Restricted L_∞-algebras

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Abstract

We give a model of restricted L_∞-algebra in a nice preadditive symmetric monoidal ∞-category $\mathcal{C}$ as an algebra over the monad $\mathcal{L}$ associated to an adjunction between $\mathcal{C}$ and the ∞-category of cocommutative bialgebras in $\mathcal{C}$, where the left adjoint lifts the free associative algebra.

If $\mathcal{C}$ is additive, we construct a canonical forgetful functor from $\mathcal{L}$-algebras in $\mathcal{C}$ to spectral Lie algebras in $\mathcal{C}$ and show that this functor is an equivalence if $\mathcal{C}$ is a $\mathbb{Q}$-linear stable ∞-category.

For every field K we construct a canonical forgetful functor from $\mathcal{L}$-algebras in connective K-modules to the ∞-category underlying a model structure on simplicial restricted Lie algebras over K.

Contents

1 Introduction

1.0.1 Motivation and basic ideas

Over a field of positive char. Lie algebras often loose their behaviour which they have over a field of char. zero. To remedy the situation one studies restricted Lie algebras, which are Lie algebras equipped with a frobenius operation.

Restricted Lie algebras in positive char. behave much like usual Lie algebras do in char. zero: By the theorem of Milnor-Moore [20] 5.18 and 6.11. every restricted Lie algebra arises as the primitive elements of its restricted enveloping Hopf algebra, which is finite dimensional if and only if the restricted Lie algebra is.

Luckily when turning to positive char. constructions in Lie theory do not only give a Lie algebra but a restricted Lie algebra. For example every associative algebra over a field of positive char. defines a restricted Lie algebra with the commutator as its Lie bracket or every algebraic group over a field of positive char. gives rise to a restricted Lie algebra structure on its tangent space.

One would like that this picture carries over to homotopy theory: When studying homotopy theory over a field of positive char., one would like to have a restricted version of L_∞-algebra which behaves much like L_∞-algebras in char. zero.

In this work we will introduce a model of restricted L_∞-algebras that is available in every nice preadditive symmetric monoidal ∞-category $\mathcal{C}$ like the ∞-category of spectra or K-module spectra over some field K and so leads to notions of spectral restricted Lie algebras and restricted L_∞-algebras over K.

We expect that results in char. zero concerning L_∞-algebras generalize to fields K of positive char. using our model of restricted L_∞-algebras.

For example we expect that every formal stack over a nice algebraic derived stack X over a field of positive char. admits a tangent restricted L_∞-algebra in the ∞-category of quasi-coherent sheaves on X generalizing a result of Hennion [12].

There is a notion of Lie algebra in the ∞-category of spectra as algebra over Ching's spectral Lie operad, i.e. the Koszul-dual operad of the shifted cocommutative cooperad in spectra, whose homology is the classical Lie operad.

To define a restricted version of spectral Lie algebras one could develop a theory of divided power Lie algebras in the ∞-category of spectra, which seem to be a reasonable model of restricted spectral Lie algebras due to a theorem of Fresse ([6] theorem 1.2.5.), according to which restricted Lie algebras over a field K are divided power Lie algebras in the category of K-vector spaces.

Another more naive model of restricted L_∞-algebra is that of a homotopy type of simplicial restricted Lie algebras over K, which form a model category and so have an underlying ∞-category.

We take a different approach to define restricted L_∞-algebras motivated by the theorem of Milnor-Moore [20] 5.18 and 6.11.:

For every field K denote Lie_K the category of restricted Lie algebras over K which are nothing than usual Lie algebras if K has char. zero.

By the theorem of Milnor-Moore [21] there is an embedding

$$\mathcal{U} : \mathrm{Lie}_K \subset \mathrm{Hopf}_K$$

of the category Lie_K of restricted Lie algebras over K into the category Hopf_K of Hopf algebras over K, where $\mathcal{U}$ sends a restricted Lie algebra to its enveloping Hopf algebra.

This way we can think of every restricted Lie algebra as a Hopf algebra, where the free restricted Lie algebra $\mathcal{L}(X)$ on a K-vector space X gets the tensoralgebra $T(X) \cong \mathcal{U}(\mathcal{L}(X))$ on X.

On the other hand every Hopf algebra Y over K gives rise to a restricted Lie algebra structure on its primitive elements $\mathcal{P}(Y)$, which is characterized by the following universal property:

The functor $\mathcal{U}$ is left adjoint to the functor $\bar{\mathcal{P}} : \mathrm{Hopf}_K \to \mathrm{Lie}_K$ that sends a Hopf algebra to its primitive elements with its natural restricted Lie algebra structure. So by adjointness the functor $T \cong \mathcal{U} \circ \mathcal{L} : \mathrm{Mod}_K \to \mathrm{Lie}_K \subset \mathrm{Hopf}_K$ that sends a K-vector space to its tensoralgebra is left adjoint to the functor $\mathcal{P} : \mathrm{Hopf}_K \to \mathrm{Mod}_K$ that sends a Hopf algebra to its primitive elements.

The theorem of Barr-Beck implies that the forgetful functor $\mathrm{Lie}_K \to \mathrm{Mod}_K$ is a monadic functor, i.e. that Lie_K is the category of algebras over the monad associated to the free restricted Lie algebra-forgetful adjunction.

As $\mathcal{U}$ is fully faithful, the unit $\mathrm{id} \to \bar{\mathcal{P}} \circ \mathcal{U}$ is an isomorphism and so gives rise to an isomorphism $\mathcal{L} \cong \bar{\mathcal{P}} \circ \mathcal{U} \circ \mathcal{L} \cong \bar{\mathcal{P}} \circ \mathrm{T}$.

Thus the free restricted Lie algebra-forgetful adjunction and the adjunction $\mathrm{T} : \mathrm{Mod}_\mathrm{K} \rightleftarrows \mathrm{Hopf}_\mathrm{K} : \mathcal{P}$ induce the same monad $\mathcal{L} \cong \mathcal{P} \circ \mathrm{T}$ on the category of K-vector spaces, whose category of algebras is Lie_K. So we get a description of the category of restricted Lie algebras as the category of algebras over the monad associated to the adjunction $\mathrm{T} : \mathrm{Mod}_\mathrm{K} \rightleftarrows \mathrm{Hopf}_\mathrm{K} : \mathcal{P}$.

We turn this description of the category of restricted Lie algebras into a definition and show that this definition makes sense in every nice preadditive symmetric monoidal ∞-category $\mathcal{C}$.

More precisely, we show that the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ admits an essentially unique lift $\mathrm{T} : \mathcal{C} \to \mathrm{Bialg}(\mathcal{C})$ to the ∞-category $\mathrm{Bialg}(\mathcal{C})$ of cocommutative bialgebras in $\mathcal{C}$ and prove that T admits a right adjoint $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathcal{C}$ (proposition 3.22 and remark 2.25).

Motivated by the theorem of Milnor-Moore we define restricted L_∞-algebras in $\mathcal{C}$ as algebras over the monad $\mathcal{L}$ on $\mathcal{C}$ associated to the adjunction $\mathrm{T} : \mathcal{C} \rightleftarrows \mathrm{Bialg}(\mathcal{C}) : \mathcal{P}$ and write $\mathrm{Lie}(\mathcal{C})$ for the ∞-category of $\mathcal{L}$-algebras in $\mathcal{C}$ (definition 2.26).

By remark 2.27 the functor $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathcal{C}$ lifts to a functor $\bar{\mathcal{P}} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Lie}(\mathcal{C})$ right adjoint to a functor $\mathcal{U} : \mathrm{Lie}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$.

Inspired by the theorem of Milnor-Moore we think of $\mathcal{U}$ as associating the enveloping bialgebra and of $\bar{\mathcal{P}}$ as associating the primitive elements.

By remark 2.27 the ∞-category $\mathrm{Lie}(\mathcal{C})$ over $\mathcal{C}$ admits the following universal property: Every lift $\mathrm{Bialg}(\mathcal{C}) \to \mathcal{D}$ of $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathcal{C}$ along a monadic functor $\mathcal{D} \to \mathcal{C}$ factors as $\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}) \to \mathcal{D}$ for a unique functor $\mathrm{Lie}(\mathcal{C}) \to \mathcal{D}$ over $\mathcal{C}$.

This may be interpreted by saying that the structure of a restricted L_∞-algebra is the finest structure the primitive elements can be endowed with.

Stated in a more axiomatically way (remark 2.28) and using that the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ uniquely lifts to cocommutative bialgebras in $\mathcal{C}$ the ∞-category $\mathrm{Lie}(\mathcal{C})$ is uniquely determined by its following relations to $\mathcal{C}$ and $\mathrm{Bialg}(\mathcal{C})$:

- We have a monadic forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathcal{C}$ with left adjoint $\mathcal{L}$.
- We have a left adjoint enveloping bialgebra functor $\mathcal{U} : \mathrm{Lie}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ such that the composition $\mathcal{U} \circ \mathcal{L} : \mathcal{C} \to \mathrm{Lie}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ lifts the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ and a weak version of the Milnor-Moore theorem holds:

  $\mathcal{U}$ restricts to a fully faithful functor $\mathcal{L}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ on free restricted L_∞-algebras.

1.0.2 Historical background and related work

There has been a long tradition to define homotopy-coherent versions of Lie algebras, which are called L_∞-algebras. Certainly differentially graded Lie algebras over a field of char. zero are the most well known structure representing L_∞-algebras over this field and play a central role in rational homotopy theory, deformation theory and derived geometry:

By Quillen (Quillen69) connected dgLie-algebras model simply connected rational homotopy types.

Expected long time by Quillen, Deligne, Drinfeld, Kapranov and others and proven by Lurie ([17]) and Pridham ([23]) L_∞-algebras over a field of char. 0 are equivalent to formal moduli problems, where Pridham also treats some extensions to positive char.

Besides the local behaviour L_∞-algebras also describe the global behaviour of a derived stack when they arise as the tangent Lie algebra: Hennion constructed an adjunction between formal stacks over a nice derived algebraic stack over a field of char. zero and L_∞-algebras in quasi-coherent sheaves over the derived algebraic stack ([12]).

A more modern model of L_∞-algebras that exists over every E_∞-ring spectrum is an algebra over the spectral Lie operad.

Discovered by Ching [4] this operad structure on the Goodwillie derivatives of the identity of the ∞-category of spectra has the Lie operad as its homology and is Koszul-dual to the shifted non-counital cocommutative cooperad in spectra.

Using the spectral Lie operad classical Lie theory lifts to stable homotopy theory and has deep connections to Goodwillie calculus by work of Camarena [1], Heuts, [13], Kjaer [15], Knudsen [16].

A further model of L_∞-algebras in positive char., which is still work in progress, are the Partition Lie algebras of Brantner and Mathew which are closely connected to the Partition complex and classify formal moduli problems over arbitrary fields.

1.0.3 Main results

We give the definition of restricted L_∞-algebras in a nice preadditive symmetric monoidal ∞-category $\mathcal{C}$ like the ∞-category of spectra or module spectra over a E_∞-ring spectrum (definition 2.26).

To define restricted L_∞-algebras in $\mathcal{C}$ we need to lift the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ to cocommutative bialgebras in $\mathcal{C}$.

We prove that there is an essentially unique such lift of the free functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ to cocommutative bialgebras in $\mathcal{C}$ (proposition 3.22).

If $\mathcal{C}$ is additionally additive, we construct a canonical forgetful functor

$$\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

from the ∞-category of restricted L_∞-algebras in $\mathcal{C}$ to the ∞-category of algebras over the spectral Lie operad, i.e. the Koszul-dual operad of the shifted cocommutative cooperad in spectra (theorem 4.2).

We show that this forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ is an equivalence if $\mathcal{C}$ is additionally a $\mathbb{Q}$-linear stable ∞-category, i.e. a stable ∞-category left tensored over $\mathrm{H}(\mathbb{Q})$-module spectra (theorem 4.5).

Given a field K we construct a canonical forgetful functor

$$\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to (\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}})_\infty$$

from the ∞-category of restricted L_∞-algebras in connective H(K)-module spectra to the ∞-category underlying a right induced model structure on the category $\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}}$ of simplicial restricted Lie algebras over K (proposition 4.34).

By the theorem of Milnor-Moore this forgetful functor restricts to an equivalence on the full subcategory of restricted L_∞-algebras, whose underlying connective H(K)-module spectrum is a K-vector space.

1.0.4 Overview and guideline how to read this work

The reader only interested in the definition of restricted L_∞-algebras and their relation to spectral Lie algebras and simplicial restricted Lie algebras should focus on sections 2.3, 4.1 and 4.3.

In section 2.3 we define restricted L_∞-algebras in a nice preadditive symmetric monoidal ∞-category $\mathcal{C}$ and study their basic properties.

For example we show that the ∞-category of restricted L_∞-algebras in $\mathcal{C}$ is presentable if $\mathcal{C}$ is presentable (remark 2.30).

Moreover we define a more general version of restricted L_∞-$\mathcal{H}$-algebras depending on a unital Hopf operad $\mathcal{H}$ in $\mathcal{C}$ that specializes to the notion of restricted L_∞-algebra if we choose $\mathcal{H}$ to be the Hopf operad, whose algebras are associative algebras.

Sections 2.1 and 2.2 provide notions needed to define restricted L_∞-algebras. In sections 2.1 we study bialgebras and Hopf algebras, in section 2.2 we study Hopf operads.

To define restricted L_∞-algebras we need to show that the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ canonically lifts to cocommutative bialgebras in $\mathcal{C}$ (proposition 3.22).

Proving this is the main goal of section 3.3, where we use techniques about cocartesian operads of section 3.1 and 6.1.

In section 4.1 we construct a forgetful functor

$$\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

from the ∞-category of restricted L_∞-algebras in a nice stable symmetric monoidal ∞-category $\mathcal{C}$ to the ∞-category of algebras over the spectral Lie operad, which we define as the Koszul-dual operad of the shifted cocommutative cooperad in spectra (theorem 4.2).

Moreover we show that this forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ is an equivalence if $\mathcal{C}$ is additionally a $\mathbb{Q}$-linear stable ∞-category, i.e. a stable ∞-category left tensored over $\mathrm{H}(\mathbb{Q})$-module spectra (theorem 4.5).

To put these constructions and proofs on a formal fundament we develop a theory of operads and cooperads in a nice symmetric monoidal ∞-category given in section 2.2 and a theory of Koszul-duality for operads and their algebras given in section 4.2.

In section 4.3 we construct a forgetful functor

$$\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to (\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}})_\infty$$

from the ∞-category of restricted L_∞-algebras in connective $\mathrm{H(K)}$-module spectra for some field K to the ∞-category underlying a right induced model structure on the category $\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}}$ of simplicial restricted Lie algebras over K (proposition. 4.34).

To show this, we use that restricted Lie algebras over K are algebras over an algebraic theory in the category of sets (remark 4.37).

This implies the existence of a right induced model structure on the category $\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}}$ which has the nice properties we need.

Section 4.4 studies the properties of algebraic theories needed to prove proposition 4.34.

In section 5 we show that every Hopf operad $\mathcal{H}$ in a symmetric monoidal ∞-category $\mathcal{D}$ endows its ∞-category of algebras with a symmetric monoidal structure such that the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{D}) \to \mathcal{D}$ gets symmetric monoidal (proposition 5.77).

This is used on the one hand in the definition of restricted L_∞-$\mathcal{H}$-algebras and on the other hand to link algebras over the spectral Lie operad in a nice stable symmetric monoidal ∞-category $\mathcal{C}$ with the symmetric monoidal ∞-category of coaugmented cocommutative coalgebras in $\mathcal{C}$ via Koszul-duality.

This relationship between spectral Lie algebras and coaugmented cocommutative coalgebras is the main ingredient to construct a forgetful functor from restricted L_∞-algebras in $\mathcal{C}$ to spectral Lie algebras in $\mathcal{C}$.

1.1 Notation and Terminologie

Fix your preferred model of ∞-categories.

By category we always mean ∞-category, by 2-category we mean $(\infty,2)$-category and by operad we mean ∞-operad.

We describe ∞-operads and $(\infty,2)$-categories purely in terms of ∞-categories, where we take Lurie's definitions found in [18] 2.1.1.10. and 4.2.1.28. but interprete them homotopy-invariant (see for example the notion of (locally) cocartesian fibration in the next subsection).

Given a category $\mathcal{C}$ denote $\mathsf{Ho}(\mathcal{C})$ its homotopy category.

Denote Cat_∞ the category of small categories and $\mathcal{S}$ the full subcategory of Cat_∞ spanned by the small spaces.

$\mathcal{S}$ and Cat_∞ admit all small limits and small colmits.

Given two small categories $\mathcal{C},\mathcal{D}$ denote $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ the category of functors $\mathcal{C}\to\mathcal{D}$ being the internal hom of $\mathsf{Ho}(\mathsf{Cat}_\infty)$.

Given a small category $\mathcal{C}$ and objects $\mathrm{X},\mathrm{Y}\in\mathcal{C}$ we write $\mathcal{C}(\mathrm{X},\mathrm{Y})$ for the space of morphisms $\mathrm{X}\to\mathrm{Y}$ in $\mathcal{C}$ that can be defined as $\mathcal{C}(\mathrm{X},\mathrm{Y}) := \{(\mathrm{X},\mathrm{Y})\}\times_{\mathcal{C}\times\mathcal{C}}\mathrm{Fun}(\Delta^1,\mathcal{C})$.

Moreover we have a natural equivalence

$$\mathsf{Cat}_\infty(\mathcal{B}\times\mathcal{C},\mathcal{D})\simeq\mathsf{Cat}_\infty(\mathcal{B},\mathrm{Fun}(\mathcal{C},\mathcal{D}))$$

for $\mathcal{B},\mathcal{C},\mathcal{D}\in\mathsf{Cat}_\infty$.

Given a small category $\mathcal{C}$ denote $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathcal{S})$ the category of presheaves on $\mathcal{C}$.

Given a category containing a morphism $\iota:\mathrm{X}\to\mathrm{Y}$, we call X a subobject of Y if $\iota:\mathrm{X}\to\mathrm{Y}$ is a monomorphism, i.e. for every $\mathrm{Z}\in\mathcal{C}$ induces a fully faithful map $\mathcal{C}(\mathrm{Z},\mathrm{X})\to\mathcal{C}(\mathrm{Z},\mathrm{Y})$.

If ι is clear from the context, we also write $\mathrm{X}\subset\mathrm{Y}$ to indicate that X is a subobject of Y via ι.

We often use this notion in the cases of a morphism of small categories and small operads, where we also use the term subcategory and suboperad.

Remark that monomorphisms are stable under pullback and thus are preserved by pullback preserving functors.

Given a full subcategory $\mathcal{K}\subset\mathsf{Cat}_\infty$ denote $\mathsf{Cat}_\infty^{\mathrm{coc}}(\mathcal{K})$ the subcategory of Cat_∞ with objects the small categories that admit colimits indexed by categories that belong to $\mathcal{K}$ and morphisms the functors that preserve these colimits.

For $\mathcal{K}=\mathsf{Cat}_\infty$ we write $\widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}$ for $\widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}(\mathcal{K})}$.

(locally) (co)cartesian morphisms and fibrations

Let $\phi : \mathcal{C} \to \mathcal{D}$ be a functor. We call a morphism $\mathsf{f} : X \to Y$ in $\mathcal{C}$ ϕ-cocartesian if the commutative square

$$\begin{array}{ccc} \mathcal{C}(Y,Z) & \longrightarrow & \mathcal{C}(X,Z) \\ \downarrow & & \downarrow \\ \mathcal{D}(\phi(Y),\phi(Z)) & \longrightarrow & \mathcal{D}(\phi(X),\phi(Z)) \end{array}$$

is a pullback square of spaces.

By the pasting law for pullbacks the following statements follow immediately from the definition:

1. Let $\mathsf{f} : X \to Y$ and $g : Y \to Z$ be morphisms of $\mathcal{C}$.
   Assume that f is ϕ-cocartesian.
   Then g is ϕ-cocartesian if and only if $g \circ \mathsf{f}$ is ϕ-cocartesian.

2. Let $\psi : \mathcal{D}' \to \mathcal{D}$ be a functor and $\phi' : \mathcal{C}' \to \mathcal{D}'$ the pullback of $\phi : \mathcal{C} \to \mathcal{D}$ along ψ.
   Let $\mathsf{f} : X \to Y$ be a morphism of $\mathcal{C}'$, whose image in $\mathcal{C}$ is ϕ-cocartesian.
   Then $\mathsf{f} : X \to Y$ is ϕ'-cocartesian.

3. Let $\varphi : \mathcal{D} \to \mathcal{E}$ be a functor and $\mathsf{f} : X \to Y$ a morphism of $\mathcal{C}$ such that $\phi(\mathsf{f})$ is φ-cocartesian.
   Then f is ϕ-cocartesian if and only if f is $\varphi \circ \phi$-cocartesian.

We call a morphism $\mathsf{f} : X \to Y$ in $\mathcal{C}$ locally ϕ-cocartesian if one of the following equivalent conditions holds:

1. $\mathsf{f} : X \to Y$ is ϕ'-cocartesian, where ϕ' denotes the pullback $\Delta^1 \times_{\mathcal{D}} \mathcal{C} \to \Delta^1$ of ϕ along $\phi(\mathsf{f})$.
2. $\mathsf{f} : X \to Y$ is a final object of the category $\{\phi(\mathsf{f})\} \times_{\mathcal{D}_{\phi(X)/}} \mathcal{C}_{X/}$.
3. For every $Z \in \mathcal{C}$ lying over the object $\phi(Y)$ composition with $\mathsf{f} : X \to Y$

   $$\{\mathrm{id}\} \times_{\mathcal{D}(\phi(Y),\phi(Y))} \mathcal{C}(Y,Z) \to \{\phi(\mathsf{f})\} \times_{\mathcal{D}(\phi(X),\phi(Y))} \mathcal{C}(X,Z)$$

   is an equivalence.

The following statements follow immediately from the definition:

Every ϕ-cocartesian morphism is locally ϕ-cocartesian.

Let $\psi : \mathcal{D}' \to \mathcal{D}$ be a functor and $\phi' : \mathcal{C}' \to \mathcal{D}'$ the pullback of $\phi : \mathcal{C} \to \mathcal{D}$ along ψ. Let $\mathsf{f} : X \to Y$ be a morphism of $\mathcal{C}'$.

Then $\mathsf{f} : X \to Y$ is locally ϕ'-cocartesian if and only if the image of f in $\mathcal{C}$ is locally ϕ-cocartesian.

We call a functor $\phi : \mathcal{C} \to \Delta^1$ a cocartesian fibration if for every object X of $\mathcal{C}$ lying over 0 there is a ϕ-cocartesian morphism $X \to Y$ in $\mathcal{C}$ such that Y lies over 1.

We call a functor $\phi : \mathcal{C} \to \mathcal{D}$ a locally cocartesian fibration if the pullback $\Delta^1 \times_{\mathcal{D}} \mathcal{C} \to \Delta^1$ along every morphism of $\mathcal{D}$ is a cocartesian fibration.

We call a functor $\phi : \mathcal{C} \to \mathcal{D}$ a cocartesian fibration if it is a locally cocartesian fibration and every locally ϕ-cocartesian morphism is ϕ-cocartesian.

We call a functor $\mathcal{C} \to \mathcal{D}$ a left fibration if it is a cocartesian fibration and all its fibers over objects of $\mathcal{D}$ are spaces.

Dually, we define (locally) cartesian morphisms, (locally) cartesian fibrations and right fibrations.

Denote

- $\mathsf{Cat}_\infty^{\mathrm{L}}$ and $\mathsf{Cat}_\infty^{\mathrm{R}}$ the wide subcategories of Cat_∞ with morphisms the left adjoint respectively right adjoint functors
- Op_∞ the category of small operads
- $\mathcal{L}$ and $\mathcal{R}$ the full subcategories of $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ spanned by the left respectively right fibrations
- $\mathrm{Cocart}, \mathrm{Cart}$ and Bicart the subcategories of $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ with objects the cocartesian fibrations, cartesian fibrations respectively bicartesian fibrations and morphisms the squares of small categories, whose top functor preserves cocartesian, cartesian, respectively both cocartesian and cartesian morphisms
- $\mathcal{U}$ the full subcategory of $\mathcal{R}$ spanned by the representable right fibrations.

  Remark 1.1. *The evaluation at the target functor* $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ *is a cartesian fibration as* Cat_∞ *admits pullbacks.*

  As left, right, cocartesian, cartesian and bicartesian fibrations and their morphisms (over a fixed category) are stable under pullback, the restrictions $\mathcal{L} \to \mathsf{Cat}_\infty, \mathcal{R} \to \mathsf{Cat}_\infty, \mathrm{Cocart} \to \mathsf{Cat}_\infty, \mathrm{Cart} \to \mathsf{Cat}_\infty$ *and* $\mathrm{Bicart} \to \mathsf{Cat}_\infty$ *of the evaluation at the target functor are cartesian fibrations.*

  Given a small category $\mathcal{C}$ we usually denote the corresponding fibers by $\mathcal{L}_\mathcal{C}, \mathcal{R}_\mathcal{C}, \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cocart}}, \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ respectively $\mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{bicart}}$.

  By proposition 6.9 the restriction $\mathcal{U} \to \mathsf{Cat}_\infty$ of the evaluation at the target functor $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathsf{Cat}_\infty$ to $\mathcal{U}$ is a cocartesian fibration and classifies the identity of Cat_∞.

Let $\mathrm{X} \to \mathrm{S}$ be a cocartesian fibration classifying a functor $\phi : \mathrm{S} \to \mathsf{Cat}_\infty$.

We call the cocartesian fibration $\mathrm{X}^{\mathrm{rev}} \to \mathrm{S}$ classifying the functor $\mathrm{S} \xrightarrow{\phi} \mathsf{Cat}_\infty \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_\infty$ the fiberwise dual of $\mathrm{X} \to \mathrm{S}$.

Given a cocartesian fibration of operads $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ the cocartesian fibration $(\mathcal{C}^\otimes)^{\mathrm{rev}} \to \mathcal{O}^\otimes$ is a cocartesian fibration of operads classifying the $\mathcal{O}^\otimes$-monoid $\mathcal{O}^\otimes \to \mathsf{Cat}_\infty \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_\infty$.

The underlying functor of $(\mathcal{C}^\otimes)^{\mathrm{rev}} \to \mathcal{O}^\otimes$ is the fiberwise dual $\mathcal{C}^{\mathrm{rev}} \to \mathcal{O}$ of $\mathcal{C} \to \mathcal{O}$.

Let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of operads, i.e. a $\mathcal{O}^\otimes$-monoidal category and $\mathcal{K} \subset \mathsf{Cat}_\infty$ a full subcategory.

We say that $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is compatible with colimits indexed by categories that belong to $\mathcal{K}$ if for every $\mathrm{X} \in \mathcal{O}$ the fiber $\mathcal{C}_\mathrm{X}$ admits colimits indexed by categories that belong to $\mathcal{K}$ and for every operation $\mathrm{h} \in \mathrm{Mul}_\mathcal{O}(\mathrm{X}_1, ..., \mathrm{X}_\mathrm{n}; \mathrm{Y})$ the induced functor $\mathcal{C}_{\mathrm{X}_1} \times ... \times \mathcal{C}_{\mathrm{X}_\mathrm{n}} \to \mathcal{C}_\mathrm{Y}$ preserves colimits indexed by categories that belong to $\mathcal{K}$ in each component.

We call $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a

- accessible $\mathcal{O}^{\otimes}$-monoidal category if for every $X \in \mathcal{O}$ the category $\mathcal{C}_X$ is accessible and for every operation $X_1, ..., X_n \to Y$ of $\mathcal{O}$ the induced functor $\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n} \to \mathcal{C}_Y$ is accessible.
- presentable $\mathcal{O}^{\otimes}$-monoidal category if $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is an accessible $\mathcal{O}^{\otimes}$-monoidal category and for every $X \in \mathcal{O}$ the category $\mathcal{C}_X$ is presentable.
- presentably $\mathcal{O}^{\otimes}$-monoidal category if for every $X \in \mathcal{O}$ the category $\mathcal{C}_X$ is presentable and for every operation $X_1, ..., X_n \to Y$ of $\mathcal{O}$ the induced functor $\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n} \to \mathcal{C}_Y$ preserves small colimits in each variable.

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1.3 Some elementary notions

1.3.1 Lax and oplax monoidal functors

Some remarks about lax and oplax $\mathcal{O}^{\otimes}$-monoidal functors.

Let $\mathcal{O}^{\otimes}$ be an operad and $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ be $\mathcal{O}^{\otimes}$-monoidal categories.

We set

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D}) := \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$$

and

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax}}(\mathcal{C}, \mathcal{D}) := \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})^{\mathrm{op}}.$$

We say that a lax $\mathcal{O}^{\otimes}$-monoidal functor $H : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ represents an oplax $\mathcal{O}^{\otimes}$-monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ or say that H corresponds to an oplax $\mathcal{O}^{\otimes}$-monoidal functor $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and write F^{rev} for H and H^{rev} for F.

For every $X \in \mathcal{O}$ we have forgetful functors

$$(-)_X : \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_X, \mathcal{D}_X)$$

and

$$(-)_X : \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}_X^{\mathrm{op}}, \mathcal{D}_X^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Fun}(\mathcal{C}_X, \mathcal{D}_X).$$

So given an oplax $\mathcal{O}^\otimes$-monoidal functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ corresponding to a lax $\mathcal{O}^\otimes$-monoidal functor $F^{rev} : (\mathcal{C}^\otimes)^{rev} \to (\mathcal{D}^\otimes)^{rev}$ we have $F_X = (F_X^{rev})^{op}$.

We have a full subcategory inclusion $\mathrm{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax}}(\mathcal{C}, \mathcal{D})$ that fits into a commutative square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}, \mathcal{D}) & \xrightarrow[\mathrm{rev}]{\simeq} & \mathrm{Fun}_{\mathcal{O}}^{\otimes}(\mathcal{C}^{rev}, \mathcal{D}^{rev})^{op} \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax}}(\mathcal{C}, \mathcal{D}) & \xrightarrow[\mathrm{rev}]{=} & \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}^{rev}, \mathcal{D}^{rev})^{op}, \end{array}$$

where the top horizontal functor takes the fiberwise dual over $\mathcal{O}^\otimes$.

1.3.2 Monoidal adjunctions

Let $\mathcal{O}^\otimes$ be an operad, $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ be $\mathcal{O}^\otimes$-monoidal categories, $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ an oplax $\mathcal{O}^\otimes$-monoidal functor corresponding to a lax $\mathcal{O}^\otimes$-monoidal functor $F^{rev} : (\mathcal{C}^\otimes)^{rev} \to (\mathcal{D}^\otimes)^{rev}$ and $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ a lax $\mathcal{O}^\otimes$-monoidal functor.

We say that $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is $\mathcal{O}^\otimes$-monoidally left adjoint to $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ or $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ is $\mathcal{O}^\otimes$-monoidally right adjoint to $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ if the lax $\mathcal{O}^\otimes$-monoidal functors

$$F^{rev} : (\mathcal{C}^\otimes)^{rev} \to (\mathcal{D}^\otimes)^{rev} \subset \mathcal{P}(\mathcal{D}^{rev})^\otimes, \qquad G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes \subset \mathcal{P}(\mathcal{C})^\otimes$$

correspond to equivalent lax $\mathcal{O}^\otimes$-monoidal functors

$$(\mathcal{C}^\otimes)^{rev} \times_{\mathcal{O}^\otimes} \mathcal{D}^\otimes \to \mathcal{O}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathcal{S}^\times.$$

Remark 1.2. *Let $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ be an oplax $\mathcal{O}^\otimes$-monoidal functor corresponding to a lax $\mathcal{O}^\otimes$-monoidal functor $F^{rev} : (\mathcal{C}^\otimes)^{rev} \to (\mathcal{D}^\otimes)^{rev}$ and $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ a lax $\mathcal{O}^\otimes$-monoidal functor corresponding to an oplax $\mathcal{O}^\otimes$-monoidal functor $G^{rev} : (\mathcal{D}^\otimes)^{rev} \to (\mathcal{C}^\otimes)^{rev}$.*

The oplax $\mathcal{O}^\otimes$-monoidal functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is $\mathcal{O}^\otimes$-monoidally left adjoint to $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ if and only if $G^{rev} : (\mathcal{D}^\otimes)^{rev} \to (\mathcal{C}^\otimes)^{rev}$ is $\mathcal{O}^\otimes$-monoidally left adjoint to $F^{rev} : (\mathcal{C}^\otimes)^{rev} \to (\mathcal{D}^\otimes)^{rev}$.

Remark 1.3.

- *$\mathcal{O}^\otimes$-monoidal left respectively right adjoints are unique if they exist.*

- *An oplax $\mathcal{O}^\otimes$-monoidal functor $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ admits a lax $\mathcal{O}^\otimes$-monoidal right adjoint $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ if and only if for all $X \in \mathcal{O}$ the induced functor $F_X : \mathcal{C}_X \to \mathcal{D}_X$ admits a right adjoint.*

  Dually a lax $\mathcal{O}^\otimes$-monoidal functor $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ admits an oplax $\mathcal{O}^\otimes$-monoidal left adjoint $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ if and only if for all $X \in \mathcal{O}$ the induced functor $G_X : \mathcal{D}_X \to \mathcal{C}_X$ admits a left adjoint.

Let $\mathcal{O}^\otimes$ be an operad and $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ be $\mathcal{O}^\otimes$-monoidal categories.

Denote

- $\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{D},\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{D},\mathcal{C})$ the full subcategory spanned by the lax $\mathcal{O}^{\otimes}$-monoidal functors $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ such that for all $X \in \mathcal{O}$ the induced functor $G_X : \mathcal{D}_X \to \mathcal{C}_X$ admits a left adjoint,

- $\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax},\mathrm{L}}(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax}}(\mathcal{C},\mathcal{D})$ the full subcategory spanned by the oplax $\mathcal{O}^{\otimes}$-monoidal functors $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ such that for all $X \in \mathcal{O}$ the induced functor $F_X : \mathcal{C}_X \to \mathcal{D}_X$ admits a right adjoint.

- $\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty}) \subset \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}$ the full subcategory spanned by the $\mathcal{O}^{\otimes}$-monoidal categories,

- $\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})^{\mathrm{R}} \subset \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})$ the wide subcategory with morphisms the lax $\mathcal{O}^{\otimes}$-monoidal functors $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ such that for all $X \in \mathcal{O}$ the induced functor $G_X : \mathcal{D}_X \to \mathcal{C}_X$ admits a left adjoint.

There is a canonical equivalence

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{D},\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax},\mathrm{L}}(\mathcal{C},\mathcal{D})^{\mathrm{op}},$$

under which left and right adjoints correspond (prop. 6.35).

There is a canonical equivalence

$$(\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})^{\mathrm{R}}$$

(prop. 6.40), under which a lax $\mathcal{O}^{\otimes}$-monoidal functor $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ corresponds to the lax $\mathcal{O}^{\otimes}$-monoidal functor $F^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ representing the oplax $\mathcal{O}^{\otimes}$-monoidal left adjoint $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ of G.

1.3.3 Preadditive, additive and stable categories

We call a category $\mathcal{C}$

- preadditive if $\mathcal{C}$ admits a zero object, finite coproducts and finite products and for every objects $A_1, ..., A_n$ for some $n \geq 2$ the canonical morphism
$$\coprod_{i=1}^{n} A_i \to \prod_{i=1}^{n} A_i$$
is an equivalence.

- additive if $\mathcal{C}$ is preadditive and for every $X \in \mathcal{C}$ the morphism
$$X \times X \xrightarrow{(\mathrm{pr}_1,\mu)} X \times X$$
is an equivalence, where $\mathrm{pr}_1 : X \times X \to X$ denotes the projection to the first factor and $\mu : X \times X \simeq X \coprod X \to X$ denotes the codiagonal.

- stable if $\mathcal{C}$ admits a zero object, finite colimits and finite limits and the suspension $\Sigma : \mathcal{C} \to \mathcal{C}$ is an equivalence.

For every category $\mathcal{D}$ with finite products the category $\mathrm{Cmon}(\mathcal{D}) \simeq \mathrm{Calg}(\mathcal{D}^{\times})$ is preadditive.

Moreover the forgetful functor $\mathrm{Cmon}(\mathcal{D}) \to \mathcal{D}$ is an equivalence if and only if $\mathcal{D}$ is preadditive.

Proof. The category $\mathrm{Cmon}(\mathcal{D}) \simeq \mathrm{Calg}(\mathcal{D}^\times)$ is preadditive as by [18] proposition 3.2.4.7. the symmetric monoidal category $\mathrm{Cmon}(\mathcal{D}) \simeq \mathrm{Calg}(\mathcal{D}^\times)$ endowed with the objectwise symmetric monoidal structure, which in this case is the cartesian structure, is cocartesian.

If $\mathcal{D}$ is preadditive, the identity of $\mathcal{D}$ uniquely lifts to an equivalence $\mathcal{D}^{\amalg} \to \mathcal{D}^\times$ of symmetric monoidal categories according to [18] corollary 2.4.1.8. Especially the forgetful functor $\mathrm{Cmon}(\mathcal{D}) \simeq \mathrm{Calg}(\mathcal{D}^\times) \simeq \mathrm{Calg}(\mathcal{D}^{\amalg}) \to \mathcal{D}$ is an equivalence by [18] proposition 2.4.1.7. and proposition 2.4.3.9.

□

For every preadditive category $\mathcal{C}$ and category $\mathcal{D}$ with finite products the forgetful functor

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{D})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$$

is an equivalence with inverse the canonical functor

$$\xi : \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\Pi}(\mathrm{Cmon}(\mathcal{C}), \mathrm{Cmon}(\mathcal{D})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{D})),$$

where we use that the forgetful functor $\mathrm{Cmon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence (see also [9] corollary 2.4. and 2.5.).

If $\mathcal{C}$ is additive, ξ induces a functor $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cgrp}(\mathcal{D}))$ inverse to the forgetful functor $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cgrp}(\mathcal{D})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$, where $\mathrm{Cgrp}(\mathcal{D}) \subset \mathrm{Cmon}(\mathcal{D})$ denotes the full subcategory spanned by the group objects, which is an additive category.

By [18] corollary 1.4.2.23. for every stable category $\mathcal{C}$ and category $\mathcal{D}$ with finite limits the forgetful functor

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})) \to \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D})$$

is an equivalence with inverse the canonical functor

$$\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{lex}}(\mathrm{Sp}(\mathcal{C}), \mathrm{Sp}(\mathcal{D})) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Sp}(\mathcal{D})),$$

where we use that the forgetful functor $\mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ is an equivalence by [18] proposition 1.4.2.21.

If $\mathcal{C}$ is preadditive, the category $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{S}))$ is preadditive being closed under finite products in the preadditive category $\mathrm{Fun}(\mathcal{C}, \mathrm{Cmon}(\mathcal{S}))$.

If $\mathcal{C}$ is additive, the category $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cgrp}(\mathcal{S}))$ is additive being closed under finite products in the additive category $\mathrm{Fun}(\mathcal{C}, \mathrm{Cgrp}(\mathcal{S}))$.

If $\mathcal{C}$ is stable, the category $\mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}, \mathrm{Sp})$ is stable being closed under finite limits and finite colimits in the stable category $\mathrm{Fun}(\mathcal{C}, \mathrm{Sp})$.

Remark 1.4. *If $\mathcal{D}$ is presentable, by prop. 4.1. [9] the category $\mathrm{Cmon}(\mathcal{D})$ is an accessible localization of the presentable category $\mathrm{Fun}(\mathcal{F}\mathrm{in}_*, \mathcal{D})$ and so itself presentable. Especially the forgetful functor $\mathrm{Cmon}(\mathcal{D}) \to \mathcal{D}$ admits a left adjoint.*

If $\mathcal{C}, \mathcal{D}$ are presentable, the equivalence

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{D})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$$

restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{L}}(\mathrm{Cmon}(\mathcal{D}), \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathrm{Cmon}(\mathcal{D})) \to \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{D}, \mathcal{C}).$$

Thus the full subcategory $\mathrm{Pr}^{\mathrm{L}}_{\mathrm{preadd}} \subset \mathrm{Pr}^{\mathrm{L}}$ *of preadditive presentable categories is a localization.*

Moreover by [9] theorem 4.6. this localization is symmetric monoidal, when Pr^{L} *is endowed with its canonical closed symmetric monoidal structure.*

This way the cartesian structures on the categories $\mathcal{S}$ *and* Cat_∞ *yield closed symmetric monoidal structures on the categories* $\mathrm{Cmon}(\mathcal{S})$ *respectively* $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ *such that the free functors* $\mathcal{S} \to \mathrm{Cmon}(\mathcal{S}), \mathsf{Cat}_\infty \to \mathrm{Cmon}(\mathsf{Cat}_\infty)$ *are symmetric monoidal.*

Moreover the opposite category involution $(-)^{\mathrm{op}}$ *on* Cat_∞ *induces a symmetric monoidal autoequivalence of the cartesian structure on* Cat_∞ *and so a symmetric monoidal autoequivalence of the closed symmetric monoidal structure on* $\mathrm{Cmon}(\mathsf{Cat}_\infty)$ *that takes a symmetric monoidal category to its fiberwise dual.*

Remark 1.5. *Every additive category* $\mathcal{C}$ *admits a canonical finite products preserving embedding* $\mathcal{C} \subset \mathcal{D}$ *into a stable category* $\mathcal{D}$.

The embedding $\mathcal{C} \subset \mathcal{D}$ *factors as embeddings* $\mathcal{C} \subset \mathcal{E} \subset \mathcal{D}$ *with an additive category* $\mathcal{E}$ *such that the embedding* $\mathcal{C} \subset \mathcal{E}$ *preserves small limits and the embedding* $\mathcal{E} \subset \mathcal{D}$ *admits a right adjoint.*

The category $\mathcal{E}$ *is closed in* $\mathcal{D}$ *under retracts. If* $\mathcal{C}$ *is idempotent complete,* $\mathcal{C}$ *is closed in* $\mathcal{E}$ *under retracts so that* $\mathcal{C}$ *is closed in* $\mathcal{D}$ *under retracts.*

If $\mathcal{C}$ *is a* $\mathcal{O}^{\otimes}$*-monoidal category for an operad* $\mathcal{O}^{\otimes}$*, the embeddings* $\mathcal{C} \subset \mathcal{E}$ *and* $\mathcal{E} \subset \mathcal{D}$ *are* $\mathcal{O}^{\otimes}$*-monoidal.*

Proof. The forgetful functor

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{S})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S})$$

is an equivalence with inverse the canonical functor

$$\phi : \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S}) \to \mathrm{Fun}^{\Pi}(\mathrm{Cmon}(\mathcal{C}), \mathrm{Cmon}(\mathcal{S})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cmon}(\mathcal{S})).$$

As $\mathcal{C}$ is additive, every object of $\mathcal{C} \simeq \mathrm{Cmon}(\mathcal{C})$ belongs to the full subcategory $\mathrm{Cgrp}(\mathcal{C}) \subset \mathrm{Cmon}(\mathcal{C})$ spanned by the group objects.

Thus the equivalence ϕ induces an equivalence

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cgrp}(\mathcal{S}))$$

invserse to the forgetful functor $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Cgrp}(\mathcal{S})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{S})$.

Replacing $\mathcal{C}$ by the additive category $\mathcal{C}^{\mathrm{op}}$ we get a canonical equivalence $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cgrp}(\mathcal{S}))$.

The left adjoint stabilization functor $\mathrm{Cgrp}(\mathcal{S}) \to \mathrm{Sp}$ is fully faithful with essential image the connective spectra and so induces a left adjoint embedding $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cgrp}(\mathcal{S})) \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$.

The Yoneda-embedding $\mathcal{C} \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ induces an embedding $\mathcal{C} \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ that preserves small limits.

So we get an embedding $\theta : \mathcal{C} \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cgrp}(\mathcal{S})) \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$.

With Sp also the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is stable as (co)limits in functor-categories are formed levelwise.

As the full subcategory $\mathrm{Fun}^{\amalg}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) = \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is closed under small colimits and limits, the category $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ is stable, too.

As $\mathrm{Cgrp}(\mathcal{S})$ is closed under retracts in Sp, $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Cgrp}(\mathcal{S}))$ is closed under retracts in $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$.

If $\mathcal{C}$ is idempotent complete, $\mathcal{C}$ is closed under retracts in $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and so in $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

The full subcategory $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is an accessible localization and so yields for every presentable category $\mathcal{B}$ a localization $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{B} \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{B}$, where $\otimes$ denotes tensorproduct of the closed symmetric monoidal structure on Pr^{L}.

There is a canonical equivalence

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{B}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{B}$$

that restricts to an equivalence

$$\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{B}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{B}$$

of presentable categories.

Via this equivalence the embedding $\theta : \mathcal{C} \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathrm{Sp})$ factors as

$$\mathcal{C} \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{S} \to \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathrm{Cgrp}(\mathcal{S}) \to \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathrm{Sp}$$

and so as

$$\mathcal{C} \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{S} \to \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathrm{Sp}.$$

If $\mathcal{C}$ is a $\mathcal{O}^{\otimes}$-monoidal category, the Yoneda-embedding $\mathcal{C} \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ gets $\mathcal{O}^{\otimes}$-monoidal, where $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ carries the $\mathcal{O}^{\otimes}$-monoidal structure given by Day-convolution.

Moreover the accessible localization $\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ is a $\mathcal{O}^{\otimes}$-monoidal localization so that the $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{C} \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ induces a $\mathcal{O}^{\otimes}$-monoidal embedding $\mathcal{C} \subset \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$.

Finally the symmetric monoidal infinite suspension functor $\mathcal{S} \to \mathrm{Sp}$ yields a $\mathcal{O}^{\otimes}$-monoidal functor

$$\mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{S} \to \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \otimes \mathrm{Sp}.$$

□

Moreover we will heavily use the following remark:

Remark 1.6. *Let $\mathcal{K} \subset \mathsf{Cat}_{\infty}$ be a full subcategory and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category compatible with colimits indexed by categories that belong to $\mathcal{K}$. Assume that for every $\mathrm{X} \in \mathcal{O}$ the fiber $\mathcal{C}_{\mathrm{X}}$ admits small colimits.*

There are $\mathcal{O}^{\otimes}$-monoidal embeddings $\mathcal{C}^{\otimes} \subset \mathcal{D}^{\otimes} \subset \mathcal{E}^{\otimes}$ such that for every $\mathrm{X} \in \mathcal{O}$ the fiber $\mathcal{E}_{\mathrm{X}}$ admits large colimits, $\mathcal{D}_{\mathrm{X}}$ is the smallest full subcategory of $\mathcal{E}_{\mathrm{X}}$ that contains $\mathcal{C}_{\mathrm{X}}$ and is closed under small colimits, the embedding $\mathcal{C}_{\mathrm{X}} \subset \mathcal{D}_{\mathrm{X}}$ admits a left adjoint and preserves colimits indexed by categories that belong to $\mathcal{K}$ and the embedding $\mathcal{C}_{\mathrm{X}} \subset \mathcal{E}_{\mathrm{X}}$ preserves small limits.

Corollary 1.7. *Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a preadditive, additive respectively stable $\mathcal{O}^{\otimes}$-monoidal category such that for every $\mathrm{X} \in \mathcal{O}$ the fiber $\mathcal{C}_{\mathrm{X}}$ admits small colimits.*

There are $\mathcal{O}^{\otimes}$-monoidal embeddings $\mathcal{C}^{\otimes} \subset \mathcal{D}^{\otimes} \subset \mathcal{E}^{\otimes}$ such that for every $\mathrm{X} \in \mathcal{O}$ the fibers $\mathcal{D}_{\mathrm{X}}, \mathcal{E}_{\mathrm{X}}$ are preadditive, additive respectively stable, $\mathcal{E}_{\mathrm{X}}$ admits large colimits, $\mathcal{D}_{\mathrm{X}}$ is the smallest full subcategory of $\mathcal{E}_{\mathrm{X}}$ that contains $\mathcal{C}_{\mathrm{X}}$ and is closed under small colimits, the embedding $\mathcal{C}_{\mathrm{X}} \subset \mathcal{D}_{\mathrm{X}}$ admits a left adjoint and the embedding $\mathcal{C}_{\mathrm{X}} \subset \mathcal{E}_{\mathrm{X}}$ preserves small limits.

Proof. Let $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a $\mathcal{O}^{\otimes}$-monoidal category. There is a $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{C}^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C})^{\otimes}$.

Given a full subcategory $\mathcal{K} \subset \mathsf{Cat}_{\infty}$ denote $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C})^{\otimes}$ the full suboperad spanned by the functors $\mathcal{C}_{\mathrm{X}}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ that preserve $\mathcal{K}$-indexed limits for some $\mathrm{X} \in \mathcal{O}$ and $\mathcal{P}_{\mathcal{K}}(\mathcal{C})^{\otimes} \subset \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ the full suboperad such that for every $\mathrm{X} \in \mathcal{O}$ the full subcategory $\mathcal{P}_{\mathcal{K}}(\mathcal{C})_{\mathrm{X}}$ is the smallest full subcategory of $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})_{\mathrm{X}}$ that contains $\mathcal{C}_{\mathrm{X}}$ and is closed under small colimits.

If $\mathcal{K}$ is empty, we drop $\mathcal{K}$ from the notation. If $\mathcal{K} = \mathsf{Cat}_{\infty}$, we have $\mathcal{P}_{\mathcal{K}}(\mathcal{C})^{\otimes} = \mathcal{C}^{\otimes}$. If $\mathcal{O}^{\otimes}$ is the trivial operad, we write $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ for $\mathcal{P}_{\mathcal{K}}(\mathcal{C})^{\otimes}$.

The $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{C}^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C})^{\otimes}$ induces a $\mathcal{O}^{\otimes}$-monoidal embedding $\mathcal{C}^{\otimes} \subset \mathcal{P}_{\mathcal{K}}(\mathcal{C})^{\otimes}$.

By remark 6.3 the embedding $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C})^{\otimes}$ induces on the fiber over every object of $\mathcal{O}$ a localization. Especially the embedding $\widehat{\mathcal{P}}_{\mathsf{Cat}_{\infty}}(\mathcal{C})^{\otimes} \subset \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ induces on the fiber over every object of $\mathcal{O}$ a localization and so restricts to an embedding $\mathcal{C}^{\otimes} = \mathcal{P}_{\mathsf{Cat}_{\infty}}(\mathcal{C})^{\otimes} \subset \mathcal{P}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ that induces on the fiber over every object of $\mathcal{O}$ a localization.

Assume that the $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes}$ is compatible with colimits indexed by categories that belong to $\mathcal{K}$. Then by prop. 6.5 the full subcategory $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C})^{\otimes}$ is a localization relative to $\mathcal{O}^{\otimes}$ and so compatible with large colimits. Thus $\mathcal{P}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ is a $\mathcal{O}^{\otimes}$-monoidal category compatible with small colimits.

For $\mathcal{K} = \mathcal{F}\mathrm{in}$ the category of small finite sets and $\mathcal{O}^{\otimes}$ the trivial operad we have $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C}) = \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})$, which is preadditive respectively additive if $\mathcal{C}$ is. In this case also $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ is preadditive respectively additive being closed under finite coproducts.
Another choice of $\mathcal{K}$ leads to the full subcategory $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C}) = \mathrm{Fun}^{\mathrm{lex}}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}})$ spanned by the finite limits preserving functors, which is stable if $\mathcal{C}$ is. In this case also $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ is stable being closed under finite colimits and arbitrary shifts.

□

2 Restricted Lie algebras

The next two subsections provide the notions needed to define restricted L_∞-algebras in a nice preadditive symmetric monoidal category $\mathcal{C}$.

In section 2.1 we define bialgebras and Hopf algebras in $\mathcal{C}$.

We show that cocommutative bialgebras in $\mathcal{C}$ can be described by monoids in the category of cocommutative coalgebras in $\mathcal{C}$ or by cocommutative coalgebras in the symmetric monoidal category of associative algebras in $\mathcal{C}$ (prop. 2.4).

In section 2.2.1 we endow the category $\mathcal{C}^\Sigma$ of symmetric sequences in $\mathcal{C}$ with a monoidal structure encoding the composition product and define operads as associative algebras in the composition product.

We show that the left action of $\mathcal{C}^\Sigma$ on itself restricts to a left action on $\mathcal{C}$ and we define algebras over an operad $\mathcal{O}$ as left modules over $\mathcal{O}$.

To define cooperads, i.e. operads in $\mathcal{C}^{\mathrm{op}}$, we cannot expect that $\mathcal{C}^{\mathrm{op}}$ is a nice symmetric monoidal category.

Thus we develop a more general composition product on $\mathcal{C}^\Sigma$ that endows $\mathcal{C}^\Sigma$ with the structure of a representable planar operad instead of a monoidal category and we define operads as associative algebras in this planar operad structure on $\mathcal{C}^\Sigma$ and cooperads as operads in $\mathcal{C}^{\mathrm{op}}$.

2.1 Bialgebras and Hopf algebras

We start with developing the basic theory of algebras, coalgebras, bialgebras and Hopf algebras.

We use the terminology of [18] and refer to this source for more details.

Given a map of operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and a $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ denote

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$$

the category of $\mathcal{O}'^{\otimes}$-algebras relative to $\mathcal{O}^{\otimes}$ and

$$\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}}$$

the category of $\mathcal{O}'^{\otimes}$-coalgebras relative to $\mathcal{O}^{\otimes}$.

- If $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ is the identity, we write $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ for $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ and $\mathrm{Coalg}_{/\mathcal{O}}(\mathcal{C})$ for $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$.
- If $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$, we write $\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})$ for $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ and $\mathrm{Coalg}_{\mathcal{O}'}(\mathcal{C})$ for $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$.
- For $\mathcal{O}^{\otimes} = \mathrm{Ass}^{\otimes}$ respectively $\mathcal{O}^{\otimes} = \mathrm{Comm}^{\otimes}$ we write $\mathrm{Alg}(\mathcal{C})$ respectively $\mathrm{Calg}(\mathcal{C})$ for $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ and $\mathrm{Coalg}(\mathcal{C})$ respectively $\mathrm{Cocoalg}(\mathcal{C})$ for $\mathrm{Coalg}_{/\mathcal{O}}(\mathcal{C})$.

The next remark follows from [18] corollary 3.2.2.5. and will be heavily used:

Remark 2.1. *Let $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of operads and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category.*

If for every $X \in \mathcal{O}$ the fiber $\mathcal{C}_X$ admits small limits, then the category $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ admits small limits that are preserved by the forgetful functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ for every $Y \in \mathcal{O}'$ lying over some $X \in \mathcal{O}$.

So dually if for every $\mathrm{X} \in \mathcal{O}$ *the fiber* $\mathcal{C}_{\mathrm{X}}$ *admits small colimits, the category* $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ *admits small colimits that are preserved by the forgetful functor* $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_{\mathrm{X}}$ *for every* $\mathrm{Y} \in \mathcal{O}'$ *lying over some* $\mathrm{X} \in \mathcal{O}$.

