bigoplus_{i=1}^{r+1} \mathcal{O}_X(e_i) \longrightarrow \mathcal{F} \longrightarrow 0.$$

By dualizing we can describe $\mathcal{F}^\vee$ as a kernel bundle of a map

$$f : \bigoplus_{i=1}^{r+1} \mathcal{O}_X(-e_i) \longrightarrow \mathcal{O}_X(-d),$$

which can be given by a matrix. Usually we will set $d = 0$ and adjust the other twists.

Remark 4.5.2. For our method it is important to determine the rank and the degree of $\mathcal{F}$ in an exact sequence $0 \longrightarrow \mathcal{F} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \xrightarrow{A} \mathcal{O}_X$. Let $X = \operatorname{Proj} S, S = K[x_1, \dots, x_n]/I$ integrally closed and $J \subseteq S$ the ideal generated by the entries of $A \neq 0$. Because of additivity the rank is just $\operatorname{rank} \mathcal{F} = n - 1$. The degree can be harder to compute:

Let $\mathcal{L} = \operatorname{im} A \subseteq \mathcal{O}_X$. The degree of $\bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i)$ is

$$\deg \bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) = -\deg X \cdot \sum_{i=1}^{n} e_i.$$

Because of the additivity of degrees we have $\deg \mathcal{F} = -\deg X \cdot \sum_{i=1}^{n} e_i - \deg \mathcal{L}$. If A is surjective (which is true if and only if J is S_+-primary), computing the degree is easy, as then $\deg \mathcal{L} = \deg \mathcal{O}_X = 0$.

Otherwise we compute $\deg \mathcal{L}$ with the Hilbert polynomial as follows. Observe that for large $n \in \mathbb{N}$ we have

$$\begin{aligned} \dim H^0(X, \mathcal{L} \otimes \mathcal{O}_X(n)) &= \chi(\mathcal{L} \otimes \mathcal{O}_X(n)) \\ &= \deg(\mathcal{L} \otimes \mathcal{O}_X(n)) + 1 - g \\ &= \deg(\mathcal{L}) + n \cdot \deg X + 1 - g. \end{aligned}$$

Now $\dim H^0(X, \mathcal{L} \otimes \mathcal{O}_X(n))$ is exactly the number of degree n elements of J, i.e. the Hilbert polynomial of the module J with indeterminate n: $\mathrm{Hilb}(J)$. We have

$$\begin{aligned}\mathrm{Hilb}(J) &= \mathrm{Hilb}(S) - \mathrm{Hilb}(S/J)\\ &= n \cdot \deg X + (1-g) - \mathrm{Hilb}(S/J).\end{aligned}$$

Thus we deduce $\deg \mathcal{L} = -\mathrm{Hilb}(S/J)$ and

$$\deg \mathcal{F} = -\deg X \cdot \sum_{i=1}^{n} e_i + \mathrm{Hilb}(S/J).$$

Lemma 4.5.3. *Let X be a scheme, $q \in \mathbb{N}_{>0}, n, m \in \mathbb{N}$, $e_1 \dots, e_n, d_1 \dots, d_m \in \mathbb{Z}$. We have*

$$\bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \otimes \bigoplus_{j=1}^{m} \mathcal{O}_X(-d_j) = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} \mathcal{O}_X(-d_j - e_i).$$

$$\mathrm{Sym}^q \left(\bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \right) = \bigoplus_{(a_1,\dots,a_n)\in\mathbb{N}^n, \sum_{i=1}^n a_i = q} \mathcal{O}_X \left(-\sum_{i=1}^{n} a_i \cdot e_i \right),$$

$$\bigwedge^q \left(\bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \right) = \bigoplus_{I \subseteq \{1,\dots,n\}, \#I = q} \mathcal{O}_X \left(-\sum_{i\in I} e_i \right).$$

Proof. Take $g_1, \dots, g_n$ as a local basis of $\bigoplus_{i=1}^n \mathcal{O}_X(-e_i)$. The lower two identifications are given by the basis $g_1^{a_1} \cdots g_n^{a_n}, \sum_{i=1}^n a_i = q$, for the symmetric power and the basis $\prod_{i\in I} g_i, \#I = q$, for the exterior power. With these bases we have locally a map for the symmetric power for each $(a_1, \dots, a_n)$ as follows

$$\mathrm{Sym}^q \left(\bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \right) \longrightarrow \bigotimes_{i=1}^{n} \mathcal{O}_X(-e_i)^{\otimes a_i} = \mathcal{O}_X \left(-\sum_{i=1}^{n} a_i \cdot e_i \right).$$

These maps are surjective and only the corresponding basis element maps to a nonzero value and they glue. The exterior power works analogously. $\square$

Lemma 4.5.4. *Let $\mathcal{F} = \ker A$ be a kernel sheaf over a smooth curve $X = \operatorname{Proj} S$. Let $A : \bigoplus_{i=1}^n \mathcal{O}_X(-e_i) \longrightarrow \bigoplus_{j=1}^m \mathcal{O}_X(-d_j)$ and assume A sits in an exact sequence as in Lemma 4.2.3. Then $\operatorname{Sym}^q \mathcal{F} = \ker A_q$, where*

$$\begin{aligned} A_q : \operatorname{Sym}^q \left(\bigoplus_{i=1}^n \mathcal{O}_X(-e_i) \right) &= \bigoplus_{a \in I} \mathcal{O}_X(-a \cdot e) \longrightarrow \\ \left(\operatorname{Sym}^{q-1} \left(\bigoplus_{i=1}^n \mathcal{O}_X(-e_i) \right) \right) \otimes \bigoplus_{j=1}^m \mathcal{O}_X(-d_j) &= \bigoplus_{(b,j) \in J} \mathcal{O}_X(-b \cdot e - d_j). \end{aligned}$$

We index the columns of A_q by the set $I = \{a = (a_1, \dots, a_n) \in \mathbb{N}^n : \sum_{i=1}^n a_i = q\}$ and the rows by the set $J = \{(b,j) = (b_1, \dots, b_n, j) \in \mathbb{N}^{n+1} : \sum_{i=1}^n b_i = q-1, 1 \le j \le m\}$. The entries are

$$A_{q,(b_1,\dots,b_n,j),(a_1,\dots,a_n)} = \begin{cases} 0, & \text{if } \exists i \in \{1,\dots,n\} : b_i > a_i \\ a_{i^*} \cdot A_{j,i^*}, & \text{otherwise; } i^* \text{ unique s.t. } a_{i^*} > b_{i^*}. \end{cases}$$

Also, $\bigwedge^q \mathcal{F} = \ker A_{\bigwedge^q}$, where

$$A_{\bigwedge^q} : \bigwedge^q \left(\bigoplus_{i=1}^n \mathcal{O}_X(-e_i) \right) \longrightarrow \left(\bigwedge^{q-1} \left(\bigoplus_{i=1}^n \mathcal{O}_X(-e_i) \right) \right) \otimes \bigoplus_{j=1}^m \mathcal{O}_X(-d_j).$$

We index the columns of $A_{\bigwedge^q}$ by the subsets $I \subseteq \{1, \dots, n\}, \#I = q$ and the rows by the set of tuples (J,j), where $J \subseteq \{1,\dots,n\}, \#J = q-1$ and $1 \le j \le m$. The entries are

$$A_{\bigwedge^q,(J,j),I} = \begin{cases} 0, & \text{if } J \not\subset I \\ \operatorname{sign}(i^*, I) \cdot A_{j,i^*}, & \text{otherwise; } i^* \text{ unique s.t. } i^* \in I \setminus J. \end{cases}$$

Here $\operatorname{sign}(i^*, I) = (-1)^{\operatorname{pos}(i^*)}$, where $\operatorname{pos}(i^*)$ gives the position of i^* in I, induced by the order of $\{1, \dots, n\}$.

Proof. The matrices A_q and $A_{\bigwedge^q}$ are the explicit descriptions of the maps in Lemma 4.2.3 when the symmetric and exterior powers of direct sums of invertible sheaves are expressed as in the previous lemma. □

Example 4.5.5. Take a plane smooth curve and over it the map $\mathcal{O}(-3)^{\oplus 2} \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}$ which is given by the matrix $A = \begin{pmatrix} x^3 & y^3 & z^2 \end{pmatrix}$. We look at the matrix for the second symmetric power as given by Lemma 4.5.4:

$$A_2 : \mathcal{O}(-6)^{\oplus 3} \oplus \mathcal{O}(-5)^{\oplus 2} \oplus \mathcal{O}(-6) \longrightarrow \mathcal{O}(-3)^{\oplus 2} \oplus \mathcal{O}(-2).$$

It has the following entries (above and to the left we have written the summands to which the respective columns and rows correspond):

$$\begin{array}{c} \\ \mathcal{O}(-3) \\ \mathcal{O}(-3) \\ \mathcal{O}(-2) \end{array} \begin{array}{c} \begin{array}{cccccc} \mathcal{O}(-6) & \mathcal{O}(-6) & \mathcal{O}(-5) & \mathcal{O}(-6) & \mathcal{O}(-5) & \mathcal{O}(-4) \end{array} \\ \begin{pmatrix} 2x^3 & y^3 & z^2 & 0 & 0 & 0 \\ 0 & x^3 & 0 & 2y^3 & z^2 & 0 \\ 0 & 0 & x^3 & 0 & y^3 & 2z^2 \end{pmatrix} \end{array}.$$

With this background we can formulate the steps we have to follow.

Algorithm 4.5.6. This algorithm decides semistability following the method of Theorem 4.4.1.

1. Start with a smooth projective curve $X = \operatorname{Proj} S$ given by an integrally closed domain S and a map $\bigoplus_{i=1}^{n} \mathcal{O}_X(-e_i) \longrightarrow \mathcal{O}_X$ described by a matrix A.

2. Compute the genus g and the degree of X (with the Hilbert polynomial of X), $\operatorname{rank}(\ker A) = n - 1$ and the slope $\mu(\ker A) = -\frac{\deg \ker A}{n-1}$. From this compute q and k as in Theorem 4.4.1.

3. Compute A_q as in Lemma 4.5.4. It's a map

$$\bigoplus_{a \in I} \mathcal{O}_X(-e'_a) \longrightarrow \bigoplus_{b \in J} \mathcal{O}_X(-d'_b)$$

   for some finite index sets I, J and some degrees as computed in the Lemma.

4. Compute the dimension d of the vector space of global sections of the kernel of the k-th twist of A_q.

5. $\ker A$ is semistable if and only if $d = 0$.

Because in practice computing the dimension of the kernel becomes a lot more resource-intensive for larger q and the corresponding k it is advisable to first try some lower powers. If we are lucky the dimension of the kernel will already be nonzero in which case we would already know that the sheaf is not semistable.

If one wants to decide semistability with Theorem 4.4.2 instead of Theorem 4.4.1 the only difference in the algorithm is to compute $(A_{\bigwedge,s})_q$ instead of just A_q, where q, s and k have the values given in Theorem 4.4.2.

4.6 Examples and computations

In this section we will give several examples. In some of these examples we relate the outcome of our algorithm with more specific methods.

Example 4.6.1. Let's start with some examples that have a positive result, i.e. cases in which the algorithm can determine that the sheaf in question is semistable.

Let $X = \operatorname{Proj} S$, $S = \mathbb{C}[x, y, z]/f$, where f is a homogeneous polynomial such that S is normal and X smooth. In this example we will consider syzygy sheafs of of the form $\operatorname{Syz}(x^n, y^n, z^n)$, which have rank 2 and degree $-3n \cdot \deg X$.

As a first example we look at $f = x^4 + y^3z + z^4$, for which X has genus 3. The theorem tells us to look at the the symmetric power $q = 7$. For $n = 1$ we have to look at $k = \lceil \frac{3qn}{2} \rceil - 1 = 10$. There are no global sections of $\operatorname{Sym}^7(\operatorname{Syz}(x, y, z))(10)$, thus $\operatorname{Syz}(x, y, z)$ is semistable. For $n = 2$, we get $k = 20$ and again semistability. If we look further at $n \leq 10$, for $n = 3$, $n = 4$, $n = 8$ and $n = 9$ we find destabilizing sections, but for $n = 5$, $n = 6$, $n = 7$ and $n = 10$ the syzygy sheaves are again semistable as we don't find global sections of the seventh symmetric power in the twists given by the theorem.

We can also look at higher degree curves, for example for $f = x^{10} + y^9z + z^{10}$ and find that the syzygy sheaves $\operatorname{Syz}(x^n, y^n, z^n)$ for $n = 1$, $n = 2$ and $n = 3$ are semistable. For this curve we have to look at symmetric power $q = 46$ and for $n = 3$ the deciding twist is already 206. This means that the resulting matrices become relatively large and the computations take a while.

Example 4.6.2. We look at $S := \mathbb{C}[x,y,z]/(x^4+y^3z+z^4)$, the corresponding smooth projective curve $X \subseteq \mathbb{P}^2$ of genus 3 and the kernel sheaf $\mathcal{F} := \operatorname{Syz}(x^3,y^3,z^2)$ of the surjective map

$$A : \mathcal{O}_X(-3)^{\oplus 2} \oplus \mathcal{O}_X(-2) \twoheadrightarrow \mathcal{O}_X, (a_1,a_2,a_3) \mapsto (a_1x^3 + a_2y^3 + a_3z^2).$$

$\mathcal{O}_X(-3)^{\oplus 2} \oplus \mathcal{O}_X(-2)$ has degree $(-3-3-2)\cdot\deg(X) = -8\cdot 4 = -32$ and rank 3, and $\mathcal{O}_X$ has degree 0 and rank 1. Thus, because the map is surjective we can use the additivity of degree and rank to determine that $\mathcal{F}$ has degree -32, rank 2 and slope -16.

The theorem tells us to look at $q = 7$ and $k = q\cdot 4 - 1 = 27$. Indeed we find global sections there, which means that it is not semistable. But we will also see that already $q = 4$ and the corresponding $k = 15$ is enough.

For the first three symmetric powers there are no destabilizing global sections, i.e. $\Gamma(X, \operatorname{Sym}^q \mathcal{F}(k))$ is empty in the twist given by $k = q\cdot 4 - 1$ (the twist coming from Theorem 4.4.1) and lower.

We can see immediately from the curve equation that for $q = 1$ we have the global section (x,z,z^2) of $\mathcal{F}(4)$. The sheaf has slope 0, thus the proper subsheaf generated by the section shows that $\mathcal{F}$ is not stable, but we have not disproven semistability yet.

Let's take the 4-th symmetric power of this situation. For every global section of $\mathcal{F}(4)$ we immediately get a global section of the symmetric power $\operatorname{Sym}^4(\mathcal{F}(4)) = \operatorname{Sym}^4(\mathcal{F})(16)$ by taking all possible products of 4 possibly repeating factors out of x, z and z^2. Consider Lemma 4.5.3.

Explicitly for the global section (x,z,z^2) we get the global section

$$v = (x^4, x^3z, x^3z^2, x^2z^2, x^2z^3, x^2z^4, xz^3, xz^4, xz^5, xz^6, z^4, z^5, z^6, z^7, z^8).$$

Again $\operatorname{Sym}^4(\mathcal{F})(16)$ has slope 0. But we have $x^4 = -y^3z - z^4$, so all entries contain the factor z, which we can divide out. This way we get a new global section

$$v' = (-y^3 - z^3, x^3, x^3z, x^2z, x^2z^2, x^2z^3, xz^2, xz^3, xz^4, xz^5, z^3, z^4, z^5, z^6, z^7)$$

of $\operatorname{Sym}^4(\mathcal{F})(15)$. But $\operatorname{Sym}^4(\mathcal{F})(15)$ has negative slope, so we have shown that $\mathcal{F}$ is not semistable. This is a way in which symmetric powers make hidden destabilization visible.

The following example relates Grothendieck's splitting principle with our approach.

Example 4.6.3. The ring $S := \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2)$ describes a quadric curve. It is isomorphic to $\mathbb{P}^1$ and thus has genus 0. On it $\operatorname{Syz}(x^2, y^2, xz, yz)$ is locally free of rank 3. The algorithm tells us that this sheaf is not semistable. Let's look at the situation from a different perspective to make sense of this.

If we twist by 3 we trivially find global sections, for example $(z, 0, -x, 0)$. This does not give a destabilizing sheaf however, because the degree of $\operatorname{Syz}(x^2, y^2, xz, yz)(3)$ is $(9 - 8) \cdot 2 > 0$. The twist by 2 does not have global sections. Global sections in that twist would directly give destabilizing subsheaves.

As a sheaf on $\mathbb{P}^1$ the syzygy sheaf $\operatorname{Syz}(x^2, y^2, xz, yz)(2)$ is a direct sum of invertible sheaves and it has degree -4. So the only possibility without global sections is

$$\operatorname{Syz}(x^2, y^2, xz, yz)(2) = \mathcal{L}^{-1} \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2},$$

where $\mathcal{L}$ is the invertible sheaf of degree one on the quadric when seen as a $\mathbb{P}^1$. Note that $\mathcal{L}^2 = \mathcal{O}_X(1)$.

From this direct sum decomposition we can already see that the second symmetric power $\operatorname{Sym}^2(\operatorname{Syz}(x^2, y^2, xz, yz))$ will have three invertible summands of highest degree namely $(\mathcal{L}^{-1}(-2))^2 = \mathcal{O}_X(-5)$ and the second exterior power $\bigwedge^2(\operatorname{Syz}(x^2, y^2, xz, yz))$ will have one such summand. The twist of interest k in these powers becomes 5 and indeed if we twist by 5 the global section space of the symmetric power turns out to be 3-dimensional and of the exterior power 1-dimensional. These correspond - in accordance with the theorem - to destabilizing subsheaves.

Remark 4.6.4. Example 4.6.3 shows that on $\mathbb{P}^1$ one can use Grothendieck's splitting theorem to check semistability, no symmetric or exterior powers have to be computed explictly.

But even if $\mathbb{P}^1$ is given as a smooth quadric it is not clear how to find global sections by only looking at the homogeneous coordinate ring. The restriction of $\operatorname{Syz}(x, y, z)$ to any smooth quadric $X \subset \mathbb{P}^2$ is isomorphic to $\mathcal{L}^{-3} \otimes \mathcal{L}^{-3}$ (hence semistable), where $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^1}(1)$ under an isomorphism $\mathbb{P}^1 \cong X$, but $\mathcal{L}$ can not be seen by looking at the global sections of $\operatorname{Syz}(x, y, z)$ only. In Example 4.6.3 we help ourselves by inferring from the direct sum that there have to exist global sections in some power.

Example 4.6.5. We have seen that any destabilizing subsheaf will be made visible by high enough symmetric powers. But computing exterior powers can be very useful if there are destabilizing subsheaves of rank 2 or higher. For a concrete example consider $S := \mathbb{C}[x,y,z]/(x^n + y^n + z^n)$ and $\mathcal{F} = \operatorname{Syz}(x^4 + y^2z^2, y^4, z^4, x^7)$ (almost any combination of polynomials of degree 4,4,4 and 7 would do). The sheaf $\mathcal{F}$ is not semistable. The subsheaf $\mathcal{E} = \operatorname{Syz}(x^4 + y^2z^2, y^4, z^4)$ is a destabilizing subsheaf of rank 2:

$$\mu(\mathcal{E}) = \frac{-12n}{2} > \frac{-19n}{3} = \mu(\mathcal{F}).$$

Because of $\mathcal{E}$'s shape as a rank 2 syzygy sheaf its second exterior power becomes $\mathcal{O}_X(-12)$ and has a global section if twisted by 12. However even for $n = 5$ the fourth symmetric power $\operatorname{Sym}^4(\mathcal{F})$ is the lowest power with a destabilizing global section. For $n = 9$ the lowest symmetric power with a destabilizing global section is $\operatorname{Sym}^{10}(\mathcal{F})$. Consider Table 4.1.

n	Genus g	$q_{\min}$	q as of Theorem 4.4.1
1	0	1	1
2	0	1	7
3	1	2	7
4	3	1	31
5	6	4	61
6	10	2	31
7	15	8	127
8	21	2	169
9	28	10	73
10	36	8	271

Table 4.1: Table detailing the situation of Example 4.6.5 for several n. $q_{\min}$ is the lowest power $q = q_{\min}$ for which $\operatorname{Sym}^q(\mathcal{F})$ has a destabilizing section. We computed this with our implementation of the algorithm.

Example 4.6.6. Smooth curves X in the projective plane have genus $g = \frac{(\deg X-1)(\deg X-2)}{2}$. So with these, we only get $g = 1, 3, 6, 10, \ldots$. But embedded in higher dimensional projective space we can find curves to work over of any genus.

A smooth curve of type (a,b) in $\mathbb{P}^1 \times \mathbb{P}^1$ has genus $g = (a-1)(b-1)$. As an explicit example look at the curve X given by the relation $f = x_0^3 y_0(y_0 + y_1) +$

$x_1^3 y_1(y_0+2y_1) = 0$ for $([x_0 : x_1], [y_0 : y_1]) \in \mathbb{P}^1 \times \mathbb{P}^1$. We can work with this in our algorithm by using the Segre embedding given by $\mathbb{C}[z_{00}, z_{01}, z_{10}, z_{11}] \longrightarrow \mathbb{C}[x_0, x_1, y_0, y_1], z_{ij} \mapsto x_i y_j$.

X is then given by generators of the ring-kernel of the Segre embedding $z_{11}z_{00} - z_{10}z_{01}$ together with the preimage of (f), generated by $z_{00}^2(z_{00} + z_{01}) + z_{10}z_{11}(z_{10} + 2z_{11})$ and $z_{00}z_{01}(z_{00} + z_{01}) + z_{11}^2(z_{10} + 2z_{11})$. By the Jacobi criterion and looking at the Hilbert polynomial we see that X is indeed a smooth curve of genus 2 and degree 5. We have a curve of type $(2, 3)$. It is smooth and thus with regards to the Segre embedding it is projectively normal [26, Exercise III.5.6(b)(3)], i.e. the ring with these three relations is normal.

As example sheaves we compute: $\mathrm{Syz}(z_{01}z_{11}+z_{00}^2, z_{01}+z_{11}, z_{10}z_{00}+z_{01}^2, z_{10})$ over X is semistable, but $\mathrm{Syz}(z_{01}z_{11} + z_{00}^2, z_{11}, z_{10}z_{00} + z_{01}^2, z_{10})$ is not.

4.7 Positive characteristic

We needed characteristic 0 to ensure that the symmetric and exterior powers of semistable sheaves are again semistable. This is not true in positive characteristic. However, in positive characteristic the Frobenius pullbacks allow us to construct a very similar algorithm.

The *Frobenius pullback* $\mathrm{F}^{e*}\mathcal{F}$ of a locally free sheaf $\mathcal{F}$ is the pullback of $\mathcal{F}$ by the e-th power of the Frobenius homomorphism $f \mapsto f^{p^e}$. There is a surjective map from $\mathrm{F}^{e*}\,\mathrm{Syz}(f_1, \ldots, f_n) \longrightarrow \mathrm{Syz}(f_1^{p^e}, \ldots, f_n^{p^e})$. It is even bijective if the f_i are primary to the irrelevant ideal S_+. Thus global sections of both sheaves are the same.