For later reference we define non-unital and augmented algebras respectively non-counital and coaugmented coalgebras over a unital operad $\mathcal{O}^{\otimes}$:

Denote $\mathrm{Surj} \subset \mathcal{F}\mathrm{in}_*$ the wide subcategory with morphisms the surjective maps. The subcategory inclusion $\mathrm{Surj} \subset \mathcal{F}\mathrm{in}_*$ exhibits Surj as an operad.

Given a unital operad $\mathcal{O}^{\otimes}$ we set $\mathcal{O}^{\otimes}_{\mathrm{nu}} := \mathrm{Surj} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{O}^{\otimes}$.

Denote

$$\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} := \mathrm{Alg}_{\mathcal{O}_{\mathrm{nu}}/\mathcal{O}}(\mathcal{C}), \quad \mathrm{Coalg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{ncu}} := \mathrm{Coalg}_{\mathcal{O}_{\mathrm{nu}}/\mathcal{O}}(\mathcal{C})$$

the category of non-unital $\mathcal{O}^{\otimes}$-algebras in $\mathcal{C}$ respectively non-counital $\mathcal{O}^{\otimes}$-coalgebras in $\mathcal{C}$.

We have a forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} = \mathrm{Alg}_{\mathcal{O}_{\mathrm{nu}}/\mathcal{O}}(\mathcal{C})$.

Remark 2.2. *Let* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a preadditive* $\mathcal{O}^{\otimes}$*-monoidal category.*

1. *Adding the tensorunit defines an embedding* $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$.

2. *This embedding is an equivalence if* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *is a idempotent complete additive* $\mathcal{O}^{\otimes}$*-monoidal category.*

3. *If for every* $\mathrm{X} \in \mathcal{O}$ *the category* $\mathcal{C}_{\mathrm{X}}$ *admits fibers, the embedding* $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$ *admits a right adjoint that takes the fiber of the augmentation.*

4. *If for every* $\mathrm{X} \in \mathcal{O}$ *the category* $\mathcal{C}_{\mathrm{X}}$ *admits small limits, the embedding* $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$ *preserves small limits.*

Proof. By [18] prop. 5.4.4.8. the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$ admits a left adjoint $\mathcal{F}$ with the following properties:

For every non-unital $\mathcal{O}^{\otimes}$-algebra X in $\mathcal{C}$ the unit $\mathrm{X} \to \mathcal{F}(\mathrm{X})$ and the unique morphism $\mathbb{1} \to \mathcal{F}(\mathrm{X})$ in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ yield an equivalence $\mathrm{X} \oplus \mathbb{1} \simeq \mathcal{F}(\mathrm{X})$ in $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$.

As the $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes}$ is compatible with the initial object, the category $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$ admits a zero object that lies over the zero object of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$. So the functor $\mathcal{F} : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ lifts to a functor $\bar{\mathcal{F}} : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$ that factors through the full subcategory $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})'_{/\mathbb{1}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$ spanned by the augmented $\mathcal{O}^{\otimes}$-algebras, whose augmentation admits a fiber in $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$.

Especially we obtain a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} & \xrightarrow{\bar{\mathcal{F}}} & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}) & \xrightarrow{-\oplus\mathbb{1}} & \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})_{/\mathbb{1}}, \end{array}$$

where the functor $- \oplus \mathbb{1}$ is the right adjoint of the forgetful functor $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})_{/\mathbb{1}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$.

The functor $\bar{\mathcal{F}} : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})'_{/\mathbb{1}}$ is left adjoint to the functor

$$\Gamma : \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})'_{/\mathbb{1}} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$$

that takes the fiber of the augmentation in $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$.

Given a non-unital $\mathcal{O}^{\otimes}$-algebra X in $\mathcal{C}$ the unit

$$X \to \Gamma(\bar{\mathcal{F}}(X)) \simeq 0 \times_{\mathcal{F}(0)} \mathcal{F}(X) \simeq 0 \times_{(0 \oplus \mathbb{1})} (X \oplus \mathbb{1})$$

is the canonical equivalence so that $\bar{\mathcal{F}}$ is fully faithful.

This shows 1. and 3., where 4. follows from remark 2.1.

2: By remark 1.5 there is a $\mathcal{O}^{\otimes}$-monoidal embedding $\mathcal{C}^{\otimes} \subset \mathcal{D}^{\otimes}$ into a stable $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{D}^{\otimes}$ such that for every $X \in \mathcal{O}$ the fiber $\mathcal{C}_X \subset \mathcal{D}_X$ is closed under finite products and retracts.

So we get a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} & \xrightarrow{\bar{\mathcal{F}}} & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})^{\mathrm{nu}} & \xrightarrow{\bar{\mathcal{F}}} & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})_{/\mathbb{1}}, \end{array}$$

where the vertical functors are fully faithful.

As $\mathcal{D}^{\otimes}$ is a stable $\mathcal{O}^{\otimes}$-monoidal category, by [18] proposition 5.4.4.10. the functor $\bar{\mathcal{F}}: \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})^{\mathrm{nu}} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})_{/\mathbb{1}}$ is an equivalence.

As for every $X \in \mathcal{O}$ the fiber $\mathcal{C}_X \subset \mathcal{D}_X$ is closed under retracts, $A \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D})^{\mathrm{nu}}$ belongs to $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$ if its image $\bar{\mathcal{F}}(A) \simeq A \oplus \mathbb{1}$ belongs to $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$ using that A is a retract of $A \oplus \mathbb{1}$ in the category $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{D})$. □

As next we endow the categories of algebras and coalgebras in a symmetric monoidal category with symmetric monoidal structures to be able to define algebras in the category of coalgebras and coalgebras in the category of algebras.

By 1.4 the category $\mathrm{Cmon}(\mathsf{Cat}_{\infty})$ admits a closed symmetric monoidal structure, whose internal hom of two symmetric monoidal categories $\mathcal{D}, \mathcal{C}$ we denote by $\mathrm{Fun}^{\otimes}(\mathcal{D}, \mathcal{C})^{\otimes}$.

As the notation suggests, the underlying category of $\mathrm{Fun}^{\otimes}(\mathcal{D}, \mathcal{C})^{\otimes}$ is $\mathrm{Fun}^{\otimes}(\mathcal{D}, \mathcal{C})$.

Moreover the opposite category involution $(-)^{\mathrm{op}}$ on Cat_{∞} induces a symmetric monoidal autoequivalence $(-)^{\mathrm{rev}}$ of $\mathrm{Cmon}(\mathsf{Cat}_{\infty})$.

By [18] proposition 2.2.4.9. the subcategory inclusion $\mathrm{Cmon}(\mathsf{Cat}_{\infty}) \subset \mathrm{Op}_{\infty}$ from symmetric monoidal categories to operads admits a left adjoint $\mathrm{Env}(-)^{\otimes}$, which assigns to an operad its enveloping symmetric monoidal category.

So for every operad $\mathcal{O}^{\otimes}$ we have a unit map of operads $\mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$.

Using the cotensoring of $\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ over Cat_{∞} composition with the unit defines an equivalence

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

of categories and not only spaces.

We set

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes} := \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathcal{C})^{\otimes}, \quad \mathrm{Coalg}_{\mathcal{O}}(\mathcal{C})^{\otimes} := (\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}^{\mathrm{rev}})^{\otimes})^{\mathrm{rev}}$$

and have a canonical symmetric monoidal equivalence

$$\mathrm{Coalg}_{\mathcal{O}}(\mathcal{C})^{\otimes} = (\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathcal{C}^{\mathrm{rev}})^{\otimes})^{\mathrm{rev}} \simeq \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O})^{\mathrm{rev}}, \mathcal{C})^{\otimes}. \quad (1)$$

Remark 2.3. *One can show (prop. 6.82) that composition with the unit* $\mathcal{O}^{\otimes} \to \mathrm{Env}(\mathcal{O})^{\otimes}$ *defines an equivalence*

$$\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathcal{C})^{\otimes} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes},$$

where $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})^{\otimes}$ *denotes the internal hom of the closed symmetric monoidal structure on* Op_{∞} *given by the Boardman-Vogt tensorproduct.*

As next we show in proposition 2.4 that for every symmetric monoidal category $\mathcal{C}$ and operads $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}$ we have a canonical equivalence

$$\mathrm{Coalg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}))^{\otimes} \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathrm{Coalg}_{\mathcal{O}}(\mathcal{C}))^{\otimes}$$

of symmetric monoidal categories and thus especially an underlying equivalence $\mathrm{Coalg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathrm{Coalg}_{\mathcal{O}}(\mathcal{C}))$.

Having this we define the category of $(\mathcal{O}, \mathcal{O}')$-bialgebras in $\mathcal{C}$ as the category

$$\mathrm{Bialg}_{\mathcal{O},\mathcal{O}'}(\mathcal{C}) := \mathrm{Coalg}_{\mathcal{O}}(\mathrm{Alg}_{\mathcal{O}'}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{O}'}(\mathrm{Coalg}_{\mathcal{O}}(\mathcal{C})).$$

Especially we write $\mathrm{Bialg}(\mathcal{C}) := \mathrm{Cocoalg}(\mathrm{Alg}(\mathcal{C})) \simeq \mathrm{Alg}(\mathrm{Cocoalg}(\mathcal{C}))$.

Note that we use the convention that $\mathrm{Bialg}(\mathcal{C})$ denotes the category of cocommutative bialgebras and not the category of bialgebras.

We use this convention as the bialgebras arising in Lie theory are cocommutative so that we will mainly deal with cocommutative bialgebras in the following chapters.

Proposition 2.4.

Let $\mathcal{O}^{\otimes}, \mathcal{O}'^{\otimes}$ *be operads and* $\mathcal{C}^{\otimes}$ *a symmetric monoidal category. There is a canonical equivalence*

$$\mathrm{Coalg}_{\mathcal{O}'}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))^{\otimes} \simeq \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coalg}_{\mathcal{O}'}(\mathcal{C}))^{\otimes}.$$

Proof. The asserted equivalence is the composition of the following canonical equivalences:

$$\begin{gathered}
\mathrm{Coalg}_{\mathcal{O}'}(\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))^{\otimes} \simeq_2 \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}')^{\mathrm{rev}}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}))^{\otimes} = \\
\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}')^{\mathrm{rev}}, \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathcal{C}))^{\otimes} \simeq \\
\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}')^{\mathrm{rev}} \otimes \mathrm{Env}(\mathcal{O}), \mathcal{C})^{\otimes} \simeq \\
\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}) \otimes \mathrm{Env}(\mathcal{O}')^{\mathrm{rev}}, \mathcal{C})^{\otimes} \simeq \\
\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}')^{\mathrm{rev}}, \mathcal{C}))^{\otimes} \simeq_2 \\
\mathrm{Fun}^{\otimes}(\mathrm{Env}(\mathcal{O}), \mathrm{Coalg}_{\mathcal{O}'}(\mathcal{C}))^{\otimes} = \mathrm{Alg}_{\mathcal{O}}(\mathrm{Coalg}_{\mathcal{O}'}(\mathcal{C}))^{\otimes}.
\end{gathered}$$

□

As next we define Hopf algebras.

Let $\mathcal{D}$ be a category with finite products and X a monoid in $\mathcal{D}$.

Denote $\mathrm{pr}_1, \mathrm{pr}_2 : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$ the projections and $\mu : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$ the multiplication of X.

We call X a group object in $\mathcal{D}$ if the canonical morphisms

$$\mathrm{X} \times \mathrm{X} \xrightarrow{(\mathrm{pr}_1, \mu)} \mathrm{X} \times \mathrm{X}, \ \mathrm{X} \times \mathrm{X} \xrightarrow{(\mu, \mathrm{pr}_2)} \mathrm{X} \times \mathrm{X}$$

are equivalences and write $\mathrm{Grp}(\mathcal{D}) \subset \mathrm{Mon}(\mathcal{D})$ for the full subcategory spanned by the group objects.

By [18] proposition 3.2.4.7. the symmetric monoidal category $\mathrm{Calg}(\mathcal{C})^{\otimes}$ is cocartesian and so dually the symmetric monoidal category $\mathrm{Cocoalg}(\mathcal{C})^{\otimes}$ is cartesian.

We refer to group objects in $\mathrm{Cocoalg}(\mathcal{C})$ as Hopf algebras in $\mathcal{C}$ and set

$$\mathrm{Hopf}(\mathcal{C}) \coloneqq \mathrm{Grp}(\mathrm{Cocoalg}(\mathcal{C})) \subset \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})) \simeq \mathrm{Bialg}(\mathcal{C}).$$

Remark 2.5.

1. *Let $\mathcal{C}, \mathcal{D}$ be categories that admit finite products and $\phi : \mathcal{C} \to \mathcal{D}$ a finite products preserving and conservative functor.*

   A monoid X of $\mathcal{C}$ is a group object of $\mathcal{C}$ if and only if the image $\phi(\mathrm{X})$ is a group object of $\mathcal{D}$.

   As the functor $\mathcal{D} \to \mathsf{Ho}(\mathcal{D})$ preserves finite products and is conservative, a monoid of $\mathcal{D}$ is a group object if and only if its image in $\mathsf{Ho}(\mathcal{D})$ is a group object.

   Hence (as it holds for 1-categories) a monoid X of $\mathcal{D}$ is a group object if and only if it admits an inverse, i.e. if there is a morphism $\mathrm{i} : \mathrm{X} \to \mathrm{X}$ in $\mathcal{D}$ such that we have commutative squares

$$\begin{array}{ccccc} \mathrm{X}\times\mathrm{X} & & \xrightarrow{\mathrm{X}\times\mathrm{i}} & & \mathrm{X}\times\mathrm{X} \\ \uparrow & & & & \downarrow{\scriptstyle\mu} \\ \mathrm{X} & \longrightarrow & * & \longrightarrow & \mathrm{X} \end{array} \qquad \begin{array}{ccccc} \mathrm{X}\times\mathrm{X} & & \xrightarrow{\mathrm{i}\times\mathrm{X}} & & \mathrm{X}\times\mathrm{X} \\ \uparrow & & & & \downarrow{\scriptstyle\mu} \\ \mathrm{X} & \longrightarrow & * & \longrightarrow & \mathrm{X} \end{array}$$

   in $\mathcal{D}$.

2. *Let $\mathcal{C}, \mathcal{D}$ be symmetric monoidal categories and $\phi : \mathcal{C} \to \mathcal{D}$ a symmetric monoidal and conservative functor.*

   A bialgebra X of $\mathcal{C}$ is a Hopf algebra of $\mathcal{C}$ if and only if the image $\phi(\mathrm{X})$ is a Hopf algebra in $\mathcal{D}$.

   Especially a bialgebra of $\mathcal{D}$ is a Hopf algebra if and only if its image in $\mathsf{Ho}(\mathcal{D})$ is a Hopf algebra.

   Hence a bialgebra X of $\mathcal{D}$ is a Hopf algebra if and only if it admits an antipode, i.e. if there is a morphism $\mathrm{i} : \mathrm{X} \to \mathrm{X}$ in $\mathrm{Cocoalg}(\mathsf{Ho}(\mathcal{D}))$ such that we have commutative squares

$$\begin{array}{ccccc} \mathrm{X}\otimes\mathrm{X} & & \xrightarrow{\mathrm{X}\otimes\mathrm{i}} & & \mathrm{X}\otimes\mathrm{X} \\ \uparrow & & & & \downarrow{\scriptstyle\mu} \\ \mathrm{X} & \longrightarrow & \mathbb{1} & \longrightarrow & \mathrm{X} \end{array} \qquad \begin{array}{ccccc} \mathrm{X}\otimes\mathrm{X} & & \xrightarrow{\mathrm{i}\otimes\mathrm{X}} & & \mathrm{X}\otimes\mathrm{X} \\ \uparrow & & & & \downarrow{\scriptstyle\mu} \\ \mathrm{X} & \longrightarrow & \mathbb{1} & \longrightarrow & \mathrm{X} \end{array}$$

   in $\mathcal{D}$.

Observation 2.6. *Let $\mathcal{D}$ be a category that admits finite products and small sifted colimits that commute with each other.*

Then the full subcategory $\mathrm{Grp}(\mathcal{D}) \subset \mathrm{Mon}(\mathcal{D})$ is closed under small sifted colimits:

Proof. Let $\mathcal{J}$ be a small sifted category. By assumption the diagonal functor $\mathcal{D} \to \mathrm{Fun}(\mathcal{J}, \mathcal{D})$ admits a left adjoint $\mathrm{colim} : \mathrm{Fun}(\mathcal{J}, \mathcal{D}) \to \mathcal{D}$ that preserves finite products as $\mathcal{J}$ is sifted.

So $\mathrm{colim} : \mathrm{Fun}(\mathcal{J}, \mathcal{D}) \to \mathcal{D}$ induces a functor

$$\mathrm{Fun}(\mathcal{J}, \mathrm{Mon}(\mathcal{D})) \simeq \mathrm{Mon}(\mathrm{Fun}(\mathcal{J}, \mathcal{D})) \to \mathrm{Mon}(\mathcal{D})$$

right adjoint to the diagonal functor that restricts to a functor

$$\mathrm{Fun}(\mathcal{J}, \mathrm{Grp}(\mathcal{D})) \simeq \mathrm{Grp}(\mathrm{Fun}(\mathcal{J}, \mathcal{D})) \to \mathrm{Grp}(\mathcal{D})$$

right adjoint to the diagonal functor.

□

Let $\mathcal{D}$ be a symmetric monoidal category compatible with small sifted colimits.

Then the category $\mathrm{Cocoalg}(\mathcal{D})$ *admits finite products being a cartesian symmetric monoidal category and small sifted colimits that commute with each other.*

So the full subcategory $\mathrm{Hopf}(\mathcal{D}) \subset \mathrm{Bialg}(\mathcal{D})$ *is closed under small sifted colimits.*

2.2 Internal operads and cooperads

2.2.1 The composition product on symmetric sequences

Denote $\Sigma \simeq \coprod_{n\geq 0} B(\Sigma_n)$ the groupoid of finite sets and bijections.

The cocartesian symmetric monoidal structure on Set restricts to a symmetric monoidal structure on Σ that exhibits Σ as the free symmetric monoidal category on the contractible category.

This follows for example from the canonical equivalence $\Sigma \simeq \mathrm{Env}(\mathrm{Triv})$, where $\mathrm{Triv}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ denotes the trivial operad and prop. 6.82.

Let $\mathcal{C}$ be a symmetric monoidal category compatible with small colimits.

The category $\mathcal{C}^{\Sigma} := \mathrm{Fun}(\Sigma, \mathcal{C}) \simeq \prod_{n\geq 0} \mathrm{Fun}(B(\Sigma_n), \mathcal{C})$ admits a symmetric monoidal structure compatible with small colimits given by Day-convolution (prop. 6.4).

We have a fully faithful symmetric monoidal functor $\mathcal{C} \to \mathcal{C}^{\Sigma}$ left adjoint to evaluation at 0 that considers an object of $\mathcal{C}$ as a symmetric sequence concentrated in degree zero.

We define the composition product on $\mathcal{C}^{\Sigma}$ as the monoidal structure on $\mathcal{C}^{\Sigma}$ corresponding to composition under the canonical equivalence

$$\Psi : \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{C}/}(\mathcal{C}^{\Sigma}, \mathcal{C}^{\Sigma}) \simeq \mathrm{Fun}^{\otimes,\mathrm{coc}}(\mathcal{S}^{\Sigma}, \mathcal{C}^{\Sigma}) \simeq \mathrm{Fun}^{\otimes}(\Sigma, \mathcal{C}^{\Sigma}) \simeq \mathcal{C}^{\Sigma}$$

of prop. 6.21 and 6.23 that evaluates at the symmetric sequence triv in $\mathcal{C}$ concentrated in degree 1 with value the tensorunit of $\mathcal{C}$.

So triv becomes the tensorunit of the composition product on $\mathcal{C}^{\Sigma}$.

For every $X \in \mathcal{C}^{\Sigma}$ we have a canonical equivalence $X \simeq \coprod_{k\geq 0} X_k \otimes_{\Sigma_k} \mathrm{triv}^{\otimes k}$, where we embed $\mathcal{C}$ into $\mathcal{C}^{\Sigma}$.

So the composition product of $X, Y \in \mathcal{C}^{\Sigma}$ is given by

$$X \circ Y \simeq (\Psi^{-1}(Y) \circ \Psi^{-1}(X))(\mathrm{triv}) \simeq \Psi^{-1}(Y)(X) \simeq \Psi^{-1}(Y)(\coprod_{k\geq 0} X_k \otimes_{\Sigma_k} \mathrm{triv}^{\otimes k})$$

$$\simeq \coprod_{k\geq 0} X_k \otimes_{\Sigma_k} \Psi^{-1}(Y)(\mathrm{triv})^{\otimes k} \simeq \coprod_{k\geq 0} X_k \otimes_{\Sigma_k} Y^{\otimes k}.$$

Thus for every $n \in \mathbb{N}$ we have a canonical equivalence

$$(X \circ Y)_n \simeq \coprod_{k\geq 0} \Big(\coprod_{n_1 \amalg \dots \amalg n_k = n} X_k \otimes \big(\bigotimes_{1\leq j\leq k} Y_{n_j} \big) \Big)_{\Sigma_k}.$$

Given a symmetric monoidal category $\mathcal{C}$ that admits small colimits (but is not neccessarily compatible with small colimits), there is at least a representable operad $(\mathcal{C}^{\Sigma})^{\otimes} \to \mathrm{Ass}^{\otimes}$ over $\mathrm{Ass}^{\otimes}$ with underlying category $\mathcal{C}^{\Sigma}$ that agrees with the composition product in the case that $\mathcal{C}$ is compatible with small colimits (constr. 5.80).

A symmetric monoidal functor $\phi : \mathcal{B} \to \mathcal{C}$ between symmetric monoidal categories that admit small colimits gives rise to a map $(\mathcal{B}^{\Sigma})^{\otimes} \to (\mathcal{C}^{\Sigma})^{\otimes}$ of representable operads over $\mathrm{Ass}^{\otimes}$ that is an embedding of operads over $\mathrm{Ass}^{\otimes}$ if the functor $\mathcal{B} \to \mathcal{C}$ is fully faithful (constr. 5.80).

Moreover if ϕ preserves small colimits, the lax monoidal functor $(\mathcal{B}^{\Sigma})^{\otimes} \to (\mathcal{C}^{\Sigma})^{\otimes}$ is monoidal by remark 5.81.

Set $\Sigma_{\geq 1} := \coprod_{n\geq 1} B(\Sigma_n)$ and $\mathcal{C}^{\Sigma_{\geq 1}} := \mathrm{Fun}(\Sigma_{\geq 1}, \mathcal{C}) \simeq \prod_{n\geq 1} \mathrm{Fun}(B(\Sigma_n), \mathcal{C})$.

We have an embedding $\mathcal{C}^{\Sigma_{\geq 1}} \subset \mathcal{C}^{\Sigma}$ left adjoint to restriction along the canonical embedding $\Sigma_{\geq 1} \subset \Sigma$ that plugs in the initial object in degree zero. If the symmetric monoidal structure on $\mathcal{C}$ is compatible with the initial object, the composition product on $\mathcal{C}^{\Sigma}$ restricts to $\mathcal{C}^{\Sigma_{\geq 1}}$.

Moreover for every $\mathcal{O}_1, ..., \mathcal{O}_n \in \mathcal{C}^{\Sigma}$ for some $n \geq 1$ and $X \in \mathcal{C}$ the composition $\mathcal{O}_1 \circ ... \circ \mathcal{O}_n \circ X$ belongs to $\mathcal{C}$.

So the representable operad $LM^{\otimes} \times_{Ass^{\otimes}} (\mathcal{C}^{\Sigma})^{\otimes} \to LM^{\otimes}$ over $LM^{\otimes}$ restricts to a representable operad over $LM^{\otimes}$ with fiber over $\mathfrak{a} \in LM$ the category $\mathcal{C}^{\Sigma}$ and with fiber over $\mathfrak{m} \in LM$ the category $\mathcal{C}$.

Given a symmetric monoidal category $\mathcal{C}$ that admits small limits we can form the representable planar operad $((\mathcal{C}^{op})^{\Sigma})^{\otimes} \to Ass^{\otimes}$ that endows $(\mathcal{C}^{op})^{\Sigma} \simeq (\mathcal{C}^{\Sigma})^{op}$ with the composition product.

We say that $((\mathcal{C}^{op})^{\Sigma})^{\otimes} \to Ass^{\otimes}$ endows $\mathcal{C}^{\Sigma}$ with the cocomposition product, which we denote by $*$.

2.2.2 Internal operads and Hopf operads

Let $\mathcal{C}$ be a symmetric monoidal category compatible with the initial object that admits small colimits.

We call associative algebras in $\mathcal{C}^{\Sigma}$ with respect to the composition product operads in $\mathcal{C}$.

We call an operad $\mathcal{O}$ in $\mathcal{C}$ non-unital if $\mathcal{O}_0$ is initial in $\mathcal{C}$.

We write $Op(\mathcal{C}) := Alg(\mathcal{C}^{\Sigma})$ and $Op^{nu}(\mathcal{C}) := \mathcal{C}^{\Sigma_{\geq 1}} \times_{\mathcal{C}^{\Sigma}} Op(\mathcal{C}) \simeq Alg(\mathcal{C}^{\Sigma_{\geq 1}})$.

Given an operad $\mathcal{O} \in Op(\mathcal{C})$ we set $Alg_{\mathcal{O}}(\mathcal{C}) := LMod_{\mathcal{O}}(\mathcal{C})$.

Dually given a symmetric monoidal category $\mathcal{C}$ compatible with the final object that admits small limits, we refer to (non-unital) operads in $\mathcal{C}^{op}$ as (non-counital) cooperads in $\mathcal{C}$ and write $CoOp(\mathcal{C}) := Op(\mathcal{C}^{op})^{op}$ and $CoOp^{ncu}(\mathcal{C}) := Op^{nu}(\mathcal{C}^{op})^{op}$.

Given a cooperad $\mathcal{Q} \in CoOp(\mathcal{C})$ we set $Coalg_{\mathcal{Q}}(\mathcal{C}) := Alg_{\mathcal{Q}}(\mathcal{C}^{op})^{op}$.

If $\mathcal{C}$ is additionally preadditive, by lemma 2.19 we have a canonical equivalence
$$CoOp^{ncu}(\mathcal{C}) \simeq Coalg(\mathcal{C}^{\Sigma_{\geq 1}}).$$
In this case given a non-counital cooperad $\mathcal{Q} \in CoOp^{ncu}(\mathcal{C}) \simeq Coalg(\mathcal{C}^{\Sigma_{\geq 1}}) \subset Coalg(\mathcal{C}^{\Sigma})$ we set
$$Coalg^{pd,conil}_{\mathcal{Q}}(\mathcal{C}) := coLMod_{\mathcal{Q}}(\mathcal{C}),$$
where we form comodules in $\mathcal{C}$ with respect to the left action of $\mathcal{C}^{\Sigma}$ on $\mathcal{C}$ induced by the composition product.

If $\mathcal{C}$ has small limits, we have a forgetful functor
$$Coalg^{pd,conil}_{\mathcal{Q}}(\mathcal{C}) \to Coalg_{\mathcal{Q}}(\mathcal{C})$$
over $\mathcal{C}$ (lemma 2.19).

Remark 2.7.

1. *The categories* $Op(\mathcal{C}), Op^{nu}(\mathcal{C})$ *admit an initial object lying over* $triv \in \mathcal{C}^{\Sigma_{\geq 1}}$*:*

$\mathcal{C}$ embeds symmetric monoidally into a symmetric monoidal category $\mathcal{C}'$ compatible with small colimits such that the initial object is preserved. So $(\mathcal{C}^{\Sigma})^{\otimes}$ embeds lax monoidally into the monoidal category $(\mathcal{C}'^{\Sigma})^{\otimes}$, whose tensorunit triv *belongs to $\mathcal{C}^{\Sigma}$.*

2. *If $\mathcal{C}$ admits a zero object, the initial object of $\mathrm{Op}^{\mathrm{nu}}(\mathcal{C})$ lying over* $\mathrm{triv} \in \mathcal{C}^{\Sigma_{\geq 1}}$ *is the zero object of the full subcategory $\{\mathrm{id}_{\mathbb{1}}\}\times_{\mathcal{C}_{\mathbb{1}/}}\mathrm{Op}^{\mathrm{nu}}(\mathcal{C}) \subset \mathrm{Op}^{\mathrm{nu}}(\mathcal{C})$ spanned by the non-unital operads $\mathcal{O}$, whose unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence (lemma 2.18).*

3. *If $\mathcal{C}$ admits a final object, $\mathrm{Op}(\mathcal{C}), \mathrm{Op}^{\mathrm{nu}}(\mathcal{C})$ admit a final object lying over the constant symmetric sequence (concentrated in positive degrees) with value the final object of $\mathcal{C}$.*

We refer to (non-unital) operads in $\mathrm{Cocoalg}(\mathcal{C})$ as (non-unital) Hopf operads in $\mathcal{C}$ and set

$$\mathrm{Op}_{\mathrm{Hopf}}(\mathcal{C}) := \mathrm{Op}(\mathrm{Cocoalg}(\mathcal{C})),\ \mathrm{Op}^{\mathrm{nu}}_{\mathrm{Hopf}}(\mathcal{C}) := \mathrm{Op}^{\mathrm{nu}}(\mathrm{Cocoalg}(\mathcal{C}))$$
$$\simeq \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}) \times_{\mathrm{Op}(\mathcal{C})} \mathrm{Op}_{\mathrm{Hopf}}(\mathcal{C}).$$

The symmetric monoidal functor $\mathrm{Cocoalg}(\mathcal{C}) \to \mathcal{C}$ yields a forgetful functor $\mathrm{Op}_{\mathrm{Hopf}}(\mathcal{C}) = \mathrm{Op}(\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{Op}(\mathcal{C})$ that sends a Hopf operad to its underlying operad.

If the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits, there is a unique left adjoint symmetric monoidal functor $\mathcal{S} \to \mathcal{C}$ that lifts to a functor $\mathcal{S} \to \mathrm{Cocoalg}(\mathcal{C})$ and so gives rise to a functor $\mathrm{Op}(\mathcal{S}) \to \mathrm{Op}_{\mathrm{Hopf}}(\mathcal{C})$.
Simlarly if the symmetric monoidal structure on $\mathcal{C}$ is compatible with finite coproducts, there is a unique left adjoint symmetric monoidal functor from finite sets to $\mathcal{C}$ that lifts to $\mathrm{Cocoalg}(\mathcal{C})$ and so gives rise to a functor from operads in finite sets to Hopf operads in $\mathcal{C}$.
By remark 2.7 3. the categories $\mathrm{Op}_{\mathrm{Hopf}}(\mathcal{C}),\ \mathrm{Op}^{\mathrm{nu}}_{\mathrm{Hopf}}(\mathcal{C})$ admit a final object as $\mathrm{Cocoalg}(\mathcal{C})$ admits a final object lying over the tensorunit of $\mathcal{C}$.

We define Comm to be the final Hopf operad in $\mathcal{C}$ and $\mathrm{Comm}^{\mathrm{nu}}$ to be the final non-unital Hopf operad in $\mathcal{C}$.
So Comm, $\mathrm{Comm}^{\mathrm{nu}}$ lie over the constant symmetric sequence in $\mathcal{C}$ (concentrated in positive degrees) with value the tensorunit of $\mathcal{C}$.

Dually if $\mathcal{C}$ admits small limits, we define Cocomm and $\mathrm{Cocomm}^{\mathrm{ncu}}$ to be Comm and $\mathrm{Comm}^{\mathrm{nu}}$, where we replace $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$.

By remark 2.7 2. the non-unital operad $\mathrm{Comm}^{\mathrm{nu}}$ and the non-counital cooperad $\mathrm{Cocomm}^{\mathrm{ncu}}$ admit a canonical augmentation respectively coaugmentation if the symmetric monoidal structure on $\mathcal{C}$ is compatible with the zero object.

Every Hopf operad $\mathcal{H}$ in $\mathcal{C}$ endows its category of algebras $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ with a symmetric monoidal structure such that the forgetful functor
$\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ is symmetric monoidal (proposition 5.77).

We define the category of bialgebras over $\mathcal{H}$ as

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) := \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}))$$

and have a canonical equivalence

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))$$

over $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \times \mathrm{Cocoalg}(\mathcal{C})$ by remark 2.9.

We call a Hopf operad $\mathcal{H}$ on $\mathcal{C}$ unital if the tensorunit of the category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ is an initial object.

If the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits so that the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint, $\mathcal{H}_0 \simeq \mathcal{H} \circ \varnothing$ is the initial object of $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$.

So a Hopf operad $\mathcal{H}$ on $\mathcal{C}$ is unital if $\mathcal{H}_0$ is canonically the tensorunit of $\mathcal{C}$. The final Hopf operad Comm is unital.

For later reference we add the following remark:

Remark 2.8. *Let $\mathcal{C}$ be a symmetric monoidal category compatible with small colimits.*

1. *The category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ admits small colimits and the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ preserves small sifted colimits.*

2. *The category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ admits small colimits, which are preserved by the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$.*

3. *If $\mathcal{C}$ is presentable, $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ and $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ are presentable.*

Proof. If the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits, the composition product on $\mathcal{C}^{\Sigma}$ defines a monoidal category compatible with small sifted colimits.

So $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ admits small sifted colimits and the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ is monadic and preserves small sifted colimits.

This guarantees that with $\mathcal{C}$ also $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ admits finite coproducts using that finite coproducts of free $\mathcal{H}$-algebras exist and every $\mathcal{H}$-algebra is the geometric realization of a diagram with values in free $\mathcal{H}$-algebras.

2. follows from remark 2.1.

Assume that $\mathcal{C}$ is presentable. Then by proposition 6.84 the category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ is accessible being the category of algebras over an accessible monad.

By proposition 5.77 the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ is symmetric monoidal so that $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ is an accessible symmetric monoidal category.

Thus by proposition 6.83 the category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ is accessible.

□

Remark 2.9. *Let $\mathcal{C}$ be a symmetric monoidal category that admits small colimits and $\mathcal{H}$ a Hopf operad on $\mathcal{C}$.*

There is a canonical equivalence

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))$$

over $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \times \mathrm{Cocoalg}(\mathcal{C})$.

Proof. We can assume that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits.

Otherwise we embed $\mathcal{C}$ into the category of presheaves $\mathcal{C}' := \mathcal{P}(\mathcal{C})$ endowed with Day-convolution and get an equivalence

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}') = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}')) \simeq \mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}'))$$

over $\mathrm{Cocoalg}(\mathcal{C}')$, whose pullback to $\mathrm{Cocoalg}(\mathcal{C})$ is the desired equivalence.

Denote $\mathcal{H}'$ the underlying operad in $\mathcal{C}$ of the Hopf operad $\mathcal{H}$.

Both forgetful functors $\mathrm{Cocoalg}(\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{Cocoalg}(\mathcal{C})$ are equivalent and are equivalences so that $\mathcal{H}$ canonically lifts to a Hopf operad in $\mathrm{Cocoalg}(\mathcal{C})$ with underlying operad $\mathcal{H}$ in $\mathrm{Cocoalg}(\mathcal{C})$.

So by functoriality the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{Alg}_{\mathcal{H}'}(\mathcal{C})$ is symmetric monoidal and so yields a functor

$$\mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}'}(\mathcal{C})).$$

As the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{Cocoalg}(\mathcal{C})$ is symmetric monoidal, with $\mathrm{Cocoalg}(\mathcal{C})$ also $\mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))$ is a cartesian symmetric monoidal category. Hence the forgetful functor

$$\mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))) \to \mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))$$

is an equivalence.

So we get a canonical functor

$$\chi : \mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C})) \simeq \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathrm{Cocoalg}(\mathcal{C}))) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$$

over $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \times \mathrm{Cocoalg}(\mathcal{C})$.

By the theorem of Barr-Beck the monadic functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ induces a monadic functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})) \to \mathrm{Cocoalg}(\mathcal{C})$.

So χ is a functor between monadic functors over $\mathrm{Cocoalg}(\mathcal{C})$ and thus an equivalence as for every $\mathrm{X} \in \mathrm{Cocoalg}(\mathcal{C})$ with image $\mathrm{X}' \in \mathcal{C}$ the object $\mathcal{H} \circ \mathrm{X}$ in $\mathrm{Cocoalg}(\mathcal{C})$ lies canonically over the object $\mathcal{H}' \circ \mathrm{X}'$ in $\mathcal{C}$.

□

2.2.3 Trivial coalgebras and primitive elements

Let $\mathcal{C}$ be a symmetric monoidal category such that $\mathcal{C}$ admits small limits and a zero object that is preserved by the tensorproduct in each component.

In this subsection we construct an adjunction

$$\mathrm{triv} : \mathcal{C} \to \mathrm{Coalg}_{\mathrm{Cocomm}^{\mathrm{ncu}}}(\mathcal{C}) : \mathrm{Prim},$$

where triv sends an object X of $\mathcal{C}$ to the non-counital cocommutative coalgebra structure on X with zero comultiplication and Prim takes the primitive elements.

Let $\mathcal{C}$ be a symmetric monoidal category such that $\mathcal{C}$ admits small colimits and the symmetric monoidal structure on $\mathcal{C}$ is compatible with the initial object.

In this case by remark 2.7 1. the category $\mathrm{Op}(\mathcal{C})$ admits an initial object lying over $\mathrm{triv} \in \mathcal{C}^{\Sigma_{\geq 1}}$.

Let $\mathcal{O} \in \mathrm{Op}(\mathcal{C})_{/\mathrm{triv}}$ be an augmented operad and $\mathcal{Q} \in \mathrm{CoOp}(\mathcal{C})_{\mathrm{triv}/}$ a coaugmented cooperad.

The augmentation of $\mathcal{O}$ gives rise to a forgetful functor

$$\mathrm{triv}_{\mathcal{O}} : \mathcal{C} \simeq \mathrm{Alg}_{\mathrm{triv}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$$

and dually the coaugmentation of $\mathcal{Q}$ gives rise to a forgetful functor

$$\mathrm{triv}_{\mathcal{Q}} : \mathcal{C} \simeq \mathrm{Coalg}_{\mathrm{triv}}(\mathcal{C}) \to \mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}).$$

By the next remark 2.10 3. the forgetful functor $\mathrm{triv}_{\mathcal{O}} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ admits a left adjoint that factors as

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \xrightarrow{\mathrm{Bar}^{\mathcal{C}}_{\mathcal{O}}(-)} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{C}) \xrightarrow{\mathrm{colim}} \mathcal{C}$$

such that for every $X \in \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ and $n \in \mathbb{N}$ the object $\mathrm{Bar}^{\mathcal{C}}_{\mathcal{O}}(X)_n$ is the object $\mathcal{O} \circ ... \circ \mathcal{O} \circ X$ representing the functor $\mathrm{Mul}_{\mathcal{C}^{\Sigma}}(\mathcal{O}, ..., \mathcal{O}, X; -)_{|\mathcal{C}} : \mathcal{C} \to \mathcal{S}$.

We apply remark 2.10 3. in the following way:

The symmetric monoidal category $\mathcal{C}$ is a (not neccessarily symmetric monoidal) localization of a symmetric monoidal category $\mathcal{C}'$ compatible with small colimits such that the initial object and the tensorunit of $\mathcal{C}'$ belong to $\mathcal{C} \subset \mathcal{C}'$. So by functoriality of the composition product the representable planar operad $\mathcal{C}^{\Sigma}$ is a (not neccessarily monoidal) localization of the monoidal category $\mathcal{C}'^{\Sigma}$ such that the tensorunit triv of $\mathcal{C}'^{\Sigma}$ belongs to $\mathcal{C}^{\Sigma} \subset \mathcal{C}'^{\Sigma}$.

We take $\mathcal{M}^{\otimes}$ to encode the canonical left action of $\mathcal{C}'^{\Sigma}$ on itself and $\mathcal{N}^{\otimes}$ to encode the canonical left action of $\mathcal{C}^{\Sigma}$ on itself.

Let $\mathcal{C}$ be a symmetric monoidal category such that $\mathcal{C}$ admits small limits and a zero object and the symmetric monoidal structure on $\mathcal{C}$ is compatible with the zero object.

Dually the forgetful functor $\mathrm{triv}_{\mathcal{Q}} : \mathcal{C} \to \mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C})$ admits a right adjoint that factors as $\mathrm{Coalg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\Delta, \mathcal{C}) \xrightarrow{\lim} \mathcal{C}$.

For $\mathcal{Q}$ the non-counital cocommutative cooperad, we call the corresponding adjunction

$$\mathrm{triv} : \mathcal{C} \to \mathrm{Coalg}_{\mathrm{Cocomm}^{\mathrm{ncu}}}(\mathcal{C}) : \mathrm{Prim}$$

the trivial cocommutative coalgebra-primitive elements adjunction.

Remark 2.10. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{D}$ a left module over $\mathcal{C}$. Let $A \to B$ be a morphism in $\mathrm{Alg}(\mathcal{C})$.*

1. *Assume first that the monoidal structure on $\mathcal{C}$ is compatible with small colimits and the left module $\mathcal{D}$ is compatible with small colimits.*

   By [18] example 4.7.3.7. the identity of $\mathrm{LMod}_A(\mathcal{D})$ factors as

$$\mathrm{LMod}_A(\mathcal{D}) \xrightarrow{\mathrm{Bar}_A(-)} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{LMod}_A(\mathcal{D})) \xrightarrow{\mathrm{colim}} \mathrm{LMod}_A(\mathcal{D})$$

   such that for every $X \in \mathrm{LMod}_A(\mathcal{D})$ and $n \in \mathbb{N}$ the left A-module $\mathrm{Bar}_A(X)_n$ is free on $A^{\otimes n} \otimes X$.

   By example 5.49 the forgetful functor $\mathrm{LMod}_B(\mathcal{D}) \to \mathrm{LMod}_A(\mathcal{D})$ admits a left adjoint $B \otimes_A - : \mathrm{LMod}_A(\mathcal{D}) \to \mathrm{LMod}_B(\mathcal{D})$.

   Denote $\mathrm{Bar}^{\mathcal{D}}_A(B, -)$ the composition

$$\mathrm{LMod}_A(\mathcal{D}) \xrightarrow{\mathrm{Bar}_A(-)} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{LMod}_A(\mathcal{D})) \xrightarrow{B \otimes_A -} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{LMod}_B(\mathcal{D})).$$

   So for every $X \in \mathrm{LMod}_A(\mathcal{D})$ and $n \in \mathbb{N}$ the left B-module $\mathrm{Bar}_A(B, X)_n$ is free on $A^{\otimes n} \otimes X$ and $B \otimes_A - : \mathrm{LMod}_A(\mathcal{D}) \to \mathrm{LMod}_B(\mathcal{D})$ factors as

$$\mathrm{LMod}_A(\mathcal{D}) \xrightarrow{\mathrm{Bar}^{\mathcal{D}}_A(B,-)} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{LMod}_B(\mathcal{D})) \xrightarrow{\mathrm{colim}} \mathrm{LMod}_B(\mathcal{D}).$$

In other words the composition

$$\mathrm{LMod_A(\mathcal{D})} \xrightarrow{\mathrm{Bar}_A^{\mathcal{D}}(B,-)} \mathrm{Fun}(\Delta^{op}, \mathrm{LMod_B(\mathcal{D})}) \xrightarrow{\mathrm{colim}} \mathrm{LMod_B(\mathcal{D})}$$

is left adjoint to the forgetful functor $\mathrm{LMod_B(\mathcal{D})} \to \mathrm{LMod_A(\mathcal{D})}$.

2. *Let now* $\mathcal{C}$ *and* $\mathcal{D}$ *be arbitrary.*

   We have a $\mathrm{LM}^\otimes$*-monoidal Yoneda-embedding* $\mathcal{D} \subset \mathcal{D}' := \mathcal{P}(\mathcal{D})$*, where* $\mathcal{P}(\mathcal{D})$ *is a left module compatible with small colimits over* $\mathcal{C}' := \mathcal{P}(\mathcal{C})$ *endowed with Day-convolution that is compatible with small colimits.*

   The functor $\mathrm{Bar}_A^{\mathcal{D}'}(B,-) : \mathrm{LMod_A}(\mathcal{D}') \to \mathrm{Fun}(\Delta^{op}, \mathrm{LMod_B}(\mathcal{D}'))$ *of 1. restricts to a functor*

$$\mathrm{Bar}_A^{\mathcal{D}}(B,-) : \mathrm{LMod_A}(\mathcal{D}) \to \mathrm{Fun}(\Delta^{op}, \mathrm{LMod_B}(\mathcal{D})).$$

   So for every $X \in \mathrm{LMod_A}(\mathcal{D})$ *and* $n \in \mathbb{N}$ *the left* B*-module* $\mathrm{Bar}_A^{\mathcal{D}}(B,X)_n$ *in* $\mathcal{D}$ *is free on* $A^{\otimes n} \otimes X \in \mathcal{D}$.

   If $\mathrm{LMod_B}(\mathcal{D})$ *admits geometric realizations, the composition*

$$\mathrm{LMod_A(\mathcal{D})} \xrightarrow{\mathrm{Bar}_A^{\mathcal{D}}(B,-)} \mathrm{Fun}(\Delta^{op}, \mathrm{LMod_B(\mathcal{D})}) \xrightarrow{\mathrm{colim}} \mathrm{LMod_B(\mathcal{D})}$$

   is left adjoint to the forgetful functor $\mathrm{LMod_B(\mathcal{D})} \to \mathrm{LMod_A(\mathcal{D})}$.

Proof. By 1. for every $Y \in \mathrm{LMod_B}(\mathcal{D})$ and $M \in \mathrm{LMod_A}(\mathcal{D})$ we have a canonical equivalence

$$\mathrm{LMod_A}(\mathcal{D})(M,Y) \simeq \mathrm{LMod_A}(\mathcal{D}')(M,Y) \simeq$$

$$\mathrm{LMod_B}(\mathcal{D}')(\mathrm{colim}(\mathrm{Bar}_A^{\mathcal{D}'}(B,M)),Y) \simeq$$

$$\lim(\mathrm{LMod_B}(\mathcal{D}')(\mathrm{Bar}_A^{\mathcal{D}'}(B,M),Y)) \simeq \lim(\mathrm{LMod_B}(\mathcal{D})(\mathrm{Bar}_A^{\mathcal{D}}(B,M),Y))$$
$$\simeq \mathrm{LMod_B}(\mathcal{D})(\mathrm{colim}(\mathrm{Bar}_A^{\mathcal{D}}(B,M)),Y).$$

□

3. *Let* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ *be a* $\mathrm{LM}^\otimes$*-monoidal category that exhibits* $\mathcal{D}$ *as a left module over* $\mathcal{C}$ *and* $\mathcal{N}^\otimes \subset \mathcal{M}^\otimes$ *a full suboperad over* $\mathrm{LM}^\otimes$.

   Set $\mathcal{B}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$ *and* $\mathcal{E} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$.

   Assume that the full subcategory inclusion $\mathcal{E} \subset \mathcal{D}$ *admits a left adjoint* L.
   In this case for every $A_1, ..., A_n \in \mathcal{B}$ *for some* $n \in \mathbb{N}$ *and* $X \in \mathcal{E}$ *the object* $L(A_1 \otimes ... \otimes A_n \otimes X) \in \mathcal{E}$ *represents the functor* $\mathrm{Mul}_{\mathcal{N}}(A_1, ..., A_n, X; -) : \mathcal{E} \to \mathcal{S}$.
   Assume that the tensorunit $\mathbb{1}$ *of* $\mathcal{C}$ *belongs to* $\mathcal{B}$.
   Let $A \in \mathrm{Alg}(\mathcal{B})_{/\mathbb{1}} \subset \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}$.

   We define $\mathrm{Bar}_A^{\mathcal{E}}(-) : \mathrm{LMod_A}(\mathcal{E}) \to \mathrm{Fun}(\Delta^{op}, \mathcal{E})$ *to be the composition*

$$\mathrm{LMod_A}(\mathcal{E}) \subset \mathrm{LMod_A}(\mathcal{D}) \xrightarrow{\mathrm{Bar}_A^{\mathcal{D}}(\mathbb{1},-)} \mathrm{Fun}(\Delta^{op}, \mathcal{D}) \xrightarrow{\mathrm{Fun}(\Delta^{op},L)} \mathrm{Fun}(\Delta^{op}, \mathcal{E}).$$

So for every $X \in \mathrm{LMod}_A(\mathcal{E})$ *and* $n \in \mathbb{N}$ *the object* $\mathrm{Bar}_A^{\mathcal{E}}(X)_n$ *is the object* $L(A^{\otimes n} \otimes X) \in \mathcal{E}$ *representing the functor* $\mathrm{Mul}_{\mathcal{N}}(A, ..., A, X; -) : \mathcal{E} \to \mathcal{S}$.

Assume that $\mathcal{D}$ *(and so also* $\mathcal{E}$*) admits geometric realizations.*

The composition

$$\mathrm{LMod}_A(\mathcal{E}) \xrightarrow{\mathrm{Bar}_A^{\mathcal{E}}(-)} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{E}) \xrightarrow{\mathrm{colim}} \mathcal{E}$$

is left adjoint to the forgetful functor $\mathcal{E} \simeq \mathrm{LMod}_{\mathbb{1}}(\mathcal{E}) \to \mathrm{LMod}_A(\mathcal{E})$.

This follows from the fact that by 2. the composition

$$\mathrm{LMod}_A(\mathcal{D}) \xrightarrow{\mathrm{Bar}_A^{\mathcal{D}}(\mathbb{1},-)} \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D}) \xrightarrow{\mathrm{colim}} \mathcal{D} \xrightarrow{L} \mathcal{E}$$

is left adjoint to the functor $\mathcal{E} \subset \mathcal{D} \simeq \mathrm{LMod}_{\mathbb{1}}(\mathcal{D}) \to \mathrm{LMod}_A(\mathcal{D})$ *that is equivalent to the functor* $\mathcal{E} \simeq \mathrm{LMod}_{\mathbb{1}}(\mathcal{E}) \to \mathrm{LMod}_A(\mathcal{E}) \subset \mathrm{LMod}_A(\mathcal{D})$.

2.2.4 Shifting operads

In this subsection we construct a shift functor on the category of operads and non-counital cooperads in a stable symmetric monoidal category $\mathcal{C}$ that admits small colimits.

We use this shift functor in the definition of the spectral Lie operad, which we define to be the Koszul-dual operad of the shifted non-counital cocommutative cooperad in spectra.

Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits.

With $\mathcal{C}$ also $\mathcal{C}^{\Sigma}$ is stable. In the following denote $[-]$ the shift functor of $\mathcal{C}^{\Sigma}$.

Let $X, Y \in \mathcal{C}^{\Sigma}$ such that X is concentrated in degree 1.

For every $n, m \in \mathbb{Z}$ we have a canonical equivalence

$$X[n] \circ Y[m] \simeq \coprod_{k \in \mathbb{N}} (X[n])_k \otimes_{\Sigma_k} (Y[m])^{\otimes k} \simeq X_1[n] \otimes Y[m] \simeq (X_1 \otimes Y)[n+m].$$

Especially we have a canonical equivalence

$$\mathrm{triv}[n] \circ \mathrm{triv}[m] \simeq \mathrm{triv}[n+m].$$

So $\mathrm{triv}[n]$ is inverse to $\mathrm{triv}[-n]$ in the composition product on $\mathcal{C}^{\Sigma}$.

Via the canonical equivalence $\mathrm{Fun}^{\otimes, \mathrm{coc}}_{\mathcal{C}/}(\mathcal{C}^{\Sigma}, \mathcal{C}^{\Sigma}) \simeq \mathcal{C}^{\Sigma}$ that evaluates at triv the object $\mathrm{triv}[n]$ corresponds to a symmetric monoidal autoequivalence α_n of $\mathcal{C}^{\Sigma}$ under $\mathcal{C}$.

Thus by the universal property of endomorphism objects we obtain a canonical monoidal autoequivalence ξ of $\mathrm{Fun}^{\otimes, \mathrm{coc}}_{\mathcal{C}/}(\mathcal{C}^{\Sigma}, \mathcal{C}^{\Sigma}) \simeq \mathcal{C}^{\Sigma}$ given by conjugation with α_n.

By remark 2.11 for every $Y \in \mathcal{C}^{\Sigma}$ and $r \in \Sigma$ we have a natural equivalence $\xi(Y)_r \simeq Y_r[(1-r)n]$.

If $\mathcal{C}$ is an arbitrary stable symmetric monoidal category that admits small colimits, $\mathcal{C}$ embeds symmetric monoidally and exact into a stable symmetric monoidal category $\mathcal{C}'$ compatible with small colimits (remark 1.6).

Thus the representable operad $(\mathcal{C}^{\Sigma})^{\otimes}$ embeds into the monoidal category $(\mathcal{C}'^{\Sigma})^{\otimes}$.

Hence the monoidal autoequivalence ξ of $\mathcal{C}'^{\Sigma}$ restricts to a monoidal autoequivalence ξ of $\mathcal{C}^{\Sigma}$ that restricts to the equivalence $(-)[n]$ on $\mathcal{C}$. Especially ξ restricts to a monoidal autoequivalence of $\mathcal{C}^{\Sigma_{\geq 1}}$.

Given a symmetric sequence Y in $\mathcal{C}$, we write $Y(n)$ for $\xi(Y)$.

ξ gives rise to autoequivalences of $\mathrm{Alg}(\mathcal{C}^{\Sigma})$ and $\mathrm{LMod}(\mathcal{C})$.

So for every operad $\mathcal{O}$ in $\mathcal{C}$ we obtain a pullback square

$$\begin{array}{ccc} \mathrm{LMod}_{\mathcal{O}(n)}(\mathcal{C}) & \xrightarrow{\simeq} & \mathrm{LMod}_{\mathcal{O}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\simeq]{(-)[-n]} & \mathcal{C} \end{array}$$

and for every cooperad $\mathcal{Q}$ in $\mathcal{C}$ we obtain a pullback square

$$\begin{array}{ccc} \mathrm{coLMod}_{\mathcal{Q}(n)}(\mathcal{C}) & \xrightarrow{\simeq} & \mathrm{coLMod}_{\mathcal{Q}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\simeq]{(-)[-n]} & \mathcal{C}. \end{array}$$

Moreover if $\mathcal{C}$ is compatible with small colimits so that the composition product on $\mathcal{C}^{\Sigma}$ defines a monoidal category, ξ gives rise to autoequivalences of

$$\mathrm{Coalg}(\mathcal{C}^{\Sigma}),\ \mathrm{Coalg}(\mathcal{C}^{\Sigma_{\geq 1}}),\ \mathrm{coLMod}(\mathcal{C}).$$

So if $\mathcal{C}$ is additionally preadditive, for every non-counital cooperad $\mathcal{Q} \in \mathrm{CoOp}^{\mathrm{ncu}}(\mathcal{C}) \simeq \mathrm{Coalg}(\mathcal{C}^{\Sigma_{\geq 1}})$ we obtain a pullback square

$$\begin{array}{ccc} \mathrm{Coalg}^{\mathrm{pd,conil}}_{\mathcal{Q}(n)}(\mathcal{C}) & \xrightarrow{\simeq} & \mathrm{Coalg}^{\mathrm{pd,conil}}_{\mathcal{Q}}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\simeq]{(-)[-n]} & \mathcal{C}. \end{array}$$

Remark 2.11. *For every* $Y \in \mathcal{C}^{\Sigma}$ *and* $r \in \Sigma$ *we have a natural equivalence*

$$\xi(Y)_r \simeq Y_r[(1-r)n].$$

Proof. The object $Y \in \mathcal{C}^{\Sigma}$ uniquely lifts to a symmetric monoidal small colimits preserving endofunctor $\bar{Y}$ of $\mathcal{C}^{\Sigma}$ under $\mathcal{C}$.