We mention the following immediate lemma.

Lemma 4.7.1. *Let $\mathcal{F}$ be a locally free sheaf over a scheme of characteristic p. Then* $\deg(\mathrm{F}^{e*}\mathcal{F}) = p^e \cdot \deg(\mathcal{F})$ *and* $\mu(\mathrm{F}^{e*}\mathcal{F}) = p^e \cdot \mu(\mathcal{F})$.

Definition 4.7.2. Let $\mathcal{F}$ be a locally free sheaf over a smooth projective curve over an algebraically closed field of characteristic p. $\mathcal{F}$ is called strongly semistable if every Frobenius pullback is semistable.

It follows directly that if $\mathcal{F}$ is strongly semistable then also $\mathrm{F}^{e*}(\mathcal{F})$ is strongly semistable for every $e \in \mathbb{N}_{>0}$.

As already mentioned in the introduction, a positive characteristic version of Theorem 4.4.1 was already proved in [12, Lemma 2.1]. Note that there is a small mistake in the statement of a corollary [12, Lemma 2.2] with regards to the necessary Frobenius pullback, which we have corrected in our statement of the theorem.

Theorem 4.7.3. *Let $\mathcal{F}$ be a locally free sheaf on a smooth projective curve X over a field of characteristic p and $r := \operatorname{rank} \mathcal{F}$.*

$\mathcal{F}$ is strongly semistable if and only if there does not exist a nontrivial global section of $\mathrm{F}^{e*}(\mathcal{F}) \otimes \mathcal{O}(k)$, for every $e \in \mathbb{N}$ and $k = \left\lceil \frac{-p^e \mu(\mathcal{F})}{\deg(X)} \right\rceil - 1$.

$\mathcal{F}$ is semistable if there does not exist a nontrivial global section of

$$\mathrm{F}^{e*}(\mathcal{F}) \otimes \mathcal{O}(k),$$

for an exponent $e \in \mathbb{N}$ with $p^e \geq (g - 1 + \deg(X))n + 1$, $n = \frac{r(r-1)}{\gcd(r, \deg \mathcal{F})}$ and $k = \left\lceil \frac{-p^e \mu(\mathcal{F})}{\deg(X)} \right\rceil - 1$.

Proof. We first prove the second assertion, in the same way as in Theorem 4.4.1. First assume that $\mathcal{F}$ is not semistable. Take a destabilizing subbundle $\mathcal{E} \subseteq \mathcal{F}$ of rank $s < r$. Then

$$\frac{\deg(\mathcal{E})}{s} = \mu(\mathcal{E}) > \mu(\mathcal{F}) = \frac{\deg(\mathcal{F})}{r}$$

and thus

$$\mu(\mathcal{E}) - \mu(\mathcal{F}) = \left(\frac{r \cdot \deg(\mathcal{E}) - s \cdot \deg(\mathcal{F})}{r \cdot s} \right) \geq \frac{\gcd(r, \deg(\mathcal{F}))}{r(r-1)} = \frac{1}{n} > 0.$$

With this we calculate

$$\begin{aligned} \mu\left(\mathrm{F}^{e*}\left(\mathcal{E}\right) \otimes \mathcal{O}(k)\right) &= p^e \mu(\mathcal{E}) + k \deg(X) \\ &= p^e \mu(\mathcal{E}) + \left(\left\lceil \frac{-p^e \mu(\mathcal{F})}{\deg(X)} \right\rceil - 1 \right) \deg(X) \\ &\geq p^e (\mu(\mathcal{E}) - \mu(\mathcal{F})) - \deg(X) \\ &\geq \frac{p^e}{n} - \deg(X) \\ &\geq \frac{(g - 1 + \deg(X))n + 1}{n} - \deg(X) \\ &= g - 1 + \frac{1}{n} \\ &> g - 1. \end{aligned}$$

Thus $\mathrm{F}^{e*}(\mathcal{E}) \otimes \mathcal{O}(k)$ has a global section by Lemma 4.1.2 and this is also a global section of $\mathrm{F}^{e*}(\mathcal{F}) \otimes \mathcal{O}(k)$.

Now to the first assertion. If $\mathcal{F}$ is not strongly semistable, some pullback will not be semistable. So for some potentially higher power e the Frobenius pullback $\mathrm{F}^{e*}(\mathcal{F})$ will have a destabilizing global section.

Now let's assume $\mathcal{F}$ is strongly semistable. Then so is $\mathrm{F}^{e*}(\mathcal{F}) \otimes \mathcal{O}(k)$. We calculate its slope

$$\begin{aligned}
\mu\left(\mathrm{F}^{e*}\left(\mathcal{F}\right) \otimes \mathcal{O}(k)\right) &= p^e\mu(\mathcal{F}) + k\deg(X) \\
&= p^e\mu(\mathcal{F}) + \left(\left\lceil \frac{-p^e\mu(\mathcal{F})}{\deg(X)} \right\rceil - 1\right)\deg(X) \\
&< p^e\mu(\mathcal{F}) + \frac{-p^e\mu(\mathcal{F})}{\deg(X)}\deg(X) \\
&= 0.
\end{aligned}$$

Being semistable with negative slope it can have no nontrivial global sections, so we are done. □

For finite fields it would be possible to give a bound on the necessary exponent to ascertain strong semistability. This is because the moduli space of strongly semistable vector bundles (with given combinatorial data) is a variety. Over a finite field there are thus only finitely many Frobenius pullbacks for any strongly semistable vector bundle until repetition. Thus it suffices to check a fixed power to determine strong semistability. For computational purposes this is not very helpful though, because the required Frobenius power is very high.

Using exterior powers like in the characteristic 0 case allows us to state the following variant of the theorem.

Theorem 4.7.4. *Let $\mathcal{F}$ be a locally free sheaf on a smooth projective curve X over a field of characteristic p and $r := \operatorname{rank}\mathcal{F}$.*

$\mathcal{F}$ is semistable if for every $s < r$ there does not exist a nontrivial global section of $\mathrm{F}^{e*}(\bigwedge^s \mathcal{F}) \otimes \mathcal{O}(k)$, where $e = \lceil \log_p((g-1+\deg(X))n+1)\rceil$, $n = \frac{r}{\gcd(r, s\deg(\mathcal{F}))}$ and $k = \left\lceil \frac{-p^e s\mu(\mathcal{F})}{\deg(X)} \right\rceil - 1$.

The same is true if we substitute the Frobenius power with the symmetric power in this theorem.

Proof. For this direction the proof in Theorem 4.4.2 works fully for characteristic p (for the Frobenius power as well as the symmetric power). A destabilizing section prohibits semistability. □

Remark 4.7.5. It is possible to check semistability for the characteristic 0 case by reduction modulo p. This is based on the fact that any destabilizing subsheaf in characteristic 0 would also occur as a destabilizing subsheaf in characteristic p. Thus we could try primes p until we find one for which the sheaf over characteristic p is semistable. Then we know that the corresponding characteristic 0 sheaf is also semistable. This process has the potential to be a computationally faster way to show semistability, because arithmetic modulo p is faster. This effect is diminished by the fact that we might have to try a lot of primes and that the degrees grow faster with Frobenius pullbacks.

Even more, if for all primes that we try the sheaves are not semistable or if we cannot decide semistability for them, then we don't know the semistability of the corresponding characteristic 0 sheaf. In particular with this method we can never decide semistability for a sheaf that is not semistable. Still reduction modulo p could be a useful part in an adaptive approach to determining semistability where you try different angles of attack at the same time.

Chapter 5

Applications and connections

5.1 The Harder-Narasimhan filtration

Definition 5.1.1. Let $\mathcal{F}$ be a locally free sheaf on a smooth projective curve X. A filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n = \mathcal{F}$$

such that $\mathcal{F}_i/\mathcal{F}_{i-1}$ is semistable and for $\lambda_i = \mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ we have

$$\lambda_1 > \lambda_2 > \ldots > \lambda_n$$

is called a *Harder-Narasimhan filtration* for $\mathcal{F}$.

Lemma 5.1.2. *Every locally free sheaf on a projective curve X has a unique Harder-Narasimhan filtration.*

Proof. See [35, Theorem 1.3.4]. □

At every step in the filtration the rank increases, thus the Harder-Narasimhan filtration has length at most the rank of the sheaf.

Let $X = \operatorname{Proj} K[x_1, \ldots, x_m]/\mathfrak{a}$ be a smooth projective curve and let $\mathcal{F} = \operatorname{Syz}(f_1, \ldots, f_n)$ be a syzygy sheaf.

A destabilizing section $s = (s_1, \ldots, s_n)$ of total degree k of $\mathcal{F}$ defines a map $\mathcal{O} \longrightarrow \mathcal{F}(k)$, which factors through a destabilizing invertible subsheaf $\mathcal{E}$ of $\mathcal{F}$. The subsheaf $\mathcal{E}$ is defined by the divisor of the zero points with multiplicity of s on the curve. Thus if the section vanishes nowhere on the curve then $\mathcal{O}$ is itself a destabilizing subsheaf of $\mathcal{F}(k)$.

Example 5.1.3. For $\mathcal{F} = \mathrm{Syz}(x^2, y^3, z^2)$ over the smooth projective curve $X = \mathbb{C}[z, y, x]/(x^2y + y^3 + z^3)$ we have a destabilizing global section $s = (y, 1, z)$ in twist 3, i.e. s is a relation of total degree 3. The section never vanishes, thus s defines a destabilizing subsheaf $\mathcal{O}(-3) \subseteq \mathcal{F}$. The line bundle $\mathcal{O}(-3)$ is semistable, so this is the full Harder-Narasimhan filtration.

Example 5.1.4. The sheaf $\mathcal{F} = \mathrm{Syz}(x^3, xy^2, z^3)$ over the smooth projective curve $X = \mathbb{C}[z, y, x]/(x^3z + xy^3 + yz^3)$ has a destabilizing global section $s = (z, y, y)$ in twist 4. The section vanishes in the projective point $P = (1 : 0 : 0)$, which lies on the curve. The multiplicity of P is 1. Thus $\mathcal{L}(P)$ is isomorphic to a destabilizing subsheaf of degree 1 mapped into $\mathcal{F}$ by the factoring of the morphism given by s. The Harder-Narasimhan filtration is $\mathcal{L}(P) \subseteq \mathcal{F}$.

Remark 5.1.5. In our algorithm, we find destabilizing sections in symmetric or exterior powers (or in the positive characteristic case Frobenius pullbacks) of the syzygy sheaf $\mathcal{F}$. In that case we know that there is a destabilizing subsheaf of $\mathcal{F}$, but we don't have a general way of getting a computational description of this destabilizing subsheaf.

In general we can't expect to explicitly find this subsheaf, but we could compute the maximal slope that occurs for a destabilizing section and as such compute some of the numerical data of the Harder-Narasimhan filtration.

The destabilizing section of maximal slope corresponds to a maximal destabilizing subsheaf $\mathcal{F}_1$. Then $\mathrm{Sym}^k(\mathcal{F}_1) \subset \mathrm{Sym}^k(\mathcal{F})$ is the maximal destabilizing subsheaf of the symmetric power (for this we need to be in characteristic 0). We can now bound the maximal slope from below by finding global sections of $\mathrm{Sym}^k(\mathcal{F}_1)(m)$ in various twists m. We can bound the maximal slope from above by showing nonexistence of global sections $\mathrm{Sym}^k(\mathcal{F})(m)$.

Alternatively we can define a sequence m_k as the minimal twists that $\mathrm{Sym}^k(\mathcal{F})(m_k)$ has global sections. Then the maximal slope is the limit of the sequence $\frac{m_k}{k}$. Since the maximal slope is rational and because the behavior from a certain k onwards becomes predictable this can be computed in finite time.