We have

$$\xi(Y) \simeq (\alpha_{-n} \circ \bar{Y} \circ \alpha_n)(\mathrm{triv}) \simeq \alpha_{-n}(\bar{Y}(\mathrm{triv}[n])) \simeq \alpha_{-n}(\bar{Y}(\mathrm{triv}))[n]$$

$$\simeq \alpha_{-n}(Y)[n] \simeq (Y \circ (\mathrm{triv}[-n]))[n] \simeq$$

$$(\coprod_{k \in \mathbb{N}} Y_k \otimes_{\Sigma_k} \mathrm{triv}[-n]^{\otimes k})[n] \simeq \coprod_{k \in \mathbb{N}} Y_k \otimes_{\Sigma_k} \mathrm{triv}^{\otimes k}[(1-k)n]$$

and thus $\xi(Y)_r \simeq Y_r \otimes_{\Sigma_r} (\Sigma_r \times \mathbb{1})[(1-r)n] \simeq Y_r[(1-r)n]$.

□

2.2.5 Truncating operads

In this subsection we define truncation of operads, which we use to construct a cofiltration of the primitive elements (prop. 4.8).

For every $n \geq 1$ denote $\Sigma_{\geq 1}^{\leq n} \subset \Sigma_{\geq 1}$ the full subcategory spanned by the sets with less or equal than n elements.

Let $\mathcal{C}$ be a symmetric monoidal category that admits small colimits and a zero object, which is preserved by the tensorproduct in each component.

The embedding $\iota : \Sigma_{\geq 1}^{\leq n} \subset \Sigma_{\geq 1}$ induces a localization $\iota^* : \mathcal{C}^{\Sigma_{\geq 1}} \rightleftarrows \mathcal{C}^{\Sigma_{\geq 1}^{\leq n}} : \iota_*$, where the left adjoint is restriction and the fully faithful right adjoint extends over degree n by the zero object.

Denote $\mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}} \subset \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}$ the full subcategory spanned by the non-unital operads in $\mathcal{C}$, whose underlying symmetric sequence vanishes over degree n, i.e. belongs to $\mathcal{C}^{\Sigma_{\geq 1}^{\leq n}} \subset \mathcal{C}^{\Sigma_{\geq 1}}$.

By lemma 2.16 the localization $\iota^* : \mathcal{C}^{\Sigma_{\geq 1}} \rightleftarrows \mathcal{C}^{\Sigma_{\geq 1}^{\leq n}} : \iota_*$ is compatible with the composition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ and so yields a localization

$$(-)_{\leq n} : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}} = \mathrm{Alg}(\mathcal{C}^{\Sigma_{\geq 1}}) \rightleftarrows \mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}} = \mathrm{Alg}(\mathcal{C}^{\Sigma_{\geq 1}^{\leq n}}).$$

This leads to the following remark:

Remark 2.12. *Let $\mathcal{C}$ be a symmetric monoidal category that admits small colimits and a zero object, which is preserved by the tensorproduct in each component.*

The embedding $\mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}} \subset \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}$ admits a left adjoint

$$(-)_{\leq n} : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}}$$

that fits into a commutative square

$$\begin{array}{ccc} \mathrm{Op}(\mathcal{C})^{\mathrm{nu}} & \longrightarrow & \mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\Sigma} & \longrightarrow & \mathcal{C}^{\Sigma_{\geq 1}^{\leq n}}, \end{array}$$

where the bottom functor is restriction.

If $\mathcal{C}$ is stable and admits totalizations, the functor $(-)_{\leq n} : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}}$ admits a left adjoint f_n.

In this case the composition $\mathsf{f}_n \circ (-)_{\leq n} : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}} \to \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}$ is left adjoint to the functor $\tau_n := (-)_{\leq n} : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Op}(\mathcal{C})_{\leq n}^{\mathrm{nu}} \subset \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}$.

Remark 2.13. *Let $\mathcal{C}$ be a symmetric monoidal category compatible with the initial object that admits small colimits.*

For every $n \geq 1$ denote $\Sigma_{\geq n} \subset \Sigma_{\geq 1}$ the full subcategory spanned by the sets with at least n elements.

The embedding $\kappa : \Sigma_{\geq n} \subset \Sigma_{\geq 1}$ induces a colocalization $\kappa_! : \mathcal{C}^{\Sigma_{\geq n}} \rightleftarrows \mathcal{C}^{\Sigma_{\geq 1}} : \kappa^*$, where the right adjoint is restriction and the fully faithful left adjoint extends under degree n by the initial object.

The embedding $\kappa_! : \mathcal{C}^{\Sigma_{\geq n}} \subset \mathcal{C}^{\Sigma_{\geq 1}}$ is $\mathcal{C}^{\Sigma_{\geq 1}}$-linear by lemma 2.17.

So for every non-unital operad $\mathcal{O}$ in $\mathcal{C}$ we get a colocalization

$$\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq n}}) \rightleftarrows \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) : \tau_{\geq n}.$$

Using truncation of operads one is able to filter algebras over an operad by the following remark:

Remark 2.14. *Let* $\mathcal{C}$ *be a stable symmetric monoidal category compatible with small colimits.*

For every $n \geq 1$ *we have an adjunction*

$$\mathsf{f}_n : \mathrm{Op}(\mathcal{C})^{nu}_{\leq n} \rightleftarrows \mathrm{Op}(\mathcal{C})^{nu} : (-)_{\leq n}.$$

For every augmented non-unital operad $\mathcal{O}$ *in* $\mathcal{C}$ *the counit* $\mathcal{E}_n : \mathsf{f}_n(\mathcal{O}_{\leq n}) \to \mathcal{O}$ *gives rise to an adjunction*

$$(\mathcal{E}_n)_! := \mathcal{O} \circ_{\mathsf{f}_n(\mathcal{O}_{\leq n})} - : \mathrm{Alg}_{\mathsf{f}_n(\mathcal{O}_{\leq n})}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) : \mathcal{E}_n^*.$$

So for every $\mathcal{O}$*-algebra* A *in* $\mathcal{C}$ *we have a counit* $\mathcal{O} \circ_{\mathsf{f}_n(\mathcal{O}_{\leq n})} A = (\mathcal{E}_n)_!(\mathcal{E}_n^*(A)) \to A$.

We have a morphism $\theta : \mathsf{f}_n(\mathcal{O}_{\leq n}) \to \mathsf{f}_{n+1}(\mathcal{O}_{\leq n+1})$ *of operads in* $\mathcal{C}$ *compatible with the counits adjoint to the morphism* $\mathcal{O}_{\leq n} \to (\mathsf{f}_{n+1}(\mathcal{O}_{\leq n+1}))_{\leq n}$ *that arises from the unit* $\mathcal{O}_{\leq n+1} \to (\mathsf{f}_{n+1}(\mathcal{O}_{\leq n+1}))_{\leq n+1}$ *by applying the functor* $\mathrm{Op}(\mathcal{C})_{\leq n+1} \subset \mathrm{Op}(\mathcal{C}) \xrightarrow{(-)_{\leq n}} \mathrm{Op}(\mathcal{C})_{\leq n}$.

So the adjunction $(\mathcal{E}_n)_! : \mathrm{Alg}_{\mathsf{f}_n(\mathcal{O}_{\leq n})}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) : \mathcal{E}_n^*$ *factors as*

$$(\mathcal{E}_{n+1})_! \circ \theta_! : \mathrm{Alg}_{\mathsf{f}_n(\mathcal{O}_{\leq n})}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathsf{f}_{n+1}(\mathcal{O}_{\leq n+1})}(\mathcal{C}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) : \theta^* \circ \mathcal{E}_{n+1}^*.$$

We have a map $\beta_n : (\mathcal{E}_n)_!(\mathcal{E}_n^*(A)) \to (\mathcal{E}_{n+1})_!(\mathcal{E}_{n+1}^*(A))$ *of* $\mathcal{O}$*-algebras in* $\mathcal{C}$ *compatible with the counits adjoint to the morphism*

$$\mathcal{E}_n^*(A) \to \mathcal{E}_n^*((\mathcal{E}_{n+1})_!(\mathcal{E}_{n+1}^*(A)))$$

that arises by applying the functor $\theta^* : \mathrm{Alg}_{\mathsf{f}_{n+1}(\mathcal{O}_{\leq n+1})}(\mathcal{C}) \to \mathrm{Alg}_{\mathsf{f}_n(\mathcal{O}_{\leq n})}(\mathcal{C})$ *to the unit* $\mathcal{E}_{n+1}^*(A) \to \mathcal{E}_{n+1}^*((\mathcal{E}_{n+1})_!(\mathcal{E}_{n+1}^*(A)))$.

By [13] remark 4.23. the morphisms β_n *promote to a filtered diagram*

$$(\mathcal{E}_1)_!(\mathcal{E}_1^*(A)) \to ... \to (\mathcal{E}_n)_!(\mathcal{E}_n^*(A)) \xrightarrow{\beta_n} (\mathcal{E}_{n+1})_!(\mathcal{E}_{n+1}^*(A)) \to ... \to A$$

of $\mathcal{O}$*-algebras in* $\mathcal{C}$, *whose colimit is* A.

This follows from the fact that the compatible maps $\mathcal{E}_n : \mathsf{f}_n(\mathcal{O}_{\leq n}) \to \mathcal{O}$ *running over all* $n \geq 1$ *exhibit* $\mathcal{O}$ *as the colimit.*

For later reference we add the following remark:

Remark 2.15. *Let* $\mathcal{C}$ *be a stable symmetric monoidal category and* $\mathcal{O}$ *a non-unital operad in* $\mathcal{C}$ *such that the unit* $\mathrm{triv} \to \mathcal{O}$ *induces an equivalence* $\mathbb{1} \simeq \mathcal{O}_1$. *By lemma 2.18 the operad* $\mathcal{O}$ *is automatically augmented.*

Let X *be a right* $\mathcal{O}$*-module in* $\mathcal{C}^{\Sigma_{\geq 1}}$, *whose underlying symmetric sequence in* $\mathcal{C}$ *is concentrated in degree* n *for some* $n \geq 1$.

Then the right $\mathcal{O}$*-module structure on* X *is trivial.*

Lemma 2.16. *Let* $\mathcal{C}$ *be a symmetric monoidal category compatible with the zero object that admits small colimits.*

For every $n \geq 1$ *the localization* $\iota^* : \mathcal{C}^{\Sigma_{\geq 1}} \rightleftarrows \mathcal{C}^{\Sigma^{\leq n}_{\geq 1}} : \iota_*$ *is compatible with the composition product on* $\mathcal{C}^{\Sigma_{\geq 1}}$.

Proof. By remark 1.6 $\mathcal{C}$ embeds symmetric monoidally into a symmetric monoidal category $\mathcal{C}'$ compatible with small colimits such that the embedding $\mathcal{C} \subset \mathcal{C}'$ preserves the zero object and admits a left adjoint L.

So we get a lax monoidal embedding $\mathcal{C}^{\Sigma_{\geq 1}} \subset \mathcal{C}'^{\Sigma_{\geq 1}}$ and an embedding $\mathcal{C}^{\Sigma_{\geq 1}^{\leq n}} \subset \mathcal{C}'^{\Sigma_{\geq 1}^{\leq n}}$.

Thus $X_1 \circ ... \circ X_k$ is the image of the corresponding composition product in $\mathcal{C}'^{\Sigma_{\geq 1}}$ under the functor $L_* : \mathcal{C}'^{\Sigma_{\geq 1}} \to \mathcal{C}^{\Sigma_{\geq 1}}$ induced by L.

A morphism of $\mathcal{C}^{\Sigma_{\geq 1}}$ is a local equivalence if and only if it induces an equivalence in degrees $\leq n$ and similar for $\mathcal{C}'$.

So the embedding $\mathcal{C}^{\Sigma_{\geq 1}} \subset \mathcal{C}'^{\Sigma_{\geq 1}}$ and its left adjoint $L_* : \mathcal{C}'^{\Sigma_{\geq 1}} \to \mathcal{C}^{\Sigma_{\geq 1}}$ preserve local equivalences.

Hence we can reduce to the case that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits.

Let $f : X \to Y$ be a morphism in $\mathcal{C}^{\Sigma_{\geq 1}}$ with $f_j : X_j \to Y_j$ an equivalence for every $j \leq n$ and $Z \in \mathcal{C}^{\Sigma_{\geq 1}}$.

We want to see that $(f \circ Z)_s$ and $(Z \circ f)_s$ are both equivalences for every $s \leq n$.

But we have

$$(f \circ Z)_s = \coprod_{k \geq 0} \Big(\coprod_{n_1 \amalg ... \amalg n_k = s} f_k \otimes \big(\bigotimes_{1 \leq j \leq k} Z_{n_j} \big) \Big)_{\Sigma_k} = \coprod_{s \geq k \geq 0} \Big(\coprod_{n_1 \amalg ... \amalg n_k = s} f_k \otimes \big(\bigotimes_{1 \leq j \leq k} Z_{n_j} \big) \Big)_{\Sigma_k}$$

and

$$(Z \circ f)_s = \coprod_{k \geq 0} \Big(\coprod_{n_1 \amalg ... \amalg n_k = s} Z_k \otimes \big(\bigotimes_{1 \leq j \leq k} f_{n_j} \big) \Big)_{\Sigma_k}.$$

□

Lemma 2.17. *Let $\mathcal{C}$ be a symmetric monoidal category compatible with the initial object that admits small colimits and let $n \in \mathbb{N}$.*

Let $X_1, ..., X_k$ be objects of $\mathcal{C}^{\Sigma_{\geq 1}}$ for some $k \geq 2$.
If $X_i \in \mathcal{C}^{\Sigma_{\geq n}}$ for some $1 \leq i \leq k$, then $X_1 \circ ... \circ X_k$ belongs to $\mathcal{C}^{\Sigma_{\geq n}}$.

Proof. By remark 1.6 $\mathcal{C}$ embeds symmetric monoidally into a symmetric monoidal category $\mathcal{C}'$ compatible with small colimits such that the embedding $\mathcal{C} \subset \mathcal{C}'$ preserves the initial object and admits a left adjoint L.

So we get a lax monoidal embedding $\mathcal{C}^{\Sigma_{\geq 1}} \subset \mathcal{C}'^{\Sigma_{\geq 1}}$ and an embedding $\mathcal{C}^{\Sigma_{\geq n}} \subset \mathcal{C}'^{\Sigma_{\geq n}}$.

Thus $X_1 \circ ... \circ X_k$ is the image of the corresponding composition product in $\mathcal{C}'^{\Sigma_{\geq 1}}$ under the functor $L_* : \mathcal{C}'^{\Sigma_{\geq 1}} \to \mathcal{C}^{\Sigma_{\geq 1}}$ induced by L.

As a left adjoint the functor L preserves the initial object so that the functor L_* restricts to a functor $\mathcal{C}'^{\Sigma_{\geq n}} \to \mathcal{C}^{\Sigma_{\geq n}}$.

Consequently we can reduce to the case that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits.

Let $X \in \mathcal{C}^{\Sigma_{\geq n}}, Y \in \mathcal{C}^{\Sigma_{\geq 1}}$. For every $s \geq 0$ we have

$$(X \circ Y)_s = \coprod_{k \geq 0} \Big(\coprod_{n_1 \amalg ... \amalg n_k = s} X_k \otimes \big(\bigotimes_{1 \leq j \leq k} Y_{n_j} \big) \Big)_{\Sigma_k} =$$

$$\coprod_{s \geq k \geq 0} \Big(\coprod_{n_1 \amalg ... \amalg n_k = s} X_k \otimes \big(\bigotimes_{1 \leq j \leq k} Y_{n_j} \big) \Big)_{\Sigma_k}$$

so that $X \circ Y \in \mathcal{C}^{\Sigma_{\geq n}}$.

Let $X \in \mathcal{C}^{\Sigma_{\geq 1}}, Y \in \mathcal{C}^{\Sigma_{\geq n}}$. For every $s \geq 0$ we have

$$(X \circ Y)_s = \coprod_{k \geq 0} \Big(\coprod_{n_1 \amalg ... \amalg n_k = s} X_k \otimes \big(\bigotimes_{1 \leq j \leq k} Y_{n_j} \big) \Big)_{\Sigma_k} =$$

$$\coprod_{k\geq 1}(\coprod_{n_1 \amalg ... \amalg n_k = s} X_k \otimes (\bigotimes_{1\leq j\leq k} Y_{n_j}))_{\Sigma_k}$$

so that $X \circ Y \in \mathcal{C}^{\Sigma_{\geq n}}$.

□

The rest of this section is devoted to the proofs of lemma 2.18 and lemma 2.19.

We start with constructing a canonical augmentation for every non-unital operad $\mathcal{O}$, whose unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence.

Lemma 2.18. *Let $\mathcal{C}$ be a symmetric monoidal category compatible with the zero object.*

The initial object of the category $\mathrm{Op}^{\mathrm{nu}}(\mathcal{C})$ that lies over $\mathrm{triv} \in \mathcal{C}^{\Sigma_{\geq 1}}$ is the zero object of the full subcategory $\{\mathrm{id}_{\mathbb{1}}\} \times_{\mathcal{C}_{\mathbb{1}/}} \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}) \subset \mathrm{Op}^{\mathrm{nu}}(\mathcal{C})$ spanned by the non-unital operads $\mathcal{O}$, whose unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence.

Proof. Set $\Sigma_{\geq 2} := \coprod_{n\geq 2} \mathrm{B}(\Sigma_n)$ and $\mathcal{C}^{\Sigma_{\geq 2}} := \mathrm{Fun}(\Sigma_{\geq 2}, \mathcal{C}) \simeq \prod_{n\geq 2} \mathrm{Fun}(\mathrm{B}(\Sigma_n), \mathcal{C})$.

Denote $\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}' \subset \mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}$ the full subcategory spanned by the symmetric sequences $\mathcal{O}$ concentrated in positive degrees under triv such that the induced morphism $\mathbb{1} \simeq \mathrm{triv}_1 \to \mathcal{O}_1$ in $\mathcal{C}$ is an equivalence.

We have a canonical equivalence $\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/} \simeq \mathcal{C}_{\mathbb{1}/} \times \mathcal{C}^{\Sigma_{\geq 2}}$ that restricts to an equivalence $\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}' \simeq \mathcal{C}^{\Sigma_{\geq 2}}$, under which triv corresponds to the initial object, which is the final object as $\mathcal{C}$ admits a zero object.

Consequently it is enough to see that the forgetful functor $\{\mathrm{id}_{\mathbb{1}}\} \times_{\mathcal{C}_{\mathbb{1}/}} \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}) \to \mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}'$ reflects the final object.

To see this we may reduce to the case that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits as $\mathcal{C}$ embeds symmetric monoidally into a symmetric monoidal category compatible with small colimits such that the embedding preserves the zero object (remark 1.6).

For $X, Y \in \mathcal{C}^{\Sigma_{\geq 1}}$ we have a natural equivalence $(X \circ Y)_1 \simeq X_1 \otimes Y_1$.

So the monoidal structure on $\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}$ induced by the composition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ restricts to a monoidal structure on $\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}'$.

Thus we have a canonical equivalence

$$\mathrm{Alg}(\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}') \simeq \mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}' \times_{\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}} \mathrm{Alg}(\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}) \simeq \mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}' \times_{\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}} \mathrm{Alg}(\mathcal{C}^{\Sigma_{\geq 1}}) \simeq$$

$$\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}' \times_{\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}} \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}) \simeq \{\mathrm{id}_{\mathbb{1}}\} \times_{\mathcal{C}_{\mathbb{1}/}} \mathrm{Op}^{\mathrm{nu}}(\mathcal{C})$$

over $\mathcal{C}^{\Sigma_{\geq 1}}_{\mathrm{triv}/}{}'$.

□

Lemma 2.19. *Let $\mathcal{C}$ be a preadditive symmetric monoidal category compatible with small colimits that admits small limits.*

The identity of $\mathcal{C}^{\Sigma}$ lifts to an oplax monoidal functor from the composition product on $\mathcal{C}^{\Sigma}$ to the cocomposition product on $\mathcal{C}^{\Sigma}$.

This oplax monoidal functor restricts to a monoidal equivalence on $\mathcal{C}^{\Sigma_{\geq 1}}$.

Thus the identity of $\mathcal{C}^\Sigma$ *yields a forgetful functor* $\mathrm{Coalg}(\mathcal{C}^\Sigma) \to \mathrm{CoOp}(\mathcal{C})$ *over* $\mathcal{C}^\Sigma$ *that restricts to an equivalence*

$$\mathrm{Coalg}(\mathcal{C}^{\Sigma_{\geq 1}}) \simeq \mathrm{CoOp}^{\mathrm{ncu}}(\mathcal{C})$$

and yields for every $\mathcal{Q} \in \mathrm{Coalg}(\mathcal{C}^\Sigma)$ *a forgetful functor*

$$\mathrm{Coalg}^{\mathrm{pd,conil}}_{\mathcal{Q}}(\mathcal{C}) \to \mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C})$$

over $\mathcal{C}$.

Proof. By remark 1.6 $\mathcal{C}^{\mathrm{op}}$ embeds symmetric monoidally into a preadditive symmetric monoidal category compatible with small colimits such that the embedding preserves small limits.

Turning to opposite categories $\mathcal{C}$ embeds symmetric monoidally into a preadditive symmetric monoidal category $\mathcal{C}'$ compatible with small limits such that the embedding $\mathcal{C} \subset \mathcal{C}'$ preserves small colimits.

The embedding $\mathcal{C} \subset \mathcal{C}'$ yields an oplax monoidal embedding $\mathcal{C}^\Sigma \subset \mathcal{C}'^\Sigma$ on cocomposition products.

The identity of $\mathcal{C}^\Sigma$ lifts to an oplax monoidal functor from the composition product to the cocomposition product if and only if the embedding $\iota : \mathcal{C}^\Sigma \subset \mathcal{C}'^\Sigma$ lifts to an oplax monoidal functor between monoidal categories from the composition product on $\mathcal{C}^\Sigma$ to the cocomposition product on $\mathcal{C}'^\Sigma$.

Moreover if this is shown, the oplax monoidal identity of $\mathcal{C}^\Sigma$ restricts to a monoidal equivalence on $\mathcal{C}^{\Sigma_{\geq 1}}$ if and only if the oplax monoidal embedding $\iota : \mathcal{C}^\Sigma \subset \mathcal{C}'^\Sigma$ restricts to a monoidal embedding $\mathcal{C}^{\Sigma_{\geq 1}} \subset \mathcal{C}'^{\Sigma_{\geq 1}}$.

In this case we get a forgetful functor

$$\mathrm{Coalg}(\mathcal{C}^\Sigma) \to \mathrm{Alg}((\mathcal{C}^{\mathrm{op}})^\Sigma)^{\mathrm{op}} = \mathrm{Op}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{CoOp}(\mathcal{C})$$

over $\mathcal{C}^\Sigma$ and a forgetful functor

$$\mathrm{LcoMod}_{\mathcal{Q}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{Q}}(\mathcal{C}'^{\mathrm{op}})^{\mathrm{op}} \simeq \mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C})$$

over $\mathcal{C}$. Moreover in this case the forgetful functor $\mathrm{Coalg}(\mathcal{C}^\Sigma) \to \mathrm{CoOp}(\mathcal{C})$ restricts to an equivalence

$$\mathrm{Coalg}(\mathcal{C}^{\Sigma_{\geq 1}}) \simeq \mathrm{Alg}((\mathcal{C}^{\mathrm{op}})^{\Sigma_{\geq 1}})^{\mathrm{op}} = \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{CoOp}^{\mathrm{ncu}}(\mathcal{C})$$

over $\mathcal{C}^{\Sigma_{\geq 1}}$.

For $\mathrm{X}, \mathrm{Y} \in \mathcal{C}^\Sigma$ the structure morphism $\iota(\mathrm{X} \otimes \mathrm{Y}) \to \iota(\mathrm{X}) \otimes \iota(\mathrm{Y})$ in $\mathcal{C}'^\Sigma$ of the embedding ι induces in degree $\mathrm{n} \in \mathbb{N}$ the canonical morphism

$$\alpha : \coprod_{k \in \mathbb{N}} \Big(\coprod_{\substack{n_1 \amalg \ldots \amalg n_k = n \\ n_i \cap n_j = \emptyset, i \neq j}} \mathrm{X}_k \otimes \big(\bigotimes_{1 \leq j \leq k} \mathrm{Y}_{n_j} \big) \Big)_{\Sigma_k} \to \prod_{k \in \mathbb{N}} \Big(\prod_{\substack{n_1 \amalg \ldots \amalg n_k = n \\ n_i \cap n_j = \emptyset, i \neq j}} \mathrm{X}_k \otimes \big(\bigotimes_{1 \leq j \leq k} \mathrm{Y}_{n_j} \big) \Big)^{\Sigma_k}$$

in $\mathcal{C}'$ induced by the norm map.

In the following we will show that α is an equivalence if Y_0 is an initial object of $\mathcal{C}$.

We will show that both canonical morphisms

$$\phi : \Big(\coprod_{\substack{n_1 \amalg \ldots \amalg n_k = n \\ n_i \cap n_j = \emptyset, i \neq j}} \mathrm{X}_k \otimes \big(\bigotimes_{1 \leq j \leq k} \mathrm{Y}_{n_j} \big) \Big)_{\Sigma_k} \to \Big(\coprod_{\substack{n_1 \amalg \ldots \amalg n_k = n \\ n_i \cap n_j = \emptyset, i \neq j}} \mathrm{X}_k \otimes \big(\bigotimes_{1 \leq j \leq k} \mathrm{Y}_{n_j} \big) \Big)^{\Sigma_k},$$

$$\Big(\coprod_{\substack{n_1 \amalg \ldots \amalg n_k = n \\ n_i \cap n_j = \emptyset, i \neq j}} \mathrm{X}_k \otimes \big(\bigotimes_{1 \leq j \leq k} \mathrm{Y}_{n_j} \big) \Big)^{\Sigma_k} \to \Big(\prod_{\substack{n_1 \amalg \ldots \amalg n_k = n \\ n_i \cap n_j = \emptyset, i \neq j}} \mathrm{X}_k \otimes \big(\bigotimes_{1 \leq j \leq k} \mathrm{Y}_{n_j} \big) \Big)^{\Sigma_k}$$

in $\mathcal{C}'$ are equivalences for every $k, n \in \mathbb{N}$ and $X, Y \in \mathcal{C}^{\Sigma}$ such that Y_0 is an initial object of $\mathcal{C}$.

We remark that for every $k, n \in \mathbb{N}$ the set of all k-tuples $(n_1, ..., n_k)$ of finite, non-empty sets that are pairwise disjoint and satisfy $n_1 \coprod ... \coprod n_k = n$ is finite.

This guarantees that for every $k, n \in \mathbb{N}$ the canonical morphism

$$\coprod_{\substack{n_1 \coprod ... \coprod n_k = n \\ n_i \cap n_j = \varnothing, i \neq j}} X_k \otimes (\bigotimes_{1 \leq j \leq k} Y_{n_j}) \to \prod_{\substack{n_1 \coprod ... \coprod n_k = n \\ n_i \cap n_j = \varnothing, i \neq j}} X_k \otimes (\bigotimes_{1 \leq j \leq k} Y_{n_j})$$

is an equivalence using that $\mathcal{C}'$ is preadditive and Y_0 is an initial object of $\mathcal{C}$.

To prove that ϕ is an equivalence, it is enough to check that the canonical Σ_k-action on $\coprod_{\substack{n_1 \coprod ... \coprod n_k = n \\ n_i \cap n_j = \varnothing, i \neq j}} \bigotimes_{1 \leq j \leq k} Y_{n_j}$ in $\mathcal{C}'$ and thus also the canonical Σ_k-action on $\coprod_{\substack{n_1 \coprod ... \coprod n_k = n \\ n_i \cap n_j = \varnothing, i \neq j}} X_k \otimes (\bigotimes_{1 \leq j \leq k} Y_{n_j})$ in $\mathcal{C}'$ is free:

The preadditivity of $\mathcal{C}$ implies that the canonical natural transformation of functors $\mathcal{C}' \to \mathrm{Fun}(B(\Sigma_k), \mathcal{C}')$ from the free to the cofree functor is an equivalence. Moreover by adjointness the free functor followed by the coinvariants and the cofree functor followed by the invariants are both canonically equivalent to the identity.

Denote W the set of k-tuples $(n_1, ..., n_k)$ of finite sets with $n_1 \coprod ... \coprod n_k = n$ and $n_i \cap n_j = \varnothing$ if $i \neq j$.

Denote $W' \subset W$ the subset of such k-tuples $(n_1, ..., n_k)$ with $n_1, ..., n_k$ not empty.

The set W carries a canonical Σ_k-action such that the canonical embedding $W \subset \Sigma^{\times k}$ gets Σ_k-equivariant, where $\Sigma^{\times k}$ carries the permutation action.

This Σ_k-action on W restricts to a free Σ_k-action on W' in the category of sets as this Σ_k-action on W' doesn't have fixpoints.

As Set is closed in Cat_∞ under finite coproducts, this Σ_k-action on W' is free in Cat_∞, too.

The functor $\Sigma^{\times k} \xrightarrow{Y^{\times k}} \mathcal{C}^{\times k} \to \mathcal{C}$ is Σ_k-equivariant and thus also the compositions $W \to \Sigma^{\times k} \xrightarrow{Y^{\times k}} \mathcal{C}^{\times k} \to \mathcal{C}$ and $W' \to \Sigma^{\times k} \xrightarrow{Y^{\times k}} \mathcal{C}^{\times k} \to \mathcal{C}$ are, where $\Sigma^{\times k}, \mathcal{C}^{\times k}$ carry the permutation actions and $\mathcal{C}$ the trivial action.

Via this Σ_k-equivariant functors we consider W, W' as Σ_k-objects of $\mathsf{Cat}_{\infty/\mathcal{C}}$ and have a Σ_k-equivariant morphism $W' \to W$ in $\mathsf{Cat}_{\infty/\mathcal{C}}$.

Being a right fibration that preserves small colimits, the forgetful functor $\mathsf{Cat}_{\infty/\mathcal{C}} \to \mathsf{Cat}_\infty$ preserves and reflects free Σ_k-objects so that the Σ_k-action on W' in $\mathsf{Cat}_{\infty/\mathcal{C}}$ is free.

We will complete the proof by constructing a canonical functor $\Psi : \mathsf{Cat}_{\infty/\mathcal{C}} \to \mathcal{C}$ that sends a functor $H : \mathcal{J} \to \mathcal{C}$ to $\mathrm{colim}(H)$ and preserves small colimits and so free Σ_k-objects.

Ψ sends the Σ_k-equivariant morphism $W' \to W$ in $\mathsf{Cat}_{\infty/\mathcal{C}}$ to a Σ_k-equivariant equivalence

$$\coprod_{\substack{n_1 \coprod ... \coprod n_k = n \\ n_i \cap n_j = \varnothing, i \neq j, n_i \neq \varnothing}} \bigotimes_{1 \leq j \leq k} Y_{n_j} \simeq \coprod_{\substack{n_1 \coprod ... \coprod n_k = n \\ n_i \cap n_j = \varnothing, i \neq j}} \bigotimes_{1 \leq j \leq k} Y_{n_j}.$$

To construct Ψ we make the following definition:

Denote $\overline{\mathsf{Cat}_{\infty/\mathcal{C}}} \to \mathsf{Cat}_\infty$ the cartesian fibration that classifies the functor $\mathrm{Fun}(-,\mathcal{C}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

The restriction of $\overline{\mathsf{Cat}_{\infty/\mathcal{C}}} \to \mathsf{Cat}_\infty$ to the wide subcategory of cartesian morphisms classifies the functor $\mathsf{Cat}_\infty(-,\mathcal{C}) \simeq \mathrm{Fun}(-,\mathcal{C})^{\simeq} : \mathsf{Cat}_\infty^{\mathrm{op}} \to \mathsf{Cat}_\infty$ and is thus canonically equivalent over Cat_∞ to the right fibration $\mathsf{Cat}_{\infty/\mathcal{C}} \to \mathsf{Cat}_\infty$.

Pulling back the full subcategory inclusion $* \to \mathsf{Cat}_\infty$ that hits the contractible category along the cartesian fibration $\overline{\mathsf{Cat}_{\infty/\mathcal{C}}} \to \mathsf{Cat}_\infty$ we get a full subcategory inclusion $\mathcal{C} \subset \overline{\mathsf{Cat}_{\infty/\mathcal{C}}}$.

This full subcategory inclusion $\mathcal{C} \subset \overline{\mathsf{Cat}_{\infty/\mathcal{C}}}$ admits a left adjoint Ψ that sends a functor $\mathrm{H} : \mathcal{J} \to \mathcal{C}$ to $\mathrm{colim}(\mathrm{H})$.

Being an object of $\overline{\mathsf{Cat}_{\infty/\mathcal{C}}}(\mathrm{H}, \mathrm{colim}(\mathrm{H})) \simeq \mathrm{Fun}(\mathcal{J},\mathcal{C})(\mathrm{H}, \delta(\mathrm{colim}(\mathrm{H})))$ the unit $\mathrm{H} \to \mathrm{colim}(\mathrm{H})$ in $\overline{\mathsf{Cat}_{\infty/\mathcal{C}}}$ corresponds to the unit $\mathrm{H} \to \delta(\mathrm{colim}(\mathrm{H}))$, where $\delta : \mathcal{C} \to \mathrm{Fun}(\mathcal{J},\mathcal{C})$ denotes the diagonal functor.

So for every $\mathrm{Z} \in \mathcal{C}$ the canonical map

$$\overline{\mathsf{Cat}_{\infty/\mathcal{C}}}(\mathrm{colim}(\mathrm{H}), \mathrm{Z}) \to \overline{\mathsf{Cat}_{\infty/\mathcal{C}}}(\mathrm{H}, \mathrm{Z})$$

is canonically equivalent to the equivalence

$$\mathcal{C}(\mathrm{colim}(\mathrm{H}), \mathrm{Z}) \to \mathrm{Fun}(\mathcal{J},\mathcal{C})(\delta(\mathrm{colim}(\mathrm{H})), \delta(\mathrm{Z})) \to \mathrm{Fun}(\mathcal{J},\mathcal{C})(\mathrm{H}, \delta(\mathrm{Z})).$$

We finally observe that the subcategory inclusion $\mathsf{Cat}_{\infty/\mathcal{C}} \subset \overline{\mathsf{Cat}_{\infty/\mathcal{C}}}$ preserves small colimits:

As the forgetful functor $\gamma : \overline{\mathsf{Cat}_{\infty/\mathcal{C}}} \to \mathsf{Cat}_\infty$ is a cartesian fibration and by definition every morphism of $\mathsf{Cat}_{\infty/\mathcal{C}}$ is γ-cartesian, this follows from the following facts:

The forgetful functor $\mathsf{Cat}_{\infty/\mathcal{C}} \to \mathsf{Cat}_\infty$ preserves small colimits and the functor $\mathrm{Fun}(-,\mathcal{C}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \mathsf{Cat}_\infty$ classified by γ preserves small limits.

□

2.3 Restricted L_∞-algebras

In this section we give the central definition of restricted L_∞-algebras and study their basic properties.

To define restricted L_∞-algebras we use the following universal property of theorem 2.20 that follows from proposition 3.22.

Theorem 2.20. *Let $\mathcal{C}$ be a symmetric monoidal category with initial tensorunit and $\mathcal{D}$ a preadditive category.*

Then the pullback

$$\mathrm{Fun}^{\amalg}(\mathcal{D},\mathcal{C}) \times_{\mathrm{Fun}(\mathcal{D},\mathcal{C})} \mathrm{Fun}(\mathcal{D},\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{Fun}^{\amalg}(\mathcal{D},\mathcal{C})$$

of the forgetful functor $\mathrm{Fun}(\mathcal{D},\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{Fun}(\mathcal{D},\mathcal{C})$ *to* $\mathrm{Fun}^{\amalg}(\mathcal{D},\mathcal{C})$ *is an equivalence.*

We apply theorem 2.20 especially to the following situation:

Let $\mathcal{H}$ be a unital Hopf operad on $\mathcal{C}$. Then composition with the forgetful functor

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})) \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$$

defines an equivalence

$$\mathrm{Fun}^{\amalg}(\mathcal{D},\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}))\times_{\mathrm{Fun}(\mathcal{D},\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}))}\mathrm{Fun}(\mathcal{D},\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})) \to \mathrm{Fun}^{\amalg}(\mathcal{D},\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})).$$

Given a commutative algebra A in a symmetric monoidal category $\mathcal{C}$ the category $\mathcal{C}_{/A}$ admits an induced symmetric monoidal structure such that we have a canonical equivalence $\mathrm{Calg}(\mathcal{C}_{/A}) \simeq \mathrm{Calg}(\mathcal{C})_{/A}$.

We define $\mathcal{C}^{\otimes}_{/A}$ as the pullback in Op_∞ of the cocartesian fibration of operads $(\mathcal{C}^{\otimes})^{\Delta^1} \to (\mathcal{C}^{\otimes})^{\{1\}}$ along $A : \mathcal{F}\mathrm{in}_* \to \mathcal{C}^{\otimes}$.

Dually given a cocommutative coalgebra A in $\mathcal{C}$ the category $\mathcal{C}_{A/}$ admits an induced symmetric monoidal structure such that we have a canonical equivalence $\mathrm{Cocoalg}(\mathcal{C}_{A/}) \simeq \mathrm{Cocoalg}(\mathcal{C})_{A/}$.

In the following we apply this to the canonical cocommutative coalgebra structure on the tensorunit of $\mathcal{C}$.

In this section let $\mathcal{C}$ be a preadditive symmetric monoidal category that admits small colimits and limits, where the colimits are preserved by the tensor product in each component, and $\mathcal{H}$ a unital Hopf operad in $\mathcal{C}$, where unital means that the tensorunit of the category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ is an initial object.

By remark 2.8 1. the category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ admits small colimits and the monadic forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ preserves small sifted colimits.

By remark 2.1 the category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}))$ admits small colimits, which are preserved by the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$.

Using theorem 2.20 we make the following definitions:

Definition 2.21.

- *The left adjoint* $\mathbb{1} \oplus - : \mathcal{C} \to \mathcal{C}_{\mathbb{1}/}$ *of the forgetful functor* $\mathcal{C}_{\mathbb{1}/} \to \mathcal{C}$ *uniquely lifts to a functor*

$$E : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C}_{\mathbb{1}/}) \simeq \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}.$$

We call E *the co-square zero extension.*

- *The free functor* $\mathcal{H} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ *uniquely lifts to a functor*

$$\mathrm{T} : \mathcal{C} \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}),$$

which we call tensoralgebra.

Given an object X of $\mathcal{C}$ the cocommutative coalgebra structure on the free $\mathcal{H}$-algebra $\mathcal{H}(\mathrm{X})$ looks the following way:

The unit of $\mathcal{H}(\mathrm{X})$ gives rise to morphisms

$$\mathcal{H}(\mathrm{X}) \simeq \mathcal{H}(\mathrm{X}) \otimes \mathbb{1} \to \mathcal{H}(\mathrm{X}) \otimes \mathcal{H}(\mathrm{X}),\ \mathcal{H}(\mathrm{X}) \simeq \mathbb{1} \otimes \mathcal{H}(\mathrm{X}) \to \mathcal{H}(\mathrm{X}) \otimes \mathcal{H}(\mathrm{X})$$

in $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ that induce a morphism $\alpha : \mathcal{H}(\mathrm{X}) \coprod \mathcal{H}(\mathrm{X}) \to \mathcal{H}(\mathrm{X}) \otimes \mathcal{H}(\mathrm{X})$ in $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$.

The diagonal $\mathrm{X} \to \mathrm{X} \oplus \mathrm{X}$ and the unique morphism $\mathrm{X} \to 0$ in $\mathcal{C}$ yield the morphims

$$\Delta : \mathcal{H}(\mathrm{X}) \to \mathcal{H}(\mathrm{X} \oplus \mathrm{X}) \simeq \mathcal{H}(\mathrm{X}) \coprod \mathcal{H}(\mathrm{X}) \xrightarrow{\alpha} \mathcal{H}(\mathrm{X}) \otimes \mathcal{H}(\mathrm{X})$$

and $\epsilon : \mathcal{H}(\mathrm{X}) \to \mathcal{H}(0) \simeq \mathbb{1}_{\mathcal{C}}$ that are the comultiplication and counit of the cocommutative coalgebra $\mathcal{H}(\mathrm{X})$.

If $\mathcal{C}$ is additive and $\mathcal{H}$ is the Hopf operad, whose algebras are associative algebras, by prop. 2.32 the tensoralgebra $\mathrm{T} : \mathcal{C} \to \mathrm{Bialg}(\mathcal{C})$ induces a functor $\mathcal{C} \to \mathrm{Hopf}(\mathcal{C})$.

Observation 2.22. *The tensoralgebra* T *factors as*

$$\mathcal{C} \xrightarrow{\mathrm{E}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathcal{F}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}),$$

where $\mathcal{F}$ *is left adjoint to the forgetful functor* $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

Proof. The category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ admits small colimits. The composition

$$\phi : \mathcal{C}_{\mathbb{1}/} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})_{\mathcal{H}(\mathbb{1})/} \xrightarrow{-\coprod_{\mathcal{H}(\mathbb{1})} \mathbb{1}} \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})_{\mathbb{1}/} \simeq \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$$

is left adjoint to the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}_{\mathbb{1}/}$.

The adjunction $\mathcal{H} : \mathcal{C} \rightleftarrows \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ factors as

$$\phi \circ (- \oplus \mathbb{1}) : \mathcal{C} \rightleftarrows \mathcal{C}_{\mathbb{1}/} \rightleftarrows \mathrm{Alg}_{\mathcal{H}}(\mathcal{C}),$$

where the right adjoints are symmetric monoidal functors so that the left adjoints are oplax symmetric monoidal functors.

So the adjunction $\phi : \mathcal{C}_{\mathbb{1}/} \rightleftarrows \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ gives rise to an adjunction

$$\mathcal{F} : \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \simeq \mathrm{Cocoalg}(\mathcal{C}_{\mathbb{1}/}) \rightleftarrows \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})).$$

The composition $\mathcal{C} \xrightarrow{\mathrm{E}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathcal{F}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ lifts the free functor $\mathcal{H} : \mathcal{C} \xrightarrow{-\otimes \mathbb{1}} \mathcal{C}_{\mathbb{1}/} \xrightarrow{\phi} \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ and is thus equivalent to T.

□

By remark 2.2 adding the tensorunit defines a localization $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ with left adjoint the functor Γ that takes the cofiber of the coaugmentation. If $\mathcal{C}$ is additive, this localization is an equivalence.

Denote

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \subset \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$$

the full subcategory spanned by the bialgebras, whose underlying coaugmented cocommutative coalgebra belongs to $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

So if $\mathcal{C}$ is additive, we have $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} = \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$.

Remark 2.23.

1. *The full subcategory* $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \subset \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ *is closed under small sifted colimits.*

2. *If* $\mathcal{C}$ *is presentable, the categories* $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}), \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ *are presentable and the embedding* $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \subset \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ *admits a left and right adjoint.*

Proof. The categories $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ and so $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}))$ admit small colimits and the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ preserves those by remark 2.8.

The forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ preserves small sifted colimits as the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}_{\mathbb{1}/}$ does by remark 2.8.

By remark 2.1 the categories $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ admit small colimits and by remark 2.2 the localization $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ preserves small colimits.

This shows 1.

If $\mathcal{C}$ is presentable, the categories $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}, \mathrm{Cocoalg}(\mathcal{C})$ and so also $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ are presentable by proposition 6.83 and remark 2.1.

By remark 2.8 the category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ is presentable.

By observation 2.22 the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ admits a left adjoint.

Hence $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ is presentable being the pullback in Pr^{R} of the right adjoint functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ along the right adjoint functor $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ and so the forgetful functor

$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}$ admits a left adjoint.

As the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ preserves small sifted colimits, by 1. the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}$ preserves small sifted colimits, too.

So by the theorem of Barr-Beck both forgetful functors $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}, \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}$ are monadic.

So we have a commutative square

$$\begin{array}{ccc}
\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} & \longrightarrow & \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} & \longrightarrow & \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/},
\end{array}$$

where both vertical functors are monadic, by 1. the top functor preserves small sifted colimits and by remark 2.2 the bottom functor preserves small colimits. Thus also the top functor preserves small colimits, which completes 2.

□

Remark 2.24.

- *For every* $X \in \mathcal{C}$ *the unit* $E(X) \to \Gamma(E(X)) \oplus \mathbb{1}$ *in* $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ *lies over the identity of* $X \oplus \mathbb{1}$.

  Thus the functor $E : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ *induces a functor*

  $$\mathrm{triv} : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$$

  that is the unique section of the forgetful functor $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \to \mathcal{C}$ *by corollary 3.24.*

  We call triv *the trivial cocommutative coalgebra functor.*

- *For every* $X \in \mathcal{C}$ *the unit* $\mathcal{H}(X) \to \Gamma(\mathcal{H}(X)) \oplus \mathbb{1}$ *in* $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ *lies over the canonical equivalence* $\oplus_{i \geq 0} \mathcal{H}_i \otimes_{\Sigma_i} X^{\otimes i} \simeq (\oplus_{i \geq 1} \mathcal{H}_i \otimes_{\Sigma_i} X^{\otimes i}) \oplus \mathbb{1}$ *in* $\mathcal{C}$.

  Thus the tensoralgebra $T : \mathcal{C} \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ *factors as* $\mathcal{C} \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \subset \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$.

By 2.2.3 the functor $\mathrm{triv} : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}$ *admits a right adjoint* Prim.

This has the following consequence:

Remark 2.25. *The functor* $T : \mathcal{C} \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ *is left adjoint to the composition* $\mathcal{P} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \xrightarrow{\mathrm{Prim}} \mathcal{C}$.

Proof. Denote Γ the left adjoint of the embedding $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

By observation 2.22 the tensoralgebra $T : \mathcal{C} \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \subset \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$ factors as $\mathcal{C} \xrightarrow{E} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathcal{F}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})$, where $\mathcal{F}$ is left adjoint to the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

For $X \in \mathcal{C}, Y \in \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ we have a canonical equivalence

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}(T(X), Y) \simeq \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}(E(X), Y) \simeq$$

$$\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}(\mathrm{triv}(X), \Gamma(Y)) \simeq \mathcal{C}(X, \mathrm{Prim}(\Gamma(Y))).$$

□

Now we are ready to give the central definition of this section:

Definition 2.26.

We define the restricted L_∞*-monad* $\mathcal{L}$ *associated to* $\mathcal{H}$ *as the monad associated to the adjunction* $T : \mathcal{C} \rightleftarrows \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} : \mathcal{P}$ *and call* $\mathcal{L}$*-algebras restricted* L_∞*-*$\mathcal{H}$*-algebras.*

We set $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{L}}(\mathcal{C})$.

We are especially interested in the case that $\mathcal{H}$ is the Hopf operad, whose algebras are associative algebras in $\mathcal{C}$.

For this choice of $\mathcal{H}$ we write $\mathrm{Lie}(\mathcal{C})$ for $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$ and call restricted L_∞-$\mathcal{H}$-algebras restricted L_∞-algebras.

If $\mathcal{C}$ is additive, by prop. 2.32 the tensoralgebra $T : \mathcal{C} \to \mathrm{Bialg}(\mathcal{C})$ induces a functor $\mathcal{C} \to \mathrm{Hopf}(\mathcal{C})$ so that $\mathcal{L}$ is the monad associated to the adjunction $T : \mathcal{C} \rightleftarrows \mathrm{Hopf}(\mathcal{C}) : \mathcal{P}$.

More generally $\mathcal{H}$ is the Hopf operad, whose algebras are E_k-algebras for some $k \geq 0$ or $k = \infty$.

By theorem 5.62 the functor $\mathcal{P} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathcal{C}$ lifts to a functor

$$\bar{\mathcal{P}} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$$

that satisfies the following universal property:

Remark 2.27. *(Universal property of* $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$*)*

Every lift $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathcal{D}$ *of* $\mathcal{P} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathcal{C}$ *along a monadic functor* $\mathcal{D} \to \mathcal{C}$ *factors as*

$$\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{D}$$

for a unique functor $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{D}$ *over* $\mathcal{C}$.

This may be interpreted by saying that the structure of a monadic restricted L_∞*-*$\mathcal{H}$*-algebra is the finest structure the primitive elements can be endowed with.*

By remark 2.23 the category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ admits geometric realizations.

Thus by the proof of [18] lemma 4.7.4.13. the functor $\bar{\mathcal{P}} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$ admits a left adjoint

$$\mathcal{U} : \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}},$$

which we call the enveloping bialgebra functor. By adjointness we have a canonical equivalence $\mathcal{U} \circ \mathcal{L} \simeq T$.

If $\mathcal{C}$ is additive and $\mathcal{H}$ is the Hopf operad, whose algebras are associative algebras, by prop. 2.32 the enveloping bialgebra $\mathcal{U} : \mathrm{Lie}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ induces a functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Hopf}(\mathcal{C})$.

Remark 2.28.

The enveloping bialgebra functor $\mathcal{U} : \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ *is unique with the following properties:*

- $\mathcal{U}$ *admits a right adjoint* $\bar{\mathcal{P}}$.
- $\mathcal{U} \circ \mathcal{L} : \mathcal{C} \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ *lifts the free functor* $\mathcal{H} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$.
- *A weak version of the Milnor-Moore theorem holds:*

  The restriction $\mathcal{L}(\mathcal{C}) \subset \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \xrightarrow{\mathcal{U}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ *to free restricted* L_∞*-algebras is fully faithful.*

Proof. By the uniqueness of lifts $\mathcal{U} \circ \mathcal{L} : \mathcal{C} \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ lifts the free functor $\mathcal{H} : \mathcal{C} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ if and only if $\mathcal{U} \circ \mathcal{L}$ is the tensoralgebra, which by adjointness is equivalent to the condition that $\bar{\mathcal{P}}$ lifts the functor $\mathcal{P}$.

The restriction $\mathcal{L}(\mathcal{C}) \subset \mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \xrightarrow{\mathcal{U}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ is fully faithful if and only if the natural transformation $\alpha : \mathcal{L} \to \bar{\mathcal{P}} \circ T$ adjoint to the unit $\mathrm{id} \to \mathcal{P} \circ T$ is an equivalence, which by theorem 5.62 is equivalent to the condition that $\bar{\mathcal{P}} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$ satifies the universal property of 2.27:

For every $X, Y \in \mathcal{C}$ the map

$$\mathrm{Lie}_{\mathcal{H}}(\mathcal{C})(\mathcal{L}(X), \mathcal{L}(Y)) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}(T(X), T(Y))$$

induced by $\mathcal{U}$ is equivalent to the map

$$\mathrm{Lie}_{\mathcal{H}}(\mathcal{C})(\mathcal{L}(X), \mathcal{L}(Y)) \simeq \mathcal{C}(X, \mathcal{L}(Y)) \to$$

$$\mathcal{C}(X, \mathcal{P}(T(Y))) \simeq \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}(T(X), T(Y))$$

induced by the morphism $\mathcal{L}(Y) \to \mathcal{P}(T(Y))$ underlying the morphism $\alpha(Y) : \mathcal{L}(Y) \to \bar{\mathcal{P}}(T(Y))$.

□

The following example given by the theorem of Milnor-Moore is the motivating example for the definition of restricted L_∞-algebras:

Example 2.29. *Let* K *be a field.*

- *Denote* Lie_K *the category of restricted Lie algebras over* K *which are nothing than usual Lie algebras if* K *has char. zero.*

  We have adjunctions $\mathcal{L} : \mathrm{Mod}_K \rightleftarrows \mathrm{Lie}_K$ *and* $\mathcal{U} : \mathrm{Lie}_K \rightleftarrows \mathrm{Hopf}_K : \bar{\mathcal{P}}$ *between* K*-vector spaces and restricted Lie algebras over* K *and restricted Lie algebras over* K *and Hopf algebras over* K*, where* $\mathcal{L}$ *denotes the free restricted Lie algebra,* $\mathcal{U}$ *the restricted enveloping Hopf algebra and* $\bar{\mathcal{P}}$ *the primitive elements with its natural structure of a restricted Lie algebra.*

  Composing both adjunctions we get the adjunction $T : \mathrm{Mod}_K \rightleftarrows \mathrm{Hopf}_K : \mathcal{P}$*, where* T *denotes the tensoralgebra and* $\mathcal{P}$ *the primitive elements.*

  By remark 4.37 the forgetful functor $\mathrm{Lie}_K \to \mathrm{Mod}_K$ *is a monadic functor.*

  By the theorem of Milnor-Moore [20] 5.18 and 6.11. the functor $\mathcal{U}$ *is fully faithful. Thus the unit* $\mathrm{id} \to \bar{\mathcal{P}} \circ \mathcal{U}$ *is an isomorphism and so gives rise to an isomorphism* $\mathcal{L} \cong \bar{\mathcal{P}} \circ \mathcal{U} \circ \mathcal{L} \cong \bar{\mathcal{P}} \circ T$.

  So the functor $\bar{\mathcal{P}} : \mathrm{Hopf}_K \to \mathrm{Lie}_K$ *exhibits* Lie_K *as the category of restricted* L_∞*-algebras in* Mod_K.

- *By taking simplical objects we get the following example:*

  Denote $\mathsf{s}\mathrm{Mod}_K$ *the category of simplicial* K*-vector spaces,* $\mathsf{s}\mathrm{Lie}_K$ *the category of simplicial restricted Lie algebras over* K *and* $\mathsf{s}\mathrm{Hopf}_K \simeq \mathrm{Hopf}(\mathsf{s}\mathrm{Mod}_K)$ *the category of simplicial Hopf algebras over* K.