5.2 Computing tight closure using semistable sheaves

Holger Brenner developed in [3], [4] and [6] a theory to fully compute solid closure in the best case or at least give inclusion/exclusion bounds and check containment in tight closure for a specified element. Since in many cases solid closure coincides with tight closure this is also a way to compute tight closure. This gives another application of semistability and shows the most direct connection between the ideal closure part and the semistability part of this thesis.

Semistability was also used in [9] to show for a specific example that tight closure does not commute with localization.

The following two theorems give complete descriptions of the tight closure in the case that the syzygy bundle of the ideal generators is semistable in characteristic 0, or strongly semistable respectively in positive characteristic.

Theorem 5.2.1. *Let R be be a two-dimensional normal standard-graded domain over an algebraically closed field K of characteristic 0. Let $I \subseteq R$ be an R_+-primary homogeneous ideal. Let $f_1, \ldots, f_n$ be a set of generators, with degrees $d_1, \ldots, d_n$ and such that $\operatorname{Syz}(f_1, \ldots, f_n)$ is semistable.*

Then

$$I^* = I + R_{\geq \frac{d_1+\ldots+d_n}{n-1}}.$$

Proof. See [4, Theorem 8.1]. □

Theorem 5.2.2. *Let A be a finitely generated normal $\mathbb{Z}$-algebra of dimension one and S_A a standard-graded flat A-algebra such that for all large enough $\mathfrak{p} \in \operatorname{Spec} A$ the fibers $S_{\kappa(\mathfrak{p})}$ are two-dimensional geometrically normal standard graded $\kappa(\mathfrak{p})$ algebras.*

Let $I = (f_1, \ldots, f_n) \subseteq S_A$ be an S_+-primary homogeneous ideal and $d_1, \ldots, d_n$ be the degrees of the generators. Let $\mathcal{F} = \operatorname{Syz}(f_1, \ldots, f_n)$ and assume that the generic fiber of $\mathcal{F}$, i.e. $\mathcal{F}_{\kappa(0)}$, is semistable.

Then for all large enough $\mathfrak{p} \in \operatorname{Spec} A$ with $\mathcal{F}_{\kappa(\mathfrak{p})}$ strongly semistable we have

$$I^*_{\kappa(\mathfrak{p})} = I_{\kappa(\mathfrak{p})} + R_{\kappa(\mathfrak{p}), \geq \frac{d_1+\ldots+d_n}{n-1}}.$$

Proof. See [4, Theorem 8.4]. □

Instead of going into detail on how these theorems work, we will describe how the same techniques can be used in a refined way in the case that the syzygy bundle is not semistable (see also [6]). In this situation the semistability algorithm from chapter 4 can be employed.

If the syzygy sheaf is not semistable one can use the Harder-Narasimhan filtration. If a sheaf is semistable its Harder-Narasimhan filtration has length 1. Starting with the filtration there is an algorithm to decide whether an element $f_0 \in R$ is in the tight closure of an R_+-primary ideal $I = (f_1, \ldots, f_n)$ over a normal two-dimensional standard-graded K-domain over an algebraically closed field. If the maximal destabilizing subsheaf of $\mathrm{Syz}(f_1, \ldots, f_n)$ is semistable it can always give a complete answer.

In order to compute the Harder-Narasimhan filtration one has to find destabilizing subsheaves. The semistability algorithm does not produce destabilizing subsheaves of the original sheaves. But it finds destabilizing sections of a symmetric power respectively Frobenius power of the sheaf.

Note that for the Frobenius power of the syzygy sheaf we have

$$\mathrm{F}^{e*} \mathrm{Syz}(f_1, \ldots, f_n) = \mathrm{Syz}(f_1^{p^e}, \ldots, f_n^{p^e}).$$

The right hand side is the syzygy sheaf of $I^{[p^e]}$. Thus by taking Frobenius powers we are actually looking at the tight closure of $I^{[p^e]}$. The containment of an element $f \in R$ in the tight closure I^* of an ideal can be determined in any Frobenius pullback $\mathrm{F}^{e*} R$, because $f \in I^*$ if and only if $f^q \in (I^{[q]})^*$.

Thus in positive characteristic in many cases we can use the semistability algorithm to compute the Harder-Narasimhan filtration of a suitable Frobenius power and with that compute the tight closure of an ideal.

For simplicity we will restrict ourselves to ideals I generated by three homogeneous elements. In that case the key to deciding whether an element is in the tight closure I^* is the following Lemma. We actually compute the solid closure $I^\star$, but remember that for positive characteristic in many cases $I^\star = I^*$, see Theorem 1.2.7.

Lemma 5.2.3. *Let R be a two-dimensional normal standard-graded domain over an algebraically closed field K of characteristic 0 or $p \gg 0$ (p large in the same sense as in Theorem 5.2.2) and $I = (f_1, f_2, f_3)$ an R_+-primary ideal. Take a power $q = p^e$, with $e \in \mathbb{N}$. Further assume that $\mathcal{F} := \mathrm{Syz}(f_1, f_2, f_3)$ is not strongly stable on $X = \mathrm{Proj}\, R$ and that there is a short exact sequence*

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{F}' \longrightarrow \mathcal{M} \longrightarrow 0,$$

with $\mathcal{F}' = \mathrm{F}^{e*}(\mathcal{F})$*. Also assume that* $\deg \mathcal{L} \geq \mu(\mathcal{F}') \geq \deg \mathcal{M}$*.*

Let $f_0 \in R$ *be an element of degree* m *which corresponds to a cycle* $c \in H^1(X, \mathcal{F}(qm))$*. By pull-back we get a cycle* $c' \in H^1(X', \mathcal{F}'(qm))$ *which has the image* $\bar{c} \in H^1(X', \mathcal{M}(qm))$*. Then* $f_0 \in I^\star$ *if and only if at least one of the following conditions hold*

1. $f_0 \in I$
2. $\deg(\mathcal{M}(qm)) \geq 0$.
3. $\deg(\mathcal{L}(qm)) \geq 0$ *and* $\bar{c} = 0$.

Proof. [6, Corollarly 3.4]. Note that $c = 0$ implies that $f_0 \in I$. □

The statement is not always true for small characteristics, because for small prime numbers the Frobenius can annihilate cohomology classes in negative degrees. In those cases we additionally have to check if this kind of annihilation occurs.

First let's do an example on how to compute solid closure in a case where we don't need the Frobenius pullback, i.e. where $q = 1$.

Example 5.2.4. Let $R = K[x, y, z]/(x^4 + y^4 + z^4)$, i.e. $X = \operatorname{Proj} R$ is the Fermat quartic over K. Let K be an algebraically closed field of characteristic $p > 0$. We want to compute the tight closure of $I = (x^3, y^3, z^3)$.

The syzygy sheaf $\mathcal{F}(4) := \mathrm{Syz}(x^3, y^3, z^3)(4)$ has a destabilizing section (x, y, z). The section is not zero anywhere on the curve, thus it directly defines a subsheaf. From this we get a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \stackrel{(x,y,z)^t}{\longrightarrow} \mathrm{Syz}(x^3, y^3, z^3)(4) \longrightarrow \mathcal{O}_X(-1) \longrightarrow 0$$

which describes the Harder-Narasimhan filtration. Following [6, Remark 4.8] the map on the right can be given by $(h_1, h_2, h_3) \mapsto \frac{yh_3 - zh_2}{x^3}$. In the sense of the previous Lemma we get $\mathcal{L} := \mathcal{O}_X(-4)$ and $\mathcal{M} := \mathcal{O}_X(-5)$.

The other important short exact sequence in this case is

$$0 \longrightarrow \mathrm{Syz}(x^3, y^3, z^3)(m) \longrightarrow \bigoplus_{i=1}^{3} \mathcal{O}_X(m-3) \stackrel{(x^3,y^3,z^3)}{\longrightarrow} \mathcal{O}_X(m) \longrightarrow 0.$$

The connecting homomorphism for the long exact sequence in cohomology maps an element $f \in R$ of degree m to $c \in H^1(X, \mathrm{Syz}(x^3, y^3, z^3)(m))$. If $c = 0$

then $f \in I$ and thus $f \in I^\star$. If $m < 4$ we have $\deg \mathcal{M}(m) < \deg \mathcal{L}(m) < 0$, so then $f \in I$ is the only way for f to be in the solid closure. If $m > 4$ we have $\deg \mathcal{M}(m) \geq 0$ thus $f \in I^\star$ because of Lemma 5.2.3.2.

The interesting case is thus $m = 4$, in which case we have to look at the cohomology class $\bar{c} \in H^1(X, \mathcal{O}_X(-1))$.

Concretely, since x^3, y^3 are parameters, we can identify the cohomology modules as Čech cohomology modules using the open covering $D(x^3), D(y^3)$. The class c would be represented by the Čech-cocycle $(\frac{f}{x^3}, \frac{f}{y^3}, 0)$. Applying cohomology to the Harder-Narasimhan quotient sequence this maps to the cocycle $-\frac{fz}{x^3y^3} \in H^1(X, \mathcal{O}_X(-1))$.

Following the Čech-cohomology construction $-\frac{fz}{x^3y^3} = 0$ is true if and only if $f \cdot z \in (x^3, y^3)$. Because $x^4 + y^4 + z^4$ is the only relation $f \cdot z \in (x^3, y^3)$ implies $f \in (x^3, y^3, z^3) = I$. Thus also for $m = 4$ nothing new is added to the solid closure. In total we have computed $I^\star = I + R_{\geq 5}$.

Next we give an example of computing the tight closure if only a Frobenius power exhibits a destabilizing global section.

Example 5.2.5. Let $X = \operatorname{Proj} K[x, y, z]/(x^4+y^4+z^4)$ be the Fermat quartic over an algebraically closed field K of characteristic 3. We want to compute the solid closure of $I = (x^3, y^5, z^7)$.

The characteristic is relatively low, but the Frobenius map in negative degree is injective on the cohomology of the structure sheaf as we will show explicitly for degree -1, i.e. we look at $H^1(X, \mathcal{O}_X(-1)) \longrightarrow H^1(X, \mathcal{O}_X(-3))$. With Čech cohomology we construct the basis

$$\frac{z}{xy}, \frac{z^2}{x^2y}, \frac{z^2}{xy^2}, \frac{z^3}{xy^3}, \frac{z^3}{x^2y^2}, \frac{z^3}{x^3y}$$

of $H^1(X, \mathcal{O}_X(-1))$. In order those are mapped to

$$\frac{z^3}{x^3y^3}, \frac{-z^2}{x^2y^3}, \frac{-z^2}{x^3y^2}, \frac{z}{x^3y}, \frac{2z}{x^2y^2}, \frac{z}{xy^3}.$$

For the second basis element for example we compute

$$\frac{z^6}{x^6y^3} = \frac{-z^2(x^4+y^4)}{x^6y^3} = \frac{-z^2}{x^2y^3}.$$

For the second equality we use that in cohomology $\frac{-z^2y}{x^6} = 0$. For the other elements similar computations are used. We can see that the basis is mapped to part of a basis and hence the whole Frobenius map is injective on $H^1(X, \mathcal{O}_X(-1))$.

For higher Frobenius powers and for more negative twists the degrees of the relevant sheaves become higher, so that the degree criterion of Mumford-Hartshorne for ampleness [25, Proposition 7.5 and Corollary 7.7] ensures that the Frobenius map is injective. It can be used similarly to the proof of [6, Corollary 6.4].