  As the functor $\bar{\mathcal{P}} : \mathrm{Hopf}_K \to \mathrm{Lie}_K$ *exhibits* Lie_K *as the category of restricted* L_∞*-algebras in* Mod_K*, the induced functor* $\mathsf{s}\bar{\mathcal{P}} : \mathsf{s}\mathrm{Hopf}_K \to \mathsf{s}\mathrm{Lie}_K$ *exhibits* $\mathsf{s}\mathrm{Lie}_K$ *as the category of restricted* L_∞*-algebras in* $\mathsf{s}\mathrm{Mod}_K$.

- *From 1. we also get the following example:*
  Assume that K *has char. zero.*
  Denote Ch_K *the symmetric monoidal category of chain complexes over* K *and* dgLie_K *the category of dg-Lie algebras over* K.
  The adjunction

  $$T : \mathsf{Ch}_K \rightleftarrows \mathrm{Hopf}(\mathsf{Ch}_K) : \mathcal{P}$$

factors as the free Lie algebra adjunction $\mathcal{L} : \mathsf{Ch}_K \rightleftarrows \mathrm{dgLie}_K$ *followed by the adjunction*

$$\mathcal{U} : \mathrm{dgLie}_K \rightleftarrows \mathrm{Hopf}(\mathsf{Ch}_K) : \bar{\mathcal{P}},$$

where $\mathcal{U}$ *takes the enveloping bialgebra and* $\bar{\mathcal{P}}$ *the primitive elements.* *As* dgLie_K *is the category of algebras over the Lie operad, the forgetful functor* $\mathrm{dgLie}_K \to \mathsf{Ch}_K$ *is monadic.*

The functor $\bar{\mathcal{P}} : \mathrm{Hopf}(\mathsf{Ch}_K) \to \mathrm{dgLie}_K$ *exhibits* dgLie_K *as the category of restricted* L_∞*-algebras in* Ch_K *as the unit* $\mathrm{id} \to \bar{\mathcal{P}} \circ \mathcal{U}$ *of the adjunction* $\mathcal{U} : \mathrm{dgLie}_K \rightleftarrows \mathrm{Hopf}(\mathsf{Ch}_K) : \bar{\mathcal{P}}$ *is an isomorphism:*

We have a symmetric monoidal functor $\chi : \mathsf{Ch}_K \to \mathrm{Mod}_K$, $A \mapsto \bigoplus_{i \in \mathbb{Z}} A_i$ *that preserves small colimits and finite limits (as it preserves kernels). Moreover* χ *is conservative as it preserves kernels and cokernels and a chain complex* A *vanishes if* $\bigoplus_{i \in \mathbb{Z}} A_i$ *does.*
χ *yields functors* $\mathrm{dgLie}_K \to \mathrm{Lie}_K$ *and* $\mathrm{Hopf}(\mathsf{Ch}_K) \to \mathrm{Hopf}_K$.
Preserving small colimits χ *yields a commutative square*

$$\begin{array}{ccc} \mathrm{dgLie}_K & \xrightarrow{\mathcal{U}} & \mathrm{Hopf}(\mathsf{Ch}_K) \\ \downarrow & & \downarrow \\ \mathrm{Lie}_K & \xrightarrow{\mathcal{U}} & \mathrm{Hopf}_K. \end{array}$$

As χ *preserves finite limits, this square induces a commutative square*

$$\begin{array}{ccc} \mathrm{Hopf}(\mathsf{Ch}_K) & \xrightarrow{\bar{\mathcal{P}}} & \mathrm{dgLie}_K \\ \downarrow & & \downarrow \\ \mathrm{Hopf}_K & \xrightarrow{\bar{\mathcal{P}}} & \mathrm{Lie}_K. \end{array}$$

So the functor $\chi : \mathsf{Ch}_K \to \mathrm{Mod}_K$ *sends the unit of the adjunction* $\mathcal{U} : \mathrm{dgLie}_K \rightleftarrows \mathrm{Hopf}(\mathsf{Ch}_K) : \bar{\mathcal{P}}$ *to the unit of the adjunction* $\mathcal{U} : \mathrm{Lie}_K \rightleftarrows \mathrm{Hopf}_K : \bar{\mathcal{P}}$*, which is an isomorphism by the theorem of Milnor Moore [20] 5.18 and 6.11.*

Remark 2.30. *If* $\mathcal{C}$ *is presentable, the category* $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$ *is presentable and the forgetful functor* $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ *is accessible.*

Proof. If $\mathcal{C}$ is presentable, by remark 2.23 the category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ is presentable.

So the right adjoint functor $\mathcal{P} : \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathcal{C}$ is accessible and the restricted L_∞-monad $\mathcal{L} \simeq \mathcal{P} \circ T$ is an accessible monad.

Thus by proposition 6.84 3. the category $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{L}}(\mathcal{C})$ is presentable and the forgetful functor $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ is accessible.

□

Remark 2.31.
If $\mathcal{C}$ *is presentable, we have a forgetful functor*

$$\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$$

right adjoint to the composition $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \xrightarrow{\mathcal{U}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$.

Proof. If $\mathcal{C}$ is presentable, by remark 2.23 the category $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}}$ is presentable and the forgetful functor $\mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ preserves small colimits and thus admits a right adjoint R by the adjoint functor theorem.

We have a commutative diagram

$$\begin{array}{ccccc} \mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) & \xrightarrow{\mathrm{R}} & \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} & \xrightarrow{\bar{\mathcal{P}}} & \mathrm{Alg}_{\mathcal{L}}(\mathcal{C}) \\ & \searrow & \downarrow{\scriptstyle \mathcal{P}} & \swarrow & \\ & & \mathcal{C}. & & \end{array}$$

The composition $\mathrm{Lie}_{\mathcal{H}}(\mathcal{C}) \xrightarrow{\mathcal{U}} \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ is left adjoint to the functor $\bar{\mathcal{P}} \circ \mathrm{R} : \mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathrm{Bialg}_{\mathcal{H}}(\mathcal{C})^{\mathrm{red}} \to \mathrm{Lie}_{\mathcal{H}}(\mathcal{C})$.

□

If $\mathcal{C}$ is additive, the enveloping bialgebra functor $\mathcal{U} : \mathrm{Lie}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ induces a functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Hopf}(\mathcal{C}) \subset \mathrm{Bialg}(\mathcal{C})$ by the next proposition 2.32:

Proposition 2.32. *If $\mathcal{C}$ is additive, the enveloping bialgebra functor*

$$\mathcal{U} : \mathrm{Lie}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C}) \simeq \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C}))$$

induces a functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Hopf}(\mathcal{C}) = \mathrm{Grp}(\mathrm{Cocoalg}(\mathcal{C})) \subset \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C}))$.

Proof. The full subcategory $\mathrm{Hopf}(\mathcal{C}) \subset \mathrm{Bialg}(\mathcal{C})$ is closed under small sifted colimits.

Hence the full subcategory of $\mathrm{Lie}(\mathcal{C})$ spanned by those restricted L_∞-algebras, whose enveloping bialgebra is a Hopf algebra, is closed under small sifted colimits.

As $\mathrm{Lie}(\mathcal{C})$ is generated under small sifted colimits by the free restricted L_∞-algebras, is is enough to see that for every $\mathrm{X} \in \mathcal{C}$ the enveloping bialgebra $\mathcal{U}(\mathcal{L}(\mathrm{X})) \simeq \mathrm{T}(\mathrm{X})$ is a Hopf algebra.

We show the following more general result without assuming that $\mathcal{C}$ is additive:

Let $\mathrm{X} \in \mathcal{C}$. Denote $\mathrm{pr}_1, \mathrm{pr}_2 : \mathrm{X} \times \mathrm{X} \to \mathrm{X}$ the projections and $\mu : \mathrm{X} \times \mathrm{X} \simeq \mathrm{X} \coprod \mathrm{X} \to \mathrm{X}$ the codiagonal.

If the canonical morphism $\alpha : \mathrm{X} \times \mathrm{X} \xrightarrow{(\mathrm{pr}_1, \mu)} \mathrm{X} \times \mathrm{X}$ is an equivalence, i.e. $\mathrm{X} \in \mathcal{C} \simeq \mathrm{Cmon}(\mathcal{C})$ is a group object, then $\mathrm{T}(\mathrm{X}) \in \mathrm{Bialg}(\mathcal{C})$ is a Hopf algebra.

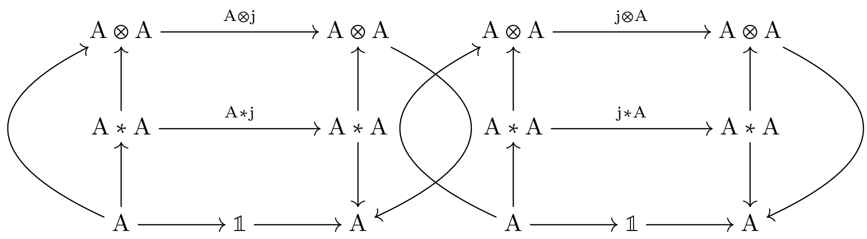
For this by remark 2.5 it is enough to check that $\mathrm{T}(\mathrm{X})$ admits an antipode.

Denote $\mathrm{i} : \mathrm{X} \simeq \mathrm{X} \times 0 \to \mathrm{X} \times \mathrm{X} \xrightarrow{\alpha^{-1}} \mathrm{X} \times \mathrm{X} \xrightarrow{\mathrm{pr}_2} \mathrm{X}$ the inverse of X.

The morphism $\mathrm{j} := \mathrm{T}(\mathrm{i}) : \mathrm{T}(\mathrm{X}) \to \mathrm{T}(\mathrm{X})$ is an antipode for $\mathrm{A} := \mathrm{T}(\mathrm{X})$ as the commutative squares

$$\begin{array}{ccccc} \mathrm{X} \times \mathrm{X} & & \xrightarrow{\mathrm{X} \times \mathrm{i}} & & \mathrm{X} \times \mathrm{X} \\ \uparrow & & & & \downarrow{\scriptstyle \mu} \\ \mathrm{X} & \longrightarrow & 0 & \longrightarrow & \mathrm{X} \end{array} \qquad \begin{array}{ccccc} \mathrm{X} \times \mathrm{X} & & \xrightarrow{\mathrm{i} \times \mathrm{X}} & & \mathrm{X} \times \mathrm{X} \\ \uparrow & & & & \downarrow{\scriptstyle \mu} \\ \mathrm{X} & \longrightarrow & 0 & \longrightarrow & \mathrm{X} \end{array}$$

in $\mathcal{C}$ give rise to commutative diagrams



in $\mathcal{C}$, where $*$ denotes the coproduct in $\mathrm{Alg}(\mathcal{C})$.

$\square$

2.4 Functoriality of restricted L_∞-algebras

In this subsection we discuss the functoriality of the category $\mathrm{Lie}(\mathcal{C})$ of restricted L_∞-algebras in a preadditive presentably symmetric monoidal category $\mathcal{C}$.

We show that a right adjoint lax symmetric monoidal functor $G : \mathcal{D} \to \mathcal{C}$ gives rise to a commutative square

$$\begin{array}{ccc} \mathrm{Lie}(\mathcal{D}) & \longrightarrow & \mathrm{Lie}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C} \end{array}$$

of right adjoints.

Moreover if G is fully faithful, symmetric monoidal and preserves filtered colimits, the induced functor $\mathrm{Lie}(\mathcal{D}) \to \mathrm{Lie}(\mathcal{C})$ is fully faithful, too.

Let $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint lax symmetric monoidal functor between preadditive presentably symmetric monoidal categories.

G gives rise to a lax symmetric monoidal functor $\mathrm{Alg}(G) : \mathrm{Alg}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C})$ that admits a left adjoint by the adjoint functor theorem.

The left adjoint $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{D})$ lifts canonically to an oplax symmetric monoidal functor and so yields a functor $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{D})$ between presentable categories that fits into a commutative square

$$\begin{array}{ccc} \mathrm{Bialg}(\mathcal{C}) & \longrightarrow & \mathrm{Bialg}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}(\mathcal{D}). \end{array} \tag{2}$$

The functor $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{D})$ admits a right adjoint G' by the adjoint functor theorem.

We have a commutative square

$$\begin{array}{ccc} \mathrm{Alg}(\mathcal{D}) & \xrightarrow{\mathrm{Alg}(G)} & \mathrm{Alg}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{G} & \mathcal{C} \end{array}$$

of right adjoints corresponding to a commutative square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}(\mathcal{D}) \end{array} \tag{3}$$

of left adjoints.

By theorem 2.20 this square yields a commutative square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{D} \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ \mathrm{Bialg}(\mathcal{C}) & \longrightarrow & \mathrm{Bialg}(\mathcal{D}) \end{array} \tag{4}$$

of left adjoints corresponding to a commutative square

$$\begin{array}{ccc} \mathrm{Bialg}(\mathcal{D}) & \xrightarrow{\mathrm{G}'} & \mathrm{Bialg}(\mathcal{C}) \\ \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle \mathcal{P}} \\ \mathcal{D} & \xrightarrow{\mathrm{G}} & \mathcal{C} \end{array}$$

of right adjoints.

By theorem 5.62 this square gives rise to a commutative square

$$\begin{array}{ccc} \mathrm{Lie}(\mathcal{D}) & \longrightarrow & \mathrm{Lie}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{D} & \longrightarrow & \mathcal{C}. \end{array} \tag{5}$$

If G preserves filtered colimits, G preserves arbitrary coproducts.

So square 3 yields a commutative square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathrm{G}} & \mathcal{C} \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{D}) & \xrightarrow{\mathrm{Alg(G)}} & \mathrm{Alg}(\mathcal{C}). \end{array}$$

If G is symmetric monoidal, the functor $\mathrm{Alg}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C})$ is symmetric monoidal so that square 2 yields a commutative square

$$\begin{array}{ccc} \mathrm{Bialg}(\mathcal{D}) & \longrightarrow & \mathrm{Bialg}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{D}) & \longrightarrow & \mathrm{Alg}(\mathcal{C}). \end{array}$$

Hence square 4 yields a commutative square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathrm{G}} & \mathcal{C} \\ \downarrow{\scriptstyle \mathrm{T}} & & \downarrow{\scriptstyle \mathrm{T}} \\ \mathrm{Bialg}(\mathcal{D}) & \xrightarrow{\mathrm{G}'} & \mathrm{Bialg}(\mathcal{C}). \end{array}$$

So we obtain a commutative square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\mathrm{G}} & \mathcal{C} \\ \downarrow{\scriptstyle \mathrm{T}} & & \downarrow{\scriptstyle \mathrm{T}} \\ \mathrm{Bialg}(\mathcal{D}) & \xrightarrow{\mathrm{G}'} & \mathrm{Bialg}(\mathcal{C}) \\ \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle \mathcal{P}} \\ \mathcal{D} & \xrightarrow{\mathrm{G}} & \mathcal{C}. \end{array}$$

Thus the monad $\mathcal{L}$ on $\mathcal{C}$ restricts to corresponding monad $\mathcal{L}$ on $\mathcal{D}$.

Hence square 5 is a pullback square so that the functor $\mathrm{Lie}(\mathcal{D}) \to \mathrm{Lie}(\mathcal{C})$ is fully faithful if $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ is.

In the following we discuss some examples for $\mathrm{G} : \mathcal{D} \to \mathcal{C}$.

Example 2.33.

- *By [9] theorem 4.6. we have symmetric monoidal localizations on the category* Pr^{L} *of presentable categories and left adjoint functors with local objects the stable, additive or preadditive presentable categories.*

  The corresponding localization functors send a presentable category $\mathcal{C}$ *to spectra objects in* $\mathcal{C}$*, abelian group objects in* $\mathcal{C}$ *respectively commutative monoids in* $\mathcal{C}$.

  So given a presentably symmetric monoidal category $\mathcal{C}$ *we obtain symmetric monoidal functors* $\mathrm{Cmon}(\mathcal{C}) \to \mathrm{Cgrp}(\mathcal{C}) \to \mathrm{Sp}(\mathcal{C})$ *with lax symmetric monoidal right adjoints* $\mathrm{Sp}(\mathcal{C}) \to \mathrm{Cgrp}(\mathcal{C}) \to \mathrm{Cmon}(\mathcal{C})$.

  So we get forgetful functors

  $$\mathrm{Lie}(\mathrm{Sp}(\mathcal{C})) \to \mathrm{Lie}(\mathrm{Cgrp}(\mathcal{C})) \to \mathrm{Lie}(\mathrm{Cmon}(\mathcal{C})).$$

  Especially we get forgetful functors

  $$\mathrm{Lie}(\mathrm{Sp}) \to \mathrm{Lie}(\mathrm{Cgrp}(\mathcal{S})) \to \mathrm{Lie}(\mathrm{Cmon}(\mathcal{S})).$$

- *Let* $\mathcal{C}$ *be a preadditive presentably symmetric monoidal category and* $\mathrm{A} \to \mathrm{B}$ *a map of commutative algebras in* $\mathcal{C}$.

  The lax symmetric monoidal forgetful functor $\mathrm{LMod}_{\mathrm{B}}(\mathcal{C}) \to \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})$ *yields a forgetful functor* $\mathrm{Lie}(\mathrm{LMod}_{\mathrm{B}}(\mathcal{C})) \to \mathrm{Lie}(\mathrm{LMod}_{\mathrm{A}}(\mathcal{C}))$.

- *The full subcategory inclusion* $\mathcal{S} \subset \mathsf{Cat}_\infty$ *admits a left adjoint that inverts all morphisms of a category and admits a right adjoint that takes the maximal subspace of a category.*

  Thus the full subcategory inclusion $\mathcal{S} \subset \mathsf{Cat}_\infty$ *gives rise to a symmetric monoidal embedding* $\mathrm{Cmon}(\mathcal{S}) \subset \mathrm{Cmon}(\mathsf{Cat}_\infty)$ *left adjoint to a lax symmetric monoidal functor* $\mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathrm{Cmon}(\mathcal{S})$ *and right adjoint to an oplax symmetric monoidal functor* $\mathrm{Cmon}(\mathsf{Cat}_\infty) \to \mathrm{Cmon}(\mathcal{S})$.

  So we get a localization $\mathrm{Lie}(\mathrm{Cmon}(\mathcal{S})) \subset \mathrm{Lie}(\mathrm{Cmon}(\mathsf{Cat}_\infty))$ *and a right adjoint functor* $\mathrm{Lie}(\mathrm{Cmon}(\mathsf{Cat}_\infty)) \to \mathrm{Lie}(\mathrm{Cmon}(\mathcal{S}))$.

3 Lifting the tensoralgebra

Given a nice preadditive symmetric monoidal category $\mathcal{C}$ we constructed a monad $\mathcal{L}$ on $\mathcal{C}$, whose algebras we called restricted L_∞-algebras (def. 2.26).

The monad $\mathcal{L}$ was the monad associated to an adjunction $T : \mathcal{C} \rightleftarrows \mathrm{Bialg}(\mathcal{C}) : \mathcal{P}$, where the left adjoint T lifts the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ (def. 2.21).

In this chapter we prove the dual of the universal property that uniquely lifts the free associative algebra functor $\mathcal{C} \to \mathrm{Alg}(\mathcal{C})$ to $\mathrm{Bialg}(\mathcal{C})$ (theorem 2.20 respectively prop. 3.22). For the case of symmetric monoidal categories proposition 3.22 makes the following statement:

Let $\mathcal{D}$ be a symmetric monoidal category with finite products and a final tensorunit and $\mathcal{C}$ a preadditive category.

Then the forgetful functor

$$\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathrm{Calg}(\mathcal{D})) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$$

from finite products preserving functors $\mathcal{C} \to \mathrm{Calg}(\mathcal{D})$ to finite products preserving functors $\mathcal{C} \to \mathcal{D}$ is an equivalence. More generally we prove a similar statement, where we replace symmetric monoidal categories by $\mathcal{O}^{\otimes}$-monoidal categories for any unital operad $\mathcal{O}^{\otimes}$.

We deduce proposition 3.22 from a universal property of the cocartesian structure (theorem 2.4.3.18. [18]) and a universal property of the cartesian structure (theorem 3.21).

The universal property of the cocartesian structure (theorem 2.4.3.18. [18]) provides a canonical equivalence

$$\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Calg}(\mathcal{D}))$$

over $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ between lax symmetric monoidal functors $\mathcal{C} \to \mathcal{D}$ and functors $\mathcal{C} \to \mathrm{Calg}(\mathcal{D})$, where $\mathcal{C}$ carries the cocartesian or equivalently cartesian structure.

The universal property of the cartesian structure (theorem 3.21) states the following:

Let $\mathcal{C}$ be a cartesian symmetric monoidal category and $\mathcal{D}$ a symmetric monoidal category with a final tensorunit.

Then the forgetful functor

$$\mathrm{Fun}^{\otimes,\mathrm{lax},\Pi}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$$

from lax symmetric monoidal finite products preserving functors $\mathcal{C} \to \mathcal{D}$ to finite products preserving functors $\mathcal{C} \to \mathcal{D}$ is an equivalence.

Also here we prove a similar statement for $\mathcal{O}^{\otimes}$-monoidal categories for any unital operad $\mathcal{O}^{\otimes}$.

The strategy to prove theorem 3.21 is as follows:

The Yoneda-embedding $\mathcal{D} \to \mathcal{P}(\mathcal{D})$ preserves finite products and is symmetric monoidal if $\mathcal{P}(\mathcal{D})$ carries the Day-convolution symmetric monoidal structure. Consequently we may replace $\mathcal{D}$ by $\mathcal{P}(\mathcal{D})$ in the statement of theorem 3.21.

By prop. 6.28 the functor-category $\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ admits a symmetric monoidal structure given by Day-convolution characterized by the following universal property:

For every symmetric monoidal category $\mathcal{B}$ we have a canonical equivalence

$$\mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{B}, \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \simeq \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{B} \times \mathcal{C}, \mathcal{P}(\mathcal{D}))$$

over $\mathrm{Fun}(\mathcal{B}, \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \simeq \mathrm{Fun}(\mathcal{B} \times \mathcal{C}, \mathcal{P}(\mathcal{D}))$.

Especially for $\mathcal{B}$ the contractible category we obtain a canonical equivalence

$$\mathrm{Calg}(\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \simeq \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$$

over $\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$.

We show that the category $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is a cocartesian symmetric monoidal category and the embedding $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is a lax symmetric monoidal embedding (corollary 3.19).

Thus the canonical equivalence $\mathrm{Calg}(\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \simeq \mathrm{Fun}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ restricts to an equivalence

$$\mathrm{Calg}(\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \simeq \mathrm{Fun}^{\otimes,\mathrm{lax},\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$$

over $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$.

Finally as $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is a cocartesian symmetric monoidal category, the forgetful functor $\mathrm{Calg}(\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \to \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is an equivalence by [18] proposition 2.4.3.9.

To prove that the category $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is a cocartesian symmetric monoidal category, we use a theory of cocartesian operads (def. 3.1) generalizing the notion of cocartesian symmetric monoidal category.

We show that $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ admits the structure of a cocartesian operad such that the embedding $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is an embedding of operads (corollary 3.19).

By remark 3.20 the full subcategory $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \subset \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ is a localization so that also by remark 3.20 the cocartesian operad structure on $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ exhibits $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$ as a cocartesian symmetric monoidal category.

3.1 Cocartesian operads

In the following section we extend the property of being cocartesian from the class of operads to the class of operads over $\mathcal{O}^{\otimes}$ for every unital operad $\mathcal{O}^{\otimes}$ and study the basic properties of cocartesian operads over $\mathcal{O}^{\otimes}$.

We refer to [19] for the notions of operad and relative (co)limits.

Let $\phi : \mathcal{O}^{\otimes} \to \mathcal{F}\mathrm{in}_*$ be a unital operad with zero object $0 \in \mathcal{O}^{\otimes}$.

Let $\mathrm{X} \in \mathcal{O}^{\otimes}$ be an object lying over $\langle \mathrm{n} \rangle \in \mathcal{F}\mathrm{in}_*$ for some $\mathrm{n} \geq 2$ corresponding to n objects $\mathrm{X}_1, ..., \mathrm{X}_\mathrm{n} \in \mathcal{O}$ so that we have n inert morphisms $\mathrm{X} \to \mathrm{X}_\mathrm{i}$ for $1 \leq \mathrm{i} \leq \mathrm{n}$ that exhibit X as the ϕ-product of the objects $\mathrm{X}_1, ..., \mathrm{X}_\mathrm{n} \in \mathcal{O}$.

For every $1 \leq \mathrm{i} \leq \mathrm{n}$ we have an active morphism $\alpha_\mathrm{i}^\mathrm{X} : \mathrm{X}_\mathrm{i} \to \mathrm{X}$ of $\mathcal{O}^{\otimes}$ corresponding to the n morphisms $\beta_{\mathrm{ij}} : \mathrm{X}_\mathrm{i} \to \mathrm{X}_\mathrm{j}$ of $\mathcal{O}^{\otimes}$ for $1 \leq \mathrm{j} \leq \mathrm{n}$ with β_{ii} the identity and β_{ij} the zero morphism.

Definition 3.1. *(cocartesian operad)*

Let $\mathcal{O}^{\otimes}$ be a unital operad.

A unital operad $\gamma : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ over $\mathcal{O}^{\otimes}$ is called cocartesian or is said to exhibit $\mathcal{C}^{\otimes}$ as a cocartesian operad over $\mathcal{O}^{\otimes}$ if the following condition is satisfied:

Let A *be an object of* $\mathcal{C}^{\otimes}$ *lying over* $\langle \mathrm{n} \rangle \in \mathcal{F}\mathrm{in}_*$ *for some* $\mathrm{n} \geq 2$ *corresponding to* n *objects* $\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n} \in \mathcal{C}$.

Then the active morphisms $\alpha_\mathrm{i}^\mathrm{A} : \mathrm{A}_\mathrm{i} \to \mathrm{A}$ *of* $\mathcal{C}^{\otimes}$ *for* $1 \leq \mathrm{i} \leq \mathrm{n}$ *exhibit* A *as the* γ*-coproduct of the objects* A_i *for* $1 \leq \mathrm{i} \leq \mathrm{n}$, *i.e. for all* $\mathrm{Z} \in \mathcal{C}^{\otimes}$ *the commutative square of spaces*

$$\begin{array}{ccc} \mathcal{C}^{\otimes}(\mathrm{A}, \mathrm{Z}) & \longrightarrow & \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{C}^{\otimes}(\mathrm{A}_\mathrm{i}, \mathrm{Z}) \\ \downarrow & & \downarrow \\ \mathcal{O}^{\otimes}(\gamma(\mathrm{A}), \gamma(\mathrm{Z})) & \longrightarrow & \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{O}^{\otimes}(\gamma(\mathrm{A}_\mathrm{i}), \gamma(\mathrm{Z})) \end{array} \tag{6}$$

induced by the morphisms $\alpha_\mathrm{i}^\mathrm{A} : \mathrm{A}_\mathrm{i} \to \mathrm{A}$ *of* $\mathcal{C}^{\otimes}$ *is a pullback square.*

If $\mathcal{O}^{\otimes} = \mathcal{F}\mathrm{in}_*$, *we call a cocartesian operad over* $\mathcal{O}^{\otimes}$ *a cocartesian operad.*

Remark 3.2.

- *Let* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a map of unital operads and* A *an object of* $\mathcal{C}^{\otimes}$ *lying over* $\langle \mathrm{n} \rangle \in \mathcal{F}\mathrm{in}_*$ *for some* $\mathrm{n} \geq 2$ *corresponding to* n *objects* $\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n} \in \mathcal{C}$.

  The active morphisms $\alpha_\mathrm{i}^\mathrm{A} : \mathrm{A}_\mathrm{i} \to \mathrm{A}$ *of* $\mathcal{C}^{\otimes}$ *for* $1 \leq \mathrm{i} \leq \mathrm{n}$ *exhibit* A *as the* γ*-coproduct of the objects* A_i *for* $1 \leq \mathrm{i} \leq \mathrm{n}$ *if and only if for all* $\mathrm{Z} \in \mathcal{C}$ *the induced commutative square of spaces*

$$\begin{array}{ccc} \mathrm{Mul}_{\mathcal{C}}(\mathrm{A}_1, \mathrm{A}_2, ..., \mathrm{A}_\mathrm{n}; \mathrm{Z}) & \longrightarrow & \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{C}(\mathrm{A}_\mathrm{i}, \mathrm{Z}) \\ \downarrow & & \downarrow \\ \mathrm{Mul}_{\mathcal{O}}(\gamma(\mathrm{A}_1), \gamma(\mathrm{A}_2), ..., \gamma(\mathrm{A}_\mathrm{n}); \gamma(\mathrm{Z})) & \longrightarrow & \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathcal{O}(\gamma(\mathrm{A}_\mathrm{i}), \gamma(\mathrm{Z})) \end{array} \tag{7}$$

  is a pullback square or equivalently for every active morphism $\mathrm{h} : \gamma(\mathrm{A}) \to \gamma(\mathrm{Z})$ *of* $\mathcal{O}^{\otimes}$ *the canonical map*

$$\{\mathrm{h}\} \times_{\mathrm{Mul}_{\mathcal{O}}(\gamma(\mathrm{A}_1),...,\gamma(\mathrm{A}_\mathrm{n});\gamma(\mathrm{Z}))} \mathrm{Mul}_{\mathcal{C}}(\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n}; \mathrm{Z}) \to$$
$$\prod_{\mathrm{i}=1}^{\mathrm{n}} \{\mathrm{h} \circ \alpha_\mathrm{i}^{\gamma(\mathrm{A})}\} \times_{\mathcal{O}(\gamma(\mathrm{A}_\mathrm{i}), \gamma(\mathrm{Z}))} \mathcal{C}(\mathrm{A}_\mathrm{i}, \mathrm{Z})$$

  induced by square 7 is an equivalence.

- *The pullback of every cocartesian operad* $\mathcal{C}^{\otimes}$ *over* $\mathcal{O}^{\otimes}$ *along any map* $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ *of unital operads is a cocartesian operad over* $\mathcal{O}'^{\otimes}$.

- *Let* $\beta : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}, \gamma : \mathcal{D}^{\otimes} \to \mathcal{E}^{\otimes}$ *be maps of unital operads.*

  Assume that $\gamma : \mathcal{D}^{\otimes} \to \mathcal{E}^{\otimes}$ *exhibits* $\mathcal{D}^{\otimes}$ *as a cocartesian operad over* $\mathcal{E}^{\otimes}$.

  Then $\beta : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *exhibits* $\mathcal{C}^{\otimes}$ *as a cocartesian operad over* $\mathcal{D}^{\otimes}$ *if and only if* $\gamma \circ \beta : \mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ *exhibits* $\mathcal{C}^{\otimes}$ *as a cocartesian operad over* $\mathcal{E}^{\otimes}$.

- *Let* $\phi : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *be a map of cocartesian operads over* $\mathcal{O}^\otimes$.

  Then ϕ *is an equivalence if and only if the underlying functor* $\mathcal{C} \to \mathcal{D}$ *of* ϕ *is an equivalence.*

  This follows from the fact that a map of operads is an equivalence if and only if it induces equivalences on all multi-mapping spaces and an essentially surjective functor on the underlying category.

- *Denote* E_0 *the reduced operad with no* n*-ary operations for* $\mathrm{n} > 1$. *Then every unital operad* $\mathcal{C}^\otimes \to \mathrm{E}_0$ *over* E_0 *is cocartesian.*

By construction [18] 2.4.3.1. we have for every category $\mathcal{C}$ a cocartesian operad $\mathcal{C}^\amalg$ with underlying category $\mathcal{C}$ such that for all unital operads $\mathcal{O}^\otimes$ the forgetful functor $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{O}, \mathcal{C})$ is an equivalence ([18] proposition 2.4.3.9.).

Proposition 3.3.

Let $\gamma : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *be a map of unital operads.*

The following conditions are equivalent:

1. γ *exhibits* $\mathcal{C}^\otimes$ *as a cocartesian operad over* $\mathcal{O}^\otimes$.
2. *The canonical commutative square of operads*

$$\begin{array}{ccc} \mathcal{C}^\otimes & \longrightarrow & \mathcal{C}^\amalg \\ \downarrow & & \downarrow \\ \mathcal{O}^\otimes & \longrightarrow & \mathcal{O}^\amalg \end{array}$$

   is a pullback square.

3. *The map of operads* $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *is the pullback of a map of cocartesian operads* $\mathcal{A}^\amalg \to \mathcal{B}^\amalg$ *along some map of operads* $\phi : \mathcal{O}^\otimes \to \mathcal{B}^\amalg$.
4. *For every unital operad* $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ *over* $\mathcal{O}^\otimes$ *the canonical functor* $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}', \mathcal{C})$ *is an equivalence.*

Proof. The commutative square of operads

$$\begin{array}{ccc} \mathcal{C}^\otimes & \longrightarrow & \mathcal{C}^\amalg \\ \downarrow & & \downarrow \\ \mathcal{O}^\otimes & \longrightarrow & \mathcal{O}^\amalg \end{array}$$

is a pullback square if and only if it induces a pullback square on the underlying categories and it induces a pullback square on all multi-mapping spaces.

The first condition is satisfied as the maps of operads $\mathcal{C}^\otimes \to \mathcal{C}^\amalg$ and $\mathcal{O}^\otimes \to \mathcal{O}^\amalg$ lift the identity.

Consequently condition 2. is equivalent to the following condition:

For all $\mathrm{n} \geq 2$ and $\mathrm{A}_1, \mathrm{A}_2, ..., \mathrm{A}_\mathrm{n} \in \mathcal{C}$ corresponding to $\mathrm{A} \in \mathcal{C}^\otimes_{\langle \mathrm{n} \rangle} \simeq \mathcal{C}^{\times \mathrm{n}}$ lying over the objects $\mathrm{X}_1, \mathrm{X}_2, ..., \mathrm{X}_\mathrm{n} \in \mathcal{O}$ corresponding to $\mathrm{X} \in \mathcal{O}^\otimes_{\langle \mathrm{n} \rangle} \simeq \mathcal{O}^{\times \mathrm{n}}$

and $Z \in \mathcal{C}$ lying over some object $Y \in \mathcal{O}$ the commutative square

$$\begin{array}{ccccc} \mathrm{Mul}_{\mathcal{C}^\otimes}(A_1, A_2, ..., A_n; Z) & \longrightarrow & \mathrm{Mul}_{\mathcal{C}^\amalg}(A_1, A_2, ..., A_n; Z) & \xrightarrow{\simeq} & \prod_{i=1}^n \mathcal{C}(A_i, Z) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Mul}_{\mathcal{O}^\otimes}(X_1, X_2, ..., X_n; Y) & \longrightarrow & \mathrm{Mul}_{\mathcal{O}^\amalg}(X_1, X_2, ..., X_n; Y) & \xrightarrow{\simeq} & \prod_{i=1}^n \mathcal{O}(X_i, Y) \end{array}$$

induced by the active morphism $\alpha_i : A_i \to A$ of $\mathcal{C}^\otimes$ and $\alpha_i : X_i \to X$ of $\mathcal{O}^\otimes$ for $1 \leq i \leq n$ is a pullback square.

But by remark 3.2 this condition is equivalent to 1.

2. trivially implies 3.

Assume that 3. holds and let $\alpha : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ be a unital operad over $\mathcal{O}^\otimes$. Then the forgetful functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_\mathcal{O}(\mathcal{O}', \mathcal{C})$ is equivalent to the canonical functor

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O} \times_\mathcal{B} \mathcal{A}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{B}}(\mathcal{A}) \to \mathrm{Fun}_\mathcal{B}(\mathcal{O}', \mathcal{A}) \simeq$$

$$\mathrm{Fun}_\mathcal{O}(\mathcal{O}', \mathcal{O} \times_\mathcal{B} \mathcal{A}) \simeq \mathrm{Fun}_\mathcal{O}(\mathcal{O}', \mathcal{C}).$$

The forgetful functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{B}}(\mathcal{A}) \to \mathrm{Fun}_\mathcal{B}(\mathcal{O}', \mathcal{A})$ is equivalent to the canonical functor $\{\phi \circ \alpha\} \times_{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{B})} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{A}) \to \{\phi \circ \alpha\} \times_{\mathrm{Fun}(\mathcal{O}', \mathcal{B})} \mathrm{Fun}(\mathcal{O}', \mathcal{A})$ and is thus an equivalence as $\mathcal{O}'^\otimes$ is unital. So 3. implies 4.

We complete the proof by showing that 4. implies 2.
Let $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ be a unital operad over $\mathcal{O}^\otimes$.
The canonical map of unital operads $\beta : \mathcal{C}^\otimes \to \mathcal{O}^\otimes \times_{\mathcal{O}^\amalg} \mathcal{C}^\amalg$ over $\mathcal{O}^\otimes$ induces a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O} \times_\mathcal{O} \mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_\mathcal{O}(\mathcal{O}', \mathcal{C}) & \xrightarrow{=} & \mathrm{Fun}_\mathcal{O}(\mathcal{O}', \mathcal{C}) \end{array}$$

By what we have proved so far, the right vertical functor of the square is an equivalence.
If we assume that 4. holds, also the left vertical functor of the square is an equivalence so that the top horizontal functor of the square is an equivalence. So by Yoneda β is an equivalence.

□

Remark 3.4.
Let $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ be a map of unital operads and $\mathcal{C} \to \mathcal{O}$ a category over $\mathcal{O}$.

Denote $\psi : \mathcal{O}'^\otimes \to \mathcal{O}^\amalg$ the unique map of operads lifting the functor $\mathcal{O}' \to \mathcal{O}$.

The forgetful functor

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O}^\otimes \times_{\mathcal{O}^\amalg} \mathcal{C}^\amalg) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_\mathcal{O}(\mathcal{O}', \mathcal{C})$$

is equivalent to the canonical functor

$$\{\psi\} \times_{\mathrm{Alg}_{\mathcal{O}'}(\mathcal{O})} \mathrm{Alg}_{\mathcal{O}'}(\mathcal{C}) \to \{\psi\} \times_{\mathrm{Fun}(\mathcal{O}', \mathcal{O})} \mathrm{Fun}(\mathcal{O}', \mathcal{C})$$

and is thus an equivalence by [18] prop. 2.4.3.9. as $\mathcal{O}'^\otimes$ is unital.

So if $\mathrm{Op}_\infty^{\mathrm{un}} \subset \mathrm{Op}_\infty$ *denotes the full subcategory spanned by the unital operads, the forgetful functor* $(\mathrm{Op}_\infty^{\mathrm{un}})_{/\mathcal{O}^\otimes} \to \mathsf{Cat}_{\infty/\mathcal{O}}$ *admits a fully faithful right adjoint.*

A unital operad over $\mathcal{O}^\otimes$ *is a local object of* $(\mathrm{Op}_\infty^{\mathrm{un}})_{/\mathcal{O}^\otimes}$ *if and only if the canonical map of unital operads* $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes \times_{\mathcal{O}^\amalg} \mathcal{O}'^\amalg$ *over* $\mathcal{O}^\otimes$ *is an equivalence.*

Hence by proposition 3.3 2. the local objects are exactly the cocartesian operads over $\mathcal{O}^\otimes$.

3.2 Cocartesian $\mathcal{O}^\otimes$-monoidal categories

As next we focus on (locally) cocartesian fibrations $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ of unital operads that exhibit $\mathcal{C}^\otimes$ as a cocartesian operad over $\mathcal{O}^\otimes$.

We use the theory of cocartesian $\mathcal{O}^\otimes$-monoidal categories, especially def. 3.13, to prove theorem 3.21.

To express that a cocartesian fibration $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ of unital operads exhibits $\mathcal{C}^\otimes$ as a cocartesian operad over $\mathcal{O}^\otimes$, we will also say that $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ exhibits $\mathcal{C}^\otimes$ as a cocartesian $\mathcal{O}^\otimes$-monoidal category or simply that $\mathcal{C}^\otimes$ is a cocartesian $\mathcal{O}^\otimes$-monoidal category.

Construction 3.5.

Let $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *be a locally cocartesian fibration of unital operads.*

Let $\mathrm{n} \geq 2$ *and* $\mathrm{X}_1, \mathrm{X}_2, ..., \mathrm{X}_\mathrm{n} \in \mathcal{O}$ *be objects corresponding to* $\mathrm{X} \in \mathcal{O}^\otimes_{\langle \mathrm{n} \rangle} \simeq \mathcal{O}^{\times \mathrm{n}}$.

Let $\mathrm{h} : \mathrm{X} \to \mathrm{Y}$ *be an active morphism of* $\mathcal{O}^\otimes$ *with* $\mathrm{Y} \in \mathcal{O}$.

For all $\mathrm{i} \in \{1, .., \mathrm{n}\}$ *set* $\mathrm{h}^\mathrm{i} := \mathrm{h} \circ \alpha_\mathrm{i}^\mathrm{X} : \mathrm{X}_\mathrm{i} \to \mathrm{X} \to \mathrm{Y}$ *and denote* $\mathrm{pr}_\mathrm{i} : \prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}} \to \mathcal{C}_{\mathrm{X}_\mathrm{i}}$ *the* i*-th projection.*

We have a canonical natural transformation $(\alpha_\mathrm{i}^\mathrm{X})_* \circ \mathrm{pr}_\mathrm{i} \to \mathrm{id}$ *of functors* $\prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}} \to \prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}}$ *that is on the* i*-th component the identity of* pr_i *and on the* j*-th component for* $\mathrm{j} \in \{1, .., \mathrm{n}\} \setminus \{\mathrm{i}\}$ *the unique natural transformation* $\mathbb{1}_{\mathrm{X}_\mathrm{j}} \to \mathrm{pr}_\mathrm{j}$ *of functors* $\prod_{\mathrm{l}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{l}} \to \mathcal{C}_{\mathrm{X}_\mathrm{j}}$ *starting at the constant functor with value the initial object* $\mathbb{1}_{\mathrm{X}_\mathrm{j}}$ *of* $\mathcal{C}_{\mathrm{X}_\mathrm{j}}$.

The canonical natural transformation $\mathrm{h}^\mathrm{i}_* \to \otimes_\mathrm{h} \circ (\alpha_\mathrm{i}^\mathrm{X})_*$ *of functors* $\mathcal{C}_{\mathrm{X}_\mathrm{i}} \to \mathcal{C}_\mathrm{Y}$ *gives rise to a natural transformation*

$$\zeta_\mathrm{i} : \mathrm{h}^\mathrm{i}_* \circ \mathrm{pr}_\mathrm{i} \to \otimes_\mathrm{h} \circ (\alpha_\mathrm{i}^\mathrm{X})_* \circ \mathrm{pr}_\mathrm{i} \to \otimes_\mathrm{h}$$

of functors $\prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}} \to \mathcal{C}_\mathrm{Y}$ *that exhibits the canonical morphism*

$$\zeta_\mathrm{i}(\mathrm{A}) : \mathrm{h}^\mathrm{i}_*(\mathrm{A}_\mathrm{i}) \to \otimes_\mathrm{h}(\mathbb{1}_{\mathrm{X}_1}, ..., \mathbb{1}_{\mathrm{X}_{\mathrm{i}-1}}, \mathrm{A}_\mathrm{i}, \mathbb{1}_{\mathrm{X}_{\mathrm{i}+1}}, ..., \mathbb{1}_{\mathrm{X}_\mathrm{n}}) \to$$

$$\otimes_\mathrm{h}(\mathrm{A}_1, \mathrm{A}_2, ..., \mathrm{A}_{\mathrm{n}-1}, \mathrm{A}_\mathrm{n})$$

in $\mathcal{C}_\mathrm{Y}$ *as natural in* $\mathrm{A} \in \mathcal{C}_\mathrm{X} \simeq \prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}}$.

Remark 3.6.

Let $\mathcal{O}^\otimes$ *be a unital operad and* $\gamma : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *a locally cocartesian fibration of operads.*

1. *The operad* $\mathcal{C}^\otimes$ *is unital if and only if for all* $\mathrm{X} \in \mathcal{O}$ *the tensorunit* $\mathbb{1}_\mathrm{X}$ *of* $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *at* X *is an initial object of the fiber* $\mathcal{C}_\mathrm{X}$.

This follows from the fact that for every $n \in \mathbb{N}$ *and every* $Y \in \mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ *corresponding to the objects* $Y_1, Y_2, ..., Y_n \in \mathcal{C}$ *and lying over the object* $X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$ *corresponding to the objects* $X_1, X_2, ..., X_n \in \mathcal{O}$ *we have a canonical equivalence*

$$\prod_{j=1}^{n} \mathcal{C}_{X_j}(\mathbb{1}_{X_j}, Y_j) \simeq \{\alpha\} \times_{\mathcal{O}^{\otimes}(\gamma(*),X)} \mathcal{C}^{\otimes}(*, Y) \simeq \mathcal{C}^{\otimes}(*, Y),$$

where $* \in \mathcal{C}^{\otimes}_{\langle 0 \rangle}$ *denotes the unique object and* $\alpha : \gamma(*) \to X$ *the unique morphism of* $\mathcal{O}^{\otimes}$.

2. γ *exhibits* $\mathcal{C}^{\otimes}$ *as a cocartesian operad over* $\mathcal{O}^{\otimes}$ *if and only if* $\mathcal{C}^{\otimes}$ *is unital and the following condition holds:*

   Let $n \geq 2$ *and* $A_1, A_2, ..., A_n \in \mathcal{C}$ *be* n *objects of* $\mathcal{C}$ *corresponding to* $A \in \mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ *lying over the objects* $X_1, X_2, ..., X_n \in \mathcal{O}$ *corresponding to* $X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$.
   Let $h : X \to Y$ *be an active morphism of* $\mathcal{O}^{\otimes}$ *with* $Y \in \mathcal{O}$. *For all* $i \in \{1, .., n\}$ *set* $h^i := h \circ \alpha_i^X : X_i \to X \to Y$.
   Then the morphisms

$$\zeta_i(A) : h^i_*(A_i) \to \otimes_h(\mathbb{1}_{X_1}, ..., \mathbb{1}_{X_{i-1}}, A_i, \mathbb{1}_{X_{i+1}}, ..., \mathbb{1}_{X_n}) \to$$
$$\otimes_h(A_1, A_2, ..., A_{n-1}, A_n)$$

   in $\mathcal{C}_Y$ *for* $i \in \{1, .., n\}$ *induced by the unique morphisms* $\mathbb{1}_{X_j} \to A_j$ *for* $j \neq i$ *and the identity of* A_i *exhibit* $\otimes_h(A_1, A_2, ..., A_{n-1}, A_n)$ *as a coproduct of the objects* $h^1_*(A_1), h^2_*(A_2), ..., h^n_*(A_n)$ *in* $\mathcal{C}_Y$.

   This follows from remark 3.2 and the fact that for every object $Z \in \mathcal{C}_Y$ *the canonical map*

$$\{h\} \times_{\mathrm{Mul}_{\mathcal{O}}(X_1, X_2, ..., X_n; Y)} \mathrm{Mul}_{\mathcal{C}}(A_1, A_2, ..., A_n; Z) \to$$
$$\prod_{i=1}^{n} (\{h^i\} \times_{\mathcal{O}(X_i, Y)} \mathcal{C}(A_i, Z))$$

   induced by square 7 is equivalent to the map

$$\mathcal{C}_Y(\otimes_h(A_1, A_2, ..., A_{n-1}, A_n), Z) \to \prod_{i=1}^{n} \mathcal{C}_Y(h^i_*(A_i), Z)$$

   induced by the morphisms $\zeta_i(A)$ *for* $1 \leq i \leq n$.

Remark 3.7.
Let $\gamma : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ *be a map of unital operads that exhibits* $\mathcal{C}^{\otimes}$ *as a cocartesian operad over* $\mathcal{O}^{\otimes}$.

Then γ *is a cocartesian fibration if and only if* γ *is a locally cocartesian fibration.*

This follows from the fact that given two locally γ*-cocartesian and active morphisms* $X \to Y$ *and* $Y \to Z \simeq \otimes_g(Y_1, ..., Y_m) \simeq \otimes_g(\otimes_{f_1}(X_i \mid i \in \varphi^{-1}(1)), ..., \otimes_{f_m}(X_i \mid i \in \varphi^{-1}(m)))$ *with* $Z \in \mathcal{C}$ *with images* $f : X' \to Y'$ *and* $g : Y' \to Z'$ *in* $\mathcal{O}^{\otimes}$ *and images* $\varphi : \langle n \rangle \to \langle m \rangle$ *and* $\langle m \rangle \to \langle 1 \rangle$ *in* $\mathcal{F}\mathrm{in}_*$ *the canonical morphism*

$$\otimes_{g \circ f}(X_1, ..., X_n) \to \otimes_g(\otimes_{f_1}(X_i \mid i \in \varphi^{-1}(1)), ..., \otimes_{f_m}(X_i \mid i \in \varphi^{-1}(m)))$$

in $\mathcal{C}_{Z'}$ is equivalent to the canonical equivalence

$$\coprod_{i=1}^{n} X_i \to \coprod_{j=1}^{m} (\coprod_{i \in \varphi^{-1}(j)} X_i).$$

Let $\mathcal{O}^\otimes$ be a unital operad and $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a $\mathcal{O}^\otimes$-monoidal category.

We call $\mathcal{C}^\otimes$ a cartesian $\mathcal{O}^\otimes$-monoidal category if the fiberwise dual of $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ relative to $\mathcal{O}^\otimes$ is a cocartesian $\mathcal{O}^\otimes$-monoidal category.

Denote

$$\mathsf{Cat}_\infty^\Sigma, \mathsf{Cat}_\infty^\Pi \subset \mathsf{Cat}_\infty$$

the subcategories with objects the categories that admit finite coproducts respectively finite products and morphisms the functors that preserve finite coproducts respectively finite products.

The opposite category involution on Cat_∞ restricts to an equivalence $\mathsf{Cat}_\infty^\Sigma \simeq \mathsf{Cat}_\infty^\Pi$.

The categories $\mathsf{Cat}_\infty^\Sigma \simeq \mathsf{Cat}_\infty^\Pi$ admit finite products which are preserved by the subcategory inclusions $\mathsf{Cat}_\infty^\Sigma, \mathsf{Cat}_\infty^\Pi \subset \mathsf{Cat}_\infty$.

Consequently the subcategory inclusion $\mathsf{Cat}_\infty^\Sigma \subset \mathsf{Cat}_\infty$ induces for every operad $\mathcal{O}^\otimes$ a subcategory inclusion $\mathrm{Mon}_\mathcal{O}(\mathsf{Cat}_\infty^\Sigma) \subset \mathrm{Mon}_\mathcal{O}(\mathsf{Cat}_\infty)$ on $\mathcal{O}^\otimes$-monoids.

Observation 3.8.

1. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal category.*

   Then $\mathcal{C}^\otimes$ is a cocartesian symmetric monoidal category if and only if $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ classifies a commutative monoid of $\mathsf{Cat}_\infty^\Sigma$.

   If $\mathcal{C}^\otimes$ is cocartesian, the tensorunit $\mathbb{1}$ of $\mathcal{C}^\otimes$ is an initial object of $\mathcal{C}$ and for all $A, B \in \mathcal{C}$ the canonical maps $A \simeq A \otimes \mathbb{1} \to A \otimes B$ and $B \simeq \mathbb{1} \otimes B \to A \otimes B$ in $\mathcal{C}$ exhibit $A \otimes B$ as the coproduct of A and B in $\mathcal{C}$.
   Thus the canonical map $(A, B) \to (A \otimes B, A \otimes B)$ in $\mathcal{C} \times \mathcal{C}$ exhibits the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ as the left adjoint of the diagonal functor $\mathcal{C} \to \mathcal{C} \times \mathcal{C}$ so that the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite coproducts.
   Hence $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ classifies a commutative monoid of $\mathsf{Cat}_\infty^\Sigma$.

   Conversely if $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ classifies a commutative monoid of $\mathsf{Cat}_\infty^\Sigma$, then the tensorunit $\mathbb{1}$ of $\mathcal{C}^\otimes$ is an initial object of $\mathcal{C}$, the category $\mathcal{C}$ admits finite coproducts and the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ preserves finite coproducts.
   Thus for all $A, B \in \mathcal{C}$ we have a canonical equivalence

   $$A \otimes B \simeq (A \coprod \mathbb{1}) \otimes (\mathbb{1} \coprod B) \simeq (A \otimes \mathbb{1}) \coprod (\mathbb{1} \otimes B) \simeq A \coprod B$$

   so that the canonical maps $A \simeq A \otimes \mathbb{1} \to A \otimes B$ and $B \simeq \mathbb{1} \otimes B \to A \otimes B$ in $\mathcal{C}$ exhibit $A \otimes B$ as the coproduct of A and B in $\mathcal{C}$.

2. *By 1. the finite products preserving subcategory inclusion*

   $$\mathsf{Cat}_\infty^\Sigma \subset \mathsf{Cat}_\infty$$

   gives rise to a fully faithful functor $\mathrm{Cmon}(\mathsf{Cat}_\infty^\Sigma) \to \mathrm{Cmon}(\mathsf{Cat}_\infty)$ with essential image the cocartesian symmetric monoidal categories.

The forgetful functor $\mathrm{Op}_\infty \to \mathsf{Cat}_\infty$ restricts to an equivalence on the full subcategory spanned by the cocartesian operads with inverse the functor that sends a category $\mathcal{C}$ to its cocartesian operad $\mathcal{C}^{\amalg}$.

By [18] remark 2.4.3.4. for every category $\mathcal{C}$ that admits finite coproducts the cocartesian operad $\mathcal{C}^{\amalg}$ is a symmetric monoidal category.

Hence the forgetful functor $\mathrm{Op}_\infty \to \mathsf{Cat}_\infty$ restricts to an equivalence $\mathrm{Cmon}(\mathsf{Cat}_\infty^{\Sigma}) \simeq \mathsf{Cat}_\infty^{\Sigma}$ so that the category $\mathsf{Cat}_\infty^{\Sigma} \simeq \mathrm{Cmon}(\mathsf{Cat}_\infty^{\Sigma})$ is preadditive being a category of commutative monoids.

Thus also the equivalent category $\mathsf{Cat}_\infty^{\Pi} \simeq \mathsf{Cat}_\infty^{\Sigma}$ is preadditive.

The next proposition 3.9 generalizes remark 3.8 1. from symmetric monoidal categories to $\mathcal{O}^{\otimes}$-monoidal categories for a unital operad $\mathcal{O}^{\otimes}$.

Proposition 3.9.

Let $\mathcal{O}^{\otimes}$ be a unital operad and $\gamma : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a cocartesian fibration of operads.

The following conditions are equivalent:

- *γ classifies a $\mathcal{O}^{\otimes}$-monoid of $\mathsf{Cat}_\infty^{\Sigma}$.*
- *$\gamma : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as a cocartesian $\mathcal{O}^{\otimes}$-monoidal category and the underlying cocartesian fibration $\mathcal{C} \to \mathcal{O}$ classifies a functor $\mathcal{O} \to \mathsf{Cat}_\infty^{\Sigma}$.*

Composing with the opposite category involution $\mathsf{Cat}_\infty^{\Sigma} \simeq \mathsf{Cat}_\infty^{\Pi}$ we get the dual statement:

The following conditions are equivalent:

- *γ classifies a $\mathcal{O}^{\otimes}$-monoid of $\mathsf{Cat}_\infty^{\Pi}$.*
- *$\gamma : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as a cartesian $\mathcal{O}^{\otimes}$-monoidal category and the underlying cocartesian fibration $\mathcal{C} \to \mathcal{O}$ classifies a functor $\mathcal{O} \to \mathsf{Cat}_\infty^{\Pi}$.*

Remark 3.10.

Assume that $\mathcal{O}^{\otimes}$ is a reduced operad different from E_0.

If $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as a cocartesian $\mathcal{O}^{\otimes}$-monoidal category, then $\mathcal{C}$ admits finite coproducts.

Hence a cocartesian fibration of operads $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as a cocartesian respectively cartesian $\mathcal{O}^{\otimes}$-monoidal category if and only if it classifies a $\mathcal{O}^{\otimes}$-monoid of $\mathsf{Cat}_\infty^{\Sigma}$ respectively $\mathsf{Cat}_\infty^{\Pi}$.

Proof. Let $\mathrm{n} \geq 2$ and $\mathrm{X}_1, \mathrm{X}_2, ..., \mathrm{X}_\mathrm{n} \in \mathcal{O}$ be objects corresponding to $\mathrm{X} \in \mathcal{O}^{\otimes}_{\langle \mathrm{n} \rangle} \simeq \mathcal{O}^{\times \mathrm{n}}$ and let $\mathrm{h} : \mathrm{X} \to \mathrm{Y}$ be an active morphism of $\mathcal{O}^{\otimes}$ with $\mathrm{Y} \in \mathcal{O}$.

For all $\mathrm{i} \in \{1, .., \mathrm{n}\}$ set $\mathrm{h}^\mathrm{i} := \mathrm{h} \circ \alpha_\mathrm{i}^\mathrm{X} : \mathrm{X}_\mathrm{i} \to \mathrm{X} \to \mathrm{Y}$ and denote $\mathrm{pr}_\mathrm{i} : \prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}} \to \mathcal{C}_{\mathrm{X}_\mathrm{i}}$ the i-th projection.

We have a natural transformation $\zeta_\mathrm{i} : \mathrm{h}^\mathrm{i}_* \circ \mathrm{pr}_\mathrm{i} \to \otimes_\mathrm{h}$ of functors $\prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}} \to \mathcal{C}_\mathrm{Y}$ that exhibits the canonical morphism

$$\zeta_\mathrm{i}(\mathrm{A}) : \mathrm{h}^\mathrm{i}_*(\mathrm{A}_\mathrm{i}) \to \otimes_\mathrm{h}(\mathbb{1}_{\mathrm{X}_1}, ..., \mathbb{1}_{\mathrm{X}_{\mathrm{i}-1}}, \mathrm{A}_\mathrm{i}, \mathbb{1}_{\mathrm{X}_{\mathrm{i}+1}}, ..., \mathbb{1}_{\mathrm{X}_\mathrm{n}}) \to$$
$$\otimes_\mathrm{h}(\mathrm{A}_1, \mathrm{A}_2, ..., \mathrm{A}_{\mathrm{n}-1}, \mathrm{A}_\mathrm{n})$$

in $\mathcal{C}_\mathrm{Y}$ as natural in $\mathrm{A} \in \mathcal{C}_\mathrm{X} \simeq \prod_{\mathrm{j}=1}^{\mathrm{n}} \mathcal{C}_{\mathrm{X}_\mathrm{j}}$.

The natural transformations $\zeta_i : h^i_* \circ pr_i \to \otimes_h$ of functors $\prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$ for $i \in \{1, .., n\}$ yield a natural transformation $\zeta : \coprod_{i=1}^n h^i_* \circ pr_i \to \otimes_h$ of functors $\prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$, where the coproduct is taken in the category of functors $\prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$.

By remark 3.6 1. and 2. the map of operads $\gamma : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ exhibits $\mathcal{C}^\otimes$ as a cocartesian $\mathcal{O}^\otimes$-monoidal category if and only if $\zeta : \coprod_{i=1}^n h^i_* \circ pr_i \to \otimes_h$ is an equivalence for all active morphisms $h : X \to Y$ of $\mathcal{O}^\otimes$ with $X \in \mathcal{O}^\otimes_{\langle n \rangle}$ for some $n \geq 2$ and $Y \in \mathcal{O}$ and for all $X \in \mathcal{O}$ the tensorunit $\mathbb{1}_X$ of $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ at X is an initial object of the fiber $\mathcal{C}_X$.

Consequently it is enough to check that for all active morphisms $h : X \to Y$ of $\mathcal{O}^\otimes$ with $X \in \mathcal{O}^\otimes_{\langle n \rangle}$ for some $n \geq 2$ and $Y \in \mathcal{O}$ the natural transformation $\zeta : \coprod_{i=1}^n h^i_* \circ pr_i \to \otimes_h$ is an equivalence if and only if the functor $\otimes_h : \prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$ preserves finite coproducts provided that we assume that the underlying cocartesian fibration $\mathcal{C} \to \mathcal{O}$ classifies a functor $\mathcal{O} \to \mathsf{Cat}_\infty^\Sigma$.

For all $j \in \{1, .., n\}$ the natural transformation $\zeta \circ (\alpha^X_j)_* : h^j_* \simeq h^j_* \circ pr_j \circ (\alpha^X_j)_* \simeq \coprod_{i=1}^n h^i_* \circ pr_i \circ (\alpha^X_j)_* \to \otimes_h \circ (\alpha^X_j)_*$ of functors $\mathcal{C}_{X_j} \to \mathcal{C}_Y$ is the canonical equivalence.

Assume that the underlying cocartesian fibration $\mathcal{C} \to \mathcal{O}$ classifies a functor $\mathcal{O} \to \mathsf{Cat}_\infty^\Sigma$.
Then the functors $h^i_* : \mathcal{C}_{X_i} \to \mathcal{C}_Y$ and thus also the functor $\coprod_{i=1}^n h^i_* \circ pr_i : \prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$ preserve finite coproducts.

As $\mathsf{Cat}_\infty^\Sigma$ is preadditive, the functor

$$(\mathrm{Fun}^{\amalg}((\alpha^X_j)_*, \mathcal{C}_Y))_{j=1}^n : \mathrm{Fun}^{\amalg}(\prod_{j=1}^n \mathcal{C}_{X_j}, \mathcal{C}_Y) \to \prod_{j=1}^n \mathrm{Fun}^{\amalg}(\mathcal{C}_{X_j}, \mathcal{C}_Y)$$

is an equivalence.

If the functor $\otimes_h : \prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$ preserves finite coproducts, the natural transformation $\zeta : \coprod_{i=1}^n h^i_* \circ pr_i \to \otimes_h$ is a morphism of

$$\mathrm{Fun}^{\amalg}(\prod_{j=1}^n \mathcal{C}_{X_j}, \mathcal{C}_Y) \simeq \prod_{j=1}^n \mathrm{Fun}^{\amalg}(\mathcal{C}_{X_j}, \mathcal{C}_Y)$$

that corresponds to the equivalence $(\zeta \circ (\alpha^X_j)_*)_{j=1}^n$ so that ζ is an equivalence.
Conversely if $\zeta : \coprod_{i=1}^n h^i_* \circ pr_i \to \otimes_h$ is an equivalence, the functor $\otimes_h : \prod_{j=1}^n \mathcal{C}_{X_j} \to \mathcal{C}_Y$ preserves finite coproducts.

□

Remark 3.11.

Let $\mathcal{C} \to \mathcal{D}$ be a cocartesian fibration.

The induced map $\mathcal{C}^{\amalg} \to \mathcal{D}^{\amalg}$ of cocartesian operads is a cocartesian fibration if and only if $\mathcal{C} \to \mathcal{D}$ classifies a functor $\mathcal{D} \to \mathsf{Cat}_\infty^\Sigma$.

Proof. By remark 3.8 2. the category $\mathsf{Cat}_\infty^\Sigma$ is preadditive so that $(\mathsf{Cat}_\infty^\Sigma)^\times$ is a cocartesian symmetric monoidal category.
Hence the forgetful functor $\mathrm{Mon}_{\mathcal{D}^{\amalg}}(\mathsf{Cat}_\infty^\Sigma) \to \mathrm{Fun}(\mathcal{D}, \mathsf{Cat}_\infty^\Sigma)$ is an equivalence.

So if the cocartesian fibration $\mathcal{C} \to \mathcal{D}$ classifies a functor $\mathcal{D} \to \mathsf{Cat}_\infty^\Sigma$, there is a unique cocartesian fibration of operads $\beta : \mathcal{C}^\otimes \to \mathcal{D}^\amalg$ lifting the cocartesian fibration $\mathcal{C} \to \mathcal{D}$ and classifying a $\mathcal{D}^\amalg$-monoid of $\mathsf{Cat}_\infty^\Sigma$.

So by proposition 3.9 β exhibits $\mathcal{C}^\otimes$ as a cocartesian $\mathcal{D}^\amalg$-monoidal category so that the operad $\mathcal{C}^\otimes$ is cocartesian.

Thus by the uniqueness of the cocartesian structure the induced map $\mathcal{C}^\amalg \to \mathcal{D}^\amalg$ of cocartesian operads is equivalent to $\mathcal{C}^\amalg \simeq \mathcal{C}^\otimes \xrightarrow{\beta} \mathcal{D}^\amalg$ and is thus a cocartesian fibration.

Conversely assume that the map $\mathcal{C}^\amalg \to \mathcal{D}^\amalg$ of cocartesian operads is a cocartesian fibration.

Then the map $\mathcal{C}^\amalg \to \mathcal{D}^\amalg$ of cocartesian operads exhibits $\mathcal{C}^\amalg$ as a cocartesian $\mathcal{D}^\amalg$-monoidal category.

So for every $X \in \mathcal{D}$ the tensorunit $\mathbb{1}_X \in \mathcal{C}_X$ is an initial object of $\mathcal{C}_X$.

As $\mathcal{D}^\amalg$ is a cocartesian operad, for every $X \in \mathcal{D}$ there is a multimorphism $h : (X, X) \to X$ of $\mathcal{D}$ corresponding to $(\mathrm{id}_X, \mathrm{id}_X)$.

So for all $A, B \in \mathcal{C}_X$ the morphisms $A \simeq \otimes_h(A, \mathbb{1}_X) \to \otimes_h(A, B)$, $B \simeq \otimes_h(\mathbb{1}_X, B) \to \otimes_h(A, B)$ in $\mathcal{C}_X$ exhibit $\otimes_h(A, B)$ as coproduct of A and B.

So the fibers $\mathcal{C}_X$ admit finite coproducts and every morphism $X \to Y$ in $\mathcal{O}$ induces a finite coproducts preserving functor $\mathcal{C}_X \to \mathcal{C}_Y$.

□

Observation 3.12.

Let $\mathcal{O}^\otimes, \mathcal{C}^\otimes, \mathcal{D}^\otimes$ be unital operads and $\gamma : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ a map of locally cocartesian fibrations of operads over $\mathcal{O}^\otimes$.

1. *The map of operads $\gamma : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ exhibits $\mathcal{C}^\otimes$ as a cocartesian operad over $\mathcal{D}^\otimes$ if and only if the following condition holds:*

   Let $n \geq 2$ and let $A_1, A_2, ..., A_n$ be objects of $\mathcal{C}$ corresponding to $A \in \mathcal{C}^\otimes_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ lying over the objects $B_1, B_2, ..., B_n$ of $\mathcal{D}$ corresponding to $B \in \mathcal{D}^\otimes_{\langle n \rangle} \simeq \mathcal{D}^{\times n}$ and lying over the objects $X_1, X_2, ..., X_n$ of $\mathcal{O}$ corresponding to $X \in \mathcal{O}^\otimes_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$.
   Let Z be an object of $\mathcal{C}$ lying over some object $W \in \mathcal{D}$ and lying over some object $Y \in \mathcal{O}$ and let $h : X \to Y$ of $\mathcal{O}^\otimes$ be an active morphism. For all $i \in \{1, .., n\}$ set $h^i := h \circ \alpha_i^X : X_i \to X \to Y$.
   Then the commutative square

$$\begin{array}{ccc} \mathcal{C}_Y(\otimes_h(A_1, A_2, ..., A_n), Z) & \longrightarrow & \prod_{i=1}^n \mathcal{C}_Y(h^i_*(A_i), Z) \\ \downarrow & & \downarrow \\ \mathcal{D}_Y(\otimes_h(B_1, B_2, ..., B_n), W) & \longrightarrow & \prod_{i=1}^n \mathcal{D}_Y(h^i_*(B_i), W) \end{array} \qquad (8)$$

   induced by the morphisms $\zeta_i(A)$ in $\mathcal{C}_Y$ and $\zeta_i(B)$ in $\mathcal{D}_Y$ for $i \in \{1, .., n\}$ is a pullback square, in other words the morphisms $\zeta_i(A)$ in $\mathcal{C}_Y$ for $i \in \{1, .., n\}$ exhibit $\otimes_h(A_1, A_2, ..., A_{n-1}, A_n)$ as a γ_Y-coproduct of the objects $h^1_*(A_1), ..., h^n_*(A_n)$.

   This follows from remark 3.2 and the fact that the fiber of the commutative square 7 over $h \in \mathrm{Mul}_{\mathcal{O}^\otimes}(X_1, ..., X_n; Y)$ is equivalent to square 8.

2. *Especially if* $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}, \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ *are unital cartesian* $\mathcal{O}^{\otimes}$*-monoidal categories,* $\gamma : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *exhibits* $\mathcal{C}^{\otimes}$ *as a cocartesian operad over* $\mathcal{D}^{\otimes}$ *if and only if the following condition* $(*)$ *holds:*

   Let $n \geq 2$ *and let* $A_1, A_2, ..., A_n$ *be objects of* $\mathcal{C}$ *corresponding to* $A \in \mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ *lying over the objects* $B_1, B_2, ..., B_n$ *of* $\mathcal{D}$ *corresponding to* $B \in \mathcal{D}^{\otimes}_{\langle n \rangle} \simeq \mathcal{D}^{\times n}$ *and lying over the objects* $X_1, X_2, ..., X_n$ *of* $\mathcal{O}$ *corresponding to* $X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$.

   Let Z *be an object of* $\mathcal{C}$ *lying over some object* $W \in \mathcal{D}$ *and lying over some object* $Y \in \mathcal{O}$ *and let* $h : X \to Y$ *of* $\mathcal{O}^{\otimes}$ *be an active morphism.*
   For all $i \in \{1, .., n\}$ *we set* $h^i := h \circ \alpha_i^X : X_i \to X \to Y$.

   For every $i \in \{1, .., n\}$ *we have a morphism* $\xi_i^A : h^i_*(A_i) \to \prod_{j=1}^n h^j_*(A_j)$ *in* $\mathcal{C}_Y$ *that is the identity on the* i*-th component and the zero morphism on every other component.*
   Then the commutative square

$$\begin{array}{ccc} \mathcal{C}_Y(\prod_{j=1}^n h^j_*(A_j), Z) & \longrightarrow & \prod_{j=1}^n \mathcal{C}_Y(h^j_*(A_j), Z) \\ \downarrow & & \downarrow \\ \mathcal{D}_Y(\prod_{j=1}^n h^j_*(B_j), W) & \longrightarrow & \prod_{j=1}^n \mathcal{D}_Y(h^j_*(B_j), W) \end{array} \tag{9}$$

   induced by the morphisms ξ_i^A *in* $\mathcal{C}_Y$ *and* ξ_i^B *in* $\mathcal{D}_Y$ *for* $i \in \{1, .., n\}$ *is a pullback square, in other words the morphisms* $\xi_i^A : h^i_*(A_i) \to \prod_{j=1}^n h^j_*(A_j)$ *in* $\mathcal{C}_Y$ *for* $i \in \{1, .., n\}$ *exhibit* $\prod_{j=1}^n h^j_*(A_j)$ *as a* γ_Y*-coproduct of the objects* $h^1_*(A_1), ..., h^n_*(A_n)$.

Definition 3.13.

Let $\mathcal{O}^{\otimes}$ *be a unital operad and* $\mathcal{C} \to \mathcal{O}, \mathcal{D} \to \mathcal{O}$ *locally cocartesian fibrations, whose fibers admit a zero object and finite products which are preserved by the induced functors on the fibers.*

Let $\beta : \mathcal{C} \to \mathcal{D}$ *be a map of locally cocartesian fibrations over* $\mathcal{O}$ *that induces on the fiber over every* $X \in \mathcal{O}$ *a finite products preserving functor.*

We say that $\beta : \mathcal{C} \to \mathcal{D}$ *exhibits* $\mathcal{C}$ *as cocartesian over* $\mathcal{D}$ *if condition* $(*)$ *from remark 3.12 2. holds.*

Remark 3.14.

Let $\mathcal{O}^{\otimes}$ *be a unital operad and* $\mathcal{C} \to \mathcal{O}, \mathcal{D} \to \mathcal{O}$ *cocartesian fibrations, whose fibers admit a zero object and finite products which are preserved by the induced functors on the fibers.*

Let $\beta : \mathcal{C} \to \mathcal{D}$ *be a map of cocartesian fibrations over* $\mathcal{O}$ *that induces on the fiber over every* $X \in \mathcal{O}$ *a finite products preserving functor.*

As the forgetful functor $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_\infty^{\Pi}) \to \mathrm{Fun}(\mathcal{O}, \mathsf{Cat}_\infty^{\Pi})$ *is an equivalence, the map* $\beta : \mathcal{C} \to \mathcal{D}$ *of cocartesian fibrations over* $\mathcal{O}$ *classifying a natural transformation of functors* $\mathcal{O} \to \mathsf{Cat}_\infty^{\Pi}$ *uniquely extends to a* $\mathcal{O}^{\otimes}$*-monoidal functor* $\gamma : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *between cartesian* $\mathcal{O}^{\otimes}$*-monoidal categories according to proposition 3.9.*

By definition the map $\beta : \mathcal{C} \to \mathcal{D}$ of cocartesian fibrations over $\mathcal{O}$ exhibits $\mathcal{C}$ as cocartesian over $\mathcal{D}$ if and only if the $\mathcal{O}^{\otimes}$-monoidal functor $\gamma : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ exhibits $\mathcal{C}^{\otimes}$ as a cocartesian operad over $\mathcal{D}^{\otimes}$.

Observation 3.15.

Let $\mathcal{O}^{\otimes}$ be a unital operad, $\mathcal{C} \to \mathcal{O}, \mathcal{D} \to \mathcal{O}$ locally cocartesian fibrations, whose fibers admit a zero object and finite products which are preserved by the induced functors on the fibers.

Let $\beta : \mathcal{C} \to \mathcal{D}$ be a map of locally cocartesian fibrations over $\mathcal{O}$ such that for all $X \in \mathcal{O}$ the induced functor $\beta_X : \mathcal{C}_X \to \mathcal{D}_X$ on the fiber over X preserves finite products and is a cartesian fibration.

Then $\beta : \mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{C}$ as cocartesian over $\mathcal{D}$ if and only if the following condition holds:

Let $n \geq 2$ and let $A_1, A_2, ..., A_n$ be objects of $\mathcal{C}$ corresponding to $A \in \mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ lying over the objects $B_1, B_2, ..., B_n$ of $\mathcal{D}$ corresponding to $B \in \mathcal{D}^{\otimes}_{\langle n \rangle} \simeq \mathcal{D}^{\times n}$ and lying over the objects $X_1, X_2, ..., X_n$ of $\mathcal{O}$ corresponding to $X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$.

Let Z be an object of $\mathcal{C}$ lying over some object $W \in \mathcal{D}$ and lying over some object $Y \in \mathcal{O}$ and let $h : X \to Y$ of $\mathcal{O}^{\otimes}$ be an active morphism.
For all $i \in \{1, .., n\}$ set $h^i := h \circ \alpha_i^X : X_i \to X \to Y$.

For every $i \in \{1, .., n\}$ we have a morphism $\xi_i^A : h^i_*(A_i) \to \prod_{j=1}^n h^j_*(A_j)$ in $\mathcal{C}_Y$ that is the identity on the i-th component and the zero morphism on every other component.
Let $\phi : \prod_{j=1}^n h^j_*(B_j) \to W$ be a morphism in $\mathcal{D}_Y$.
For all $i \in \{1, .., n\}$ we set $\phi_i := \phi \circ \xi_i^B : h^i_*(B_i) \to \prod_{j=1}^n h^j_*(B_j) \to W$.

Then the canonical functor

$$\varrho : (\mathcal{C}_Y)_{\prod_{j=1}^n h^j_*(B_j)}(\prod_{j=1}^n h^j_*(A_j), \phi^*(Z)) \to$$

$$\prod_{j=1}^n (\mathcal{C}_Y)_{h^j_*(B_j)}((\xi_j^B)^*(\prod_{j=1}^n h^j_*(A_j)), (\xi_j^B)^*(\phi^*(Z))) \to$$

$$\prod_{j=1}^n (\mathcal{C}_Y)_{h^j_*(B_j)}(h^j_*(A_j), \phi_j^*(Z))$$

induced by the functor

$$((\xi_j^B)^*)_{j=1}^n : (\mathcal{C}_Y)_{\prod_{j=1}^n h^j_*(B_j)} \to \prod_{j=1}^n (\mathcal{C}_Y)_{h^j_*(B_j)}$$

and the morphism $h^j_*(A_j) \to (\xi_j^B)^*(\prod_{j=1}^n h^j_*(A_j))$ in $(\mathcal{C}_Y)_{h^j_*(B_j)}$ corresponding to ξ_j^A is an equivalence.

This follows from the fact that square 9 induces on the fiber over $\phi \in \mathcal{D}_Y(\prod_{j=1}^n h^j_*(B_j), W)$ the functor ϱ.

3.3 A universal property of the cartesian structure

In this section we use the results about cocartesian operads of the previous section to prove a universal property of the cartesian structure (theorem 3.21).

We start with fixing some notation:

Denote

- $\mathsf{Cat}_\infty^\Pi \subset \mathsf{Cat}_\infty$ the subcategory with objects the small categories that admit finite products and with morphisms the functors that preserve finite products.
- $\mathsf{Cat}_\infty^* \subset \mathsf{Cat}_\infty$ the subcategory with objects the small categories that admit a final object and with morphisms the functors that preserve the final object.

The categories Cat_∞^Π and Cat_∞^* admit small limits which are preserved by the subcategory inclusions to Cat_∞.

Moreover Cat_∞^Π is preadditive by observation 3.8 and Cat_∞^* admits a zero object.

Denote

- $(\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R} \to \mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$ the pullback of the bicartesian fibration
$$\mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)$$
along the functor
$$\theta : \mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^* \subset \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty \xrightarrow{(-)^{\mathrm{op}} \times \mathrm{id}} \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty \xrightarrow{\times} \mathsf{Cat}_\infty$$
and
- $\Xi \subset (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}$ the full subcategory spanned by the triples $(\mathcal{C}, \mathcal{D}, \mathcal{F})$ consisting of small categories $\mathcal{C} \in \mathsf{Cat}_\infty^\Pi$, $\mathcal{D} \in \mathsf{Cat}_\infty^*$ and a right fibration $\mathcal{F} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ classifying a functor $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$ adjoint to a functor $\mathcal{C} \to \mathcal{P}(\mathcal{D})$ that preserves finite products.

The full subcategory $\mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ is closed under finite products so that $\mathcal{R}$ admits finite products which are preserved by the functor $\mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)$.

The functor $\theta : \mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^* \to \mathsf{Cat}_\infty$ preserves finite products as the functor $\times : \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ preserves finite products being the right adjoint of the diagonal functor $\mathsf{Cat}_\infty \to \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty$.

Thus the pullback $(\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}$ admits finite products which are preserved by the projections.

Remark 3.16.

1. *The full subcategory Ξ is closed under finite products in* $(\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}$.
2. *The category $\Xi \subset (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}$ admits a zero object.*

Proof. 1.: The final object of $(\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}$ belongs to Ξ :

The identity of the contractible category classifies the unique finite products preserving functor $* \to \mathcal{S}$ starting at the contractible category.

The full subcategory Ξ is closed under twofold products in $(\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}$:

Given small categories $\mathcal{C}, \mathcal{C}' \in \mathsf{Cat}_\infty^\Pi$ and $\mathcal{D}, \mathcal{D}' \in \mathsf{Cat}_\infty^*$ and right fibrations $\mathcal{F} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{F}' \to \mathcal{C}'^{\mathrm{op}} \times \mathcal{D}'$ classifying functors

$$\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}, \mathcal{C}' \times \mathcal{D}'^{\mathrm{op}} \to \mathcal{S}$$

adjoint to functors $\psi : \mathcal{C} \to \mathcal{P}(\mathcal{D})$ respectively $\phi : \mathcal{C}' \to \mathcal{P}(\mathcal{D}')$ the right fibration

$$\mathcal{F} \times \mathcal{F}' \to (\mathcal{C}^{\mathrm{op}} \times \mathcal{D}) \times (\mathcal{C}'^{\mathrm{op}} \times \mathcal{D}') \simeq (\mathcal{C} \times \mathcal{C}')^{\mathrm{op}} \times (\mathcal{D} \times \mathcal{D}')$$

classifies the functor $(\mathcal{C} \times \mathcal{C}') \times (\mathcal{D} \times \mathcal{D}')^{\mathrm{op}} \to \mathcal{S}$ adjoint to the functor

$$\mathcal{C} \times \mathcal{C}' \xrightarrow{\psi \times \phi} \mathcal{P}(\mathcal{D}) \times \mathcal{P}(\mathcal{D}') \to \mathcal{P}(\mathcal{D} \times \mathcal{D}').$$

So it is enough to see that the canonical functor $\mathcal{P}(\mathcal{D}) \times \mathcal{P}(\mathcal{D}') \to \mathcal{P}(\mathcal{D} \times \mathcal{D}')$ preserves finite products:

As the canonical functor $\mathcal{P}(\mathcal{D}) \times \mathcal{P}(\mathcal{D}') \to \mathcal{P}(\mathcal{D} \times \mathcal{D}')$ is natural in the small categories $\mathcal{D}, \mathcal{D}'$ and limits in presheaf categories are formed levelwise, this follows from the fact that the product functor $\times : \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ preserves finite products being the right adjoint of the diagonal functor $\mathcal{S} \to \mathcal{S} \times \mathcal{S}$.

2.: For arbitrary categories $\mathcal{C} \in \mathsf{Cat}_\infty^\Pi, \mathcal{D} \in \mathsf{Cat}_\infty^*$ and every right fibration $\mathcal{F} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ we have a canonical equivalence

$$(\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R}((*, *, \mathrm{id}_*), (\mathcal{C}, \mathcal{D}, \mathcal{F})) \simeq$$
$$(\mathsf{Cat}_\infty^\Pi(*, \mathcal{C}) \times \mathsf{Cat}_\infty^*(*, \mathcal{D})) \times_{\mathsf{Cat}_\infty(*, \mathcal{C}^{\mathrm{op}} \times \mathcal{D})} \mathcal{R}(\mathrm{id}_*, \mathcal{F}) \simeq$$
$$\mathcal{S}(*, \mathcal{F}_{*_\mathcal{C}, *_\mathcal{D}}) \simeq \mathcal{F}_{*_\mathcal{C}, *_\mathcal{D}}$$

of spaces, where $*_\mathcal{C}, *_\mathcal{D}$ denote the final objects of $\mathcal{C}$ respectively $\mathcal{D}$. □

By remark 3.16 we get a symmetric monoidal functor

$$\Xi^\times \subset ((\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R})^\times \to (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times.$$

Let $\mathcal{O}^\otimes$ be a unital operad, $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ a cartesian $\mathcal{O}^\otimes$-monoidal category classifying a map of operads $\psi : \mathcal{O}^\otimes \to (\mathsf{Cat}_\infty^\Pi)^\times$ and $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ a $\mathcal{O}^\otimes$-monoidal category classifying a map of operads $\phi : \mathcal{O}^\otimes \to (\mathsf{Cat}_\infty^*)^\times$.

By prop. 6.1 the symmetric monoidal functor

$$((\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R})^\times \to (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times$$

is a cocartesian fibration.

Denote

- $\mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}))^\otimes := \mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^\otimes$ the pullback of the cocartesian fibration

$$((\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R})^\times \to (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times$$

of operads along the map of operads $\mathcal{O}^\otimes \xrightarrow{(\psi,\phi)} (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times$.

- $\mathrm{Fun}^\Pi(\mathcal{C},\mathcal{P}(\mathcal{D}))^\otimes \subset \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}))^\otimes$ the pullback of the symmetric monoidal functor

$$\Xi^\times \subset ((\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R})^\times \to (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times$$

along the map of operads $\mathcal{O}^\otimes \xrightarrow{(\psi,\phi)} (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times$.

For every $X \in \mathcal{O}$ we have a canonical equivalence

$$\mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}))^\otimes_X \simeq \mathrm{Fun}(\mathcal{C}_X,\mathcal{P}(\mathcal{D}_X))$$

that restricts to an equivalence $\mathrm{Fun}^\Pi(\mathcal{C},\mathcal{D})^\otimes_X \simeq \mathrm{Fun}^\Pi(\mathcal{C}_X,\mathcal{P}(\mathcal{D}_X))$.

The next proposition 3.17 tells us that the functor

$$\Xi \subset (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R} \to \mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$$

exhibits Ξ as cocartesian over $\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$.

Thus by remark 3.14 the symmetric monoidal functor

$$\Xi^\times \subset ((\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R})^\times \to (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*)^\times$$

exhibits $\Xi^\times$ as a cocartesian operad over $(\mathsf{Cat}_\infty^\Pi)^\times \times_{\mathcal{F}\mathrm{in}_*} (\mathsf{Cat}_\infty^*)^\times$ so that the map of operads

$$\mathrm{Fun}^\Pi(\mathcal{C},\mathcal{P}(\mathcal{D}))^\otimes \to \mathcal{O}^\otimes$$

exhibits $\mathrm{Fun}^\Pi(\mathcal{C},\mathcal{P}(\mathcal{D}))^\otimes$ as a cocartesian operad over $\mathcal{O}^\otimes$.

Proposition 3.17. *The functor*

$$\Xi \subset (\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*) \times_{\mathsf{Cat}_\infty} \mathcal{R} \to \mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$$

exhibits Ξ *as cocartesian over* $\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$.

Proof. Let $n \geq 2$ and $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n, \mathcal{C} \in \mathsf{Cat}_\infty^\Pi$, $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n, \mathcal{D} \in \mathsf{Cat}_\infty^*$ be categories and $\mathcal{F}_i \to \mathcal{C}_i^{\mathrm{op}} \times \mathcal{D}_i$ for $1 \leq i \leq n$ and $\mathcal{G} \to \mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ be right fibrations that belong to Ξ.

For $1 \leq i \leq n$ denote

$$\xi_i = (\xi_i^1, \xi_i^2) : (\mathcal{C}_i, \mathcal{D}_i) \to (\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j) \simeq \prod_{j=1}^n (\mathcal{C}_j, \mathcal{D}_j)$$

the morphism in $\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$ and

$$\vartheta_i : (\mathcal{C}_i, \mathcal{D}_i, \mathcal{F}_i) \to (\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j, \prod_{j=1}^n \mathcal{F}_j) \simeq \prod_{j=1}^n (\mathcal{C}_j, \mathcal{D}_j, \mathcal{F}_j)$$

the morphism in Ξ that are the identity on the i-th component and the zero morphism on every other component.

The morphism

$$\vartheta_i : (\mathcal{C}_i, \mathcal{D}_i, \mathcal{F}_i) \to (\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j, \prod_{j=1}^n \mathcal{F}_j) \simeq \prod_{j=1}^n (\mathcal{C}_j, \mathcal{D}_j, \mathcal{F}_j)$$

in Ξ lies over the morphism

$$\xi_i = (\xi_i^1, \xi_i^2) : (\mathcal{C}_i, \mathcal{D}_i) \to (\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j) \simeq \prod_{j=1}^n (\mathcal{C}_j, \mathcal{D}_j)$$

in $\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$ and thus corresponds to a morphism

$$\varpi_i : \mathcal{F}_i \to \xi_i^*(\prod_{j=1}^n \mathcal{F}_j) \simeq ((\xi_i^1)^{op} \times \xi_i^2)^*(\prod_{j=1}^n \mathcal{F}_j)$$

in the fiber $\Xi_{(\mathcal{C}_i, \mathcal{D}_i)} \simeq \mathcal{R}_{\mathcal{C}_i^{op} \times \mathcal{D}_i}$.

ϖ_i induces on the fiber over a pair $(X, Y) \in \mathcal{C}_i^{op} \times \mathcal{D}_i$ the map of spaces

$$(\varpi_i)_{X,Y} : (\mathcal{F}_i)_{X,Y} \to \prod_{j=1}^n (\mathcal{F}_j)_{(\xi_i^1(X))_j, (\xi_i^2(Y))_j}$$

that is the identity on the i-th component and the unique morphism to the contractible space $(\mathcal{F}_j)_{*_\mathcal{C}, *_\mathcal{D}}$ on every other component.

Thus

$$(\varpi_i)_{X,Y} : (\mathcal{F}_i)_{X,Y} \to \prod_{j=1}^n (\mathcal{F}_j)_{(\xi_i^1(X))_j, (\xi_i^2(Y))_j} \simeq (\mathcal{F}_i)_{X,Y}$$

is equivalent to the identity.

By remark 3.12 we have to show that for all morphisms $h = (f, g) : (\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j) \to (\mathcal{C}, \mathcal{D})$ in $\mathsf{Cat}_\infty^\Pi \times \mathsf{Cat}_\infty^*$ the canonical map

$$\varrho : \Xi_{(\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j)}(\prod_{j=1}^n \mathcal{F}_j, h^*(\mathcal{G})) \xrightarrow{\chi} \prod_{i=1}^n \Xi_{(\mathcal{C}_i, \mathcal{D}_i)}(\xi_i^*(\prod_{j=1}^n \mathcal{F}_j), \xi_i^*(h^*(\mathcal{G})))$$

$$\to \prod_{i=1}^n \Xi_{(\mathcal{C}_i, \mathcal{D}_i}(\mathcal{F}_i, (h \circ \xi_i)^*(\mathcal{G}))$$

induced by the functor

$$((\xi_i)^*)_{i=1}^n : \Xi_{(\prod_{j=1}^n \mathcal{C}_j, \prod_{j=1}^n \mathcal{D}_j)} \to \prod_{i=1}^n \Xi_{(\mathcal{C}_i, \mathcal{D}_i)}$$

and the morphism $\varpi_i : \mathcal{F}_i \to \xi_i^*(\prod_{j=1}^n \mathcal{F}_j)$ in $\Xi_{(\mathcal{C}_i, \mathcal{D}_i)} \simeq \mathcal{R}_{\mathcal{C}_i^{op} \times \mathcal{D}_i}$ is an equivalence.

As ϖ_i is an equivalence, it remains to see that χ is an equivalence.

χ is equivalent to the canonical map

$$\chi' : \mathcal{R}_{(\prod_{j=1}^n \mathcal{C}_j)^{op} \times \prod_{j=1}^n \mathcal{D}_j}(\prod_{j=1}^n \mathcal{F}_j, (f^{op} \times g)^*(\mathcal{G})) \to$$

$$\prod_{i=1}^n \mathcal{R}_{\mathcal{C}_i^{op} \times \mathcal{D}_i}(((\xi_i^1)^{op} \times \xi_i^2)^*(\prod_{j=1}^n \mathcal{F}_j), ((\xi_i^1)^{op} \times \xi_i^2)^*((f^{op} \times g)^*(\mathcal{G})))$$

induced by the functor

$$(((\xi_i^1)^{op} \times \xi_i^2)^*)_{i=1}^n : \mathcal{R}_{(\prod_{j=1}^n \mathcal{C}_j)^{op} \times \prod_{j=1}^n \mathcal{D}_j} \to \prod_{i=1}^n \mathcal{R}_{\mathcal{C}_i^{op} \times \mathcal{D}_i}.$$

For every $1 \leq i \leq n$ the right fibrations $\mathcal{F}_i \to \mathcal{C}_i^{op} \times \mathcal{D}_i$ respectively $\mathcal{G} \to \mathcal{C}^{op} \times \mathcal{D}$ classify functors $F_i : \mathcal{C}_i \times \mathcal{D}_i^{op} \to \mathcal{S}$ respectively $G : \mathcal{C} \times \mathcal{D}^{op} \to \mathcal{S}$ that are adjoint to finite products preserving functors $H_i : \mathcal{C}_i \to \mathcal{P}(\mathcal{D}_j)$ respectively $M : \mathcal{C} \to \mathcal{P}(\mathcal{D})$.

Denote $\sigma : (\prod_{j=1}^n \mathcal{C}_j)^{op} \times \prod_{j=1}^n \mathcal{D}_j \simeq \prod_{j=1}^n (\mathcal{C}_j^{op} \times \mathcal{D}_j)$ the canonical functor that permutes the factors.

The right fibration $\prod_{j=1}^n \mathcal{F}_j \to \prod_{j=1}^n (\mathcal{C}_j^{op} \times \mathcal{D}_j) \simeq (\prod_{j=1}^n \mathcal{C}_j)^{op} \times \prod_{j=1}^n \mathcal{D}_j$ is classified by the functor

$$\prod_{j=1}^n \mathcal{C}_j \times (\prod_{j=1}^n \mathcal{D}_j)^{op} \simeq \prod_{j=1}^n (\mathcal{C}_j \times \mathcal{D}_j^{op}) \xrightarrow{\prod_{j=1}^n F_j} \prod_{j=1}^n \mathcal{S} \xrightarrow{\times} \mathcal{S}$$

that is adjoint to the finite products preserving functor

$\prod_{j=1}^n \mathcal{C}_j \xrightarrow{\prod_{j=1}^n H_j} \prod_{j=1}^n \mathcal{P}(\mathcal{D}_j) \xrightarrow{\alpha} \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)$.

The functor $g : \prod_{j=1}^n \mathcal{D}_j \to \mathcal{D}$ induces a functor $g^* : \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)$ that preserves small limits.

χ' is equivalent to the canonical map

$$\mathcal{P}((\prod_{j=1}^n \mathcal{C}_j)^{op} \times \prod_{j=1}^n \mathcal{D}_j)(\times \circ (\prod_{j=1}^n F_j) \circ \sigma, G \circ (f^{op} \times g)) \to$$

$$\prod_{i=1}^n \mathcal{P}(\mathcal{C}_i^{op} \times \mathcal{D}_i)(\times \circ (\prod_{j=1}^n F_j) \circ \sigma \circ ((\xi_i^1)^{op} \times \xi_i^2), G \circ (f^{op} \times g) \circ ((\xi_i^1)^{op} \times \xi_i^2))$$

induced by the functor $(((\xi_i^1)^{op} \times \xi_i^2)^*)_{i=1}^n : \mathcal{P}((\prod_{j=1}^n \mathcal{C}_j)^{op} \times \prod_{j=1}^n \mathcal{D}_j) \to \prod_{i=1}^n \mathcal{P}(\mathcal{C}_i^{op} \times \mathcal{D}_i)$ and so equivalent to the composition

$$\mathrm{Fun}(\prod_{j=1}^n \mathcal{C}_j, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^n H_j), g^* \circ M \circ f) \xrightarrow{\psi}$$

$$\prod_{i=1}^n \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^n H_j) \circ \xi_i^1, g^* \circ M \circ f \circ \xi_i^1) \xrightarrow{\phi}$$

$$\prod_{i=1}^n \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\mathcal{D}_i))((\xi_i^2)^* \circ \alpha \circ (\prod_{j=1}^n H_j) \circ \xi_i^1, (\xi_i^2)^* \circ g^* \circ M \circ f \circ \xi_i^1),$$

where ψ is induced by the functor

$$(\mathrm{Fun}(\xi_i^1, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)))_{i=1}^n : \mathrm{Fun}(\prod_{j=1}^n \mathcal{C}_j, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)) \to \prod_{i=1}^n \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j))$$

and ϕ by the functor

$$\prod_{i=1}^n \mathrm{Fun}(\mathcal{C}_i, (\xi_i^2)^*) : \prod_{i=1}^n \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)) \to \prod_{i=1}^n \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\mathcal{D}_i)).$$

The functor $(\mathrm{Fun}(\xi_i^1, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)))_{i=1}^n$ restricts to the functor $(\mathrm{Fun}^{\Pi}(\xi_i^1, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)))_{i=1}^n$:

$$\mathrm{Fun}^{\Pi}(\prod_{j=1}^n \mathcal{C}_j, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)) \to \prod_{i=1}^n \mathrm{Fun}^{\Pi}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)),$$

which is an equivalence.

As both functors $\alpha \circ (\prod_{j=1}^n H_j)$ and $g^* \circ M \circ f$ preserve finite products, the map

$$\psi : \mathrm{Fun}(\prod_{j=1}^n \mathcal{C}_j, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^n H_j), g^* \circ M \circ f) \to$$

$$\prod_{i=1}^{n} \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^{n} H_j) \circ \xi_i^1, g^* \circ M \circ f \circ \xi_i^1)$$

induced by the functor $(\mathrm{Fun}(\xi_i^1, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)))_{i=1}^n$ is equivalent to the map

$$\psi' : \mathrm{Fun}^{\Pi}(\prod_{j=1}^{n} \mathcal{C}_j, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^{n} H_j), g^* \circ M \circ f) \to$$

$$\prod_{i=1}^{n} \mathrm{Fun}^{\Pi}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^{n} H_j) \circ \xi_i^1, g^* \circ M \circ f \circ \xi_i^1)$$

induced by the equivalence $(\mathrm{Fun}^{\Pi}(\xi_i^1, \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)))_{i=1}^n$:

$$\mathrm{Fun}^{\Pi}(\prod_{j=1}^{n} \mathcal{C}_j, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j)) \to \prod_{i=1}^{n} \mathrm{Fun}^{\Pi}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j)).$$

Thus ψ is an equivalence.

To see that ϕ is an equivalence, it is enough to check that for all $1 \leq i \leq n$ the canonical map

$$\zeta : \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j))(\alpha \circ (\prod_{j=1}^{n} H_j) \circ \xi_i^1, g^* \circ M \circ f \circ \xi_i^1) \to$$

$$\mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\mathcal{D}_i))((\xi_i^2)^* \circ \alpha \circ (\prod_{j=1}^{n} H_j) \circ \xi_i^1, (\xi_i^2)^* \circ g^* \circ M \circ f \circ \xi_i^1)$$

is an equivalence.

For $1 \leq i \leq n$ denote $\varsigma_i : \mathcal{P}(\mathcal{D}_i) \to \prod_{j=1}^n \mathcal{P}(\mathcal{D}_j)$ the morphism in Cat_∞^* that is the identity on the i-th component and the zero morphism on every other component.

Denote $\mathrm{pr}_i : \prod_{j=1}^n \mathcal{D}_j \to \mathcal{D}_i$ the i-th projection and $\mathrm{pr}_i^* : \mathcal{P}(\mathcal{D}_i) \to \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)$ the induced functor.

We have canonical equivalences $(\prod_{j=1}^n H_j) \circ \xi_i^1 \simeq \varsigma_i \circ H_i$ and $\alpha \circ \varsigma_i \simeq \mathrm{pr}_i^*$ that give rise to an equivalence $\alpha \circ (\prod_{j=1}^n H_j) \circ \xi_i^1 \simeq \alpha \circ \varsigma_i \circ H_i \simeq \mathrm{pr}_i^* \circ H_i$.

Set $\iota := \xi_i^2 : \mathcal{D}_i \to \prod_{j=1}^n \mathcal{D}_j$. By lemma 3.18 ι is fully faithful and right adjoint to the projection $\mathrm{pr}_i : \prod_{j=1}^n \mathcal{D}_j \to \mathcal{D}_i$.

Denote $\eta : \mathrm{id} \to \iota \circ \mathrm{pr}_i$ the unit and $\varepsilon : \mathrm{pr}_i \circ \iota \to \mathrm{id}$ the counit of this adjunction.

The counit $\varepsilon : \mathrm{pr}_i \circ \iota \to \mathrm{id}$ is an equivalence as ι is fully faithful.

The adjunction $\mathrm{pr}_i : \prod_{j=1}^n \mathcal{D}_j \rightleftarrows \mathcal{D}_i : \iota$ gives rise to an adjunction

$$\mathrm{pr}_i^* : \mathcal{P}(\mathcal{D}_i) \rightleftarrows \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j) : \iota^*,$$

whose unit is $\varepsilon^* : \mathrm{id} \to (\mathrm{pr}_i \circ \iota)^* \simeq \iota^* \circ \mathrm{pr}_i^*$ and counit is $\eta^* : \mathrm{pr}_i^* \circ \iota^* \simeq (\iota \circ \mathrm{pr}_i)^* \to \mathrm{id}$.

Hence $\mathrm{pr}_i^* : \mathcal{P}(\mathcal{D}_i) \to \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j)$ is fully faithful, i.e. the adjunction $\mathrm{pr}_i^* : \mathcal{P}(\mathcal{D}_i) \rightleftarrows \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j) : \iota^*$ is a colocalization.

The colocalization $\mathrm{pr}_i^* : \mathcal{P}(\mathcal{D}_i) \rightleftarrows \mathcal{P}(\prod_{j=1}^n \mathcal{D}_j) : \iota^*$ gives rise to a colocalization

$$\mathrm{Fun}(\mathcal{C}_i, \mathrm{pr}_i^*) : \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\mathcal{D}_i)) \rightleftarrows \mathrm{Fun}(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j)) : \mathrm{Fun}(\mathcal{C}_i, \iota^*)$$

$$= \mathrm{Fun}(\mathcal{C}_i, (\xi_i^2)^*).$$

The equivalence $\alpha \circ (\prod_{j=1}^{n} H_j) \circ \xi_i^1 \simeq pr_i^* \circ H_i$ guarantees that $\alpha \circ (\prod_{j=1}^{n} H_j) \circ \xi_i^1$ belongs to the essential image of the fully faithful functor $Fun(\mathcal{C}_i, pr_i^*) : Fun(\mathcal{C}_i, \mathcal{P}(\mathcal{D}_i)) \to Fun(\mathcal{C}_i, \mathcal{P}(\prod_{j=1}^{n} \mathcal{D}_j))$ so that ζ is an equivalence.

□

To prove proposition 3.17 we used the following lemma:

Lemma 3.18. *Let* $n \in \mathbb{N}$ *and* $\mathcal{D}_1, ..., \mathcal{D}_n$ *be small categories that admit a final object* $*_1, *_2, ...$ *respectively* $*_n$.

Denote $pr_i : \prod_{j=1}^{n} \mathcal{D}_j \to \mathcal{D}_i$ *the i-th projection for* $1 \leq i \leq n$ *and*

$$\iota : \mathcal{D}_i \simeq (\prod_{j=1}^{i-1} *_j) \times \mathcal{D}_i \times (\prod_{j=i+1}^{n} *_j) \to \prod_{j=1}^{n} \mathcal{D}_j$$

the fully faithful functor, which is the identity of $\mathcal{D}_i$ *on the i-th component and the constant functor with image the final object on every other component.*

The functor ι *is right adjoint to* pr_i.

Proof. Set $\mathcal{D} := \prod_{j=1}^{n} \mathcal{D}_j$ and denote $\delta : \mathcal{D}_j \to Fun(\mathcal{D}, \mathcal{D}_j)$ the diagonal functor.

Denote $\eta : id_{\mathcal{D}} \to \iota \circ pr_i$ the morphism in $Fun(\mathcal{D}, \prod_{j=1}^{n} \mathcal{D}_j) \simeq \prod_{j=1}^{n} Fun(\mathcal{D}, \mathcal{D}_j)$ corresponding to the n natural transformations $\alpha_j : pr_j \to pr_j \circ \iota \circ pr_i$ of functors $\mathcal{D} \to \mathcal{D}_j$ for $1 \leq j \leq n$ with $\alpha_i : pr_i \to pr_i \circ \iota \circ pr_i \simeq pr_i$ the identity and $\alpha_j : pr_j \to pr_j \circ \iota \circ pr_i \simeq \delta(*_j)$ for $j \neq i$ the unique morphism.

So the composition $pr_i \circ \eta : pr_i \to pr_i \circ \iota \circ pr_i \simeq pr_i$ is equivalent to the identity.

To complete the proof it is enough to show that $\eta \circ \iota \circ pr_i : \iota \circ pr_i \to \iota \circ pr_i \circ \iota \circ pr_i$ is an equivalence.

The morphism $\eta \circ \iota \circ pr_i$ of $Fun(\mathcal{D}, \prod_{j=1}^{n} \mathcal{D}_j) \simeq \prod_{j=1}^{n} Fun(\mathcal{D}, \mathcal{D}_j)$ corresponds to the n natural transformations

$$\beta_j : pr_j \circ \iota \circ pr_i \to pr_j \circ \iota \circ pr_i \circ \iota \circ pr_i$$

of functors $\mathcal{D} \to \mathcal{D}_j$ for $1 \leq j \leq n$ with $\beta_i : pr_i \circ \iota \circ pr_i \to pr_i \circ \iota \circ pr_i \circ \iota \circ pr_i \simeq pr_i \circ \iota \circ pr_i$ the identity and $\beta_j : pr_j \circ \iota \circ pr_i \simeq \delta(*_j) \to pr_j \circ \iota \circ pr_i \circ \iota \circ pr_i \simeq \delta(*_j)$ for $j \neq i$ the identity.

Thus $\eta \circ \iota \circ pr_i : \iota \circ pr_i \to \iota \circ pr_i \circ \iota \circ pr_i$ is an equivalence.

□

Corollary 3.19. *Let* $\mathcal{O}^{\otimes}$ *be a unital operad,* $\mathcal{C}^{\otimes}$ *a* $\mathcal{O}^{\otimes}$*-monoidal category and* $\mathcal{D}^{\otimes}$ *a* $\mathcal{O}^{\otimes}$*-monoidal category compatible with small colimits.*

The map of operads

$$Fun^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))^{\otimes} \to \mathcal{O}^{\otimes}$$

exhibits $Fun^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))^{\otimes}$ *as a cocartesian operad over* $\mathcal{O}^{\otimes}$.

Proof. The operad $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}(\mathcal{D}))^{\otimes}$ over $\mathcal{O}^{\otimes}$ is the pullback of the symmetric monoidal functor

$$\Xi^{\times} \subset ((\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*}) \times_{\mathsf{Cat}_{\infty}} \mathcal{R})^{\times} \to (\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*})^{\times}$$

along the map of operads $\mathcal{O}^{\otimes} \to (\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*})^{\times}$.

By proposition 3.17 the functor

$$\Xi \subset (\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*}) \times_{\mathsf{Cat}_{\infty}} \mathcal{R} \to \mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*}$$

exhibits Ξ as cocartesian over $\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*}$ so that by remark 3.14 the symmetric monoidal functor

$$\Xi^{\times} \subset ((\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*}) \times_{\mathsf{Cat}_{\infty}} \mathcal{R})^{\times} \to (\mathsf{Cat}_{\infty}^{\Pi} \times \mathsf{Cat}_{\infty}^{*})^{\times}$$

exhibits $\Xi^{\times}$ as a cocartesian operad over $(\mathsf{Cat}_{\infty}^{\Pi})^{\times} \times_{\mathcal{F}\mathrm{in}_*} (\mathsf{Cat}_{\infty}^{*})^{\times}$.

So the pullback

$$\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}(\mathcal{D}))^{\otimes} \to \mathcal{O}^{\otimes}$$

exhibits $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}(\mathcal{D}))^{\otimes}$ as a cocartesian operad over $\mathcal{O}^{\otimes}$.

□

Remark 3.20.

For every small categories $\mathcal{C},\mathcal{D}$ the full subcategory $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{S}) \subset \mathrm{Fun}(\mathcal{C},\mathcal{S})$ and thus also the full subcategory

$$\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{S})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}(\mathcal{D})) \subset \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D})) \simeq$$

$$\mathrm{Fun}(\mathcal{D}^{\mathrm{op}},\mathrm{Fun}(\mathcal{C},\mathcal{S}))$$

is a localization.

Thus by corollary 6.54 the map of operads $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}(\mathcal{D}))^{\otimes} \to \mathcal{O}^{\otimes}$ is a locally cocartesian fibration.

As the map of operads $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}_(\mathcal{D}))^{\otimes} \to \mathcal{O}^{\otimes}$ exhibits $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}_(\mathcal{D}))^{\otimes}$ as a cocartesian operad over $\mathcal{O}^{\otimes}$, by remark 3.7 the locally cocartesian fibration $\mathrm{Fun}^{\Pi}(\mathcal{C},\mathcal{P}(\mathcal{D}))^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration.

Given a category $\mathcal{B}$ and categories $\mathcal{C} \to \mathcal{B}, \mathcal{D} \to \mathcal{B}$ over $\mathcal{B}$ denote

$$\mathrm{Fun}_{\mathcal{B}}^{\Pi}(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}_{\mathcal{B}}(\mathcal{C},\mathcal{D})$$

the full subcategory spanned by the functors over $\mathcal{B}$ that induce on the fiber over every object X of $\mathcal{B}$ a functor that preserves finite products.

Given an operad $\mathcal{O}^{\otimes}$ and $\mathcal{O}^{\otimes}$-monoidal categories $\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}$ denote

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax},\Pi}(\mathcal{C},\mathcal{D}) \subset \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C},\mathcal{D})$$

the full subcategory spanned by the lax $\mathcal{O}^{\otimes}$-monoidal functors that induce on the fiber over every object X of $\mathcal{O}$ a functor that preserves finite products.

Now we are ready to prove the desired universal property of the cartesian structure.

Theorem 3.21. *Let $\mathcal{O}^{\otimes}$ be a unital operad.*

Let $\mathcal{C}^{\otimes}$ be a cartesian $\mathcal{O}^{\otimes}$-monoidal category such that $\mathcal{C} \to \mathcal{O}$ classifies a functor $\mathcal{O} \to \mathsf{Cat}_{\infty}^{\Pi}$ and $\mathcal{D}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category.
Assume that the tensorunit of $\mathcal{D}^{\otimes}$ is a final object of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{D})$.

The forgetful functor

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax},\Pi}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{C}, \mathcal{D})$$

is an equivalence.

Proof. Consider the following commutative square:

$$\begin{array}{ccc} \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}) & \longrightarrow & \mathrm{Fun}_{\mathcal{O}}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \end{array}$$

As the square is a pullback square, it is enough to show that the right vertical functor

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \to \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))$$

in the diagram gets an equivalence after pulling back to the full subcategory

$$\mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \subset \mathrm{Fun}_{\mathcal{O}}(\mathcal{C}, \mathcal{P}(\mathcal{D})).$$

We have a pullback square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax},\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})) & \longrightarrow & \mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{lax}}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})) & \longrightarrow & \mathrm{Fun}_{\mathcal{O}}(\mathcal{C}, \mathcal{P}(\mathcal{D})) \end{array}$$

that is equivalent to the pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) & \longrightarrow & \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \end{array}$$

by prop. 6.28.
Consequently we have to see that the forgetful functor

$$\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D})))$$

is an equivalence.