Using our semistability algorithm we find a destabilizing section $(2z^{13} + 2z^9y^4 + zy^{12}, z^5yx + 2zy^5x, x)$ of $\mathcal{F}(22)$, the twist by 22 of the syzygy sheaf $\mathcal{F} := \mathrm{Syz}(x^9, y^{15}, z^{21}) = \mathrm{F}^* \mathrm{Syz}(x^3, y^5, z^7)$. The section is not zero anywhere on the curve. If it were we would need $x = 0$ which implies $z = \sqrt[4]{-1}y$, which means $2z^{13} + 2z^9y^4 + zy^{12} = 2z^{13} = 0$, thus $y = z = 0$ and the origin is not in the projective space. Thus the section generates a subsheaf, sitting in the following exact sequence

$$0 \longrightarrow \mathcal{O}_X(-22) \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X(-23) \longrightarrow 0.$$

To decide whether an element f of degree m is in I^* we look at f^3 which has degree $3m$. If $3m \geq 23$, then $f \in I^*$. Thus $R_{\geq 8} \in I^*$. If $3m < 22$, then $f \in I^* \Leftrightarrow f \in I$. Thus $I^* = I + R_{\geq 8}$.

We can use the same destabilizing section to compute the tight closure of $I^{[3]}$. For that purpose we take again an element f of degree m. If $m \geq 23$, then $f \in (I^{[3]})^*$. If $m < 22$, then $f \in (I^{[3]})^* \Leftrightarrow f \in I^{[3]}$. For $m = 22$, we have to decide whether $\bar{c} = \frac{f \cdot x}{x^9 y^{15}} = \frac{f}{x^8 y^{15}} = 0 \in H^1(X, \mathcal{O}_X(-1))$. This is the case exactly if $f \in (x^8, y^{15})$. Thus degree 22 elements are in $(I^{[3]})^*$ exactly if they are multiples of x^8 or are in $I^{[3]}$ already (which also contains y^{15}). This means that $(I^{[3]})^* = I^{[3]} + x^8 \cdot R_{\geq 14} + R_{\geq 23}$.

By taking powers of the relation we can in the same way compute the tight closure for any $I^{[3^e]}$.

Chapter 6

Implementation

You will find our implementations of the main algorithm and the necessary subroutines together with an explanation on how to use them online at `https://github.com/JonathanSteinbuch/sheafstability`.

6.1 Outline

Let's take a look at the most important classes and their purpose in the implementation.

```
template<typename Scalar>
```

The polynomials and matrices work with three different types of scalars. So far those are the integers, the rationals and $\mathbb{F}_p$ for prime numbers $p < 65535$. For fast performance and ease of implementation the scalar type is specified using templates. This tells the compiler to automatically generate all the structures and methods for all three scalar types.

The integers and rationals are implemented by `mpz_class` and `mpq_class` respectively of the GNU multiprecision library [21]. The fields $\mathbb{F}_p$ are implemented by the class `numbermodulo`.

```
class PolyRing
```

This class implements polynomial rings and more generally quotient rings of polynomial rings with an ideal. For this it contains a pointer to the base ring, which is set to the ring itself if the instance implements a basic polynomial ring.

The class contains a rich representation of the monomials of the ring which includes a list of all monomials and a list of all elements of a monomial basis (See Definition 6.3.2) up to some degree (which is automatically increased when necessary). Also it includes a lookup table which contains for every monomial its linear combination with elements of the monomial basis. In addition many operations on monomials are implemented. These structures allow fast arithmetic on polynomials and also give fast access to a list of all elements of the monomial basis of a given degree which is important for constructing the degree matrix. However, building the lists and lookup tables takes considerable time, which for the large degrees occurring in the Frobenius pullbacks can limit the performance.

The class also contains a simple version of Buchberger's algorithm in case the user doesn't provide the quotient ideal in Gröbner basis form and the Hilbert polynomial computation (Algorithm 6.3.9).

```
class Poly
```

An instance of this class represents a single polynomial. The data consists of a pointer to the `PolyRing` of which the polynomial is an element of and of a `std::map<mBmonomial, Scalar>` which represents a finite set of monomials together with a map into the Scalar type. The monomials are stored just by the position they occur in in the monomial order. This allows the monomials in the polynomial to be automatically stored in their order which is convenient for Gröbner basis operations.

The class implements many arithmetic operations on polynomials. The arithmetic operations are broken down to operations on monomials which are implemented in the PolyRing. Two other notable methods of the class are a constructor which reads a polynomial from a string and a function `getMatrix` which generates the matrix for multiplying the polynomial with another polynomial of a given degree as laid out in Remark 6.2.1.

```
class DensePolyMatrix
```

This is a dense representation of a matrix of polynomials, dense being the opposite of a sparse implementation and meaning that every entry is explicitly specified. The data is thus simply stored in a `vector` of `vectors` with each entry being a `Poly`. In addition we store the degrees coming in and out of the matrix making it suitable to represent a map of (twisted) sheaves.

Notable member functions are methods constructing the symmetric and exterior power matrices as in Lemma 4.5.4 and the power matrices for the Frobenius pullback. Also of importance is the `kernel` method. To compute the kernel in a given degree, the method generates the degree matrix, has the kernel of it computed and retransforms the kernel into a matrix of polynomials.

`class SparsePolyMatrix`

This is a sparse reimplementation of the `DensePolyMatrix`. It was born after realizing that even the matrices of polynomials can sometimes become rather large. It lacks some of the additional features of the dense implementation, but the core is the same as above.

`class SparseScalarMatrix`

This class is the central linear algebra workhorse. It contains many methods for facilitating the analysis and echelonization of a sparse matrix. The detailed workings of the class are explained in Section 6.4.

`class ReaderAndCaller`

This class reads the input file and determines which scalar type to work with depending on the given characteristic and the necessity of fractions. In the characteristic 0 case - if the user hasn't expressly requested computations over $\mathbb{Q}$ - the GNU multiprecision integer type will be used, unless the leading coefficients of the Gröbner basis can't be chosen as 1 without fractions occurring in the other coefficients.

In the end a derived class of `class JobType` will be instantiated depending on the job given in the input file.

`class JobType`

This is the base class for the job handler classes. It's sole virtual member function is called `doJob` and is implemented by the derived classes depending on the job they are a front for.

Currently the derived classes are the following.

- `class SemistabilityJob`. This class encapsulates the main algorithm in all steps, including calling the functions for checking smoothness, computing degree and genus of the curve and the powers of the sheaves.
- `class PowersJob`. This computes just the matrices whose kernel is the symmetric or exterior power or the Frobenius pullback.
- `class SemistabilityJob`. This computes just the kernel of a (power) sheaf in a certain twist.

Of note is also the `struct JobOptionsType` which holds the options read from the input file and is passed by the `ReaderAndCaller` class.

6.2 Main algorithm implementation details

We need to compute the global sections of $(\operatorname{Sym}^q \bigwedge^s \mathcal{F}) \otimes \mathcal{O}(k)$, for some $q, s \in \mathbb{N}_{\geq 0}, k \in \mathbb{Z}$ over a curve $X = \operatorname{Proj} S$, with S a graded integrally closed algebra of finite type over K.

Remark 6.2.1. As laid out in Lemma 4.5.4 and Algorithm 4.5.6 we construct a matrix A_q which sits in an exact sequence

$$0 \longrightarrow \operatorname{Sym}^q \mathcal{F} \otimes \mathcal{O}_X(k) \longrightarrow \bigoplus_{a \in I} \mathcal{O}_X(k - a \cdot e) \xrightarrow{A_q} \bigoplus_{(b,j) \in J} \mathcal{O}_X(k - b \cdot e - d_j).$$

We want to compute the dimension of the vector space of global sections of the kernel of A_q. You will find the dimension as an entry in the Betti table of the module presented by A_q, for which there are implementations in many computer algebra systems. For very simple cases we did this in Macaulay2[22] and CoCoA[1]. This approach proved to be inefficient for anything but the most simple examples, though. It turned out to be way faster to only compute the correct degree case as follows.

We apply the global section functor to the exact sequence and get the exact sequence

$$0 \longrightarrow \Gamma(X, \operatorname{Sym}^q \mathcal{F} \otimes \mathcal{O}_X(k)) \longrightarrow \bigoplus_{a \in I} S_{k - a \cdot e} \xrightarrow{A_q} \bigoplus_{(b,j) \in J} S_{k - b \cdot e - d_j}.$$

For a fixed $d \in \mathbb{Z}$ the elements of S_d form a finite dimensional vector space, with a basis given by the degree d monomials of R which are not multiples of leading monomials of a Gröbner basis of the defining ideal of S (fix any degree-respecting monomial order).

All we have to compute is thus the kernel of a matrix $B_q(k)$ (computed from A_q) over a field, a linear algebra problem.

Example 6.2.2. Look at the map $\mathcal{O}_X(-4)^3 \oplus \mathcal{O}_X(-7) \longrightarrow \mathcal{O}_X$ over $X = \mathrm{Proj}(\mathbb{C}[x,y,z]/(x^9+y^9+z^9))$ from Example 4.6.5 which is given by the matrix $A = \begin{pmatrix} z^2y^2+x^4 & y^4 & z^4 & x^7 \end{pmatrix}$.

In the first power the theorem tells us to look at the twist 6, thus we get a map $\mathcal{O}_X(2)^3 \oplus \mathcal{O}_X(-1) \longrightarrow \mathcal{O}_X(6)$. The global sections of $\mathcal{O}_X(2)^3 \oplus \mathcal{O}_X(-1)$ are given by a tuple of three degree 2 elements and one degree -1 element. The degree 2 elements have a monomial basis $z^2, zy, zx, y^2, yx, x^2$ with 6 generators. The only element in negative degrees is 0. On the other hand, there are 28 monomials of degree 6. In the relevant twist 6 we get thus a 28x18-matrix $B(6)$ with entries in $\mathbb{C}$, see Table 6.1.

Consider Table 6.2 for the sizes of the resulting matrices as q grows. Note that $q = 73$ is the power from Theorem 4.4.1.

Remark 6.2.3. The resulting matrices from Remark 6.2.1 have an enormous size, prohibiting dense representations in computer memory. Fortunately only few entries are nonzero. If A is an $m \times n$ matrix the matrix A_q has only n nonzero entries in each row, while it has $\binom{q+n-1}{n-1}$ columns. The matrix $B_q(k)$ is even more sparse assuming the polynomial entries of A are sparse in the sense that they are made up of relatively few monomials compared to all monomials in their degree. We can also see this phenomenon in the matrix of Table 6.1. There every column only has as many nonzero entries as the corresponding polynomial has nonzero coefficients.

To do any useful computations it is thus very important to store the matrices in a sparse matrix format. This means that only the nonzero entries and their positions are stored. This has not only the benefit of requiring less memory, it also means that we only need to iterate over the nonzero-entries in every reduction step. We will perform matrix factorization - i.e. the process of factoring the matrix into triangle matrices and unitary matrices - in order to compute the kernel. There is a lot of potential for optimization in the factorization of sparse matrices because naive implementations tend to

	z^2	zy	zx	y^2	yx	x^2	z^2	zy	zx	y^2	yx	x^2	z^2	zy	zx	y^2	yx	x^2
z^6													1					
z^5y														1				
z^5x															1			
z^4y^2	1															1		
z^4yx																	1	
z^4x^2																		1
z^3y^3		1																
z^3y^2x			1															
z^3yx^2																		
z^3x^3																		
z^2y^4				1			1											
z^2y^3x					1													
$z^2y^2x^2$						1												
z^2yx^3																		
z^2x^4	1																	
zy^5								1										
zy^4x									1									
zy^3x^2																		
zy^2x^3																		
zyx^4		1																
zx^5			1															
y^6										1								
y^5x											1							
y^4x^2												1						
y^3x^3																		
y^2x^4				1														
yx^5					1													
x^6						1												

Table 6.1: The matrix $B(6)$ from Example 6.2.2. We have written the corresponding monomial basis elements on the top and to the left. 0-entries have been omitted.

introduce unnecessarily many additional nonzero entries making the matrix less sparse in the process. It's important to reduce the rows in a good order and to choose good pivot elements.