This follows from proposition 3.3 4. as $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{P}(\mathcal{D}))^{\otimes}$ is a cocartesian operad over $\mathcal{O}^{\otimes}$ by proposition 3.17 using that the $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{D}^{\otimes} \to \mathcal{P}(\mathcal{D})^{\otimes}$ preserves final objects and tensorunits. □

As next we deduce prop. 3.22 from a universal property of the cocartesian structure (theorem 2.4.3.18. [18]) and the universal property of the cartesian structure of theorem 3.21.

Proposition 3.22.

Let $\mathcal{O}^{\otimes}$ be a unital operad, $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category, whose tensorunit is a final object of $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$, and $\mathcal{D}$ a preadditive category.

The functor

$$\mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{O} \times \mathcal{D}, \mathcal{C}) \times_{\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}))} \mathrm{Fun}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})) \to \mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{O} \times \mathcal{D}, \mathcal{C})$$

is an equivalence.

Remark 3.23. *If for all $\mathrm{X} \in \mathcal{O}$ the fiber $\mathcal{C}_{\mathrm{X}}$ admits finite products, the categories $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}), \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ admit finite products which are formed levelwise.*

So the canonical equivalence

$$\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{O}}(\mathcal{O} \times \mathcal{D}, \mathcal{C})$$

restricts to an equivalence

$$\mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{O} \times \mathcal{D}, \mathcal{C}).$$

Thus the forgetful functor

$$\mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) \times_{\mathrm{Fun}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}))} \mathrm{Fun}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$$

$$\to \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}))$$

is an equivalence.

Hence by remark 1.3.3 the functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ induces an equivalence

$$\mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})) \to \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})).$$

Proof. As $\mathcal{D}$ is preadditive, we have a canonical equivalence $\mathcal{D}^{\amalg} \simeq \mathcal{D}^{\times}$ so that the $\mathcal{O}^{\otimes}$-monoidal category $\alpha : \mathcal{O}^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{D}^{\amalg} \to \mathcal{O}^{\otimes}$ is cartesian.

So by theorem 3.21 the forgetful functor

$$\mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{O} \times \mathcal{D}, \mathcal{C}) \times_{\mathrm{Fun}_{\mathcal{O}}(\mathcal{O} \times \mathcal{D}, \mathcal{C})} \mathrm{Alg}_{\mathcal{O} \times \mathcal{D}/\mathcal{O}}(\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}^{\Pi}(\mathcal{O} \times \mathcal{D}, \mathcal{C})$$

is an equivalence.

Denote β the constant functor $\mathcal{D} \to \mathrm{Alg}_{\mathcal{O}}(\mathcal{O})$ with value the identity of $\mathcal{O}^{\otimes}$. By [18] theorem 2.4.3.18. we have a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O} \times \mathcal{D}/\mathcal{O}}(\mathcal{C}) \simeq \{\alpha\} \times_{\mathrm{Alg}_{\mathcal{O} \times \mathcal{D}}(\mathcal{O})} \mathrm{Alg}_{\mathcal{O} \times \mathcal{D}}(\mathcal{C}) \simeq$$

$$\{\beta\} \times_{\mathrm{Fun}(\mathcal{D}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{O}))} \mathrm{Fun}(\mathcal{D}, \mathrm{Alg}_{\mathcal{O}}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}))$$

that fits into a commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}\times\mathcal{D}/\mathcal{O}}(\mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}\times\mathcal{D}, \mathcal{C}) & \longrightarrow & \mathrm{Fun}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})). \end{array}$$

□

The following corollary 3.24 generalizes an important statement in deformation theory ([18] theorem 7.3.4.7.) from stable to preadditive $\mathcal{O}^{\otimes}$-monoidal categories:

Corollary 3.24. *Let $\mathcal{O}^{\otimes}$ be a unital operad and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a preadditive $\mathcal{O}^{\otimes}$-monoidal category.*

The forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ induces an equivalence

$$\mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}) \to \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}).$$

Thus for every preadditive category $\mathcal{D}$ and every finite products preserving functor $\mathcal{D} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})$ the category $\mathrm{Fun}_{\mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}})$ is contractible.

Proof. We have a pullback square

$$\begin{array}{ccc} \mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}) & \longrightarrow & \mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{P}(\mathcal{C}))^{\mathrm{nu}}) \\ \downarrow & & \downarrow \\ \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) & \longrightarrow & \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{P}(\mathcal{C}))). \end{array}$$

Consequently we may replace $\mathcal{C}^{\otimes}$ by $\mathcal{P}(\mathcal{C})^{\otimes}$ in the statement we want to prove and so may assume that $\mathcal{C}^{\otimes}$ is compatible with finite coproducts.

Denote $\mathcal{C}'^{\otimes} := \mathcal{P}^{\Sigma}(\mathcal{C})^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}$ the full subcategory of $\mathcal{P}(\mathcal{C})^{\otimes}$ spanned by the presheaves on $\mathcal{C}_{\mathrm{X}}$ for some $\mathrm{X} \in \mathcal{O}$ that preserve finite products.

As $\mathcal{C}^{\otimes}$ is compatible with finite coproducts, $\mathcal{P}^{\Sigma}(\mathcal{C})^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}$ is an accessible $\mathcal{O}^{\otimes}$-monoidal localization.

For every $\mathrm{X} \in \mathcal{O}$ we have a canonical equivalence $\mathcal{P}^{\Sigma}(\mathcal{C})_{\mathrm{X}} \simeq \mathcal{P}^{\Sigma}(\mathcal{C}_{\mathrm{X}}) \simeq \mathrm{Fun}^{\Pi}((\mathcal{C}_{\mathrm{X}})^{\mathrm{op}}, \mathrm{Cmon}(\mathcal{S}))$ so that $\mathcal{P}^{\Sigma}(\mathcal{C})^{\otimes}$ is a preadditive presentably $\mathcal{O}^{\otimes}$-monoidal category.

We have a pullback square

$$\begin{array}{ccc} \mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}) & \longrightarrow & \mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}')^{\mathrm{nu}}) \\ \downarrow & & \downarrow \\ \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C})) & \longrightarrow & \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O}, \mathcal{C}')). \end{array}$$

Consequently we may replace $\mathcal{C}^{\otimes}$ by $\mathcal{C}'^{\otimes}$ in the statement we want to prove and so can assume that $\mathcal{C}^{\otimes}$ is a preadditive presentably $\mathcal{O}^{\otimes}$-monoidal category, especially that for every $\mathrm{X} \in \mathcal{O}$ the fiber $\mathcal{C}_{\mathrm{X}}$ admits finite limits.

By 2.1 we have a colocalization

$$\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \rightleftarrows \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$$

and commutative squares

$$\begin{array}{ccc} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) & \xrightarrow{\mathbb{1}\oplus -} & \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}} \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} & \longrightarrow & \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}} & \xrightarrow{0\times_{\mathbb{1}} -} & \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}). \end{array}$$

Let $\mathcal{D}$ be a preadditive category.

By Yoneda applied to the homotopy category of the category of small preadditive categories it is enough to show that the functor

$$\mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}})) \to \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})))$$

induces a bijection on equivalence classes.

We have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Cmon}(\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}})) & \longrightarrow & \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Cmon}(\mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}))) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}) & \xrightarrow{\psi} & \mathrm{Fun}^{\Pi}(\mathcal{D}, \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})). \end{array}$$

The functor $\mathbb{1} \oplus - : \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}}$ is right adjoint to the forgetful functor $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})$.

So by proposition 3.22 the functor $\mathbb{1} \oplus - : \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}}$ lifts to a functor $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}_{/\mathbb{1}})$.

The composition $\mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$ is a section of the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})$.

As the forgetful functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})$ admits a section, ψ admits a section and is thus essentially surjective.

Let $\varphi, \varphi' : \mathcal{D} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}}$ be functors such that both compositions $\mathcal{D} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})$ are equivalent to some finite products preserving functor H.

Both compositions

$$\mathcal{D} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}} \to \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}}$$

are equivalent to the finite products preserving functor

$$\mathcal{D} \xrightarrow{\mathrm{H}} \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C}) \xrightarrow{\mathbb{1}\oplus -} \mathrm{Fun}_{\mathcal{O}}(\mathcal{O},\mathcal{C})_{/\mathbb{1}}.$$

Thus by proposition 3.22 both functors $\mathcal{D} \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})^{\mathrm{nu}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})_{/\mathbb{1}}$ are equivalent so that φ, φ' are equivalent.

□

4 Comparison results

In section 2.3 we defined restricted L_∞-algebras in a nice preadditive symmetric monoidal category $\mathcal{C}$ (def. 2.26).

If $\mathcal{C}$ is stable, we have a notion of Lie algebra in $\mathcal{C}$ as algebra over the spectral Lie operad, which we define as the Koszul dual operad of the non-counital cocommutative cooperad in spectra.

Moreover given a field K we have the notion of restricted Lie algebra over K which is nothing than a Lie algebra over K if K has char. zero.

In this section we relate restricted L_∞-algebras in a nice stable symmetric monoidal category $\mathcal{C}$ to Lie algebras in $\mathcal{C}$ over the spectral Lie operad and to simplicial restricted Lie algebras over a field K :

We construct a forgetful functor

$$\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

from the ∞-category of restricted L_∞-algebras in $\mathcal{C}$ to the ∞-category of algebras over the spectral Lie operad (theorem 4.2) and we construct a forgetful functor

$$\mathrm{Lie}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to (\mathsf{s}\mathrm{Lie}_{\mathrm{K}}^{\mathrm{res}})_\infty$$

from the ∞-category of restricted L_∞-algebras in connective H(K)-module spectra to the ∞-category underlying a right induced model structure on the category $\mathsf{s}\mathrm{Lie}_{\mathrm{K}}^{\mathrm{res}}$ of simplicial restricted Lie algebras over K (prop. 4.34).

Moreover we show that the forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ is an equivalence if $\mathcal{C}$ is a $\mathbb{Q}$-linear stable ∞-category, i.e. a stable ∞-category left tensored over $\mathrm{H}(\mathbb{Q})$-module spectra (theorem 4.5).

Besides this we will see that the spectral tangent Lie algebra refines to a restricted L_∞-algebra (example 4.3).

4.1 Comparison to spectral Lie algebras

Given a nice stable symmetric monoidal category $\mathcal{C}$ we defined restricted L_∞-algebras by their relation to cocommutative bialgebras (def. 2.26) expressed by the enveloping bialgebra-primitive elements adjunction

$$\mathcal{U} : \mathrm{Lie}(\mathcal{C}) \rightleftarrows \mathrm{Bialg}(\mathcal{C}) : \bar{\mathcal{P}}.$$

Mimicing the classical Koszul duality between the Lie operad and the cocommutative cooperad we construct an adjunction

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{pd,conil}}, \tag{10}$$

between spectral Lie algebras in $\mathcal{C}$ and conilpotent divided power coalgebras in $\mathcal{C}$, where the left adjoint takes the homology and the right adjoint takes the tangent Lie algebra.

To relate both notions of Lie algebras we show (theorem 4.2) that adjunction 10 gives rise to an adjunction

$$\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Bialg}(\mathcal{C}) : \mathfrak{P}, \tag{11}$$

where the right adjoint lifts the functor $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C}$ (constructed in 2.2.3).

Then by the universal property of $\mathrm{Lie}(\mathcal{C})$ (remark 2.27) the right adjoint $\mathfrak{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ of adjunction 11 factors as

$$\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

for a unique functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ over $\mathcal{C}$.

To construct adjunction 10, we use Koszul-duality for operads and their algebras which we treat in the next section.

We start with constructing adjunction 10.

We define the spectral Lie operad $\mathrm{Lie} := (\mathrm{Cocomm}^{\mathrm{ncu}})^{\vee}(-1)$ as the negative shift of the Koszul-dual of the non-counital cocommutative cooperad $\mathrm{Cocomm}^{\mathrm{ncu}}$.

By [5] the spectral operad Lie has its homology the classical Lie operad.

By 2.2.4 the operadic shift gives rise to an equivalence $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{Lie}(1)}(\mathcal{C})$ with underlying functor the shift functor.

By prop. 4.21 we have an adjunction

$$\mathrm{Alg}_{\mathrm{Lie}(1)}(\mathcal{C}) = \mathrm{LMod}_{\mathrm{Lie}(1)}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{dp,conil}} = \mathrm{coLMod}_{\mathrm{Cocomm}^{\mathrm{ncu}}}(\mathcal{C}).$$

So we get an adjunction $\theta : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{dp,conil}}$, where the left adjoint lifts the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathrm{triv}\circ_{\mathrm{Lie}}-} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C}$ and the right adjoint lifts the functor $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{dp,conil}} \xrightarrow{\mathrm{triv}\circ^{\mathrm{Cocomm}^{\mathrm{ncu}}}-} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$ by prop. 4.21.

Composing this adjunction

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{dp,conil}}$$

with the left adjoint forgetful functor $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{dp,conil}} \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ of lemma 2.19 we get an adjunction

$$\mathcal{H} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} : \gamma,$$

where the left adjoint lifts the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathrm{triv}\circ_{\mathrm{Lie}}-} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C}$ and the right adjoint lifts the functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$, where the functor Prim is constructed in 2.2.3.

We call $\mathcal{H}$ the Lie-homology and γ the tangent Lie algebra functor.

By remark 4.19 we have a canonical equivalence

$$\theta \circ \mathrm{triv}_{\mathrm{Lie}} \simeq (\mathrm{Cocomm}^{\mathrm{ncu}} \circ_{\mathrm{triv}} -) \circ \Sigma$$

that leads to a canonical equivalence $\mathcal{H} \circ \mathrm{triv}_{\mathrm{Lie}} \simeq \mathrm{S} \circ \Sigma$, where S denotes the composition $\mathcal{C} \xrightarrow{\mathrm{Cocomm}^{\mathrm{ncu}}\circ_{\mathrm{triv}}-} \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{dp,conil}} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

We call S the symmetric functor.

As next we observe how the tangent Lie algebra functor gives rise to a lift of the primitive elements to Lie algebras:

For this we use the following fact (remark 4.17):

Let $\mathcal{C}$ be a stable category and $\mathcal{D}$ a category that admits geometric realizations, finite products and totalizations and $\phi : \mathcal{D} \to \mathcal{C}$ a conservative functor that preserves geometric realizations, finite products and totalizations.

Then the Bar-Cobar adjunction $\mathrm{Bar} : \mathrm{Mon}(\mathcal{D}) \rightleftarrows \mathcal{D}_* : \mathrm{Cobar}$ for the cartesian structure on $\mathcal{D}$ is an equivalence.

Observation 4.1. *Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits that admits totalizations.*

Let $\mathcal{D}$ be a category that admits geometric realizations, finite products and totalizations and $\phi : \mathcal{D} \to \mathcal{C}$ a conservative functor that preserves geometric realizations, finite products and totalizations.

There is a correspondence between functors

$$\psi : \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \to \mathcal{D}$$

lifting the functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$ *and functors*

$$\Psi : \mathrm{Bialg}(\mathcal{C}) \to \mathcal{D}$$

lifting the functor $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C}$*:*

ψ *gives rise to a functor*

$$\mathrm{Bialg}(\mathcal{C}) \simeq \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \xrightarrow{\mathrm{Mon}(\psi)} \mathrm{Mon}(\mathcal{D}) \xrightarrow{\mathrm{Bar}} \mathcal{D}$$

lifting the functor $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C}$.

Conversely Ψ *yields a functor*

$$\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Cobar}} \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \simeq \mathrm{Bialg}(\mathcal{C}) \xrightarrow{\Psi} \mathcal{D}$$

lifting the functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$.

The functors $\psi \mapsto \mathrm{Bar} \circ \mathrm{Mon}(\psi)$ *and* $\Psi \mapsto \Psi \circ \mathrm{Cobar}$ *are inverse to each other.*

Proof. The composition

$$\mathrm{Bialg}(\mathcal{C}) \simeq \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \xrightarrow{\mathrm{Mon(Cobar)}} \mathrm{Mon}(\mathrm{Bialg}(\mathcal{C})) \xrightarrow{\mathrm{Mon}(\Psi)}$$

$$\mathrm{Mon}(\mathcal{D}) \xrightarrow{\mathrm{Bar}} \mathcal{D}$$

is equivalent to the functor

$$\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\Psi} \mathcal{D} \xrightarrow{\mathrm{Cobar}} \mathrm{Mon}(\mathcal{D}) \xrightarrow{\mathrm{Bar}} \mathcal{D}$$

and so equivalent to Ψ.

The composition

$$\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Cobar}} \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \xrightarrow{\mathrm{Mon}(\psi)} \mathrm{Mon}(\mathcal{D}) \xrightarrow{\mathrm{Bar}} \mathcal{D}$$

is equivalent to the functor

$$\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\psi} \mathcal{D} \xrightarrow{\mathrm{Cobar}} \mathrm{Mon}(\mathcal{D}) \xrightarrow{\mathrm{Bar}} \mathcal{D}$$

and so equivalent to ψ.

□

Applying observation 4.1 to the adjunction $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ we get the following theorem:

Theorem 4.2. *Let $\mathcal{C}$ be an additive symmetric monoidal category compatible with small colimits such that $\mathcal{C}$ admits small limits.*

There is a forgetful functor

$$\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

over $\mathcal{C}$.

Proof. First assume that $\mathcal{C}$ is stable.

The right adjoint γ of the adjunction $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ lifts the functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$ and so by remark 4.17 factors as

$$\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Cobar}} \mathrm{Bialg}(\mathcal{C}) \xrightarrow{\mathfrak{P}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

for a lift $\mathfrak{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ of the primitives $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathcal{C}$.

Lifting the primitives the functor $\mathfrak{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ factors as

$$\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

for a unique functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ over $\mathcal{C}$ due to the universal property of $\mathrm{Lie}(\mathcal{C})$ (remark 2.27).

If $\mathcal{C}$ is additive, by remark 1.5 there is a symmetric monoidal embedding $\mathcal{C} \subset \mathcal{D}$ into a stable symmetric monoidal category $\mathcal{D}$ such that $\mathcal{C}$ is closed in $\mathcal{D}$ under finite products and retracts.

Moreover the symmetric monoidal embedding $\mathcal{C} \subset \mathcal{D}$ factors as symmetric monoidal embeddings $\mathcal{C} \subset \mathcal{E} \subset \mathcal{D}$ with an additive symmetric monoidal category $\mathcal{E}$ such that the embedding $\mathcal{C} \subset \mathcal{E}$ preserves small limits and the embedding $\mathcal{E} \subset \mathcal{D}$ admits a right adjoint R.

The functor R induces a functor $\mathrm{R}' : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{D}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{E})$.

The composition

$$\mathrm{Bialg}(\mathcal{E}) \subset \mathrm{Bialg}(\mathcal{D}) \xrightarrow{\mathfrak{P}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{D}) \xrightarrow{\mathrm{R}'} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{E})$$

lifts the functor

$$\mathrm{Bialg}(\mathcal{E}) \subset \mathrm{Bialg}(\mathcal{D}) \xrightarrow{\mathcal{P}} \mathcal{D} \xrightarrow{\mathrm{R}} \mathcal{E}.$$

As $\mathcal{E}$ is closed in $\mathcal{D}$ under small colimits, the functor $\mathcal{E} \subset \mathcal{D} \xrightarrow{\mathrm{T}} \mathrm{Bialg}(\mathcal{D})$ is equivalent to the functor $\mathcal{E} \xrightarrow{\mathrm{T}} \mathrm{Bialg}(\mathcal{E}) \subset \mathrm{Bialg}(\mathcal{D})$, where T denote the corresponding tensoralgebra functors.

Thus the functor $\mathrm{Bialg}(\mathcal{E}) \subset \mathrm{Bialg}(\mathcal{D}) \xrightarrow{\mathcal{P}} \mathcal{D} \xrightarrow{\mathrm{R}} \mathcal{E}$ is right adjoint to the functor $\mathrm{T} : \mathcal{E} \to \mathrm{Bialg}(\mathcal{E})$ and so equivalent to the primitives for $\mathcal{E}$.

As $\mathcal{C}$ is closed in $\mathcal{E}$ under small limits, the composition

$$\mathrm{Bialg}(\mathcal{E}) \subset \mathrm{Bialg}(\mathcal{D}) \xrightarrow{\mathfrak{P}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{D}) \xrightarrow{\mathrm{R}'} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{E})$$

lifting the primitives $\mathrm{Bialg}(\mathcal{E}) \to \mathcal{E}$ induces a functor $\mathfrak{P}' : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ that lifts the primitives $\mathrm{Bialg}(\mathcal{C}) \to \mathcal{C}$.

Lifting the primitives the functor $\mathfrak{P}' : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ factors as

$$\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

for a unique functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ over $\mathcal{C}$ due to the universal property of $\mathrm{Lie}(\mathcal{C})$ (remark 2.27).

□

Example 4.3. *Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits such that $\mathcal{C}$ admits small limits.*

By prop. 4.2 there is a forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ *over* $\mathcal{C}$.

The tangent Lie algebra functor $\gamma : \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ *lifts to* $\mathrm{Lie}(\mathcal{C})$ *along this forgetful functor* $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$.

Proof. The right adjoint γ of the adjunction $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ lifts the functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$ and so due to remark 4.17 factors as

$$\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Cobar}} \mathrm{Bialg}(\mathcal{C}) \xrightarrow{\mathfrak{P}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$$

for a lift $\mathfrak{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ of the primitives $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}) \to \mathcal{C}$.

By the universal property of $\mathrm{Lie}(\mathcal{C})$ (remark 2.27) the functor $\mathfrak{P}$: $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ factors as $\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ for a unique functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ over $\mathcal{C}$.

The functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Cobar}} \mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C})$ lifts the functor $\gamma : \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$.

□

As next we describe the left adjoint $\mathfrak{U}$ of the primitive elements $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ more explicitely and show that $\mathfrak{U}$ satisfies a version of the Poincare-Birkhoff-Witt theorem if $\mathcal{C}$ is a $\mathbb{Q}$-linear stable category (remark 4.4).

We defined the primitives $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ as the composition

$$\mathrm{Bialg}(\mathcal{C}) \simeq \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \xrightarrow{\mathrm{Mon}(\gamma)} \mathrm{Mon}(\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})) \xrightarrow{\mathrm{Bar}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}).$$

By lemma 4.18 the left adjoint $\mathcal{H} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ of γ preserves finite products and so yields a functor

$$\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathrm{Cobar}} \mathrm{Mon}(\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})) \xrightarrow{\mathrm{Mon}(\mathcal{H})} \mathrm{Bialg}(\mathcal{C})$$

that is left adjoint to the primitives $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$.

Remark 4.4. *If* $\mathcal{C}$ *is a* $\mathbb{Q}$*-linear stable category, the functor* $\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ *satisfies a version of the Poincare-Birkhoff-Witt theorem:*

The functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathfrak{U}} \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ *factors canonically as* $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\mathrm{S}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$, *where* S *denotes the symmetric functor defined as the composition* $\mathrm{Cocomm}^{\mathrm{ncu}} \circ - : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{pd,conil}} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

Proof. If $\mathcal{C}$ is a $\mathbb{Q}$-linear stable category, by [7] proposition 1.7.2. loops of Lie algebras are trivial, i.e. the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ factors canonically as $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\mathrm{triv}_{\mathrm{Lie}}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$.

By construction the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathfrak{U}} \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ factors as

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\Omega} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathcal{H}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$$

and so as $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\mathrm{triv}_{\mathrm{Lie}}} \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathcal{H}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

Using the canonical equivalence $\mathcal{H} \circ \mathrm{triv}_{\mathrm{Lie}} \simeq \mathrm{S} \circ \Sigma$ of functors $\mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ we find that the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathfrak{U}} \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ is equivalent to $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C} \xrightarrow{\mathrm{S}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

□

As next we show that the functor $\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Hopf}(\mathcal{C}) \subset \mathrm{Bialg}(\mathcal{C})$ is fully faithful if $\mathcal{C}$ is a $\mathbb{Q}$-linear stable category (theorem 4.5).

This was before shown in [8] theorem 4.2.4. for dg-categories but with some gaps in the proof. We give a different and complete proof, which arose from a discussion with Gijs Heuts, to whom we are especially grateful.

Theorem 4.5 implies that the forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ is an equivalence. To prove this, we need to check that for every $\mathrm{X} \in \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ the unit $\mathrm{X} \to \mathfrak{P}(\mathfrak{U}(\mathrm{X}))$ is an equivalence.

For every $\mathrm{Y} \in \mathcal{C}$ the canonical equivalence

$$\mathrm{Y} \simeq \mathrm{triv} \circ_{\mathrm{Cocomm}^{\mathrm{ncu}}} (\mathrm{Cocomm}^{\mathrm{ncu}} \circ \mathrm{Y})$$

in $\mathcal{C}$ is adjoint to a morphism $\mathrm{triv}(\mathrm{Y}) \to \mathrm{Cocomm}^{\mathrm{ncu}} \circ \mathrm{Y}$ in $\mathrm{Cocoalg}(\mathcal{C})^{\mathrm{pd,conil}}$ that lies over a morphism $\mathrm{triv}(\mathrm{Y}) \to \mathrm{S}(\mathrm{Y})$ in $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ adjoint to a morphism $\mathrm{Y} \to \mathrm{Prim}(\mathrm{S}(\mathrm{Y}))$ in $\mathcal{C}$.

By the version of the Poincare-Birkhoff-Witt theorem of remark 4.4 the unit $\mathrm{X} \to \mathfrak{P}(\mathfrak{U}(\mathrm{X}))$ in $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ lies over the morphism $\mathrm{X} \to \mathrm{Prim}(\mathrm{S}(\mathrm{X}))$ in $\mathcal{C}$.

Consequently we need to see that for every $\mathrm{Y} \in \mathcal{C}$ the canonical morphism $\mathrm{Y} \to \mathrm{Prim}(\mathrm{S}(\mathrm{Y}))$ is an equivalence, which we prove in proposition 4.7.

Theorem 4.5. *Let $\mathcal{C}$ be a stable presentably $\mathbb{Q}$-linear symmetric monoidal category.*

The functor $\mathfrak{U} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Bialg}(\mathcal{C})$ is fully faithful.

So the forgetful functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ is an equivalence.

Proof. Let $\mathrm{X} \in \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ with underlying object $\mathrm{X}' \in \mathcal{C}$.

We want to see that the unit $\mathrm{X} \to \mathfrak{P}(\mathfrak{U}(\mathrm{X}))$ is an equivalence or equivalently that its image $\alpha : \mathrm{X}' \to \mathrm{Prim}(\mathfrak{U}(\mathrm{X}))$ in $\mathcal{C}$ is an equivalence.

By the Poincare-Birkhoff-Witt theorem of remark 4.4 the morphism α is equivalent to the canonical morphism $\mathrm{X}' \to \mathrm{Prim}(\mathrm{S}(\mathrm{X}'))$ in $\mathcal{C}$ that is an equivalence by proposition 4.7.

□

The rest of this chapter is devoted to the proof of proposition 4.7.

We deduce proposition 4.7 from proposition 4.8 and lemma 4.12.

Proposition 4.8 provides a cofiltration of the primitive elements which allows us to show that for every $\mathrm{Y} \in \mathcal{C}$ the canonical morphism $\mathrm{Y} \to \mathrm{Prim}(\mathrm{S}(\mathrm{Y}))$ has a vanishing cofiber using the calculations of lemma 4.12.

The idea to prove that $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ embeds fully faithful into $\mathrm{Bialg}(\mathcal{C})$ via $\mathfrak{U}$ by constructing a cofiltration of the primitive elements is from [8] prop. A. 8.2.3. used in theorem 4.2.4.

To prove proposition 4.7 we make the following definition:

Let $\mathcal{Q}$ be a non-counital cooperad in $\mathcal{C}$, whose counit induces an equivalence $\mathcal{Q}_1 \simeq \mathbb{1}$ with Koszul-dual operad $\mathcal{Q}^\vee$.

For every $n \in \mathbb{N}$ the canonical map of operads $\mathcal{Q}^\vee \to \tau_n(\mathcal{Q}^\vee)$ makes $\tau_n(\mathcal{Q}^\vee)$ to a right module over $\mathcal{Q}^\vee$ in $\mathcal{C}^{\Sigma_{\geq 1}}$.

Denote $\mathcal{N}_n$ the right $\mathcal{Q}$-comodule in $\mathcal{C}^{\Sigma_{\geq 1}}$ Koszul-dual to $\tau_n(\mathcal{Q}^\vee)$.

We especially use this for $\mathcal{Q} = \mathrm{Cocomm}^{\mathrm{ncu}}$ and $\mathcal{Q}^\vee = \mathrm{Lie}(1)$ and $\mathcal{C}$ the stable symmetric monoidal category of spectra.

The canonical map of operads $\mathcal{Q}^\vee \to \tau_n(\mathcal{Q}^\vee)$ considered as a map of right modules over $\mathcal{Q}^\vee$ in $\mathcal{C}^{\Sigma_{\geq 1}}$ is Koszul-dual to a map $\mathrm{triv} \to \mathcal{N}_n$ of right comodules over $\mathcal{Q}$ in $\mathcal{C}^{\Sigma_{\geq 1}}$ that induces an equivalence $\mathbb{1} \simeq (\mathcal{N}_n)_1$.

The canonical map of operads $\tau_n(\mathcal{Q}^\vee) \to \tau_{n-1}(\mathcal{Q}^\vee)$ under $\mathcal{Q}^\vee$ considered as a map of right modules over $\mathcal{Q}^\vee$ in $\mathcal{C}^{\Sigma_{\geq 1}}$ is Koszul-dual to a map $\mathcal{N}_n \to \mathcal{N}_{n-1}$ of right comodules over $\mathcal{Q}$ in $\mathcal{C}^{\Sigma_{\geq 1}}$.

Remark 4.6.

The fiber in $\mathrm{RMod}_{\mathcal{Q}^\vee}(\mathcal{C}^{\Sigma_{\geq 1}})$ *of the canonical map* $\tau_n(\mathcal{Q}^\vee) \to \tau_{n-1}(\mathcal{Q}^\vee)$ *is the trivial right* $\mathcal{Q}^\vee$*-module on the symmetric sequence concentrated in degree* n *with value* $\mathcal{Q}^\vee_n$ *by remark 2.15.*

Thus the fiber in $\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}})$ *of the map* $\mathcal{N}_n \to \mathcal{N}_{n-1}$ *is the cofree right* $\mathcal{Q}$*-comodule on the symmetric sequence concentrated in degree* n *with value* $\mathcal{Q}^\vee_n$ *(see the end of the proof of prop. 4.22 for the statement that (co)free right (co)modules correspond to trivial ones under Koszul-duality).*

Proposition 4.7. *Let* $\mathcal{C}$ *be a stable presentably* $\mathbb{Q}$*-linear symmetric monoidal category.*

For every $X \in \mathcal{C}$ *the canonical morphism* $X \to \mathrm{Prim}(S(X))$ *in* $\mathcal{C}$ *is an equivalence.*

Proof. By proposition 4.8 applied to the cooperad $\mathrm{Cocomm}^{\mathrm{ncu}}$ the canonical morphism $X \to \mathrm{Prim}(S(X))$ factors as $\alpha : X \to \lim_{1 \leq n} \mathcal{N}_n \circ X \simeq \mathrm{Prim}(S(X))$, where the compositon $\alpha_n : X \xrightarrow{\alpha} \lim_{1 \leq n} \mathcal{N}_n \circ X \to \mathcal{N}_n \circ X$ is the canonical morphism $X \simeq \mathrm{triv} \circ X \to \mathcal{N}_n \circ X$ induced by the morphism $\mathrm{triv} \to \mathcal{N}_n$ that induces an equivalence $\mathbb{1} \simeq (\mathcal{N}_n)_1$.

The cofiber of α is the limit of the induced diagram

$$... \to \mathrm{cofib}(\alpha_n) \to \mathrm{cofib}(\alpha_{n-1}) \to ... \to \mathrm{cofib}(\alpha_1).$$

Consequently α is an equivalence if for every $n > 1$ the induced morphism $\mathrm{cofib}(\alpha_n) \to \mathrm{cofib}(\alpha_{n-1})$ is the zero morphism.

The morphism α_n is the canonical morphism

$$X \simeq (\mathcal{N}_n)_1 \otimes X \to \coprod_{k \geq 1} (\mathcal{N}_n)_k \otimes_{\Sigma_k} X^{\otimes k}$$

so that the cofiber of α_n is given by $\prod_{k \geq 2} (\mathcal{N}_n)_k \otimes_{\Sigma_k} X^{\otimes k}$.

The canonical morphism $\mathrm{cofib}(\alpha_n) \to \mathrm{cofib}(\alpha_{n-1})$ is the morphism

$$\coprod_{k \geq 2} (\mathcal{N}_n)_k \otimes_{\Sigma_k} X^{\otimes k} \to \coprod_{k \geq 2} (\mathcal{N}_{n-1})_k \otimes_{\Sigma_k} X^{\otimes k}$$

induced by the morphisms $(\mathcal{N}_n)_k \to (\mathcal{N}_{n-1})_k$ for $k \geq 2$.

So the result follows from lemma 4.12.

□

Proposition 4.8. *Let $\mathcal{C}$ be a $\mathbb{Q}$-linear stable presentably symmetric monoidal category, $\mathcal{Q}$ a non-counital cooperad in $\mathcal{C}$, whose counit induces an equivalence $\mathcal{Q}_1 \simeq \mathbb{1}$ with Koszul-dual operad $\mathcal{Q}^\vee$. Let A be a $\mathcal{Q}$-coalgebra in $\mathcal{C}$ and $X \in \mathcal{C}$.*

There is a canonical equivalence

$$\mathrm{triv} *^{\mathcal{Q}} A \simeq \lim_{1\leq n} \mathcal{N}_n \circ^{\mathcal{Q}} A$$

in $\mathcal{C}$ and so especially a canonical equivalence

$$\mathrm{triv} *^{\mathcal{Q}} (\mathcal{Q} \circ X) \simeq \lim_{1\leq n} \mathcal{N}_n \circ^{\mathcal{Q}} (\mathcal{Q} \circ X) \simeq \lim_{1\leq n} \mathcal{N}_n \circ X.$$

in $\mathcal{C}$.

Proposition 4.8 follows immediately from lemma 4.9 2. and lemma 4.11, which we will prove in the following:

Lemma 4.9. *Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits that admits small limits and let $n \in \mathbb{N}$.*

1. *Let $\mathcal{O}$ be a non-unital operad in $\mathcal{C}$, whose unit induces an equivalence $\mathbb{1} \simeq \mathcal{O}_1$ with Koszul-dual cooperad $\mathcal{O}^\vee$.*

   The canonical map of cooperads $\tau_n(\mathcal{O}^\vee) \to \mathcal{O}^\vee$ makes $\tau_n(\mathcal{O}^\vee)$ to a right comodule over $\mathcal{O}^\vee$ in $\mathcal{C}^{\Sigma_{\geq 1}}$ that gives rise to a right $\mathcal{O}$-module $\mathcal{M}_n$ in $\mathcal{C}^{\Sigma_{\geq 1}}$ via Koszul-duality.

   For every $\mathcal{O}$-algebra A in $\mathcal{C}$ there is a canonical equivalence

   $$\mathrm{triv} \circ_{\mathcal{O}} A \simeq \mathrm{colim}_{1\leq n} \mathcal{M}_n \circ_{\mathcal{O}} A$$

   in $\mathcal{C}$.

2. *Let $\mathcal{Q}$ be a non-counital cooperad in $\mathcal{C}$, whose counit induces an equivalence $\mathcal{Q}_1 \simeq \mathbb{1}$ with Koszul-dual operad $\mathcal{Q}^\vee$. Let A be a $\mathcal{Q}$-coalgebra in $\mathcal{C}$.*

   There is a canonical equivalence

   $$\mathrm{triv} *^{\mathcal{Q}} A \simeq \lim_{1\leq n} \mathcal{N}_n *^{\mathcal{Q}} A$$

   in $\mathcal{C}$.

Proof. 1: By remark 2.14 there is a canonical equivalence

$$\mathrm{colim}_{1\leq n} \mathcal{O} \circ_{f_n(\mathcal{O}_{\leq n})} A \to A$$

in $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$.

Applying the left adjoint functor $\mathrm{triv} \circ_{\mathcal{O}} - : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ to this equivalence we get a canonical equivalence

$$\mathrm{colim}_{1\leq n} \mathrm{triv} \circ_{\mathcal{O}} (\mathcal{O} \circ_{f_n(\mathcal{O}_{\leq n})} A) \to \mathrm{triv} \circ_{\mathcal{O}} A$$

in $\mathcal{C}$. There is a canonical equivalence

$$\mathrm{triv} \circ_{\mathcal{O}} (\mathcal{O} \circ_{f_n(\mathcal{O}_{\leq n})} A) \simeq \mathrm{triv} \circ_{f_n(\mathcal{O}_{\leq n})} A \simeq (\mathrm{triv} \circ_{f_n(\mathcal{O}_{\leq n})} \mathcal{O}) \circ_{\mathcal{O}} A$$

in $\mathcal{C}$. By remark 4.10 the right $\mathcal{O}$-module $\mathrm{triv} \circ_{f_n(\mathcal{O}_{\leq n})} \mathcal{O}$ is Koszul-dual to the right $\mathcal{O}^\vee$-comodule $\tau_n(\mathcal{O}^\vee)$. This completes 1.

2: By the dual of remark 1.6 we have symmetric monoidal embeddings $\mathcal{C} \subset \mathcal{C}' \subset \mathcal{C}''$ such that $\mathcal{C}', \mathcal{C}''$ are stable symmetric monoidal categories compatible with small respectively large limits, the embedding $\mathcal{C} \subset \mathcal{C}'$ admits a right adjoint R, the embedding $\mathcal{C}' \subset \mathcal{C}''$ preserves small limits and the embedding $\mathcal{C} \subset \mathcal{C}''$ preserves small colimits.

The symmetric monoidal embeddings $\mathcal{C} \subset \mathcal{C}' \subset \mathcal{C}''$ yield embeddings $\mathrm{Op}(\mathcal{C})^{\mathrm{nu}} \subset \mathrm{Op}(\mathcal{C}'')^{\mathrm{nu}}$ and $\mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}} \subset \mathrm{CoOp}(\mathcal{C}'')^{\mathrm{ncu}}$.

As the embedding $\mathcal{C} \subset \mathcal{C}''$ preserves small colimits, the Koszul-duality equivalence $\mathrm{Op}(\mathcal{C}'')^{\mathrm{nu}}_{/\mathrm{triv}} \simeq \mathrm{CoOp}(\mathcal{C}'')^{\mathrm{ncu}}_{\mathrm{triv}/}$ restricts to the Koszul-duality equivalence $\mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}} \simeq \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/}$ and the Koszul-duality equivalence

$$(-) \circ_{\mathcal{Q}^\vee} \mathrm{triv} : \mathrm{RMod}_{\mathcal{Q}^\vee}(\mathcal{C}''^{\Sigma_1}) \simeq \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}''^{\Sigma_1}) : (-) *^{\mathcal{Q}} \mathrm{triv}$$

restricts to the corresponding equivalence for $\mathcal{C}$.

Thus the object $\mathcal{M}_n$ of $\mathcal{C}$ gets the similarly defined object in $\mathcal{C}''$.

By the dual version of 1. applied to $\mathcal{C}''$ and the images of $\mathcal{Q}$ and A in $\mathcal{C}''$ there is a canonical equivalence

$$\theta : \mathrm{triv} *^{\mathcal{Q}}_{\mathcal{C}''} \mathrm{A} \simeq \lim_{1 \leq n} \mathcal{N}_n *^{\mathcal{Q}}_{\mathcal{C}''} \mathrm{A}$$

in $\mathcal{C}''$, where we use the cocomposition product and limit in $\mathcal{C}''$ as indicated.

As $\mathcal{C}'$ is closed under small limits in $\mathcal{C}''$ and the embedding $\mathcal{C}'^{\Sigma} \subset \mathcal{C}''^{\Sigma}$ is monoidal, the equivalence θ is an equivalence

$$\theta : \mathrm{triv} *^{\mathcal{Q}}_{\mathcal{C}'} \mathrm{A} \simeq \lim_{1 \leq n} \mathcal{N}_n *^{\mathcal{Q}}_{\mathcal{C}'} \mathrm{A}$$

in $\mathcal{C}'$, where we take the cocomposition product and limit of $\mathcal{C}'$.

Applying the right adjoint functor $\mathrm{R} : \mathcal{C}' \to \mathcal{C}$ we get a canonical equivalence

$$\mathrm{triv} *^{\mathcal{Q}}_{\mathcal{C}} \mathrm{A} \simeq \mathrm{R}(\mathrm{triv} *^{\mathcal{Q}}_{\mathcal{C}'} \mathrm{A}) \simeq \lim_{1 \leq n} \mathrm{R}(\mathcal{N}_n *^{\mathcal{Q}}_{\mathcal{C}'} \mathrm{A}) \simeq \lim_{1 \leq n} \mathcal{N}_n *^{\mathcal{Q}}_{\mathcal{C}} \mathrm{A}$$

in $\mathcal{C}$, where the cocomposition product and limit is formed in $\mathcal{C}$ (see remark 2.10 for the definition of relative tensorproducts in representable planar operads).

□

Remark 4.10. *Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits and let $n \in \mathbb{N}$.*

Let $\mathcal{O}$ be a non-unital operad in $\mathcal{C}$, whose unit induces an equivalence $\mathbb{1} \simeq \mathcal{O}_1$ with Koszul dual cooperad $\mathcal{O}^\vee$.

The canonical map of cooperads $\tau_n(\mathcal{O}^\vee) \to \mathcal{O}^\vee$ makes $\tau_n(\mathcal{O}^\vee)$ to a right comodule over $\mathcal{O}^\vee$ in $\mathcal{C}^{\Sigma_{\geq 1}}$.

The right $\mathcal{O}$-module $\mathrm{triv} \circ_{\mathsf{f}_n(\mathcal{O}_{\leq n})} \mathcal{O}$ in $\mathcal{C}^{\Sigma_{\geq 1}}$ is Koszul-dual to the right $\mathcal{O}^\vee$-comodule $\tau_n(\mathcal{O}^\vee)$.

Proof. Denote $\psi : \mathsf{f}_n(\mathcal{O}_{\leq n}) \to \mathcal{O}$ the canonical map of operads.

By remark 4.20 we have a commutative square of categories

$$\begin{array}{ccc}
\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) & \longrightarrow & \mathrm{coRMod}_{\mathcal{O}^\vee}(\mathcal{C}^{\Sigma_{\geq 1}}) \\
\downarrow \psi^* & & \downarrow (\psi^\vee)^* \\
\mathrm{RMod}_{\mathsf{f}_n(\mathcal{O}_{\leq n})}(\mathcal{C}^{\Sigma_{\geq 1}}) & \longrightarrow & \mathrm{coRMod}_{\tau_n(\mathcal{O}^\vee)}(\mathcal{C}^{\Sigma_{\geq 1}}) \\
\downarrow \psi_* & & \downarrow (\psi^\vee)_* \\
\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) & \longrightarrow & \mathrm{coRMod}_{\mathcal{O}^\vee}(\mathcal{C}^{\Sigma_{\geq 1}}).
\end{array}$$

The diagonal of this square sends the augmentation $\mathcal{O} \to \mathrm{triv}$ to both objects we want to identify.

□

Lemma 4.11. *Let $\mathcal{C}$ be a $\mathbb{Q}$-linear stable presentably symmetric monoidal category, $\mathcal{Q}$ a non-counital cooperad in $\mathcal{C}$, whose counit induces an equivalence $\mathcal{Q}_1 \simeq \mathbb{1}$ and A a $\mathcal{Q}$-coalgebra in $\mathcal{C}$.*

For every $n \in \mathbb{N}$ the canonical morphism

$$\mathcal{N}_n \circ^{\mathcal{Q}} A \to \mathcal{N}_n *^{\mathcal{Q}} A$$

in $\mathcal{C}$ is an equivalence.

Proof. We show this by induction on $n \geq 1$.

The category $\mathcal{C}^{\Sigma}$ is stable and for every $X \in \mathcal{C}^{\Sigma}$ the functor $(-) \circ X : \mathcal{C}^{\Sigma} \to \mathcal{C}^{\Sigma}$ is exact.

Hence the category $\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma})$ is stable and the forgetful functor $\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma}) \to \mathcal{C}^{\Sigma}$ is exact. So the functors $(-) \circ^{\mathcal{Q}} A : \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma}) \to \mathcal{C}^{\Sigma}$, $(-) *^{\mathcal{Q}} A : \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma}) \to \mathcal{C}^{\Sigma}$ are exact.

By remark 4.6 the fiber F in the stable category $\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}})$ and thus in $\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma})$ of the morphism $\mathcal{N}_{n+1} \to \mathcal{N}_n$ is the cofree right $\mathcal{Q}$-comodule on a symmetric sequence in $\mathcal{C}$ concentrated in some degree.

We have a commutative square

$$\begin{array}{ccccc}
F \circ^{\mathcal{Q}} A & \longrightarrow & \mathcal{N}_{n+1} \circ^{\mathcal{Q}} A & \longrightarrow & \mathcal{N}_n \circ^{\mathcal{Q}} A \\
\downarrow & & \downarrow & & \downarrow \\
F *^{\mathcal{Q}} A & \longrightarrow & \mathcal{N}_{n+1} *^{\mathcal{Q}} A & \longrightarrow & \mathcal{N}_n *^{\mathcal{Q}} A
\end{array}$$

in $\mathcal{C}$, where both horizontal morphisms are fiber sequences.

So by induction we are reduced to show that the canonical morphism

$$F \circ^{\mathcal{Q}} A \to F *^{\mathcal{Q}} A$$

is an equivalence.

For $n = 1$ the object $\mathcal{N}_1$ is $\mathcal{Q}$ considered as a right comodule over itself, i.e. the cofree right $\mathcal{Q}$-comodule on the symmetric sequence in $\mathcal{C}$ concentrated in degree 1 with value the tensorunit of $\mathcal{C}$.

Consequently we need to see that the canonical morphism

$$\alpha : Z \circ^{\mathcal{Q}} A \to Z *^{\mathcal{Q}} A$$

in $\mathcal{C}$ is an equivalence if Z is the cofree right $\mathcal{Q}$-comodule on a symmetric sequence X in $\mathcal{C}$ concentrated in some degree $k \geq 1$.

As Z is the cofree right $\mathcal{Q}$-comodule on X, by lemma 2.19 the morphism α is the canonical morphism

$$X \circ A \simeq Z \circ^{\mathcal{Q}} A \to Z *^{\mathcal{Q}} A \simeq X * A$$

in $\mathcal{C}$.

As X is concentrated in degree $k \geq 1$, the last morphism is the norm map $(X_k \otimes A^{\otimes k})_{\Sigma_k} \to (X_k \otimes A^{\otimes k})^{\Sigma_k}$ in $\mathcal{C}$ that is an equivalence as $\mathcal{C}$ is a $\mathbb{Q}$-linear category.

□

Lemma 4.12. *For every* $d > 1$ *and* $n \geq 1$ *the rational homology of* $(\mathcal{N}_n)_d$ *is concentrated in degree* $1 - n$.

Especially for every $d > 1$ *the canonical map of spectra* $(\mathcal{N}_n)_d \to (\mathcal{N}_{n-1})_d$ *is the zero map in* Sp^{Σ_d} *(as it induces the zero map on rational homology).*

Proof. We will show the following:

1. For every $d > 1$ the rational homology of $\mathcal{N}^n_d$ is concentrated in degrees $\geq 1 - n$.
2. For every $d > 1$ the rational homology of $\mathcal{N}^n_d$ is concentrated in degrees $\leq 1 - n$.

1: By remark 4.6 the fiber F of the canonical map $\mathcal{N}_n \to \mathcal{N}_{n-1}$ in $coRMod_{Cocomm^{ncu}}(Sp^\Sigma)$ is the cofree right $Cocomm^{ncu}$-comodule on the symmetric sequence concentrated in degree n with value $Lie(1)_n \simeq Lie_n[1-n]$. So we have that

$$F_d \simeq (Lie_n[1-n] \otimes (Cocomm^{ncu})^{\otimes n})_d \simeq \coprod_{d_1+...+d_n=d} Lie_n[1-n] \otimes \Sigma_d \times_{(\Sigma_{d_1} \times ... \times \Sigma_{d_n})} \mathbb{1}$$

has homology concentrated in degree $1 - n$.

Using the fiber sequence $F_d \to (\mathcal{N}_n)_d \to (\mathcal{N}_{n-1})_d$ statement 1. follows by induction on $n \geq 1$, where the case $n = 1$ follows from $\mathcal{N}_1 \simeq Cocomm^{ncu}$ and that $\mathbb{1}$ is connective.

2: Denote $\tau_{>n}(Lie(1))$ the fiber of the canonical map $Lie(1) \to \tau_n(Lie(1))$ of operads considered as a map of right $Lie(1)$-modules in Sp^Σ.

Applying the exact functor $(-) \circ_{Lie(1)} triv : RMod_{Lie(1)}(Sp^\Sigma) \to Sp^\Sigma$ we get a fiber sequence $W \to triv \to \mathcal{N}_n$ and so for every $d > 1$ a fiber sequence $W_d \to 0 \to (\mathcal{N}_n)_d$ so that we have $W_d \simeq \Omega((\mathcal{N}_n)_d)$.

Hence 2. is equivalent to the condition that for every $d > 1$ the rational homology of $W_d = (\tau_{>n}(Lie(1)) \circ_{Lie(1)} triv)_d$ is concentrated in degrees $\leq -n$.

By remark 4.13 W_d is the colimit of a filtered diagram $\tau_{>n}(Lie(1)) \simeq D_0 \to ... \to D_k \to ....$ such that the cofiber C_k of the morphism $D_{k-1} \to D_k$ is equivalent to the k-th shift of

$$colim_{f \in (\mathcal{F}in^{k+2}_d)_{ndeg}} (\tau_{>n}(Lie(1))_{I_1} \otimes \bigotimes_{i \in I_1} \tau_{>n}(Lie(1))_{f_1^{-1}(i)} \otimes \bigotimes_{i \in I_2} Lie(1)_{f_2^{-1}(i)} \otimes ... \otimes \bigotimes_{i \in I_k} Lie(1)_{f_k^{-1}(i)} \otimes \bigotimes_{i \in I_{k+1}} triv_{f_{k+1}^{-1}(i)}),$$

where $(\mathcal{F}in^{k+2}_d)_{ndeg} \subset (\mathcal{F}in^{k+2}_d)$ denotes the full subcategory spanned by the sequences of maps of finite sets $f : J \xrightarrow{f_{k+1}} I_{k+1} \xrightarrow{f_k} ... \xrightarrow{f_1} I_1$ of length $k + 2$ such that no map in the sequence is a bijection.

So C_k is equivalent to the k-th shift of

$$colim_{f \in (\mathcal{F}in^{k+2}_d)'_{ndeg}} (Lie_{I_1}[1- | I_1 |] \otimes \bigotimes_{i \in I_1} Lie_{f_1^{-1}(i)}[1- | f_1^{-1}(i) |] \otimes ... \otimes \bigotimes_{i \in I_k} Lie_{f_k^{-1}(i)}[1- | f_k^{-1}(i) |] \otimes \bigotimes_{i \in I_{k+1}} triv_{f_{k+1}^{-1}(i)}),$$

where $(\mathcal{F}in^{k+2}_d)'_{ndeg} \subset (\mathcal{F}in^{k+2}_d)$ denotes the full subcategory spanned by the sequences of maps of finite sets $f : J \xrightarrow{f_{k+1}} I_{k+1} \xrightarrow{f_k} ... \xrightarrow{f_1} I_1$ of length

$k+2$ such that all maps in the sequence are surjections, no map in the sequence is a bijection and the cardinality $\mid I_1 \mid$ of I_1 is larger than n.

For every $f \in (\mathcal{F}\mathrm{in}_d^{k+2})'_{\mathrm{ndeg}}$ and $1 \le j \le k$ the object $\bigotimes_{i \in I_j} \mathrm{Lie}_{f_j^{-1}(i)}[1- \mid f_j^{-1}(i) \mid]$ has homology concentrated in negative degrees such that

$$\mathrm{Lie}_{I_1}[1- \mid I_1 \mid] \otimes \bigotimes_{i \in I_1} \mathrm{Lie}_{f_1^{-1}(i)}[1- \mid f_1^{-1}(i) \mid] \otimes ...$$

$$\otimes \bigotimes_{i \in I_k} \mathrm{Lie}_{f_k^{-1}(i)}[1- \mid f_k^{-1}(i) \mid] \otimes \bigotimes_{i \in I_{k+1}} \mathrm{triv}_{f_{k+1}^{-1}(i)}$$

has homology concentrated in degrees $\le -n-k$.

Thus the cofiber C_k has homology concentrated in degrees $\le -n$.

So by induction on k the homology of D_k is concentrated in degrees $\le -n$, where the case $k=1$ follows from the fact that $\mathcal{D}_0 \simeq \tau_{>n}(\mathrm{Lie}(1))$ has homology concentrated in degrees $\le -n$.

Thus $(\tau_{>n}(\mathrm{Lie}(1)) \circ_{\mathrm{Lie}(1)} \mathrm{triv})_d \simeq \mathrm{colim}_{k \ge 1} D_k$ has homology concentrated in degrees $\le -n$.

□

Remark 4.13. *Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits.*

For every $n \in \mathbb{N}$ and finite set J denote $\mathcal{F}\mathrm{in}_J^n$ the groupoid with objects sequences of maps of finite sets $f : J \xrightarrow{f_{n-1}} I_{n-1} \xrightarrow{f_{n-2}} ... \xrightarrow{f_1} I_1$ of length n and the evident isomorphisms.

Given symmetric sequences $\mathcal{O}_1, ..., \mathcal{O}_n$ for some $n \ge 2$ the composition product $(\mathcal{O}_1 \circ ... \circ \mathcal{O}_n)_d$ at degree $d \ge 0$ is canonically equivalent to the colimit

$$\mathrm{colim}_{f \in \mathcal{F}\mathrm{in}_d^n}((\mathcal{O}_1)_{I_1} \otimes \bigotimes_{i \in I_1} (\mathcal{O}_2)_{f_1^{-1}(i)} \otimes ... \otimes \bigotimes_{i \in I_{n-1}} (\mathcal{O}_n)_{f_{n-1}^{-1}(i)})$$

(see [3] def. 2.12. for a more detailed treatment).

Let $\mathcal{O}$ be an operad in $\mathcal{C}$ with $\mathcal{O}_0$ the zero object and unit $\mathbb{1} \to \mathcal{O}_1$ an equivalence.

Given a left $\mathcal{O}$-module X and right $\mathcal{O}$-module Y in $\mathcal{C}^{\Sigma_{\ge 1}}$ the object $X \circ_{\mathcal{O}} Y$ in $\mathcal{C}$ is the geometric realization of the simplicial Bar-construction $\mathcal{B}(X, \mathcal{O}, Y) : \Delta^{\mathrm{op}} \to \mathcal{C}$ that sends n to $X \circ \mathcal{O}^{\circ n} \circ Y$.

For every $n \in \mathbb{N}$ denote $\mathcal{B}(X, \mathcal{O}, Y)^n$ the colimit over the restriction $\Delta_{\le n}^{\mathrm{op}} \subset \Delta^{\mathrm{op}} \to \mathcal{C}$, where $\Delta_{\le n} \subset \Delta$ is the full subcategory spanned by the objects $[r]$ with $r \le n$.

We have induced maps $\alpha^n : \mathcal{B}(X, \mathcal{O}, Y)^n \to \mathcal{B}(X, \mathcal{O}, Y)^{n+1}$ that form a filtered diagram $X \circ Y \simeq \mathcal{B}(X, \mathcal{O}, Y)^0 \to ... \to \mathcal{B}(X, \mathcal{O}, Y)^n \to ...$, whose colimit is the geometric realization of $\mathcal{B}(X, \mathcal{O}, Y)$, i.e. $X \circ_{\mathcal{O}} Y$.

Denote $L_n \to \mathcal{B}(X, \mathcal{O}, Y)_n \simeq X \circ \mathcal{O}^{\circ n} \circ Y$ the n-th latching object of $\mathcal{B}(X, \mathcal{O}, Y)$ defined as the colimit of the restriction of the functor

$(\Delta^{\mathrm{op}})_{/[n]} \to \Delta^{\mathrm{op}} \xrightarrow{\mathcal{B}(X,\mathcal{O},Y)} \mathcal{C}$ to the full subcategory spanned by the surjective maps $[n] \to [k]$ in Δ with $k \ne n$.

By [18] remark 1.2.4.3. there is a canonical equivalence

$$X \circ \mathcal{O}^{\circ n} \circ Y \simeq \mathcal{B}(X, \mathcal{O}, Y)_n \simeq L_n \oplus \mathrm{cofib}(\alpha^{n-1})[-n]$$

and so a canonical equivalence $(X \circ \mathcal{O}^{\circ n} \circ Y)_d \simeq (L_n)_d \oplus \mathrm{cofib}(\alpha_d^{n-1})[-n]$.

Under the equivalence $(X \circ \mathcal{O}^{\circ n} \circ Y)_d \simeq$

$$\mathrm{colim}_{f \in \mathcal{F}in_d^{n+2}} (X_{I_1} \otimes \bigotimes_{i \in I_1} X_{f_1^{-1}(i)} \otimes ... \otimes \bigotimes_{i \in I_n} \mathcal{O}_{f_n^{-1}(i)} \otimes \bigotimes_{i \in I_{n+1}} Y_{f_{n+1}^{-1}(i)})$$

the summand $(L_n)_d$ *corresponds to the summand*

$$\mathrm{colim}_{f \in (\mathcal{F}in_d^{n+2})_{\mathrm{deg}}} (X_{I_1} \otimes \bigotimes_{i \in I_1} X_{f_1^{-1}(i)} \otimes ... \otimes \bigotimes_{i \in I_n} \mathcal{O}_{f_n^{-1}(i)} \otimes \bigotimes_{i \in I_{n+1}} Y_{f_{n+1}^{-1}(i)}),$$

where $(\mathcal{F}in_d^{n+2})_{\mathrm{deg}} \subset (\mathcal{F}in_d^{n+2})$ *denotes the full subcategory spanned by the sequences of maps of finite sets* $f : J \xrightarrow{f_{n+1}} I_{n+1} \xrightarrow{f_n} ... \xrightarrow{f_1} I_1$ *of length* $n+2$ *such that at least one of the maps is a bijection and the shifted cofiber* $\mathrm{cofib}(\alpha_d^{n-1})[-n]$ *corresponds to the summand*

$$\mathrm{colim}_{f \in (\mathcal{F}in_d^{n+2})_{\mathrm{ndeg}}} (X_{I_1} \otimes \bigotimes_{i \in I_1} X_{f_1^{-1}(i)} \otimes ... \otimes \bigotimes_{i \in I_n} \mathcal{O}_{f_n^{-1}(i)} \otimes \bigotimes_{i \in I_{n+1}} Y_{f_{n+1}^{-1}(i)}),$$

where $(\mathcal{F}in_d^{n+2})_{\mathrm{ndeg}} \subset (\mathcal{F}in_d^{n+2})$ *denotes the full subcategory spanned by the sequences of maps of finite sets* $f : J \xrightarrow{f_{n+1}} I_{n+1} \xrightarrow{f_n} ... \xrightarrow{f_1} I_1$ *of length* $n+2$ *such that no map in the sequence is a bijection.*

Remark 4.14. *Let* $\mathcal{C}$ *be a bicomplete preadditive symmetric monoidal category compatible with small colimits.*

We expect that the category $\mathcal{C}^{\Sigma}$ *carries another monoidal structure* $\circ'$ *different from the composition product with* $X \circ' Y \simeq \coprod_{k \in \mathbb{N}} (X_k \otimes Y^{\otimes k})^{\Sigma_k}$ *for* $X, Y \in \mathcal{C}^{\Sigma}$. *But we are unable to construct this monoidal structure here.*

This monoidal structure restricts to $\mathcal{C}^{\Sigma_{\geq 1}}$ *and gives rise to a left action of* $\mathcal{C}^{\Sigma}$ *on itself that restricts to a left action of* $\mathcal{C}^{\Sigma}$ *on* $\mathcal{C}$.

The identity of $\mathcal{C}^{\Sigma}$ *lifts to a lax monoidal functor* α *from this monoidal structure on* $\mathcal{C}^{\Sigma}$ *to the monoidal structure on* $\mathcal{C}^{\Sigma}$ *given by composition product, whose structure map*

$$X \circ Y \simeq \coprod_{k \in \mathbb{N}} (X_k \otimes Y^{\otimes k})_{\Sigma_k} \to X \circ' Y \simeq \coprod_{k \in \mathbb{N}} (X_k \otimes Y^{\otimes k})^{\Sigma_k}$$

is the norm map.

Similar as in the proof of lemma 2.19 the lax monoidal functor α *restricts to a monoidal functor on* $\mathcal{C}^{\Sigma_{\geq 1}}$.

Hence via this monoidal structure the composition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ *acts on* $\mathcal{C}$ *by* $X \circ' Y \simeq \coprod_{k \in \mathbb{N}} (X_k \otimes Y^{\otimes k})^{\Sigma_k}$ *for* $X \in \mathcal{C}^{\Sigma}, Y \in \mathcal{C}$.

Given a non-unital operad $\mathcal{O}$ *in* $\mathcal{C}$ *a left* $\mathcal{O}$*-module in* $\mathcal{C}$ *with respect to this action is a* $\mathcal{O}$*-algebra with divided powers and given a non-counital cooperad* $\mathcal{Q}$ *in* $\mathcal{C}$ *a left* $\mathcal{Q}$*-comodule in* $\mathcal{C}$ *is a conilpotent* $\mathcal{Q}$*-coalgebra.*

Applying Koszul-duality to this modified left actions of $\mathcal{C}^{\Sigma_{\geq 1}}$ *on* $\mathcal{C}$ *we get an adjunction*

$$\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{pd}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{conil}}$$

and by composing with the forgetful functor an adjunction

$$\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{pd}}(\mathcal{C}) \rightleftarrows \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{conil}} \rightleftarrows \mathrm{Cocoalg}(\mathcal{C}), \tag{12}$$

where the right adjoint lifts the functor $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$.

The right adjoint of adjunction 12 yields a functor

$$\mathrm{Bialg}(\mathcal{C}) = \mathrm{Mon}(\mathrm{Cocoalg}(\mathcal{C})) \xrightarrow{\theta} \mathrm{Mon}(\mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{pd}}(\mathcal{C})) \xrightarrow{\mathrm{Bar}} \mathrm{Alg}_{\mathrm{Lie}}^{\mathrm{pd}}(\mathcal{C}).$$

As the forgetful functor $\mathrm{Alg}^{\mathrm{pd}}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C}$ *does not commute with geometric realizations, it does not follow that the functor* Bar *is an equivalence that forgets to the shift functor of* $\mathcal{C}$.

We conjecture the following:

Conjecture 4.15. *For every* $\mathrm{X} \in \mathrm{Bialg}(\mathcal{C})$ *the canonical morphism* $\Sigma(\theta(\mathrm{X})) \to \mathrm{Bar}(\theta(\mathrm{X}))$ *is an equivalence.*

If conjecture 4.15 holds, the functor $\mathrm{Bar} \circ \theta : \mathrm{Bialg}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{pd}}_{\mathrm{Lie}}(\mathcal{C})$ *lifts the primitives* $\mathrm{Bialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\mathrm{Prim}} \mathcal{C}$.

So by the universal property of $\mathrm{Lie}(\mathcal{C})$ *(remark 2.27) the functor* $\mathrm{Bar} \circ \theta$ *factors as*

$$\mathrm{Bialg}(\mathcal{C}) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{pd}}_{\mathrm{Lie}}(\mathcal{C})$$

for a unique functor $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{pd}}_{\mathrm{Lie}}(\mathcal{C})$ *over* $\mathcal{C}$.

This makes it possible to ask the following question:

Question 4.16. *Is the forgetful functor* $\mathrm{Lie}(\mathcal{C}) \to \mathrm{Alg}^{\mathrm{pd}}_{\mathrm{Lie}}(\mathcal{C})$ *an equivalence if* $\mathcal{C}$ *is the category of* K*-module spectra for some field* K*?*

This question is motivated by [6] theorem 1.2.5. of Fresse, according to which restricted Lie algebras over a field K *are divided power Lie algebras in the category of* K*-vector spaces.*

Remark 4.17. *Let* $\mathcal{C}$ *be a stable category and* $\mathcal{D}$ *a category that admits geometric realizations, finite products and totalizations.*

Let $\phi : \mathcal{D} \to \mathcal{C}$ *be a conservative functor that preserves geometric realizations, finite products and totalizations.*

The adjunction $\mathrm{Bar} : \mathrm{Mon}(\mathcal{D}) \rightleftarrows \mathcal{D}_* : \mathrm{Cobar}$ *is an equivalence.*

Proof. As $\phi : \mathcal{D} \to \mathcal{C}$ preserves geometric realizations, finite products and totalizations, the functor ϕ sends the unit and counit of the adjunction $\mathrm{Bar} : \mathrm{Mon}(\mathcal{D}) \rightleftarrows \mathcal{D}_* : \mathrm{Cobar}$ to the unit respectively counit of the adjunction $\mathrm{Bar} : \mathrm{Mon}(\mathcal{C}) \rightleftarrows \mathcal{C}_* : \mathrm{Cobar}$.

As ϕ is conservative, it is enough to see that the adjunction $\mathrm{Bar} : \mathrm{Mon}(\mathcal{C}) \rightleftarrows \mathcal{C}_* : \mathrm{Cobar}$ is an equivalence.

As $\mathcal{C}$ is stable, the forgetful functor $\mathrm{Mon}(\mathcal{C}) \simeq \mathrm{Alg}(\mathcal{C}^{\times}) \simeq \mathrm{Alg}(\mathcal{C}^{\amalg}) \to \mathcal{C}$ is an equivalence.

The functor $\mathrm{Bar} : \mathrm{Mon}(\mathcal{C}) \simeq \mathcal{C} \to \mathcal{C}_* \simeq \mathcal{C}$ is equivalent to the functor $\Sigma : \mathcal{C} \simeq \mathcal{C}$ as for every $\mathrm{A} \in \mathcal{C} \simeq \mathrm{Cmon}(\mathcal{C})$ the relative tensorproduct $\mathrm{Bar}(\mathrm{A}) = 0 \otimes_{\mathrm{A}} 0$ is the coproduct of $\mathrm{A} \to 0$ with itself in the category $\mathrm{Calg}(\mathrm{Mod}_{\mathrm{A}}(\mathcal{C}^{\times})) \simeq \mathrm{Cmon}(\mathcal{C})_{\mathrm{A}/} \simeq \mathcal{C}_{\mathrm{A}/}$.

□

Lemma 4.18. *The functor* $\mathcal{H} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ *preserves finite products.*

Proof. In the following we consider $\mathbb{N}, \mathbb{Z}$ as categories by viewing them as posets.

Denote $\gamma : \mathrm{Fun}(\mathbb{Z}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \to \prod_{\mathbb{Z}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ the functor that sends a filtered object A to its associated graded object $(\mathrm{A_i}/\mathrm{A_{i-1}})_{\mathrm{i} \in \mathbb{Z}}$.

γ restricts to a functor $\beta : \mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \to \prod_{\mathbb{N}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

The functor $\mathcal{H} : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ admits a functorial filtration:

By [16] 3.1. the functor $\mathcal{H}$ factors as

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathcal{H}'} \mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \xrightarrow{\mathrm{colim}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$$

and the composition

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \xrightarrow{\mathcal{H}'} \mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \xrightarrow{\beta} \prod_{\mathbb{N}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \xrightarrow{\oplus} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$$

factors as the forgetful functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C}$ followed by the shifted symmetric functor $\mathrm{S} \circ \Sigma : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$.

The functor $\mathrm{colim} : \mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ is symmetric monoidal when $\mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/})$ carries the Day-convolution symmetric monoidal structure.

Consequently it is enough to see that the functor $\mathcal{H}' : \mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/})$ is symmetric monoidal when $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C})$ carries the cartesian structure and $\mathrm{Fun}(\mathbb{N}, \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/})$ the Day-convolution.

The functor γ and so its restriction β and the functor $\oplus : \prod_{\mathbb{N}} \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ are symmetric monoidal with respect to Day-convolution.

As $\oplus \circ \beta$ is conservative, the assertion follows from the fact that the forgetful functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \to \mathcal{C}$, shift functor and symmetric functor $\mathrm{S} : \mathcal{C} \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ preserve finite products:

As the object-wise symmetric monoidal structure on $\mathrm{Calg}(\mathcal{C})$ is cocartesian and $\mathcal{C}$ is preadditive, the free commutative algebra functor $\mathcal{C} \to \mathrm{Calg}(\mathcal{C})$ is symmetric monoidal when $\mathcal{C}$ carries the cartesian structure and $\mathrm{Calg}(\mathcal{C})$ the object-wise symmetric monoidal structure.

Thus the free commutative algebra functor $\mathcal{C} \to \mathrm{Calg}(\mathcal{C})$ lifts to a symmetric monoidal functor $\mathcal{C} \to \mathrm{Cobialg}(\mathcal{C})$, where $\mathcal{C}$ carries the cartesian structure and $\mathrm{Cobialg}(\mathcal{C})$ the object-wise symmetric monoidal structure.

So the composition $\mathrm{S} : \mathcal{C} \to \mathrm{Cobialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ preserves finite products.

□

4.2 Derived Koszul duality

In this section we prove the statements about Koszul-duality we used in the last section.

We start with constructing a Bar-Cobar adjunction between augmented associative algebras and coaugmented coassociative coalgebras in a monoidal category that admits geometric realizations and totalizations (prop. 4.23).

Then we extend this Bar-Cobar adjunction to a Bar-Cobar adjunction between modules and comodules (prop. 4.23).

Finally we apply these Bar-Cobar adjunctions to the composition product on some nice preadditive symmetric monoidal category to obtain Koszul-duality adjunctions between augmented operads and coaugmented cooperads and their algebras and coalgebras (4.21).

We show that these Koszul-duality adjunctions are equivalences under reasonable conditions (prop. 4.22).

The results about Koszul-duality are extensions of results of [18] 5.2. and are inspired by [8].

We start with presenting the results we will prove:

Given a monoidal category $\mathcal{C}$ that admits geometric realizations we have a functor

$$\mathrm{Bar} : \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}} \to \mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/}, \ \mathrm{A} \mapsto \mathbb{1} \otimes_{\mathrm{A}} \mathbb{1} = \mathrm{colim}_{\mathrm{n} \in \Delta^{\mathrm{op}}} \mathrm{A}^{\otimes \mathrm{n}},$$

where the comultiplication of $\mathbb{1} \otimes_{\mathrm{A}} \mathbb{1}$ is given by the morphism

$$\mathbb{1} \otimes_{\mathrm{A}} \mathbb{1} \simeq \mathbb{1} \otimes_{\mathrm{A}} \mathrm{A} \otimes_{\mathrm{A}} \mathbb{1} \to (\mathbb{1} \otimes_{\mathrm{A}} \mathbb{1}) \otimes (\mathbb{1} \otimes_{\mathrm{A}} \mathbb{1}) \simeq \mathbb{1} \otimes_{\mathrm{A}} \mathbb{1} \otimes_{\mathrm{A}} \mathbb{1}$$

induced by the augmentation of $\mathrm{A} \in \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}$.

Let $\mathcal{C}, \mathcal{E}$ be monoidal categories and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{E})$-bimodule such that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ admit geometric realizations.

Given $\mathrm{A} \in \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}, \mathrm{B} \in \mathrm{Alg}(\mathcal{E})_{/\mathbb{1}}$ the functor

$$\mathrm{triv}_{\mathrm{A},\mathrm{B}} : \mathcal{D} \simeq {}_{\mathbb{1}}\mathrm{BMod}_{\mathbb{1}}(\mathcal{D}) \to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})$$

that forgets along the maps of algebras $\mathrm{A} \to \mathbb{1}, \mathrm{B} \to \mathbb{1}$ admits a left adjoint

$$\mathbb{1} \otimes_{\mathrm{A}} - \otimes_{\mathrm{B}} \mathbb{1} : {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to \mathcal{D}, \ \mathrm{X} \mapsto \mathbb{1} \otimes_{\mathrm{A}} \mathrm{X} \otimes_{\mathrm{B}} \mathbb{1} \simeq \mathrm{colim}_{\mathrm{n} \in \Delta^{\mathrm{op}}} (\mathrm{A}^{\otimes \mathrm{n}} \otimes \mathrm{X} \otimes \mathrm{B}^{\otimes \mathrm{n}})$$

by remark 2.10.

This left adjoint $\mathbb{1} \otimes_{\mathrm{A}} - \otimes_{\mathrm{B}} \mathbb{1}$ lifts to a functor

$${}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to {}_{\mathrm{Bar}(\mathrm{A})}\mathrm{coBMod}_{\mathrm{Bar}(\mathrm{B})}(\mathcal{D}).$$

Given $\mathrm{X} \in {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})$ the biaction

$$\mathbb{1} \otimes_{\mathrm{A}} \mathrm{X} \otimes_{\mathrm{B}} \mathbb{1} \simeq \mathbb{1} \otimes_{\mathrm{A}} \mathrm{A} \otimes_{\mathrm{A}} \mathrm{X} \otimes_{\mathrm{B}} \mathrm{B} \otimes_{\mathrm{B}} \mathbb{1} \to (\mathbb{1} \otimes_{\mathrm{A}} \mathbb{1}) \otimes (\mathbb{1} \otimes_{\mathrm{A}} \mathrm{X} \otimes_{\mathrm{B}} \mathbb{1}) \otimes (\mathbb{1} \otimes_{\mathrm{B}} \mathbb{1}) \simeq$$
$$\mathbb{1} \otimes_{\mathrm{A}} \mathbb{1} \otimes_{\mathrm{A}} \mathrm{X} \otimes_{\mathrm{B}} \mathbb{1} \otimes_{\mathrm{B}} \mathbb{1}$$

is induced by the augmentations of A, B.

More precisely there is a commutative square

$$\begin{array}{ccc} \mathrm{BMod}(\mathcal{D}) & \longrightarrow & \mathrm{coBMod}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}} \times \mathrm{Alg}(\mathcal{E})_{/\mathbb{1}} & \longrightarrow & \mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/} \times \mathrm{Coalg}(\mathcal{E})_{\mathbb{1}/} \end{array} \qquad (13)$$

that induces on the fiber over every $A \in \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}, B \in \mathrm{Alg}(\mathcal{E})_{/\mathbb{1}}$ a functor

$$_A\mathrm{BMod}_B(\mathcal{D}) \to {}_{\mathrm{Bar}(A)}\mathrm{coBMod}_{\mathrm{Bar}(B)}(\mathcal{D})$$

lifting the functor $\mathbb{1} \otimes_A - \otimes_B \mathbb{1} : {}_A\mathrm{BMod}_B(\mathcal{D}) \to \mathcal{D}$.

Dually let $\mathcal{C}, \mathcal{E}$ be monoidal categories and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{E})$-bimodule such that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ admit totalizations.

Then by replacing $\mathcal{C}, \mathcal{D}, \mathcal{E}$ by $\mathcal{C}^{op}, \mathcal{D}^{op}, \mathcal{E}^{op}$ and turning to opposite categories we obtain a functor

$$\mathrm{Cobar} : \mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/} \to \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}, A \mapsto \mathbb{1} \otimes^A \mathbb{1} := \lim_{n \in \Delta^{op}} A^{\otimes n}$$

and a commutative square

$$\begin{array}{ccc} \mathrm{coBMod}(\mathcal{D}) & \longrightarrow & \mathrm{BMod}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/} \times \mathrm{Coalg}(\mathcal{E})_{\mathbb{1}/} & \longrightarrow & \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}} \times \mathrm{Alg}(\mathcal{E})_{/\mathbb{1}} \end{array}$$

that induces on the fiber over every $A \in \mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/}$, $B \in \mathrm{Coalg}(\mathcal{E})_{\mathbb{1}/}$ a functor

$$_A\mathrm{coBMod}_B(\mathcal{D}) \to {}_{\mathrm{Cobar}(A)}\mathrm{BMod}_{\mathrm{Cobar}(B)}(\mathcal{D})$$

lifting the functor

$$\mathbb{1} \otimes^A - \otimes^B \mathbb{1} : {}_A\mathrm{coBMod}_B(\mathcal{D}) \to \mathcal{D}, \; X \mapsto \lim_{n \in \Delta^{op}} (A^{\otimes n} \otimes X \otimes B^{\otimes n}).$$

If $\mathcal{C}, \mathcal{D}, \mathcal{E}$ admit realizations and totalizations, we have adjunctions

$$\mathrm{Bar} : \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}} \rightleftarrows \mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/} : \mathrm{Cobar}, \quad \mathrm{BMod}(\mathcal{D}) \rightleftarrows \mathrm{coBMod}(\mathcal{D})$$

and square 13 is a map of adjunctions, where the left vertical functor is a cartesian fibration and the right vertical functor is a cocartesian fibration.

So given $A \in \mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}, B \in \mathrm{Alg}(\mathcal{E})_{/\mathbb{1}}$ and morphisms $\mathrm{Bar}(A) \to A'$, $\mathrm{Bar}(B) \to B'$ in $\mathrm{Coalg}(\mathcal{C})_{\mathbb{1}/}$ respectively $\mathrm{Coalg}(\mathcal{E})_{\mathbb{1}/}$ square 13 induces an adjunction

$$_A\mathrm{BMod}_B(\mathcal{D}) \rightleftarrows {}_{A'}\mathrm{coBMod}_{B'}(\mathcal{D}).$$

Remark 4.19. *If the $(\mathcal{C}, \mathcal{E})$-bimodule structure on $\mathcal{D}$ is compatible with geometric realizations, for every $Y \in \mathcal{D}$ the canonical morphism*

$$\mathbb{1} \otimes_A \mathrm{triv}_{A,B}(Y) \otimes_B \mathbb{1} \simeq \mathrm{colim}_{n \in \Delta^{op}} (A^{\otimes n} \otimes Y \otimes B^{\otimes n}) \to \mathrm{Bar}(A) \otimes Y \otimes \mathrm{Bar}(B) \simeq$$

$$\mathrm{colim}_{n \in \Delta^{op}} (A^{\otimes n}) \otimes Y \otimes \mathrm{colim}_{n \in \Delta^{op}} (B^{\otimes n})$$

is an equivalence.

So in this case the functor ${}_A\mathrm{BMod}_B(\mathcal{D}) \to {}_{\mathrm{Bar}(A)}\mathrm{coBMod}_{\mathrm{Bar}(B)}(\mathcal{D})$ exhibits the category ${}_{\mathrm{Bar}(A)}\mathrm{coBMod}_{\mathrm{Bar}(B)}(\mathcal{D})$ as the category of coalgebras over the comonad associated to the adjunction $\mathbb{1} \otimes_A - \otimes_B \mathbb{1} : {}_A\mathrm{BMod}_B(\mathcal{D}) \rightleftarrows \mathcal{D} : \mathrm{triv}_{A,B}$.

Remark 4.20. *Let $\mathcal{C}, \mathcal{E}$ be monoidal categories and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{E})$-bimodule such that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ admit geometric realizations.*

Square 13 is a map of cocartesian fibrations:

Given morphisms $A \to A'$ in $\mathrm{Alg}(\mathcal{C})_{/\mathbb{1}}$ and $B \to B'$ in $\mathrm{Alg}(\mathcal{E})_{/\mathbb{1}}$ and $X \in {}_A\mathrm{BMod}_B(\mathcal{D})$ the induced morphism $\mathbb{1} \otimes_{A'} (A' \otimes_A X \otimes_B B') \otimes_{B'} \mathbb{1} \to \mathbb{1} \otimes_A X \otimes_B \mathbb{1}$ lies over the canonical equivalence in $\mathcal{D}$.

Given a symmetric monoidal category $\mathcal{C}$ compatible with small colimits the category $\mathcal{C}^{\Sigma}$ is a monoidal category endowed with the composition product and so endows $\mathcal{C}^{\Sigma}$ with a bimodule structure over itself that restricts to a bimodule structure on $\mathcal{C}$ over $\mathcal{C}^{\Sigma}$, where $\mathcal{C}^{\Sigma}$ acts trivially on $\mathcal{C}$ from the right. This bimodule structure on $\mathcal{C}$ over $\mathcal{C}^{\Sigma}$ restricts to a bimodule structure on $\mathcal{C}$ over $\mathcal{C}^{\Sigma_{\geq 1}}$.

If $\mathcal{C}$ admits totalizations, we can form the Bar-Cobar adjunctions for $\mathcal{C}^{\Sigma_{\geq 1}}$.

If $\mathcal{C}$ has small colimits (but the symmetric monoidal structure on $\mathcal{C}$ is not neccessarily compatible with small colimits), the composition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ does not define a monoidal category but a representable planar operad.

Thus we cannot form the Bar-Cobar adjunctions for $\mathcal{C}^{\Sigma_{\geq 1}}$ directly.

But by embedding $\mathcal{C}$ symmetric monoidally into a preadditive symmetric monoidal category compatible with small colimits that admits totalizations we can construct the Bar-Cobar adjunctions for $\mathcal{C}^{\Sigma_{\geq 1}}$ in this more general case by the following proposition 4.21:

Proposition 4.21. *Let $\mathcal{C}$ be a preadditive symmetric monoidal category that admits small colimits and small limits.*

1. *There is an adjunction*

$$(-)^{\vee} := \mathrm{Bar} : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}} \simeq \mathrm{Alg}(\mathcal{C}^{\Sigma_{\geq 1}})_{/\mathrm{triv}} \rightleftarrows \mathrm{Coalg}(\mathcal{C}^{\Sigma_{\geq 1}})_{\mathrm{triv}/}$$
$$\simeq \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/} : (-)^{\vee} := \mathrm{Cobar}.$$

2. *We have an adjunction* $\mathrm{RMod}(\mathcal{C}^{\Sigma_{\geq 1}}) \rightleftarrows \mathrm{coRMod}(\mathcal{C}^{\Sigma_{\geq 1}})$ *and a map of adjunctions*

$$\begin{array}{ccc} \mathrm{RMod}(\mathcal{C}^{\Sigma_{\geq 1}}) & \longrightarrow & \mathrm{coRMod}(\mathcal{C}^{\Sigma_{\geq 1}}) \\ \downarrow & & \downarrow \\ \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}} & \longrightarrow & \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/}. \end{array}$$

Given an augmented non-unital operad $\mathcal{O}$ in $\mathcal{C}$ and a morphism $\mathcal{O}^{\vee} \to \mathcal{Q}$ of coaugmented cooperads in $\mathcal{C}$ this square induces an adjunction

$$\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) \rightleftarrows \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}),$$

where the left adjoint lifts the functor $(-) \circ_{\mathcal{O}} \mathrm{triv} : \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) \to \mathcal{C}$ *left adjoint to the trivial right $\mathcal{O}$-module functor and dually the right adjoint lifts the functor* $(-) *^{\mathcal{Q}} \mathrm{triv} : \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}) \to \mathcal{C}$ *right adjoint to the trivial right $\mathcal{Q}$-comodule functor.*

3. *If the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits, there is an adjunction* $\mathrm{LMod}(\mathcal{C}) \rightleftarrows \mathrm{coLMod}(\mathcal{C})$ *and a map of adjunctions*

$$\begin{array}{ccc} \mathrm{LMod}(\mathcal{C}) & \longrightarrow & \mathrm{coLMod}(\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}} & \longrightarrow & \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/}. \end{array}$$

Given an augmented non-unital operad $\mathcal{O}$ *in* $\mathcal{C}$ *and a morphism* $\mathcal{O}^\vee \to \mathcal{Q}$ *of coaugmented cooperads in* $\mathcal{C}$ *this square induces an adjunction*

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) = \mathrm{LMod}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \mathrm{Coalg}^{\mathrm{pd}}_{\mathcal{Q}}(\mathcal{C}) = \mathrm{coLMod}_{\mathcal{Q}}(\mathcal{C}),$$

where the left adjoint lifts the functor $\mathrm{triv} \circ_{\mathcal{O}} (-) : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ *left adjoint to the trivial* $\mathcal{O}$*-algebra functor and the right adjoint lifts the functor* $\mathrm{triv} \circ^{\mathcal{Q}} (-) : \mathrm{Coalg}^{\mathrm{pd}}_{\mathcal{Q}}(\mathcal{C}) \to \mathcal{C}$ *right adjoint to the trivial divided power* $\mathcal{Q}$*-coalgebra functor.*

Proof. If the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits, the compositon product on $\mathcal{C}^\Sigma$ defines a monoidal structure.

In this case the statements 1.,2.,3. follow from prop. 4.23.

Otherwise there are symmetric monoidal embeddings $\mathcal{C} \subset \mathcal{C}' \subset \mathcal{C}''$ with preadditive symmetric monoidal categories $\mathcal{C}', \mathcal{C}''$ compatible with small colimits such that the embedding $\mathcal{C} \subset \mathcal{C}'$ admits a left adjoint L, the embedding $\mathcal{C}' \subset \mathcal{C}''$ preserves small colimits and the embedding $\mathcal{C} \subset \mathcal{C}''$ preserves small limits.

The embeddings $\mathcal{C} \subset \mathcal{C}', \mathcal{C}' \subset \mathcal{C}''$ induce embeddings

$$\mathrm{Op}^{\mathrm{nu}}(\mathcal{C}) \subset \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}') \subset \mathrm{Op}^{\mathrm{nu}}(\mathcal{C}''),\ \mathrm{CoOp}^{\mathrm{ncu}}(\mathcal{C}) \subset \mathrm{CoOp}^{\mathrm{ncu}}(\mathcal{C}') \subset \mathrm{CoOp}^{\mathrm{ncu}}(\mathcal{C}''),$$

$$\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) \subset \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}'^{\Sigma_{\geq 1}}) \subset \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}''^{\Sigma_{\geq 1}}),$$
$$\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}) \subset \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}'^{\Sigma_{\geq 1}}) \subset \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}''^{\Sigma_{\geq 1}}).$$

The induced right adjoint embedding $\mathcal{C}^{\Sigma_{\geq 1}} \subset \mathcal{C}'^{\Sigma_{\geq 1}}$ is monoidal with respect to cocomposition product. Thus the embeddings

$$\mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/} \subset \mathrm{CoOp}(\mathcal{C}')^{\mathrm{ncu}}_{\mathrm{triv}/},\ \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}) \subset \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}'^{\Sigma_{\geq 1}})$$

admit left adjoints L' respectively L'' that forget to the functor $\mathrm{L}^{\Sigma_{\geq 1}}$.

As the symmetric monoidal structure on $\mathcal{C}''$ is compatible with small colimits, we have the adjunctions of 1. and 2. for $\mathcal{C}''$.

The right adjoints

$$(-)^\vee : \mathrm{CoOp}(\mathcal{C}'')^{\mathrm{ncu}}_{\mathrm{triv}/} \to \mathrm{Op}(\mathcal{C}'')^{\mathrm{nu}}_{/\mathrm{triv}},\ \mathrm{X} \mapsto \mathrm{triv} *^{\mathrm{X}} \mathrm{triv}$$

$$(-) *^{\mathcal{Q}} \mathrm{triv} : \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}''^{\Sigma_{\geq 1}}) \to \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}''^{\Sigma_{\geq 1}})$$

restrict to functors $(-)^\vee : \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/} \to \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}}$,

$$(-) *^{\mathcal{Q}} \mathrm{triv} : \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}) \to \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}})$$

as $\mathcal{C}$ is closed in $\mathcal{C}''$ under small limits.

The left adjoints

$$(-)^\vee : \mathrm{Op}(\mathcal{C}'')^{\mathrm{nu}}_{/\mathrm{triv}} \to \mathrm{CoOp}(\mathcal{C}'')^{\mathrm{ncu}}_{\mathrm{triv}/},$$

$$(-) \circ_{\mathcal{O}} \mathrm{triv} : \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}''^{\Sigma_{\geq 1}}) \to \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}''^{\Sigma_{\geq 1}})$$

restrict to functors $(-)^\vee : \mathrm{Op}(\mathcal{C}')^{\mathrm{nu}}_{/\mathrm{triv}} \to \mathrm{CoOp}(\mathcal{C}')^{\mathrm{ncu}}_{\mathrm{triv}/}$ and

$(-) \circ_{\mathcal{O}} \mathrm{triv} : \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}'^{\Sigma_{\geq 1}}) \to \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}'^{\Sigma_{\geq 1}})$ as $\mathcal{C}'$ is closed in $\mathcal{C}''$ under small colimits.

Thus the composition

$$\mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}} \subset \mathrm{Op}(\mathcal{C}')^{\mathrm{nu}}_{/\mathrm{triv}} \xrightarrow{(-)^\vee} \mathrm{CoOp}(\mathcal{C}')^{\mathrm{ncu}}_{\mathrm{triv}/} \xrightarrow{\mathrm{L}'} \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/}$$

is left adjoint to the functor $(-)^\vee : \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/} \to \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}}$ and the composition

$$\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) \subset \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}'^{\Sigma_{\geq 1}}) \xrightarrow{(-)\circ_{\mathcal{O}}\mathrm{triv}} \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}'^{\Sigma_{\geq 1}}) \xrightarrow{L''}$$
$$\mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}})$$

is left adjoint to the functor $(-)*^{\mathcal{Q}}\mathrm{triv} : \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}) \to \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}})$. □

Proposition 4.22. *Let $\mathcal{C}$ be a stable symmetric monoidal category compatible with small colimits that admits small limits.*

1. *For every non-unital operad $\mathcal{O}$ in $\mathcal{C}$, whose unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence, the unit $\mathcal{O} \to (\mathcal{O}^\vee)^\vee$ of the Koszul-duality adjunction*
$$(-)^\vee : \mathrm{Op}(\mathcal{C})^{\mathrm{nu}}_{/\mathrm{triv}} \rightleftarrows \mathrm{CoOp}(\mathcal{C})^{\mathrm{ncu}}_{\mathrm{triv}/} : (-)^\vee$$
is an equivalence and for every non-counital cooperad $\mathcal{Q}$ in $\mathcal{C}$, whose counit $\mathcal{Q}_1 \to \mathbb{1}$ is an equivalence, the counit $(\mathcal{Q}^\vee)^\vee \to \mathcal{Q}$ is an equivalence.

2. *For every non-unital operad $\mathcal{O}$ in $\mathcal{C}$, whose unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence, the Koszul-duality adjunction*
$$(-) \circ_{\mathcal{O}} \mathrm{triv} : \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) \rightleftarrows \mathrm{coRMod}_{\mathcal{O}^\vee}(\mathcal{C}^{\Sigma_{\geq 1}}) : (-) *^{\mathcal{O}^\vee} \mathrm{triv}$$
is an equivalence.

Proof. The unit of the adjunction of 1. applied to an operad $\mathcal{O}$ is equivalent to the unit of the adjunction of 2. applied to $\mathcal{O}$ considered as module over itself.

So 1. follows from 2. and the following statement, where we use that the counit $\mathcal{O}_1^\vee \to \mathbb{1}$ is an equivalence if the unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence (which follows from the proof of lemma 2.18):

For every non-counital cooperad $\mathcal{Q}$ in $\mathcal{C}$, whose counit $\mathcal{Q}_1 \to \mathbb{1}$ is an equivalence, the Koszul-duality adjunction

$$(-) \circ_{\mathcal{Q}^\vee} \mathrm{triv} : \mathrm{RMod}_{\mathcal{Q}^\vee}(\mathcal{C}^{\Sigma_{\geq 1}}) \rightleftarrows \mathrm{coRMod}_{\mathcal{Q}}(\mathcal{C}^{\Sigma_{\geq 1}}) : (-) *^{\mathcal{Q}} \mathrm{triv} \tag{14}$$

induced by forgetting along the counit $(\mathcal{Q}^\vee)^\vee \to \mathcal{Q}$ is an equivalence.

So we need to see that the unit and counit of the adjunction of 2. and of the adjunction 14 are equivalences.

We will show that the unit η of the adjunction of 2. is an equivalence. The case of the counit of adjunction 2. is dual and the other cases are similar.

We will show that for every $k \geq 1$ the following statement $(*)$ holds:

For every $n \geq 1$ and every right $\mathcal{O}$-module X that vanishes under degree n, i.e. that belongs to $\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq n}})$, the morphim

$$(\eta_X)_k : X_k \to ((X \circ_{\mathcal{O}} \mathrm{triv}) *^{\mathcal{O}^\vee} \mathrm{triv})_k,$$

is an equivalence.

By lemma 2.16 the object $X \circ_{\mathcal{O}} \mathrm{triv}$ and so $(X \circ_{\mathcal{O}} \mathrm{triv}) *^{\mathcal{O}^\vee} \mathrm{triv}$ belongs to $\mathcal{C}^{\Sigma_{\geq n}}$. Hence η_k is an equivalence if $n > k$.

Consequently it remains to show $(*)$ for $n \leq k$, which we do by descending induction on n.

By remark 2.13 there is an $X' \in \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq n+1}})$ and a morphism $X' \to X$ in $\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq n}})$ that induces an equivalence in degree larger than n.

The cofiber X'' in the stable category $\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}})$ of the morphism $X' \to X$ is the trivial right $\mathcal{O}$-module concentrated in degree n with value X_n (remark 2.15), where we use that the unit $\mathbb{1} \to \mathcal{O}_1$ is an equivalence.

We have a commutative square

$$\begin{array}{ccccc} X' & \longrightarrow & X & \longrightarrow & X'' \\ \downarrow{\scriptstyle \eta_{X'}} & & \downarrow{\scriptstyle \eta_X} & & \downarrow{\scriptstyle \eta_{X''}} \\ (X' \circ_{\mathcal{O}} \mathrm{triv}) *^{\mathcal{O}^\vee} \mathrm{triv} & \longrightarrow & (X \circ_{\mathcal{O}} \mathrm{triv}) *^{\mathcal{O}^\vee} \mathrm{triv} & \longrightarrow & (X'' \circ_{\mathcal{O}} \mathrm{triv}) *^{\mathcal{O}^\vee} \mathrm{triv} \end{array}$$

in $\mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}})$, where bottom and top morphisms are cofiber sequences.

So by our induction hypothesis we are reduced to show that η_X is an equivalence if X carries the trivial right $\mathcal{O}$-module structure.

By proposition 4.21 the left adjoint of adjunction 2. lifts the functor $(-) \circ_{\mathcal{O}} \mathrm{triv} : \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}}) \to \mathcal{C}$ left adjoint to the trivial right $\mathcal{O}$-module functor.

So by adjointness the right adjoint

$$(-) *_{\mathcal{O}^\vee} \mathrm{triv} : \mathrm{coRMod}_{\mathcal{O}^\vee}(\mathcal{C}^{\Sigma_{\geq 1}}) \to \mathrm{RMod}_{\mathcal{O}}(\mathcal{C}^{\Sigma_{\geq 1}})$$

of adjunction 2. sends cofree right $\mathcal{O}^\vee$-comodules in $\mathcal{C}^{\Sigma_{\geq 1}}$ to trivial right $\mathcal{O}$-modules in $\mathcal{C}^{\Sigma_{\geq 1}}$.

On the other hand by remark 4.20 the left adjoint of adjunction 2. sends trivial right $\mathcal{O}$-modules in $\mathcal{C}^{\Sigma_{\geq 1}}$ to cofree right $\mathcal{O}^\vee$-comodules in $\mathcal{C}^{\Sigma_{\geq 1}}$ as the composition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ preserves small sifted colimits in each component.

The dual statement about the counit of adjunction 2. follows from the fact that the composition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ also preserves small sifted limits in each component as it is equivalent via the norm map to the cocomposition product on $\mathcal{C}^{\Sigma_{\geq 1}}$ by lemma 2.19.

□

The rest of this section is devoted to the proof of prop. 4.23, in which we construct the Bar-Cobar-adjunction for associative algebras and bimodules.

We start with preparing the proof of proposition 4.23.

Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be categories.

We call a right fibration $\mathcal{C} \to \mathcal{D} \times \mathcal{E}$ left representable if for every $X \in \mathcal{D}$ the right fibration $\{X\} \times_{\mathcal{D}} \mathcal{C} \to \mathcal{E}$ is representable, equivalently the category $\{X\} \times_{\mathcal{D}} \mathcal{C}$ admits a final object.

In other words a right fibration $\mathcal{C} \to \mathcal{D} \times \mathcal{E}$ is left representable if it classifies a functor $\mathcal{D}^{op} \times \mathcal{E}^{op} \to \mathcal{S}$ adjoint to a functor $\mathcal{D}^{op} \to \mathcal{P}(\mathcal{E})$ that factors through $\mathcal{E}$.

We call a right fibration $\mathcal{C} \to \mathcal{D} \times \mathcal{E}$ right representable if the right fibration $\mathcal{C} \to \mathcal{D} \times \mathcal{E} \simeq \mathcal{E} \times \mathcal{D}$ is left representable.

So a left and right representable right fibration $\mathcal{C} \to \mathcal{D} \times \mathcal{E}$ classifies functors $F^{op} : \mathcal{D}^{op} \to \mathcal{E} \subset \mathcal{P}(\mathcal{E})$ and $G : \mathcal{E}^{op} \to \mathcal{D} \subset \mathcal{P}(\mathcal{D})$ such that $F : \mathcal{D} \to \mathcal{E}^{op}$ is left adjoint to $G : \mathcal{E}^{op} \to \mathcal{D}$.

Given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{C}', \mathcal{D}', \mathcal{E}'$ and left representable right fibrations $\mathcal{C} \to \mathcal{D} \times \mathcal{E}, \mathcal{C}' \to \mathcal{D}' \times \mathcal{E}'$ we call a commutative square

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{D} \times \mathcal{E} & \longrightarrow & \mathcal{D}' \times \mathcal{E}' \end{array} \tag{15}$$

a map of left representable right fibrations if for every $X \in \mathcal{D}$ the induced functor $\{X\} \times_{\mathcal{D}} \mathcal{C} \to \{X\} \times_{\mathcal{D}'} \mathcal{C}'$ preserves final objects.

The left representable right fibration $\mathcal{C} \to \mathcal{D} \times \mathcal{E}$ classifies a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ adjoint to a functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{E} \subset \mathcal{P}(\mathcal{E})$ and similar for $\mathcal{C}' \to \mathcal{D}' \times \mathcal{E}'$.

Square 15 classifies a natural transformation from the functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ to the functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{D}'^{\mathrm{op}} \times \mathcal{E}'^{\mathrm{op}} \to \mathcal{S}$ adjoint to a natural transformation from the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{E} \subset \mathcal{P}(\mathcal{E})$ to the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{D}'^{\mathrm{op}} \to \mathcal{E}' \subset \mathcal{P}(\mathcal{E}') \to \mathcal{P}(\mathcal{E})$ adjoint to a natural transformation α from the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{E} \subset \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}')$ to the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{D}'^{\mathrm{op}} \to \mathcal{E}' \subset \mathcal{P}(\mathcal{E}')$.

The functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{E} \subset \mathcal{P}(\mathcal{E}) \to \mathcal{P}(\mathcal{E}')$ factors as $\mathcal{D}^{\mathrm{op}} \to \mathcal{E} \to \mathcal{E}' \subset \mathcal{P}(\mathcal{E}')$ so that α induces a natural transformation β from the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{E} \to \mathcal{E}'$ to the functor $\mathcal{D}^{\mathrm{op}} \to \mathcal{D}'^{\mathrm{op}} \to \mathcal{E}'$.

Square 15 is a map of left representable right fibrations if and only if β is an equivalence.

Similarly we define maps of right representable right fibrations.

If $\mathcal{C} \to \mathcal{D} \times \mathcal{E}, \mathcal{C}' \to \mathcal{D}' \times \mathcal{E}'$ are left and right representable right fibrations, square 15 is a map of left and right representable right fibrations if and only if β^{op} is an equivalence and defines a map of adjunctions from the adjunction $\mathcal{D} \rightleftarrows \mathcal{E}^{\mathrm{op}}$ to the adjunction $\mathcal{D}' \rightleftarrows \mathcal{E}'^{\mathrm{op}}$.

For every category $\mathcal{C}$ the twisted arrow-category $\mathrm{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is a left and right representable right fibration classifying the identity adjunction of $\mathcal{C}$.

Given an operad $\mathcal{O}^{\otimes}$ and $\mathcal{O}^{\otimes}$-monoidal categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ we call a $\mathcal{O}^{\otimes}$ monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes}$ a left (right) representable right fibration of $\mathcal{O}^{\otimes}$-monoidal categories if it induces on the fiber over every $X \in \mathcal{O}$ a left representable right fibration.

The functor $\mathrm{Tw}(-) : \mathsf{Cat}_{\infty} \to \mathsf{Cat}_{\infty}$ that sends a category to its twisted arrow-category preserves finite products and so yields for every $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{Tw}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes} \times (\mathcal{C}^{\otimes})^{\mathrm{rev}}$.

To define Koszul-duality we study for $\mathcal{O}^{\otimes} = \mathrm{Ass}^{\otimes}, \mathrm{BM}^{\otimes}$ under which conditions a left (right) representable right fibration $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes}$ of $\mathcal{O}^{\otimes}$-monoidal categories induces a left (right) representable right fibration $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{E})$.

We first remark that $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{D}) \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{E})$ is a right fibration:

Given a $\mathcal{O}^{\otimes}$ monoidal functor $\mathcal{A}^{\otimes} \to \mathcal{B}^{\otimes}$ that induces on the fiber over every $X \in \mathcal{O}$ a right fibration, the induced functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{A}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{B})$ is a right fibration:

If the commutative square

$$\begin{array}{ccc} (\mathcal{A}^{\otimes})^{\Delta^1} & \longrightarrow & (\mathcal{B}^{\otimes})^{\Delta^1} \\ \downarrow & & \downarrow \\ (\mathcal{A}^{\otimes})^{\{1\}} & \longrightarrow & (\mathcal{B}^{\otimes})^{\{1\}} \end{array}$$

of $\mathcal{O}^{\otimes}$ monoidal categories yields on every $X \in \mathcal{O}$ a pullback square, it is a pullback square of $\mathcal{O}^{\otimes}$-monoidal categories and so gives rise to a pullback square

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{A})) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{B})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\{1\}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{A})) & \longrightarrow & \mathrm{Fun}(\{1\}, \mathrm{Alg}_{/\mathcal{O}}(\mathcal{B})). \end{array}$$

Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ be $\mathrm{BM}^{\otimes}$-monoidal categories and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \times_{\mathrm{BM}^{\otimes}} \mathcal{E}^{\otimes}$ a $\mathrm{BM}^{\otimes}$-monoidal functor.

Let $A \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})_{/\mathbb{1}}, B \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{b}})_{/\mathbb{1}}$ with images $A' \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})_{/\mathbb{1}}, B' \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}})_{/\mathbb{1}}$ and $\mathbb{1} \in \mathrm{Alg}(\mathcal{E}_{\mathfrak{a}})_{/\mathbb{1}}$, $\mathbb{1} \in \mathrm{Alg}(\mathcal{E}_{\mathfrak{b}})_{/\mathbb{1}}$.

For every $Y \in \mathcal{E}_{\mathfrak{m}} \simeq {}_{\mathbb{1}}\mathrm{BMod}_{\mathbb{1}}(\mathcal{E}_{\mathfrak{m}})$ corresponding to $\bar{Y} \in {}_{\mathbb{1}}\mathrm{BMod}_{\mathbb{1}}(\mathcal{E}_{\mathfrak{m}})$ the pullback $\mathrm{BM}^{\otimes} \times_{\mathcal{E}^{\otimes}} \mathcal{C}^{\otimes}$ along $\bar{Y}$ is a $\mathrm{BM}^{\otimes}$-monoidal category and we have a canonical equivalence

$$ {}_{A}\mathrm{BMod}_{B}(\{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} \mathcal{C}_{\mathfrak{m}}) \simeq \{\bar{Y}\} \times_{{}_{\mathbb{1}}\mathrm{BMod}_{\mathbb{1}}(\mathcal{E}_{\mathfrak{m}})} {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}}) \simeq $$

$$ \{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}}). $$

The forgetful functor $\{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} \mathcal{C}_{\mathfrak{m}} \to \{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}})$ factors as the forgetful functor

$$ \{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} \mathcal{C}_{\mathfrak{m}} \to {}_{A}\mathrm{BMod}_{B}(\{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} \mathcal{C}_{\mathfrak{m}}) \simeq \{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}}) $$

and thus preserves final objects if $\{Y\} \times_{\mathcal{E}_{\mathfrak{m}}} \mathcal{C}_{\mathfrak{m}}$ admits a final object.

Consequently if the functor $\mathcal{C}_{\mathfrak{m}} \to \mathcal{D}_{\mathfrak{m}} \times \mathcal{E}_{\mathfrak{m}}$ is a right representable right fibration, the induced functor ${}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}}) \to {}_{A'}\mathrm{BMod}_{B'}(\mathcal{D}_{\mathfrak{m}}) \times \mathcal{E}_{\mathfrak{m}}$ is and the forgetful functor $\mathcal{C}_{\mathfrak{m}} \simeq {}_{\mathbb{1}}\mathrm{BMod}_{\mathbb{1}}(\mathcal{C}_{\mathfrak{m}}) \to {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}})$ is a map of such.

Let $A \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})_{/\mathbb{1}}$. As the functor

$$ \{\mathbb{1}\} \times_{\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}^{\mathrm{op}})} \mathrm{Alg}(\mathrm{Tw}(\mathcal{C}_{\mathfrak{a}})) \to \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}) $$

is a right fibration, there is a unique map $A' \to \mathbb{1}$ in $\mathrm{Alg}(\mathrm{Tw}(\mathcal{C}_{\mathfrak{a}}))$ lying over the map $A \to \mathbb{1}$ in $\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$ and lying over the identity of $\mathbb{1}$ in $\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}^{\mathrm{op}})$.

Similarly for $B \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{b}})_{/\mathbb{1}}$ there is a unique map $B' \to \mathbb{1}$ in $\mathrm{Alg}(\mathrm{Tw}(\mathcal{C}_{\mathfrak{b}}))$ lying over the map $B \to \mathbb{1}$ in $\mathrm{Alg}(\mathcal{C}_{\mathfrak{b}})$ and lying over the identity of $\mathbb{1}$ in $\mathrm{Alg}(\mathcal{C}_{\mathfrak{b}}^{\mathrm{op}})$.

Thus the induced functor ${}_{A'}\mathrm{BMod}_{B'}(\mathrm{Tw}(\mathcal{C}_{\mathfrak{m}})) \to {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}}) \times \mathcal{C}_{\mathfrak{m}}^{\mathrm{op}}$ is a right representable right fibration classifying a functor $\theta : \mathcal{C}_{\mathfrak{m}} \to {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}})$ and the forgetful functor $\mathrm{Tw}(\mathcal{C}_{\mathfrak{m}}) \simeq {}_{\mathbb{1}}\mathrm{BMod}_{\mathbb{1}}(\mathrm{Tw}(\mathcal{C}_{\mathfrak{m}})) \to {}_{A'}\mathrm{BMod}_{B'}(\mathrm{Tw}(\mathcal{C}_{\mathfrak{m}}))$ is a map of such classifying an equivalence between θ and the forgetful functor $\mathrm{triv}_{A,B} : \mathcal{C}_{\mathfrak{m}} \to {}_{A}\mathrm{BMod}_{B}(\mathcal{C}_{\mathfrak{m}})$.

If $\mathcal{C}_\mathfrak{m}, \mathcal{C}_\mathfrak{a}, \mathcal{C}_\mathfrak{b}$ admit geometric realizations, by lemma 4.29 and remark 2.10 the functor

$$_{A'}\mathrm{BMod}_{B'}(\mathrm{Tw}(\mathcal{C}_\mathfrak{m})) \to {}_A\mathrm{BMod}_B(\mathcal{C}_\mathfrak{m}) \times \mathcal{C}_\mathfrak{m}^{\mathrm{op}}$$

is a left and right representable right fibration classifying an adjunction $\mathbb{1} \otimes_A - \otimes_B \mathbb{1} : {}_A\mathrm{BMod}_B(\mathcal{C}_\mathfrak{m}) \rightleftarrows \mathcal{C}_\mathfrak{m} : \mathrm{triv}_A$, where the left adjoint sends $X \in {}_A\mathrm{BMod}_B(\mathcal{C}_\mathfrak{m})$ to $\mathbb{1} \otimes_A X \otimes_B \mathbb{1} \simeq \mathrm{colim}_{n \in \Delta^{\mathrm{op}}}(A^{\otimes n} \otimes X \otimes B^{\otimes n})$ by remark 2.10.

As next we need some facts about bimodules:

Let $\mathcal{C}, \mathcal{E}$ be monoidal categories and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{E})$-bimodule classified by a $\mathrm{BM}^\otimes$-monoidal category $\mathcal{M}^\otimes \to \mathrm{BM}^\otimes$. Let $A \in \mathrm{Alg}(\mathcal{C})$ and $B \in \mathrm{Alg}(\mathcal{E})$.