There are extremely well optimized algorithms for sparse matrix factorization - but only for floating point values. One floating-point-algorithm we tried out is SuiteSparseQR[16]. However, it doesn't seem practical (or maybe even possible) to control the cumulative floating point error in a way that let's us with certainty distinguish a kernel of dimension 1 from a kernel of dimension 0.

Thus for an implementation of the algorithm we need to be able to do sparse exact value matrix triangularization. Unfortunately most exact value matrix factorization implementations only work on dense matrices (for example it is implemented in Normaliz[14]). We considered using the sparse implementation in Bradford Hovinen's LELA[31], but it has only an optimized algorithm for matrices of the type occuring in Faugère's F4-algorithm

q	A_q	k	$B_q(k)$	$\dim\ker B_q(k)$	Δt
1	1×4	6	28×18	0	< 1ms
2	4×10	12	156×99	0	< 1ms
3	10×20	18	501×343	0	2ms
4	20×35	25	1401×1153	0	8ms
5	35×56	31	2848×2433	0	30ms
6	56×84	37	5190×4551	0	98ms
7	84×120	44	9474×8763	0	404ms
8	120×165	50	14889×13891	0	1s
9	165×220	56	22339×20985	0	4s
10	220×286	63	34219×32865	2	11s
11	286×364	69	47718×45930	0	27s
12	364×455	75	64845×62551	2	69s
13	455×560	82	90234×88066	128	169s
$\vdots$					
16	816×969	101	196743×193608	452	1743s
$\vdots$					
73	67525×70300	462	very large	?	?

Table 6.2: This table accompanying Example 6.2.2 lists the sizes of various matrices A_q, the degree k considered, and the size of the matrix $B_q(k)$. Δt is the time to compute the kernel with our implementation on our computer. Since the actual computation time varies between computers and because there may be future optimizations these runtime numbers are only meant to illustrate the general trend.

and it proved difficult to use.

Because we didn't find a suitable implementation for integer matrix triangularization that suited our needs we implemented our own version of the Gauss algorithm for sparse matrices in C++.

Remark 6.2.4. For some of our results we need to work over an algebraically closed field. Of course we can't actually represent complex numbers or any other uncountable field in computer memory. However if all involved coefficients in the input (which are the generators of the defining ideal I of the base ring and the matrix A) are in $\mathbb{Q}$ any resulting kernel sections will also just have coefficients in $\mathbb{Q}$. Even more, if the leading coefficients of a Gröbner basis of I are units in $\mathbb{Z}$ and all entries of A have coefficients only in $\mathbb{Z}$ we can do the whole kernel computation with only values in $\mathbb{Z}$. Even though we then have to be a bit careful in the Gauss algorithm this is still faster and uses less memory than a representation in $\mathbb{Q}$.

Of course it would be possible to work with $\mathbb{Q}$ adjoint with a finite number of additional elements of $\mathbb{C}$, but we haven't explicitly implemented this. Introducing an additional variable to the base polynomial ring and the necessary defining equations would be relatively easy but very costly.

Remark 6.2.5. The performance of the algorithm depends a lot on the input. We list some of the characteristics and how they affect performance.

- Degree d and genus g of the curve $X = \mathrm{Proj}(S)$ are important in two ways. Firstly, we have the symmetric power exponent $q = (g-1+d)n+1$ as of Theorem 4.4.1. The higher q is, the larger the matrix A_q becomes, as A_q is an $m \cdot \binom{q+l-2}{l-2} \times \binom{q+l-1}{l-1}$ matrix, where $m \times l$ are the dimensions of A.

  Additionally d and g affect the size of the matrix $B_q(k)$: The Hilbert polynomial of X is $\mathrm{Hilb}(S)(t) = t \cdot d + (1-g)$. For large enough t the values of $\mathrm{Hilb}(S)(t)$ are the dimension of the vector space of degree t elements of S. Thus again larger d are very costly here, while g is only in the constant coefficient of the Hilbert polynomial and so doesn't affect the size as much.

- The dimensions of the $m \times l$ matrix A affect the size of A_q directly as seen in the previous point. But it also goes into the rank and degree and thus the slope of $\mathcal{F}$. Recall again Theorem 4.4.1 and look at $n = \frac{r(r-1)}{\gcd(r, \deg \mathcal{F})}$, which multiplies into q. At best this is $r-1$ and at worst $r(r-1)$. Thus $r = l - m$ goes into the exponent q linearly at best and quadratic in the worst case.

  The twist is computed as $k = \left\lceil \frac{-q\mu(\mathcal{F})}{\deg X} \right\rceil - 1$. Here the degree of $\mathcal{F}$ enters, which is computed from the degrees of the entries of A. High degree entries lead to a high twist. A higher twist means higher degree of the monomials determining $B_q(k)$, thus there are more of them and thus $B_q(k)$ has a larger size.

- The number of monomials used for the polynomials is also a strong factor - the more dense the polynomials, the more space and time is needed. For the entries of A this was explained in Remark 6.2.3.

But for the Gröbner basis elements of $\mathfrak{a}$, where $\mathfrak{a}$ is the defining ideal of the curve, the same is true. If the result from a multiplication in the matrix A_q is a leading monomial of an element f in the reduced Gröbner basis, then it will be represented in $B_q(k)$ in the rows corresponding to all the other monomials of f. If there are more monomials with nonzero coefficients then the matrix will be less sparse. Thus we can say that the more "general" the curve is, i.e. the more nonzero coefficients there are in the defining polynomials, the harder it is to compute with.

- The coefficients of the polynomials involved also play a minor role. Because of the way the integer Gauss algorithm works, during the triangularization the absolute values of the entries in $B_q(k)$ will generally increase a lot. Because of this fixed length integer data types don't suffice as data containers and we need to use multiprecision integer data types, i.e. integer data types with an arbitrary length. If we start with large values we will need even more digits to store the entries, which also increases the time used to handle them.

Remark 6.2.6. For convenience of use, helpful future improvements to our implementation might include the following.

- An included feature to check whether the input ring is normal and to automatically work over the normalization if it isn't. So far we only check smoothness, and that the scheme is a curve at all, which are relatively easy to check.
- The ability to embed any sheaf as a kernel sheaf automatically.
- Further performance improvements to be able to check more and more difficult sheaves for semistability.

6.3 Hilbert polynomial and monomial bases

In our computations we need to compute the degree d and the genus g of a curve. This can be done with the Hilbert polynomial.function To a projective curve $X = \operatorname{Proj} S$ the Hilbert polynomial $\operatorname{Hilb}(X)(n) = \operatorname{Hilb} S(n)$ is of the form $d \cdot n + (1 - g)$.

Gröbner bases come into play on the one hand for the computation of the Hilbert polynomial and on the other hand in computing the monomial basis for writing the degree matrix. For the description of the degree matrix see Remark 6.2.1.

Monomial bases

For computations we will need a *graded monomial order*, i.e. a total well-ordering on the monomials that respects multiplication and for which monomials of higher degree are always bigger in the order. For our purposes we will choose the *degree reverse-lexicographic order*. In the degree reverse-lexicographic order, *degrevlex* for short, we have $x^\nu > x^\mu$ for two monomials with multivariate exponents $\nu = (\nu_1, \ldots, \nu_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ if one of the following is true

1. $\sum_{i=1}^n \nu_i > \sum_{i=1}^n \mu_i$
2. $\sum_{i=1}^n \nu_i = \sum_{i=1}^n \mu_i$ and $\nu_i < \mu_i$ for the largest $i \in \{1, \ldots, n\}$ such that $\nu_i \neq \mu_i$.

Thus for example in three variables x, y, z the ordering begins as follows: $1 < z < y < x < z^2 < yz < xz < y^2 < xy < x^2 < z^3 < \ldots$

Definition 6.3.1. Let $I \subseteq R$ be an ideal in an affine ring $R = K[x_1, \ldots, x_n]$. Fix a monomial order. A *Gröbner basis* for I is a family $f_1, \ldots, f_m$ of generators of I such that for every $f \in I$ there is an $i \in \{1, \ldots, m\}$ such that the leading monomial of f_i divides the leading monomial of f.

Every ideal admits a Gröbner basis for any monomial order, which can for example be found using Buchberger's algorithm [15].

Gröbner bases allow us to check whether an element is in the ideal or not by using the division algorithm, beginning at the leading monomial and reducing if the leading monomial is divisible by a leading monomial of the Gröbner basis.

In particular they allow us to decide whether we need a given monomial in a monomial basis of an affine ring just by checking the leading monomials of the Gröbner basis of the defining ideal.

Definition 6.3.2. Let $R = K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be an affine ring. A *monomial basis* of R is a family of monomials of $K[x_1, \ldots, x_n]$ such that the residue classes form a K-vector space basis of R.

Proposition 6.3.3. *Let $R = K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be an affine ring together with a monomial order on $K[x_1, \ldots, x_n]$, such that $f_1, \ldots, f_m$ form a Gröbner Basis. Then the family $\mathfrak{X}$ of all monomials that are not divisible by a leading monomial $\mathrm{LM}(f_1), \ldots \mathrm{LM}(f_m)$ is a monomial basis of R.*

Proof. Let $P = \sum a_\nu x^\nu$ be a nonzero linear combination of monomials, i.e. a polynomial. P being mapped to 0 in R means that $P \in (f_1, \ldots, f_m)$. P can only be an element in the ideal if the leading monomial is divisible by a leading monomial $\mathrm{LM}(f_i)$ of the Gröbner basis. This means that the monomial family $\mathfrak{X}$ maps to a linearly independent family in R. For any polynomial the remainder after division by $f_1, \ldots, f_m$ is a linear combination of monomials not divisible by the leading monomials, thus the monomial family $\mathfrak{X}$ even maps to a K-basis. □

As usual we will often identify the elements of R with their representing polynomials if no confusion is likely to arise. Consequently we will also talk about a monomial basis and the vector space basis it represents as if they were the same thing.

If R is graded, a monomial basis $\mathfrak{X}$ is compatible with the grading $R = \bigoplus_{d \in D} R_d$ in the sense that $\mathfrak{X}_d := \mathfrak{X} \cap R_d$ is a vector space basis of R_d. If R is standard graded all the graded components R_d are finite-dimensional.

Proposition 6.3.4. *Let $R = K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a standard graded affine ring with monomial basis $\mathfrak{X}$. Fix $i \in \mathbb{N}$ and an element $f \in R$ of degree d. To each monomial x^ν in $\mathfrak{X}_i$ write the product $f \cdot x^\nu$ as a linear combination of monomial basis elements $f \cdot x^\nu = \sum_{x^\mu \in \mathfrak{X}_{d+i}} a_{\mu,\nu} x^\mu$. Then the multiplication map $f \cdot - : R_i \longrightarrow R_{d+i}$ is described by the matrix with entries $a_{\mu,\nu}$.*

Hilbert polynomial

Definition 6.3.5. Let S be a standard graded, finitely generated algebra over a polynomial ring R. The *Hilbert function* is defined as $\varphi_S : \mathbb{Z} \longrightarrow \mathbb{N}, d \mapsto \dim S_d$. The *Hilbert polynomial* is the unique polynomial $\mathrm{Hilb}(S) \in \mathbb{Q}[n]$ such that for large $d \gg 0$ we have $\mathrm{Hilb}(S)(d) = \varphi_S(d)$.

The Hilbert polynomial has as degree the dimension of $\operatorname{Proj} S$. Thus for curves the Hilbert polynomial is linear.

For us the Hilbert polynomial is important in several ways. It can be used to determine the size of the monomial basis in each degree - at least after a certain degree. Also we can use it to define a meaningful degree and genus of a curve with it as follows.

Definition 6.3.6. If $X = \operatorname{Proj} S$ is a projective curve and S finitely generated over a polynomial ring, then $\operatorname{Hilb}(S) = dX + (1 - g)$, where d is the *degree* of the curve and g is the (arithmetic) *genus* of the curve.

For a plane curve defined by a homogeneous polynomial f this definition of degree is consistent with the degree of f.

The remainder of this section is devoted to the algorithm we use to determine the Hilbert polynomial.

Lemma 6.3.7. *Let $S = K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ be a standard graded affine ring such that $f_1, \ldots, f_m$ are a Gröbner basis with respect to a fixed monomial order. Then $S' := K[x_1, \ldots, x_n]/(\operatorname{LM}(f_1), \ldots, \operatorname{LM}(f_m))$ has the same Hilbert function and therefore the same Hilbert polynomial as S.*

Proof. [18, Theorem 15.26]. □

For the following construction we write $l(\nu) := \sum_{i=1}^{n} \nu_i$, where ν_i is the i-th component of the tuple $\nu \in \mathbb{N}^n$.

Proposition 6.3.8. *Let $S = K[x_1, \ldots, x_n]/(x^{\nu_1}, \ldots, x^{\nu_m})$ be a standard graded ring where $x^{\nu_j} = \prod_{i=1}^{n} x_i^{\nu_{ji}}$. The exponents of the multiples of a monomial x^{ν_i} form an integer cone C_i with base point ν_i and edges extending in positive direction parallel to all axes. The ideal $(x^{\nu_1}, \ldots, x^{\nu_m})$ is the vector space generated by the monomials with exponents in the union $U := \bigcup_{i=1}^{m} C_i$. Thus S can be identified with the vector space of all monomials with exponents in the complement $D := \mathbb{N}^n \setminus U$. The degree d component of S is then the vector space generated by the monomials with exponents in $D \cap L_d$, where $L_d := \{\nu \in \mathbb{N}^n | l(\nu) = d\}$ is the hyperplane of degree d exponents.*

Hence, the Hilbert function of S is given by $P(d) = \#(D \cap L_d)$.

We have

$$\#(C_i \cap L_d) = \binom{n - 1 + d - l(\nu)}{n - 1}.$$

For $d \geq l(\nu_i) - (n - 1)$ this is a polynomial in d. The intersection of two cones generated by exponents $\mu \in \mathbb{N}^n$ and $\nu \in \mathbb{N}^n$ is generated by the exponent $(\max(\mu_1, \nu_1), \ldots, \max(\mu_n, \nu_n))$. The set U can be constructed by a linear combination of cones, by subtracting the intersections of the C_i and resulting cones recursively. The complement D is the cone to the monomial

1 subtracted by U, thus it can also be constructed as a linear combination of cones. Thus at least for $d \geq \left(\sum_{i=1}^{n} \max_{j=1}^{m} \nu_{ji}\right) - (n-1)$ the Hilbert function coincides with the linear combination of the corresponding polynomials. This describes a method to compute the Hilbert polynomial for a monomial algebra.

Algorithm 6.3.9. As before let $S = K[x_1, \ldots, x_n]/(x^{\nu_1}, \ldots, x^{\nu_m})$ be a standard graded ring. Let $M_0 = \{1\}$, $M_1 = \{x^{\nu_1}, \ldots, x^{\nu_m}\}$ and for $i > 1$ we define $M_i = \{\text{lcm}(x^\mu, x^\nu) | x^\mu, x^\nu \in M_{i-1}, \mu \neq \nu\}$. Let $M = \bigcup_{i \in \mathbb{N}} M_i$. Because all least common multiples have exponent at most $\sum_{i=1}^{n} \max_{j=1}^{m} \nu_{ji}$ this is a finite set.

Compute a function $f : M \longrightarrow \mathbb{Z}$ as follows. Set $f(1) = 1$. Take any $x^\nu \in M$ from the monomials with minimal degree that have not yet assigned a value. Set

$$f(x^\nu) = - \sum_{x^\mu \in M \text{ for which } x^\mu | x^\nu} f(x^\mu).$$

This is well-defined because all divisors have lower degree.

Then

$$\text{Hilb}(S)(d) = \sum_{x^\nu \in M} f(x^\nu) \cdot \binom{n-1+d-l(\nu)}{n-1}.$$

Example 6.3.10. Let's look at $\mathbb{Q}[w, z, y, x]/(-wz + yx, w^2y + 2wy^2 + zx^2 + x^3, wy^2 + z^2x + zx^2 + 2y^3)$. First we have the full cone C_0 generated by 1. The monomials of degree d in this cone are counted by

$$\#(C_0 \cap L_d) = \binom{3+d}{3} = 1 + \frac{11}{6}d + d^2 + \frac{1}{6}d^3.$$

The leading monomials are yx, x^3, y^3. Then for M_1 this means we substract the respective cones C_1, C_2, C_3 from C_0, the monomial counts are given by $\#(C_1 \cap L_d) = \binom{1+d}{3} = -\frac{1}{6}d + \frac{1}{6}d^3$ for yx and $\#(C_2 \cap L_d) = \#(C_3 \cap L_d) = \binom{d}{3} = \frac{1}{3}d - \frac{1}{2}d^2 + \frac{1}{6}d^3$ for x^3 and y^3 respectively.

Their first intersections are generated by y^3x, yx^3 and y^3x^3 which form M_2. Because y^3x^3 is a common multiple of all generators, y^3x^3 is already the maximal element of M. Its contribution is cancelled out, because the contribution of all other monomials adds up to 0. We are left with two contributions of $\#(C_1 \cap C_2 \cap L_d) = \#(C_1 \cap C_3 \cap L_d) = \binom{d-1}{3} = -1 + \frac{11}{6}d - d^2 + \frac{1}{6}d^3$ one for each of y^3x and yx^3.

So $M = \{1, yx, x^3, y^3, y^3x, yx^3, y^3x^3\}$ and f is given by

x^ν	1	yx	x^3	y^3	y^3x	yx^3	y^3x^3
$f(x^\nu)$	1	-1	-1	-1	1	1	0

In total this leaves us with

$$\begin{aligned}\mathrm{Hilb}(X)(d) &= \binom{3+d}{3} - \binom{1+d}{3} - 2\binom{d}{3} + 2\binom{d-1}{3} \\ &= -1 + 5d\end{aligned}$$

Because the intersection generated by y^3x^3 was canceled out, the biggest relevant intersection generators y^3x and yx^3 have degree 4. Hence, the Hilbert polynomial already agrees for $d \geq 4 - 3 = 1$ with the Hilbert function and not only for $d \geq 6 - 3 = 3$ as expected from the proposition.

6.4 Sparse linear algebra

We have seen that the central computational step of our algorithm is computing the rank of a large sparse matrix. There are many ways to store the information of a sparse matrix. Important to us are the following aspects.

As the primary data structure, we use a vector of rows. Here a vector is a `std::vector` datatype, i.e. an array with length defined at runtime. Each row is an ordered map, with the keys indicating columns and the values being the entries of the matrix at the respective row and column. Specifically the matrix is stored in the following datatype.

```
std::vector<boost::container::flat_map<unsigned int, Scalar>
    >
```

The `boost::container::flat_map` datatype behaves outwardly in the same way as the `std::map`, but is actually internally not implemented as a tree, but as a vector. This has the advantage that there is much less overhead in general, but in particular for iterating over the row. This comes at the cost of having a worse complexity for deleting or inserting elements, i.e. when an entry switches states between being zero or nonzero. Because we have very few nonzero-entries in each row the flat map performs better, which we also confirmed with run time comparisons. In the following we refer to making a zero entry nonzero as *filling* the entry, so that it becomes *filled* or nonzero. The converse action is then called *emptying*.

In this data structure it is cumbersome to find all nonzero entries in a column. In fact one would have to iterate over all rows and in each row perform a binary search for an element with a key matching the column, which would give us a complexity of $\mathcal{O}(n \cdot \log m)$, where n is the number of rows and m is the typical number of nonzero entries in a column. For this reason we add another data structure, which is a vector of the columns, and for each column an ordered set of all rows with nonzero entries in that column. In total this gives us the following.

```
std::vector<boost::container::flat_set<unsigned int> >
```

Note that we don't save the value again, as we can find that relatively efficiently from the row data structure when we know in which rows to look. However, we have to maintain the integrity of both data structures, whenever we fill or empty an entry.

We conclude that the sparse matrix data structure in our implementation has the following properties

1. The total memory usage is linear in the number of nonzero matrix entries.

2. Iterating over all the nonzero entries of a row is linear in the number of nonzero row entries.

3. Iterating over all the nonzero entries of a column is proportional to the number of nonzero entries in the column times the logarithm of the typical number of entries in the affected rows.

4. Filling or emptying an entry is linear in the number of all nonzero entries in the affected row and the affected column.

Any sparse matrix manipulation can only be efficient as long as we keep the matrix sparse. Because of this we take several measures to try to do as little filling as possible during the Gaussian elimination. The key to this is the choice of good pivot elements and good ordering for the rows and columns.

The pivot element of an elimination step is the entry of the matrix that is chosen to reduce all the entries of active rows in the same column to zero. If we work from top to bottom the pivot element of the i-th elimination step is the i-th diagonal element and all rows below it are active. Of course working from top to bottom is seldom the best strategy.

There are mainly two aspects that make a good pivot element. The first aspect is that it makes the subsequent arithmetic as easy as possible, for example because it divides many of the other entries in its column. The second aspect is the change in the number of nonzero entries that occurs, i.e. how much fill is produced in an elimination step with the respective pivot element. Because we have seen above that the number of nonzero entries is crucial to the performance we will focus on the second aspect. A good common heuristic for the introduced number of nonzero entries is the number of nonzero entries in the pivot row itself.

To the point of a good ordering it is useful to mention the *symbolic defect*. It counts the rows or columns that we can determine to be able to be reduced to zero simultaneously just by looking at the symbolic structure the matrix has when not looking at the actual values of the entries but only at the filled and empty status. The easiest case is of course if a row or column is completely empty as that means it is a zero row respectively column and can be disregarded entirely. In general a symbolic defect occurs if there is a collection of n rows (respectively columns) with filled entries in less than n columns (respectively rows).

The symbolic defect can be found using an algorithm that tries to fill all entries on the diagonal by swapping rows and backtracks if that is not directly possible. If it is not at all possible it marks a column as symbolically defective and swaps in another column to continue.

The following example illustrates this algorithm.

Example 6.4.1. Consider the following matrix where $*$ stands for a filled entry and 0 for an empty entry.

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 & * \\ 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 & 0 \end{pmatrix}.$$

We try to fill the diagonal entries beginning from the top left. The first four diagonal entries are already filled, so there is nothing to do for those. For the fifth diagonal entry we try to find a filled entry in the fifth column. The only one of these is in the second row which is already used, so we backtrack to the second diagonal entry to see if we could have used another row there.

For this purpose we make a backtracking depth first search and remember the sequence of rows visited in that way. We start by filling the sequence with $(5, 2)$ for the rows we already looked at. Then we look for an element in column 2 after row 2. We find that in row 6, so we add 6 to the sequence, which now is $(5, 2, 6)$. Because 6 is after 5 we can use a cyclic permutation of the rows in the sequence to get the new matrix

$$\begin{pmatrix} * & 0 & 0 & 0 & 0 & * \\ 0 & * & * & 0 & 0 & 0 \\ 0 & 0 & * & 0 & 0 & 0 \\ * & 0 & 0 & * & 0 & 0 \\ 0 & * & 0 & 0 & * & 0 \\ 0 & 0 & * & 0 & 0 & 0 \end{pmatrix}.$$

This leaves the sixth diagonal entry to be filled. Again we have to start a sequence $(6, 1)$. However, the only other entry in column 1 is in row 4 which is smaller than 6. So we add 4 to the sequence and have to continue looking in column 4. Column 4 however has no further entries, so we backtrack again and remove 4 from the sequence. There are no further entries in column 1, so we have to backtrack again, remove the 1 from the sequence and see no further entries in column 6, so we even remove the 6 from the sequence. When we see that the sequence is empty we know that we have a symbolic defect with column 6, so we would mark column 6 and if there were anymore columns, we would swap in the next column.

Overall we have fully analyzed if the matrix has nonzero kernel (it does) without introducing any fill, without even looking at the actual values. A further advantage of the described algorithm is that even if we don't find sufficient symbolic defect to determine that the kernel is nonzero, we have already ordered the rows such that the diagonals are filled, which is a useful first step for the Gauss algorithm.

Algorithm 6.4.2. The following pseudocode describes the full algorithm.

1. Array lookaheadprogress[Number of Columns] $= \{0, \ldots, 0\}$.
2. For every column c do:
   (a) Add c to empty sequence.
   (b) While sequence is nonempty do:

i. Look at the element e at the end of the sequence.

ii. If there exists a filled entry in column e with row $d \geq e$ then add d to the sequence and permute all rows in the sequence cyclically. Continue the outer loop with next column $c + 1$.

iii. Else if there exists a filled entry in column e with row $d >$ lookaheadprogress[c] then if d is not yet in the sequence add d to the sequence. On the other hand if d is already in the sequence, then set lookaheadprogress[c] $= d$ instead. Afterwards continue with the inner loop.

iv. If there does not exist such an entry then remove e from the sequence and set lookaheadprogress of the previous element in the sequence to e and continue with the inner loop.

(c) If the sequence is empty here it means we have not found a suitable permutation sequence for this column, so we mark the column, swap it with an unvisited column and increase the symbolic defect counter.

We can modify the algorithm to preferentially choose rows with few filled entries so that the element in the diagonal is already an acceptable pivot element. We will do this but we will also still check at each elimination step if there is a suitable row with fewer filled entries, because this can change significantly during the elimination process.

The algorithm here follows the `ANALYSE` phase described in [17]. The other two main steps are the factorization step, which is the application of the Gauss algorithm to get a triagonal matrix and the solving step, which uses a basis for all possible combinations of the free columns and fills in the dependent values according to the triangular form.

The book [17], like most sparse matrix literature, describes sparse matrix methods from the point of view of people working with floating point values. But many of the methods are still applicable here. What is added on top of that is the need to handle large integers, as after repeated multiplication with the pivot elements some of the entries will become quite large. We try to mitigate the growth of the entries by dividing out any common factors that occur in rows as much as possible. The big integers as they occur are stored in the containers provided by the gnu multiprecision library [21].

6.5 Using the software library

Get the source code by cloning the git repository on a linux system with the commands

```
git clone
   https://github.com/JonathanSteinbuch/sheafstability.git
cd sheafstability
```

To compile the program you need to have the GNU Multiprecision Library (`https://gmplib.org/`) and the Boost Program Options (`https://www.boost.org/`) installed.

To compile just run the command

```
make
```

in the main directory.

The basic usage is as follows. In the main directory run:

```
stability --input-file="input.txt" --output-file="output.txt"
```

There are several command line options which we will list later. In most cases using the default should be fine.

6.5.1 Format of the input and output file

The first line of the input file has to contain the name of the routine, that you want to use. There are the following accessible routines:

1. `semistability`,
2. `powers`,
3. `kernel`.

The `semistability` routine, is the implementation of the main algorithm.

Example 6.5.1. Instead of formally describing how the input and output file are formed, it is easier to just give an example. The following input file tells the program to decide whether the sheaf given by $\mathrm{Syz}(x^4 + y^2z^2, y^4, z^4, x^7)$ over $\mathbb{Q}[x, y, z]/z^5 + y^5 + x^5$ is semistable.

```
semistability
characteristic: 0
variables: "x", "y", "z"
relations: z^5+y^5+x^5
matrix: {{x^4+y^2z^2,y^4,z^4,x^7}}
```

The output file will contain the following:

```
1
{ExteriorPower => 1, SymmetricPower => 4, Twist => 25}
```

The 1 in the first line means that the sheaf is not semistable. The next line describes where a destabilizing subsheaf has been found.

The other two routines give direct access to subroutines of the main algorithm.

The routine named `powers` computes the defining matrix for the power $\mathrm{Sym}^q(\mathrm{Ext}^s(\mathcal{F}))$ of a kernel sheaf $\mathcal{F}$.

Example 6.5.2. To use it provide an input file of the following form:

```
powers
characteristic: 0
variables: "x", "y", "z"
relations: z^5+x^5+y^5
matrix: {{x^3, y^3, z^2}}
spower: 2
exteriorpower: 1
```

In this case we compute the defining matrix for

$$\mathrm{Sym}^2(\mathrm{Ext}^1(\mathrm{Syz}(x^3, y^3, z^2)))$$

over the curve defined by $\mathbb{Q}[x, y, z]/z^5 + y^5 + x^5$.

The output will be

```
    |  -6 -6 -5  -6 -5  -4
----| --- -- -- --- -- ---
  -3| 2x3 y3 z2   0  0   0
  -3|   0 x3  0 2y3 z2   0
  -2|   0  0 x3   0 y3 2z2
```

which is a representation of the matrix A_2:

$$\begin{pmatrix} 2x^3 & y^3 & z^2 & 0 & 0 & 0 \\ 0 & x^3 & 0 & 2y^3 & z^2 & 0 \\ 0 & 0 & x^3 & 0 & y^3 & 2z^2 \end{pmatrix}.$$

The numbers above and to the left are the twists of the summands in the kernel sequence as in Example 4.5.5.

Example 6.5.3. If you change the characteristic to positive, then the `powers` routine computes the Frobenius pull-back instead. For example the following produces the defining matrix for the Frobenius pull-back $\mathrm{F}^* \mathcal{F}$ in characteristic 3.

```
powers
characteristic: 3
variables: "x", "y", "z"
relations: z^5+x^5+y^5
matrix: {{x^3, y^3, z^2}}
spower: 3
exteriorpower: 1
```

The result is the following.

```
    |            -9  -9  -6
----| ----------- -- --
   0| 2z5x4+2y5x4  y9  z6
```

It represents the matrix $\begin{pmatrix} x^9 & y^9 & z^6 \end{pmatrix}$. Note that $x^9 = 2z^5x^4 + 2y^5x^4$ because of the relation and that x^9 is not in the monomial basis that the program chose, which is why a seemingly more complicated polynomial was output.

Also note that `spower` should be a power of the characteristic, otherwise the output has nothing to do with Frobenius pull-backs. A value of `spower: 9` would compute $\mathrm{F}^{2*} \mathcal{F}$.

With the `kernel` routine we can directly get global sections of the sheaf $\mathrm{Sym}^q(\mathrm{Ext}^s(\mathcal{F}))(k)$ corresponding to a kernel sheaf $\mathcal{F}$.

To use it provide an input file of the following form:

```
kernel
characteristic: 0
variables: "x", "y", "z"
relations: z^5+x^5+y^5
matrix: {{x^3, y^3, z^2}}
spower: 2
exteriorpower: 1
twist: 9
```

In this case we compute global sections of $\mathrm{Sym}^2(\mathrm{Ext}^1(\mathrm{Syz}(x^3, y^3, z^2)))(9)$ over

$\mathbb{Q}[x, y, z]/z^5 + y^5 + x^5$. The output file will contain the matrix

$$\begin{pmatrix} -z^2y \\ 0 \\ 2yx^3 \\ -z^2x \\ 2y^3x \\ z^3yx \end{pmatrix}.$$

The first line of the output file is 0 if there are no global sections. If you specifiy the program option to compute the full kernel (with the option `-f`) it will have the dimension of the space of global sections. By default it's just one if there is a global section.

The result for the full kernel will be

$$\begin{pmatrix} -z^2y & yx^2 & y^3 \\ 0 & 2y^3 & -2x^3 \\ 2yx^3 & 2z^3y & 0 \\ -z^2x & -x^3 & -y^2x \\ 2y^3x & 0 & -2z^3x \\ z^3yx & -zyx^3 & zy^3x \end{pmatrix}.$$

The columns are a generating set of minimal length for the global sections in twist 9.

6.5.2 Notes on the polynomial format

The polynomials are output in a short form where an integer directly after a variable means that that variable is taken to the power of the integer. So for example `-5z32y9x` is the same as `-5*z^32*y^9*x^1`. If you prefer a longer form there is a command line option (`-p`) for that. You can also enter polynomials in short form in the input file if you want. The parser is relatively robust in that regard.

You can input variables in many forms. The only restrictions are that the first character has to be a letter from the alphabet and that a variable descriptor can not be contained in full in another. So `z_{0,1}` is a totally fine variable descriptor, for example.

6.5.3 Command line options

There are several command line options to change the behavior of the program or to change the output format.

verbosity The verbosity level from 0 to 3. The default is 1, 0 is silent unless there is an error.

input-file Input file name.

output-file Output file name.

exterior-powers (`-e`) Set if you want to use exterior powers in the algorithm, i.e. Theorem 4.4.2.

stop-unstable (`-s`) Stop immediately if a destabilizing section has been found.

linear-progression (`-l`) Linear progression of powers instead of quadratic increase of powers in the attempt to find destabilizing sections. Can be used to make sure no lower powers with destabilizing sections are missed.

pre-analyze (`-a`) Use a symbolical analysis step to capture symbolic rank defects before the Gauß algorithm.

forceQ (`-q`) Force computations over $\mathbb{Q}$ (in characteristic 0). By default $\mathbb{Z}$ is used where possible.

output-for-M2 (`-m`) Write output matrices in a format easily readable by Macaulay2.

output-for-latex (`-x`) Write output matrices in a format easily usable in LaTeX[38].

compute-full-kernel (`-f`) Always compute the full kernels and not just whether the kernel is nonzero.

long-form-polys (`-p`) Output polynomials in a less condensed format.

Example 6.5.4. To compute the problem described in `input.txt` with high verbosity, and output the polynomials in a longer format and stop as soon as we find a destabilizing section we would execute the following command on the command line.

```
stability --input-file="input.txt"
   --output-file="output.txt" --verbosity=3 -p -s
```

6.5.4 Macaulay2 package

In addition to the usual way to access the program from the command line, there is a Macaulay2 package that you can use for the same purpose. So far this is mainly a proof of concept and the output is much less verbose. But it should be easily expandable should the need arise.

Example 6.5.5. The following is a minimal working Macaulay2 code example to use our library to decide semistability. To run it you need SheafStability.m2 and the sheafstability executable on the same path.

```
loadPackage "SheafStability";

R = QQ[x,y,z,MonomialOrder=>GRevLex]
I = ideal(z^5+x^5+y^5)
S = R/I
M = matrix{{x^3, y^3, z^2}}

x = computeSemistability(M)
```

In this case, since the corresponding kernel sheaf is semistable the variable `x` just contains the value `true`.

If the result was that the sheaf is not semistable the variable `x` would be a list with three entries. First the value `false`, then a hash table with the keys `ExteriorPower`, `SymmetricPower` and `Twist` pointing to the location of a destabilizing section and lastly a matrix containing an actual destabilizing section.

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