We have the category ${}_A\mathrm{BMod}_B(\mathcal{D})$ of (A, B)-bimodules in $\mathcal{D}$.
We write $\mathrm{BMod}_A(\mathcal{D})$ for ${}_A\mathrm{BMod}_A(\mathcal{D})$.

If $\mathcal{M}^\otimes \to \mathrm{BM}^\otimes$ is compatible with geometric realizations, we have a $\mathrm{BM}^\otimes$-monoidal category ${}_A\mathrm{BMod}_B(\mathcal{M})^\otimes \to \mathrm{BM}^\otimes$ compatible with geometric realizations that exhibits ${}_A\mathrm{BMod}_B(\mathcal{D})$ as bitensored over the monoidal categories $\mathrm{BMod}_A(\mathcal{C}), \mathrm{BMod}_B(\mathcal{E})$, where the actions and monoidal structures are given by the relative tensorproduct.
Moreover we have a $\mathrm{BM}^\otimes$-monoidal functor ${}_A\mathrm{BMod}_B(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$ with underlying functor ${}_A\mathrm{BMod}_B(\mathcal{D}) \to \mathcal{D}$ and underlying monoidal functors $\mathrm{BMod}_A(\mathcal{C})^\otimes \to \mathcal{C}^\otimes$, $\mathrm{BMod}_B(\mathcal{E})^\otimes \to \mathcal{E}^\otimes$.
A $\mathrm{BM}^\otimes$-monoidal functor $F : \mathcal{M}^\otimes \to \mathcal{M}'^\otimes$ that induces on the fiber over every object of BM a functor that preserves geometric realizations gives rise to a commutative square

$$\begin{array}{ccc} {}_A\mathrm{BMod}_B(\mathcal{M})^\otimes & \longrightarrow & {}_{F(A)}\mathrm{BMod}_{F(B)}(\mathcal{M}')^\otimes \\ \downarrow & & \downarrow \\ \mathcal{M}^\otimes & \longrightarrow & \mathcal{M}'^\otimes \end{array}$$

of $\mathrm{BM}^\otimes$-monoidal categories with underlying commutative squares the evident ones.

If $\mathcal{M}^\otimes \to \mathrm{BM}^\otimes$ is not compatible with geometric realizations, we embed $\mathcal{M}^\otimes$ into the $\mathrm{BM}^\otimes$-monoidal category $\mathcal{M}'^\otimes := \mathcal{P}(\mathcal{M})^\otimes$ compatible with geometric realizations via the $\mathrm{BM}^\otimes$-monoidal Yoneda-embedding and write ${}_A\mathrm{BMod}_B(\mathcal{M})^\otimes \subset {}_A\mathrm{BMod}_B(\mathcal{M}')^\otimes$ for the full suboperad spanned by the objects of ${}_A\mathrm{BMod}_B(\mathcal{D}), \mathrm{BMod}_A(\mathcal{C}), \mathrm{BMod}_B(\mathcal{E})$.
So ${}_A\mathrm{BMod}_B(\mathcal{M})^\otimes$ is an operad over $\mathrm{BM}^\otimes$.

This definition extends the former one: If $\mathcal{M}^\otimes \to \mathrm{BM}^\otimes$ is compatible with geometric realizations, the $\mathrm{BM}^\otimes$-monoidal Yoneda-embedding $\mathcal{M}^\otimes \subset \mathcal{M}'^\otimes$ yields an embedding ${}_A\mathrm{BMod}_B(\mathcal{M})^\otimes \subset {}_A\mathrm{BMod}_B(\mathcal{M}')^\otimes$ of operads over $\mathrm{BM}^\otimes$, where we use the former definition of ${}_A\mathrm{BMod}_B(\mathcal{M})^\otimes$:

If $\mathcal{M}^\otimes \to \mathrm{BM}^\otimes$ is compatible with geometric realizations, the $\mathrm{BM}^\otimes$-monoidal Yoneda-embedding factors as $\mathrm{BM}^\otimes$-monoidal embeddings $\mathcal{M}^\otimes \subset \mathcal{M}''^\otimes \subset \mathcal{M}'^\otimes$, where $\mathcal{M}''^\otimes \to \mathrm{BM}^\otimes$ is a $\mathrm{BM}^\otimes$-monoidal localization of $\mathcal{M}'^\otimes = \mathcal{P}(\mathcal{M})^\otimes$ (and so a $\mathrm{BM}^\otimes$-monoidal category compatible with small colimits) and the embedding $\mathcal{M}^\otimes \subset \mathcal{M}''^\otimes$ induces on the fiber over every object of BM a functor that preserves geometric realizations.
The $\mathrm{BM}^\otimes$-monoidal localization $\mathcal{M}''^\otimes \subset \mathcal{M}'^\otimes$ yields a $\mathrm{BM}^\otimes$-monoidal localization ${}_A\mathrm{BMod}_B(\mathcal{M}'')^\otimes \subset {}_A\mathrm{BMod}_B(\mathcal{M}')^\otimes$.

The $BM^{\otimes}$-monoidal embedding $\mathcal{M}^{\otimes} \subset \mathcal{M}''^{\otimes}$ yields a $BM^{\otimes}$-monoidal embedding ${}_A BMod_B(\mathcal{M})^{\otimes} \subset {}_A BMod_B(\mathcal{M}'')^{\otimes}$.

Let $\mathcal{C}$ be a monoidal category and $A \in Alg(\mathcal{C})$.

By cor. 3.4.1.7. [18] there is a canonical equivalence $Alg(BMod_A(\mathcal{C})) \simeq Alg(\mathcal{C})_{A/}$ compatible with the forgetful functors to $Alg(\mathcal{C})$.

Now we are ready to state the main proposition concerning Koszul-duality:

Proposition 4.23. *Let $\mathcal{C}, \mathcal{E}$ be monoidal categories and $\mathcal{D}$ a $(\mathcal{C}, \mathcal{E})$-bimodule.*

Assume that the tensorunit of $\mathcal{C}$ and $\mathcal{E}$ is a final object and that $\mathcal{C}, \mathcal{D}, \mathcal{E}$ admit geometric realizations.

The functors

$$BMod(Tw(\mathcal{D})) \to BMod(\mathcal{D}) \times BMod(\mathcal{D}^{op}),$$

$$Alg(Tw(\mathcal{C})) \to Alg(\mathcal{C}) \times Alg(\mathcal{C}^{op}), \; Alg(Tw(\mathcal{E})) \to Alg(\mathcal{E}) \times Alg(\mathcal{E}^{op})$$

are left representable right fibrations and both forgetful functors

$$BMod(Tw(\mathcal{D})) \to Alg(Tw(\mathcal{C})), \; BMod(Tw(\mathcal{D})) \to Alg(Tw(\mathcal{E}))$$

are maps of such.

As the tensorunit of $\mathcal{C}$ is a final object, the canonical monoidal functor $\{\mathbb{1}\} \times_{\mathcal{C}^{op}} Tw(\mathcal{C}) \simeq \mathcal{C}_{/\mathbb{1}} \to \mathcal{C}$ is an equivalence and thus induces an equivalence

$$\{\mathbb{1}\} \times_{Alg(\mathcal{C}^{op})} Alg(Tw(\mathcal{C})) \simeq Alg(\{\mathbb{1}\} \times_{\mathcal{C}^{op}} Tw(\mathcal{C})) \to Alg(\mathcal{C}).$$

Thus every $A \in Alg(\mathcal{C})$ uniquely lifts to an object A' of $Alg(Tw(\mathcal{C}))$ lying over the initial algebra $\mathbb{1} \in Alg(\mathcal{C}^{op})$.
Similarly every $B \in Alg(\mathcal{E})$ uniquely lifts to an object B' of $Alg(Tw(\mathcal{E}))$ lying over the initial algebra $\mathbb{1} \in Alg(\mathcal{E}^{op})$.

There is a canonical map

$$\begin{array}{ccc} {}_A BMod_B(\mathcal{D}) \times_{BMod(\mathcal{D})} BMod(Tw(\mathcal{D})) & \longrightarrow & {}_{A'} BMod_{B'}(Tw(\mathcal{D})) \\ \downarrow & & \downarrow \\ {}_A BMod_B(\mathcal{D}) \times BMod(\mathcal{D}^{op}) & \longrightarrow & {}_A BMod_B(\mathcal{D}) \times \mathcal{D}^{op} \end{array}$$

of left representable right fibrations.

Remark 4.24. *The left representable right fibrations*

$$BMod(Tw(\mathcal{D})) \to BMod(\mathcal{D}) \times BMod(\mathcal{D}^{op}),$$

$$Alg(Tw(\mathcal{C})) \to Alg(\mathcal{C}) \times Alg(\mathcal{C}^{op}), \; Alg(Tw(\mathcal{E})) \to Alg(\mathcal{E}) \times Alg(\mathcal{E}^{op})$$

classify functors

$$BMod(\mathcal{D})^{op} \to BMod(\mathcal{D}^{op}), \; Alg(\mathcal{C})^{op} \to Alg(\mathcal{C}^{op}), \; Alg(\mathcal{E})^{op} \to Alg(\mathcal{E}^{op})$$

and the forgetful functor

$$BMod(Tw(\mathcal{D})) \to Alg(Tw(\mathcal{C})) \times Alg(Tw(\mathcal{E}))$$

classifies a commutative square

$$\begin{array}{ccc} \mathrm{BMod}(\mathcal{D})^{\mathrm{op}} & \longrightarrow & \mathrm{BMod}(\mathcal{D}^{\mathrm{op}}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} \times \mathrm{Alg}(\mathcal{E})^{\mathrm{op}} & \longrightarrow & \mathrm{Alg}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Alg}(\mathcal{E})^{\mathrm{op}}. \end{array}$$

So turning to opposite categories we obtain a commutative square

$$\begin{array}{ccc} \mathrm{BMod}(\mathcal{D}) & \longrightarrow & \mathrm{CoBMod}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{E}) & \longrightarrow & \mathrm{Coalg}(\mathcal{C}) \times \mathrm{Coalg}(\mathcal{E}) \end{array}$$

that induces on the fiber over every $\mathrm{A} \in \mathrm{Alg}(\mathcal{C}),\ \mathrm{B} \in \mathrm{Alg}(\mathcal{E})$ *a functor*

$$\theta : {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to {}_{\mathrm{Bar(A)}}\mathrm{CoBMod}_{\mathrm{Bar(B)}}(\mathcal{D}).$$

The map

$$\begin{array}{ccc} {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \times_{\mathrm{BMod}(\mathcal{D})} \mathrm{BMod}(\mathrm{Tw}(\mathcal{D})) & \longrightarrow & {}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathrm{Tw}(\mathcal{D})) \\ \downarrow & & \downarrow \\ {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \times \mathrm{BMod}(\mathcal{D}^{\mathrm{op}}) & \longrightarrow & {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \times \mathcal{D}^{\mathrm{op}} \end{array}$$

of left representable right fibrations classifies an equivalence between the functor

$${}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to \mathrm{BMod}(\mathcal{D}) \to \mathrm{CoBMod}(\mathcal{D}) \to \mathcal{D}$$

being equivalent to the functor

$${}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \xrightarrow{\theta} {}_{\mathrm{Bar(A)}}\mathrm{CoBMod}_{\mathrm{Bar(B)}}(\mathcal{D}) \to \mathcal{D}$$

and the functor $\mathbb{1} \otimes_{\mathrm{A}} - \otimes_{\mathrm{B}} \mathbb{1} : {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to \mathcal{D}$.

Proof. Denote $\mathcal{M}^{\otimes} \to \mathrm{BM}^{\otimes}$ the $\mathrm{BM}^{\otimes}$-monoidal category classifying the $(\mathcal{C}, \mathcal{E})$-bimodule $\mathcal{D}$.

By lemma 4.25 the $\mathrm{BM}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes} := \mathcal{P}(\mathcal{M})^{\otimes}$ gives rise to a left representable right fibration

$$\mathcal{X}_{\mathcal{M}}^{\otimes} := (\mathcal{M}^{\otimes})^{\mathrm{rev}} \times_{(\mathcal{M}'^{\otimes})^{\mathrm{rev}}} \mathrm{Tw}(\mathcal{M}')^{\otimes} \to \mathcal{M}'^{\otimes} \times_{\mathrm{BM}^{\otimes}} (\mathcal{M}^{\otimes})^{\mathrm{rev}}$$

of $\mathrm{BM}^{\otimes}$-monoidal categories.

By lemma 4.25 1. it is enough to see that the right fibrations

$$\mathrm{BMod}(\mathcal{X}_{\mathcal{D}}) \to \mathrm{BMod}(\mathcal{D}') \times \mathrm{BMod}(\mathcal{D}^{\mathrm{op}}),\ \mathrm{Alg}(\mathcal{X}_{\mathcal{C}}) \to \mathrm{Alg}(\mathcal{C}') \times \mathrm{Alg}(\mathcal{C}^{\mathrm{op}}),$$
$$\mathrm{Alg}(\mathcal{X}_{\mathcal{E}}) \to \mathrm{Alg}(\mathcal{E}') \times \mathrm{Alg}(\mathcal{E}^{\mathrm{op}})$$

are left representable and both forgetful functors

$$\mathrm{BMod}(\mathcal{X}_{\mathcal{D}}) \to \mathrm{Alg}(\mathcal{X}_{\mathcal{C}}),\ \mathrm{BMod}(\mathcal{X}_{\mathcal{D}}) \to \mathrm{Alg}(\mathcal{X}_{\mathcal{E}})$$

are maps of such.

By lemma 4.25 2. the $\mathrm{BM}^{\otimes}$-monoidal co-Yoneda-embedding $\mathcal{M}^{\otimes} \subset \mathcal{N}^{\otimes} := (\mathcal{P}(\mathcal{M}^{\mathrm{rev}})^{\otimes})^{\mathrm{rev}}$ yields an embedding $\mathcal{X}^{\otimes}_{\mathcal{M}} \to \mathcal{X}^{\otimes}_{\mathcal{N}}$ of left representable right fibrations of $\mathrm{BM}^{\otimes}$-monoidal categories.

As $\mathcal{N}^{\otimes}$ is compatible with totalizations, by lemma 4.25 3. the $\mathrm{BM}^{\otimes}$-monoidal category $\mathcal{X}^{\otimes}_{\mathcal{N}} \to \mathrm{BM}^{\otimes}$ is compatible with geometric realizations and the $\mathrm{BM}^{\otimes}$-monoidal functor $\mathcal{X}^{\otimes}_{\mathcal{N}} \to \mathcal{N}'^{\otimes} \times_{\mathrm{BM}^{\otimes}} (\mathcal{N}^{\otimes})^{\mathrm{rev}}$ induces on the fiber over every object of BM a functor that preserves geometric realizations.

Moreover the $\mathrm{BM}^{\otimes}$-monoidal category $\mathcal{P}(\mathcal{M}^{\mathrm{rev}})^{\otimes} = (\mathcal{N}^{\otimes})^{\mathrm{rev}} \to \mathrm{BM}^{\otimes}$ yields on the fiber over every object of BM a category that admits totalizations and the $\mathrm{BM}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{M}^{\mathrm{rev}} \to \mathcal{P}(\mathcal{M}^{\mathrm{rev}})$ preserves fiberwise totalizations.

As the tensorunit of $\mathcal{C}, \mathcal{E}$ is a final object, the tensorunit of $\mathcal{C}', \mathcal{E}'$ also is. So the canonical functors $\{\mathbb{1}\} \times_{\mathcal{C}^{\mathrm{op}}} \mathcal{X}_{\mathcal{C}} \simeq \mathcal{C}'_{/\mathbb{1}} \to \mathcal{C}', \{\mathbb{1}\} \times_{\mathcal{E}^{\mathrm{op}}} \mathcal{X}_{\mathcal{E}} \simeq \mathcal{E}'_{/\mathbb{1}} \to \mathcal{E}'$ are equivalences.

Thus the assertion follows from proposition 4.26.

□

Lemma 4.25. *Let $\mathcal{O}^{\otimes}$ be an operad, $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category that induces on the fiber over every $\mathrm{X} \in \mathcal{O}$ a category that admits geometric realizations and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal functor that induces on the fiber over every $\mathrm{X} \in \mathcal{O}$ a functor that preserves geometric realizations.*

The $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{C}^{\otimes} \subset \mathcal{C}'^{\otimes} := \mathcal{P}(\mathcal{C})^{\otimes}$ gives rise to a right fibration $\mathcal{X}^{\otimes}_{\mathcal{C}} := (\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{(\mathcal{C}'^{\otimes})^{\mathrm{rev}}} \mathrm{Tw}(\mathcal{C}')^{\otimes} \to \mathcal{C}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\otimes})^{\mathrm{rev}}$ of $\mathcal{O}^{\otimes}$-monoidal categories.

1. *The right fibration $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Tw}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}^{\mathrm{rev}})$ is left representable if the right fibration $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{X}_{\mathcal{C}}) \to \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}') \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}^{\mathrm{rev}})$ is left representable.*
   Given a map of operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ the induced functor $\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Tw}(\mathcal{C})) \to \mathrm{Alg}_{/\mathcal{O}'}(\mathrm{Tw}(\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}))$ is a map of left representable right fibrations if the functor $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{X}_{\mathcal{C}}) \to \mathrm{Alg}_{/\mathcal{O}'}(\mathcal{X}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}})$ is.

2. *The $\mathcal{O}^{\otimes}$-monoidal functor $\mathcal{X}^{\otimes}_{\mathcal{C}} \to \mathcal{C}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\otimes})^{\mathrm{rev}}$ is a left representable right fibration of $\mathcal{O}^{\otimes}$-monoidal categories and the $\mathcal{O}^{\otimes}$-monoidal functor $\mathcal{X}^{\otimes}_{\mathcal{C}} \to \mathcal{X}^{\otimes}_{\mathcal{D}}$ is a map of left representable right fibrations of $\mathcal{O}^{\otimes}$-monoidal categories.*

3. *If $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is compatible with totalizations, the $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{X}^{\otimes}_{\mathcal{C}} \to \mathcal{O}^{\otimes}$ is compatible with geometric realizations and the $\mathcal{O}^{\otimes}$-monoidal functor $\mathcal{X}^{\otimes}_{\mathcal{C}} \to \mathcal{C}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\otimes})^{\mathrm{rev}}$ induces on the fiber over every $\mathrm{X} \in \mathcal{O}$ a functor that preserves geometric realizations.*

Proof. 1: The $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{C}^{\otimes} \subset \mathcal{C}'^{\otimes} := \mathcal{P}(\mathcal{C})^{\otimes}$ yields a $\mathcal{O}^{\otimes}$-monoidal equivalence

$$\mathrm{Tw}(\mathcal{C})^{\otimes} \simeq (\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\otimes})^{\mathrm{rev}}) \times_{(\mathcal{C}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}'^{\otimes})^{\mathrm{rev}})} \mathrm{Tw}(\mathcal{C}')^{\otimes}$$

over $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\otimes})^{\mathrm{rev}}$.

So we have a $\mathcal{O}^{\otimes}$-monoidal equivalence $\mathrm{Tw}(\mathcal{C})^{\otimes} \simeq \mathcal{C}^{\otimes} \times_{\mathcal{C}'^{\otimes}} \mathcal{X}^{\otimes}_{\mathcal{C}}$ over $\mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{C}^{\otimes})^{\mathrm{rev}}$ that gives rise to an equivalence

$$\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Tw}(\mathcal{C})) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}')} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{X}_{\mathcal{C}})$$

over $\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}) \times \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}^{\mathrm{rev}})$.

So for every $\mathrm{Y} \in \mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})$ we obtain a canonical equivalence

$$\{\mathrm{Y}\} \times_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C})} \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Tw}(\mathcal{C})) \simeq \{\mathrm{Y}\} \times_{\mathrm{Alg}_{/\mathcal{O}}(\mathcal{C}')} \mathrm{Alg}_{/\mathcal{O}}(\mathcal{X}_{\mathcal{C}}).$$

2: Every category $\mathcal{B}$ is a localization of a category $\mathcal{B}'$ that admits small colimits.

We show that the right fibration $\mathcal{X}_{\mathcal{B}} := \mathcal{B}^{\mathrm{op}} \times_{\mathcal{B}'^{\mathrm{op}}} \mathrm{Tw}(\mathcal{B}') \to \mathcal{B}' \times \mathcal{B}^{\mathrm{op}}$ is left representable and for every functor $\mathrm{A} \to \mathcal{B}$ preserving geometric realizations between categories that admit geometric realizations the induced functor $\mathcal{X}_{\mathcal{A}} \to \mathcal{X}_{\mathcal{B}}$ is a map of left representable right fibrations:

The full subcategory inclusion $\mathcal{B} \subset \mathcal{B}'$ admits a left adjoint L so that the opposite embedding $\mathcal{B}^{\mathrm{op}} \subset \mathcal{B}'^{\mathrm{op}}$ is a left adjoint functor.

So with the right fibration $\{\mathrm{Z}\} \times_{\mathcal{B}'} \mathrm{Tw}(\mathcal{B}') \simeq (\mathcal{B}'^{\mathrm{op}})_{/\mathrm{Z}} \to \mathcal{B}'^{\mathrm{op}}$ for $\mathrm{Z} \in \mathcal{B}'$ also its pullback $\{\mathrm{Z}\} \times_{\mathcal{B}'} \mathcal{X}_{\mathcal{B}} \to \mathcal{B}^{\mathrm{op}}$ along the left adjoint functor $\mathcal{B}^{\mathrm{op}} \subset \mathcal{B}'^{\mathrm{op}}$ is representable.

Thus the left representable right fibration $\mathcal{X}_{\mathcal{B}} \to \mathcal{B}' \times \mathcal{B}^{\mathrm{op}}$ classifies a functor $\mathcal{B}'^{\mathrm{op}} \times \mathcal{B} \to \mathcal{S}$ adjoint to the functor $\mathrm{L}^{\mathrm{op}} : \mathcal{B}'^{\mathrm{op}} \to \mathcal{B}^{\mathrm{op}} \subset \mathcal{P}(\mathcal{B}^{\mathrm{op}})$.

As the functor $\mathrm{F} : \mathrm{A} \to \mathcal{B}$ preserves geometric realizations, the canonical natural transformation $\mathrm{L}_{\mathcal{B}} \circ \mathrm{F}' \to \mathrm{F} \circ \mathrm{L}_{\mathcal{A}}$ is an equivalence so that the functor $\mathcal{X}_{\mathcal{A}} \to \mathcal{X}_{\mathcal{B}}$ is a map of left representable right fibrations.

3: If $\mathcal{B}$ admits totalizations, the right fibration $\alpha : \mathcal{X}_{\mathcal{B}} \to \mathcal{B}' \times \mathcal{B}^{\mathrm{op}}$ admits geometric realizations that are preserved by α :

α classifies the functor $\beta : \mathcal{B}'^{\mathrm{op}} \times \mathcal{B} \subset \mathcal{B}'^{\mathrm{op}} \times \mathcal{B}' \xrightarrow{\mathcal{B}'(-,-)} \mathcal{S}$.

As the Yoneda-embedding $\mathcal{B} \subset \mathcal{B}'$ preserves small limits and the mapping space functor $\mathcal{B}'(-,-) : \mathcal{B}'^{\mathrm{op}} \times \mathcal{B}' \to \widehat{\mathcal{S}}$ preserves small limits in each component, β preserves small sifted limits and so especially totalizations.

Thus by cor. 5.2.2.37. [18] the category $\mathcal{X}_{\mathcal{B}}$ admits geometric realizations that are preserved and detected by α provided that $\mathcal{B}$ admits totalizations.

Thus if the $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes}$ is compatible with totalizations, the $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{X}_{\mathcal{C}}^{\otimes}$ is compatible with geometric realizations.

□

The main ingredient in proposition 4.23 is the next proposition:

Proposition 4.26. *Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ be $\mathrm{BM}^{\otimes}$-monoidal categories and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}, \mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ be $\mathrm{BM}^{\otimes}$-monoidal functors.*

Let $\mathcal{C}'^{\otimes} \subset \mathcal{C}^{\otimes}, \mathcal{D}'^{\otimes} \subset \mathcal{D}^{\otimes}, \mathcal{E}'^{\otimes} \subset \mathcal{E}^{\otimes}$ be full $\mathrm{BM}^{\otimes}$-monoidal subcategories such that the $\mathrm{BM}^{\otimes}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}, \mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ restrict to functors $\mathcal{C}' \to \mathcal{D}', \mathcal{C}' \to \mathcal{E}'$.

Assume that the following conditions hold:

1. *$\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ are compatible with geometric realizations and the $\mathrm{BM}^{\otimes}$-monoidal functor $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ induces on the fiber over every object of BM a functor that preserves geometric realizations.*
2. *$\mathcal{E}_{\mathfrak{a}}, \mathcal{E}_{\mathfrak{b}}, \mathcal{E}_{\mathfrak{m}}$ admit totalizations and $\mathcal{E}'_{\mathfrak{a}}, \mathcal{E}'_{\mathfrak{b}}, \mathcal{E}'_{\mathfrak{m}}$ are closed under totalizations.*
3. *The functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \times_{\mathrm{BM}^{\otimes}} \mathcal{E}^{\otimes}$, $\mathcal{C}'^{\otimes} \to \mathcal{D}'^{\otimes} \times_{\mathrm{BM}^{\otimes}} \mathcal{E}'^{\otimes}$ are left representable right fibrations of $\mathrm{BM}^{\otimes}$-monoidal categories and the embedding $\mathcal{C}'^{\otimes} \subset \mathcal{C}^{\otimes}$ is a map of such.*

4. *The canonical functors* $\{\mathbb{1}\} \times_{\mathcal{E}'_{\mathfrak{a}}} \mathcal{C}'_{\mathfrak{a}} \to \mathcal{D}'_{\mathfrak{a}}, \ \{\mathbb{1}\} \times_{\mathcal{E}'_{\mathfrak{b}}} \mathcal{C}'_{\mathfrak{b}} \to \mathcal{D}'_{\mathfrak{b}}$ *are equivalences.*

- *The functors*

$$\mathrm{BMod}(\mathcal{C}'_{\mathfrak{m}}) \to \mathrm{BMod}(\mathcal{D}'_{\mathfrak{m}}) \times \mathrm{BMod}(\mathcal{E}'_{\mathfrak{m}}),$$

$$\mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}}) \to \mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}}) \times \mathrm{Alg}(\mathcal{E}'_{\mathfrak{a}}),$$
$$\mathrm{Alg}(\mathcal{C}'_{\mathfrak{b}}) \to \mathrm{Alg}(\mathcal{D}'_{\mathfrak{b}}) \times \mathrm{Alg}(\mathcal{E}'_{\mathfrak{b}})$$

are left representable right fibrations and the forgetful functors $\mathrm{BMod}(\mathcal{C}'_{\mathfrak{m}}) \to \mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}}), \ \mathrm{BMod}(\mathcal{C}'_{\mathfrak{m}}) \to \mathrm{Alg}(\mathcal{C}'_{\mathfrak{b}})$ *are maps of such.*

- *The monoidal equivalence* $\{\mathbb{1}\} \times_{\mathcal{E}'_{\mathfrak{a}}} \mathcal{C}'_{\mathfrak{a}} \to \mathcal{D}'_{\mathfrak{a}}$ *of 4. induces an equivalence*

$$\{\mathbb{1}\} \times_{\mathrm{Alg}(\mathcal{E}'_{\mathfrak{a}})} \mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}}) \simeq \mathrm{Alg}(\{\mathbb{1}\} \times_{\mathcal{E}'_{\mathfrak{a}}} \mathcal{C}'_{\mathfrak{a}}) \to \mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}})$$

and similar for $\mathfrak{b} \in \mathrm{BM}$.
Thus every $\mathrm{A} \in \mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}}), \ \mathrm{B} \in \mathrm{Alg}(\mathcal{D}'_{\mathfrak{b}})$ *lift to objects* $\mathrm{A}' \in \mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}})$ *respectively* $\mathrm{B}' \in \mathrm{Alg}(\mathcal{C}'_{\mathfrak{b}})$ *lying over the initial algebras of* $\mathrm{Alg}(\mathcal{E}'_{\mathfrak{a}})$ *respectively* $\mathrm{Alg}(\mathcal{E}'_{\mathfrak{b}})$.

There is a canonical map

$$\begin{array}{ccc} {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}) \times_{\mathrm{BMod}(\mathcal{D}'_{\mathfrak{m}})} \mathrm{BMod}(\mathcal{C}'_{\mathfrak{m}}) & \longrightarrow & {}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{m}}) \\ \downarrow & & \downarrow \\ {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}) \times \mathrm{BMod}(\mathcal{E}'_{\mathfrak{m}}) & \longrightarrow & {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}) \times \mathcal{E}'_{\mathfrak{m}} \end{array}$$

of left representable right fibrations.

Proof. Let $\mathrm{A} \in \mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}}), \ \mathrm{B} \in \mathrm{Alg}(\mathcal{D}'_{\mathfrak{b}})$.
The monoidal equivalence $\{\mathbb{1}\} \times_{\mathcal{E}'_{\mathfrak{a}}} \mathcal{C}'_{\mathfrak{a}} \to \mathcal{D}'_{\mathfrak{a}}$ induces an equivalence

$$\{\mathbb{1}\} \times_{\mathrm{Alg}(\mathcal{E}'_{\mathfrak{a}})} \mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}}) \simeq \mathrm{Alg}(\{\mathbb{1}\} \times_{\mathcal{E}'_{\mathfrak{a}}} \mathcal{C}'_{\mathfrak{a}}) \to \mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}})$$

and similar for $\mathfrak{b} \in \mathrm{BM}$.

Thus $\mathrm{A} \in \mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}}), \ \mathrm{B} \in \mathrm{Alg}(\mathcal{D}'_{\mathfrak{b}})$ lift to objects $\mathrm{A}' \in \mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}})$ respectively $\mathrm{B}' \in \mathrm{Alg}(\mathcal{C}'_{\mathfrak{b}})$ lying over the initial algebras of $\mathrm{Alg}(\mathcal{E}'_{\mathfrak{a}})$ respectively $\mathrm{Alg}(\mathcal{E}'_{\mathfrak{b}})$.

Condition 1. guarantees that there is a $\mathrm{BM}^{\otimes}$-monoidal functor ${}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C})^{\otimes} \to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})^{\otimes}$.

Denote $\bar{\mathrm{A}}, \bar{\mathrm{B}}$ the initial algebras of $\mathrm{BMod}_{\mathrm{A}}(\mathcal{D}_{\mathfrak{a}})$ respectively $\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{b}})$ lying over $\mathrm{A} \in \mathcal{D}_{\mathfrak{a}}$ respectively $\mathrm{B} \in \mathcal{D}_{\mathfrak{b}}$ so that the forgetful functor $\alpha : {}_{\bar{\mathrm{A}}}\mathrm{BMod}_{\bar{\mathrm{B}}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{m}})) \to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{m}})$ is an equivalence.

We remark that α is equivalent to the functor induced by the lax $\mathrm{BM}^{\otimes}$-monoidal forgetful functor ${}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{m}})^{\otimes} \to \mathcal{D}_{\mathfrak{m}}^{\otimes}$.

Moreover under the equivalence α every $\mathrm{X} \in {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{m}})$ corresponds to a $\mathrm{BM}^{\otimes}$-monoidal functor $\bar{\mathrm{X}} : \mathrm{BM}^{\otimes} \to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})^{\otimes}$, whose pullbacks to $\mathrm{Ass}^{\otimes}$ are the monoidal functors $\bar{\mathrm{A}} : \mathrm{Ass}^{\otimes} \to \mathrm{BMod}_{\mathrm{A}}(\mathcal{D}_{\mathfrak{a}})^{\otimes}$ respectively $\bar{\mathrm{B}} : \mathrm{Ass}^{\otimes} \to \mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{b}})^{\otimes}$.

By lemma 4.28 condition 4. implies that the forgetful functors

$$\{\bar{A}\} \times_{\mathrm{Alg}(\mathrm{BMod}_A(\mathcal{D}'_{\mathfrak{a}}))} \mathrm{Alg}(\mathrm{BMod}_{A'}(\mathcal{C}'_{\mathfrak{a}})) \to \{A\} \times_{\mathrm{Alg}(\mathcal{D}'_{\mathfrak{a}})} \mathrm{Alg}(\mathcal{C}'_{\mathfrak{a}}),$$

$$\{\bar{B}\} \times_{\mathrm{Alg}(\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{b}}))} \mathrm{Alg}(\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{b}})) \to \{B\} \times_{\mathrm{Alg}(\mathcal{D}'_{\mathfrak{b}})} \mathrm{Alg}(\mathcal{C}'_{\mathfrak{b}}),$$

$$\gamma : {}_{\bar{A}}\mathrm{BMod}_{\bar{B}}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}})) \times_{\mathrm{BMod}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{m}}))$$
$$\to {}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}}) \times_{\mathrm{BMod}(\mathcal{D}'_{\mathfrak{m}})} \mathrm{BMod}(\mathcal{C}'_{\mathfrak{m}})$$

over ${}_{\bar{A}}\mathrm{BMod}_{\bar{B}}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}})) \times \mathrm{BMod}(\mathcal{E}'_{\mathfrak{m}}) \simeq {}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}}) \times \mathrm{BMod}(\mathcal{E}'_{\mathfrak{m}})$ are equivalences.

Consequently it is enough to see that the categories

$$\{\bar{A}\} \times_{\mathrm{Alg}(\mathrm{BMod}_A(\mathcal{D}'_{\mathfrak{a}}))} \mathrm{Alg}(\mathrm{BMod}_{A'}(\mathcal{C}'_{\mathfrak{a}})),$$

$$\{\bar{B}\} \times_{\mathrm{Alg}(\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{b}}))} \mathrm{Alg}(\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{b}}))$$

admit a final object, the right fibration

$${}_{\bar{A}}\mathrm{BMod}_{\bar{B}}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}})) \times_{\mathrm{BMod}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{m}})) \to$$
$${}_{\bar{A}}\mathrm{BMod}_{\bar{B}}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}})) \times \mathrm{BMod}(\mathcal{E}'_{\mathfrak{m}})$$

is left representable and for every $X \in {}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}})$ both forgetful functors

$$\{\bar{X}\} \times_{\mathrm{BMod}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{m}})) \to$$
$$\{\bar{A}\} \times_{\mathrm{Alg}(\mathrm{BMod}_A(\mathcal{D}'_{\mathfrak{a}}))} \mathrm{Alg}(\mathrm{BMod}_{A'}(\mathcal{C}'_{\mathfrak{a}})),$$
$$\{\bar{X}\} \times_{\mathrm{BMod}({}_A\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{m}})) \to$$
$$\{\bar{B}\} \times_{\mathrm{Alg}(\mathrm{BMod}_B(\mathcal{D}'_{\mathfrak{b}}))} \mathrm{Alg}(\mathrm{BMod}_{B'}(\mathcal{C}'_{\mathfrak{b}}))$$

preserve final objects.

Using 2. and 3., by proposition 4.29 the $\mathrm{BM}^{\otimes}$-monoidal functor

$${}_{A'}\mathrm{BMod}_{B'}(\mathcal{C})^{\otimes} \to {}_A\mathrm{BMod}_B(\mathcal{D})^{\otimes}$$

is a left representable right fibration of $\mathrm{BM}^{\otimes}$-monoidal categories.

So the categories

$$\{X\} \times_{{}_A\mathrm{BMod}_B(\mathcal{D}_{\mathfrak{m}})} {}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{m}}),\ \{A\} \times_{\mathrm{BMod}_A(\mathcal{D}_{\mathfrak{a}})} \mathrm{BMod}_{A'}(\mathcal{C}_{\mathfrak{a}}),$$
$$\{B\} \times_{\mathrm{BMod}_B(\mathcal{D}_{\mathfrak{b}})} \mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{b}})$$

admit a final object.

The pullback

$$\mathrm{BM}^{\otimes} \times_{{}_A\mathrm{BMod}_B(\mathcal{D})^{\otimes}} {}_{A'}\mathrm{BMod}_{B'}(\mathcal{C})^{\otimes}$$

along $\bar{X}$ is a $\mathrm{BM}^{\otimes}$-monoidal category, whose fiber over $\mathfrak{a} \in \mathrm{BM}$ is the pullback $\mathrm{Ass}^{\otimes} \times_{\mathrm{BMod}_A(\mathcal{D}_{\mathfrak{a}})^{\otimes}} \mathrm{BMod}_{A'}(\mathcal{C}_{\mathfrak{a}})^{\otimes}$ along $\bar{A}$ and whose fiber over $\mathfrak{b} \in \mathrm{BM}$ is the pullback $\mathrm{Ass}^{\otimes} \times_{\mathrm{BMod}_B(\mathcal{D}_{\mathfrak{b}})^{\otimes}} \mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{b}})^{\otimes}$ along $\bar{B}$.

Consequently the final objects of the categories

$$\{X\} \times_{{}_A\mathrm{BMod}_B(\mathcal{D}_{\mathfrak{m}})} {}_{A'}\mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{m}}),\ \{A\} \times_{\mathrm{BMod}_A(\mathcal{D}_{\mathfrak{a}})} \mathrm{BMod}_{A'}(\mathcal{C}_{\mathfrak{a}}),$$
$$\{B\} \times_{\mathrm{BMod}_B(\mathcal{D}_{\mathfrak{b}})} \mathrm{BMod}_{B'}(\mathcal{C}_{\mathfrak{b}})$$

lift to final objects of the categories

$$\mathrm{BMod}(\{\mathrm{X}\} \times_{{}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{m}})} {}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}_{\mathfrak{m}})) \simeq$$

$$\{\bar{\mathrm{X}}\} \times_{\mathrm{BMod}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{m}}))} \mathrm{BMod}({}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}_{\mathfrak{m}})),$$

$$\mathrm{Alg}\{\mathrm{A}\}\times_{\mathrm{BMod}_{\mathrm{A}}(\mathcal{D}_{\mathfrak{a}})}\mathrm{BMod}_{\mathrm{A}'}(\mathcal{C}_{\mathfrak{a}})) \simeq \{\bar{\mathrm{A}}\}\times_{\mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}}(\mathcal{D}_{\mathfrak{a}}))}\mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}'}(\mathcal{C}_{\mathfrak{a}}))$$

$$\mathrm{Alg}\{\mathrm{B}\}\times_{\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{b}})}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}_{\mathfrak{b}})) \simeq \{\bar{\mathrm{B}}\}\times_{\mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}_{\mathfrak{b}}))}\mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}_{\mathfrak{b}}))$$

that belong to the full subcategories

$$\{\bar{\mathrm{X}}\} \times_{\mathrm{BMod}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{m}})),$$

$$\{\bar{\mathrm{A}}\} \times_{\mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}}(\mathcal{D}'_{\mathfrak{a}}))} \mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}'}(\mathcal{C}'_{\mathfrak{a}})),$$

$$\{\bar{\mathrm{B}}\} \times_{\mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{b}}))} \mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{b}})) :$$

This follows from proposition 4.29 2. using condition 2. and 3.

Especially the final object of

$$\{\bar{\mathrm{X}}\} \times_{\mathrm{BMod}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{m}})),$$

lies over the final objects of

$$\{\bar{\mathrm{A}}\} \times_{\mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}}(\mathcal{D}'_{\mathfrak{a}}))} \mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}'}(\mathcal{C}'_{\mathfrak{a}}))$$

and

$$\{\bar{\mathrm{B}}\} \times_{\mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{b}}))} \mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{b}})).$$

Composing the inverse of the equivalence

$$\gamma : {}_{\bar{\mathrm{A}}}\mathrm{BMod}_{\bar{\mathrm{B}}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}})) \times_{\mathrm{BMod}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{m}}))$$

$$\to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}) \times_{\mathrm{BMod}(\mathcal{D}'_{\mathfrak{m}})} \mathrm{BMod}(\mathcal{C}'_{\mathfrak{m}})$$

with the forgetful functor

$${}_{\bar{\mathrm{A}}}\mathrm{BMod}_{\bar{\mathrm{B}}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}})) \times_{\mathrm{BMod}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}}))} \mathrm{BMod}({}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{m}})) \to$$

$${}_{\bar{\mathrm{A}}}\mathrm{BMod}_{\bar{\mathrm{B}}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}})) \times_{{}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}'_{\mathfrak{m}})} {}_{\mathrm{A}'}\mathrm{BMod}_{\mathrm{B}'}(\mathcal{C}'_{\mathfrak{m}})$$

we get the desired map of left representable right fibrations.

□

As next we prove three lemmata used in the proof of proposition 4.26:

Lemma 4.27. *Let $\mathcal{C}, \mathcal{E}$ be monoidal categories, $\mathcal{D}$ a $(\mathcal{C}, \mathcal{E})$-bimodule and* $\mathrm{A} \in \mathrm{Alg}(\mathcal{C}), \mathrm{B} \in \mathrm{Alg}(\mathcal{E})$.

We have a pullback square

$$\begin{array}{ccc} \mathrm{BMod}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})) & \longrightarrow & \mathrm{BMod}(\mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}}(\mathcal{C})) \times \mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}}(\mathcal{E})) & \longrightarrow & \mathrm{Alg}(\mathcal{C}) \times \mathrm{Alg}(\mathcal{E}) \end{array} \qquad (16)$$

that induces on the fiber over every $\mathrm{X} \in \mathrm{Alg}(\mathrm{BMod}_{\mathrm{A}}(\mathcal{C})), \mathrm{Y} \in \mathrm{Alg}(\mathrm{BMod}_{\mathrm{B}}(\mathcal{E}))$ *the forgetful functor*

$$\alpha : {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})) \to {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D}),$$

which is an equivalence.

Proof. As square 16 is a map of cartesian fibrations, it is enough to see that α is an equivalence.

We first reduce to the case that the $(\mathcal{C}, \mathcal{E})$-bimodule $\mathcal{D}$ is compatible with geometric realizations:

The $\mathrm{BM}^{\otimes}$-monoidal Yoneda-embedding $\mathcal{D} \to \mathcal{D}' := \mathcal{P}(\mathcal{D})$ yields a forgetful functor ${}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}')) \to {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D}')$ over $\mathcal{D}'$, whose pullback to $\mathcal{D}$ is α.

So we can assume that the $(\mathcal{C}, \mathcal{E})$-bimodule $\mathcal{D}$ is compatible with geometric realizations.

In this case we have a $\mathrm{BM}^{\otimes}$-monoidal category ${}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})^{\otimes}$ compatible with geometric realizations.

Thus the categories ${}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D})$, ${}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})$, ${}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}))$ admit geometric realizations that are preserved by the forgetful functors

$${}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D}) \to \mathcal{D},\ {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to \mathcal{D},\ {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})) \to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}).$$

Consequently the functors

$${}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D}) \to \mathcal{D},\ {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})) \to {}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D}) \to \mathcal{D}$$

are monadic with left adjoints

$$\mathcal{D} \to {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D}),\ \mathrm{Z} \mapsto \mathrm{X} \otimes \mathrm{Z} \otimes \mathrm{Y}$$

$$\mathcal{D} \to {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})),\ \mathrm{Z} \mapsto \mathrm{X} \otimes_{\mathrm{A}} (\mathrm{A} \otimes \mathrm{Z} \otimes \mathrm{A}) \otimes_{\mathrm{A}} \mathrm{Y}$$

that are canonically equivalent, i.e. equivalent compatible with the units:

Both units

$$\mathrm{Z} \simeq \mathbb{1} \otimes \mathrm{Z} \otimes \mathbb{1} \to \mathrm{X} \otimes \mathrm{Z} \otimes \mathrm{Y},\ \ \mathrm{Z} \simeq \mathbb{1} \otimes \mathrm{Z} \otimes \mathbb{1} \to \mathrm{A} \otimes \mathrm{Z} \otimes \mathrm{A} \to \mathrm{X} \otimes_{\mathrm{A}} (\mathrm{A} \otimes \mathrm{Z} \otimes \mathrm{A}) \otimes_{\mathrm{A}} \mathrm{Y}$$

are equivalent as the morphism $\mathrm{A} \otimes \mathrm{Z} \otimes \mathrm{A} \to \mathrm{X} \otimes_{\mathrm{A}} (\mathrm{A} \otimes \mathrm{Z} \otimes \mathrm{A}) \otimes_{\mathrm{A}} \mathrm{Y} \simeq \mathrm{X} \otimes \mathrm{Z} \otimes \mathrm{Y}$ arises by tensoring the identity of Z with the morphisms $\mathrm{A} \to \mathrm{X}, \mathrm{A} \to \mathrm{Y}$.

Thus we have a commutative triangle

$$\begin{array}{ccc} {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}({}_{\mathrm{A}}\mathrm{BMod}_{\mathrm{B}}(\mathcal{D})) & \xrightarrow{\alpha} & {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D}) \\ & \searrow \quad \swarrow & \\ & \mathcal{D} & \end{array}$$

between monadic functors over $\mathcal{D}$ that induces an equivalence on monads.

So α is an equivalence.

□

Lemma 4.28.

1. *Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ be monoidal categories and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \times_{\mathrm{Ass}^{\otimes}} \mathcal{E}^{\otimes}$ a right fibration of monoidal categories.*

   Assume that the canonical monoidal functor $\{\mathbb{1}\} \times_{\mathcal{E}^{\otimes}} \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is an equivalence.

   Let $\mathrm{X} \in \mathrm{Alg}(\mathcal{C})$ lying over $\mathrm{Y} \in \mathrm{Alg}(\mathcal{D})$ and $\mathbb{1} \in \mathrm{Alg}(\mathcal{E})$.

Then the commutative square

$$\begin{array}{ccc} \mathrm{Alg}(\mathrm{BMod}_{\mathrm{X}}(\mathcal{C})) & \longrightarrow & \mathrm{Alg}(\mathrm{BMod}_{\mathrm{Y}}(\mathcal{D})) \times \mathrm{Alg}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}(\mathcal{D}) \times \mathrm{Alg}(\mathcal{E}) \end{array} \tag{17}$$

is a pullback square.

2. *Let* $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ *be* $\mathrm{BM}^{\otimes}$*-monoidal categories and* $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes} \times_{\mathrm{BM}^{\otimes}} \mathcal{E}^{\otimes}$ *a right fibration of* $\mathrm{BM}^{\otimes}$*-monoidal categories.*

   Assume that the canonical monoidal functors

$$\{\mathbb{1}\} \times_{\mathcal{E}_{\mathfrak{a}}} \mathcal{C}_{\mathfrak{a}} \to \mathcal{D}_{\mathfrak{a}}, \ \{\mathbb{1}\} \times_{\mathcal{E}_{\mathfrak{b}}} \mathcal{C}_{\mathfrak{b}} \to \mathcal{D}_{\mathfrak{b}}$$

   are equivalences.

   Let $\mathrm{X} \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}), \ \mathrm{X}' \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{b}})$ *lying over* $\mathrm{Y} \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})$ *respectively* $\mathrm{Y}' \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}})$ *and* $\mathbb{1} \in \mathrm{Alg}(\mathcal{E}_{\mathfrak{a}})$ *respectively* $\mathbb{1} \in \mathrm{Alg}(\mathcal{E}_{\mathfrak{b}})$.

   Then the commutative square

$$\begin{array}{ccc} \mathrm{BMod}({}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{X}'}(\mathcal{C}_{\mathfrak{m}})) & \longrightarrow & \mathrm{BMod}({}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}})) \\ \downarrow & & \downarrow \\ \mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}) & \longrightarrow & \mathrm{BMod}(\mathcal{D}_{\mathfrak{m}}) \end{array} \tag{18}$$

   is a pullback square.

Proof. 1: Square 17 is equivalent to the square

$$\begin{array}{ccc} \mathrm{Alg}(\mathcal{C})_{\mathrm{X}/} & \longrightarrow & \mathrm{Alg}(\mathcal{D})_{\mathrm{Y}/} \times \mathrm{Alg}(\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}(\mathcal{D}) \times \mathrm{Alg}(\mathcal{E}). \end{array}$$

As both vertical functors in this square are left fibrations, this square is a pullback square if and only if for every $\mathrm{B} \in \mathrm{Alg}(\mathcal{C})$ lying over $\mathrm{B}' \in \mathrm{Alg}(\mathcal{D})$ and $\mathrm{B}'' \in \mathrm{Alg}(\mathcal{E})$ the induced map

$$\alpha : \mathrm{Alg}(\mathcal{C})(\mathrm{X},\mathrm{B}) \to \mathrm{Alg}(\mathcal{D})(\mathrm{Y},\mathrm{B}') \times \mathrm{Alg}(\mathcal{E})(\mathbb{1},\mathrm{B}'')$$

is an equivalence.

Using that the functor $\mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{D}) \times \mathrm{Alg}(\mathcal{E})$ is a right fibration, α is an equivalence if and only if X is an initial object of the fiber $\{(\mathrm{Y},\mathbb{1})\} \times_{(\mathrm{Alg}(\mathcal{D})\times\mathrm{Alg}(\mathcal{E}))} \mathrm{Alg}(\mathcal{C})$

The monoidal functor $\{\mathbb{1}\}\times_{\mathcal{E}}\mathcal{C} \to \mathcal{D}$ and thus also the functor $\mathrm{Alg}(\{\mathbb{1}\}\times_{\mathcal{E}} \mathcal{C}) \simeq \{\mathbb{1}\} \times_{\mathrm{Alg}(\mathcal{E})} \mathrm{Alg}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{D})$ are equivalences so that the fiber $\{(\mathrm{Y},\mathbb{1})\} \times_{(\mathrm{Alg}(\mathcal{D})\times\mathrm{Alg}(\mathcal{E}))} \mathrm{Alg}(\mathcal{C})$ is contractible.

2: Set

$$\mathrm{W}^{\mathcal{C}}_{\mathrm{X},\mathrm{X}'} := (\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})_{\mathrm{X}/} \times \mathrm{Alg}(\mathcal{C}_{\mathfrak{b}})_{\mathrm{X}'/}) \times_{(\mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})\times\mathrm{Alg}(\mathcal{C}_{\mathfrak{b}}))} \mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}),$$

$$\mathrm{W}^{\mathcal{D}}_{\mathrm{Y},\mathrm{Y}'} := (\mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})_{\mathrm{Y}/} \times \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}})_{\mathrm{Y}'/}) \times_{(\mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})\times\mathrm{Alg}(\mathcal{D}_{\mathfrak{b}}))} \mathrm{BMod}(\mathcal{D}_{\mathfrak{m}}).$$

By lemma 4.27 square 18 is equivalent to the square

$$\begin{array}{ccc} \mathrm{W}^{\mathcal{C}}_{\mathrm{X},\mathrm{X}'} & \longrightarrow & \mathrm{W}^{\mathcal{D}}_{\mathrm{Y},\mathrm{Y}'} \\ \downarrow & & \downarrow \\ \mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}) & \longrightarrow & \mathrm{BMod}(\mathcal{D}_{\mathfrak{m}}), \end{array}$$

whose composition with the pullback square

$$\begin{array}{ccc} \mathrm{W}^{\mathcal{D}}_{\mathrm{Y},\mathrm{Y}'} & \longrightarrow & \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})_{\mathrm{Y}/} \times \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}})_{\mathrm{Y}'/} \\ \downarrow & & \downarrow \\ \mathrm{BMod}(\mathcal{D}_{\mathfrak{m}}) & \longrightarrow & \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}}) \times \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}}) \end{array}$$

is the composition of pullback squares

$$\begin{array}{ccccc} \mathrm{W}^{\mathcal{C}}_{\mathrm{X},\mathrm{X}'} & \longrightarrow & \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})_{\mathrm{X}/} \times \mathrm{Alg}(\mathcal{C}_{\mathfrak{b}})_{\mathrm{X}'/} & \longrightarrow & \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})_{\mathrm{Y}/} \times \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}})_{\mathrm{Y}'/} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{BMod}(\mathcal{C}_{\mathfrak{m}}) & \longrightarrow & \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}}) \times \mathrm{Alg}(\mathcal{C}_{\mathfrak{b}}) & \longrightarrow & \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}}) \times \mathrm{Alg}(\mathcal{D}_{\mathfrak{b}}), \end{array}$$

where the right hand square is a pullback square by 1.

□

Lemma 4.29. *Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ be $\mathrm{BM}^{\otimes}$-monoidal categories and $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}, \mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ be $\mathrm{BM}^{\otimes}$-monoidal functors. Assume that $\mathcal{E}_{\mathfrak{a}}, \mathcal{E}_{\mathfrak{m}}$ admit totalizations.*

Let $\mathrm{X}, \mathrm{X}' \in \mathrm{Alg}(\mathcal{C}_{\mathfrak{a}})$ lying over $\mathrm{Y}, \mathrm{Y}' \in \mathrm{Alg}(\mathcal{D}_{\mathfrak{a}})$ and $\mathbb{1} \in \mathrm{Alg}(\mathcal{E}_{\mathfrak{a}})$.

Assume that the functor $\mathcal{C} \to \mathcal{D} \times \mathcal{E}$ is a left representable right fibration classifying a functor $\mathcal{D}^{\mathrm{op}} \times \mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ adjoint to a functor $\phi : \mathcal{D}^{\mathrm{op}} \to \mathcal{E} \subset \mathcal{P}(\mathcal{E})$.

1. *The functor*

   $$\gamma : {}_{\mathrm{X}}\mathrm{BMod}_{\mathrm{X}'}(\mathcal{C}_{\mathfrak{m}}) \to {}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}}) \times \mathcal{E}_{\mathfrak{m}}$$

   is a left representable right fibration.

2. *γ classifies a functor ${}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}})^{\mathrm{op}} \times \mathcal{E}_{\mathfrak{m}}^{\mathrm{op}} \to \mathcal{S}$ adjoint to a functor $\theta :_{\mathrm{Y}} \mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}})^{\mathrm{op}} \to \mathcal{E}_{\mathfrak{m}} \subset \mathcal{P}(\mathcal{E}_{\mathfrak{m}})$ that sends an object $\mathrm{A} \in_{\mathrm{Y}} \mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}})$ to the totalization of a cosimplicial object R of $\mathcal{E}_{\mathfrak{m}}$ that takes values in the essential image of ϕ.*

Proof. We want to see that θ factors through $\mathcal{E}_{\mathfrak{m}}$.

Denote $\mathrm{V} : {}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}}) \to \mathcal{D}_{\mathfrak{m}}$ the forgetful functor and $\mathrm{F} : \mathcal{D}_{\mathfrak{m}} \to {}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}})$ the free functor.

The forgetful functor $\mathrm{V} : {}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}}) \to \mathcal{D}_{\mathfrak{m}}$ is monadic so that every object A of ${}_{\mathrm{Y}}\mathrm{BMod}_{\mathrm{Y}'}(\mathcal{D}_{\mathfrak{m}})$ is the geometric realization of a V-split simplicial object W such that for every $\mathrm{n} \in \mathbb{N}$ there is a $\mathrm{Z} \in \mathcal{D}_{\mathfrak{m}}$ and an equivalence $\mathrm{W}_{\mathrm{n}} \simeq \mathrm{F}(\mathrm{Z})$.

As $\mathcal{E}_{\mathfrak{m}}$ is closed in $\mathcal{P}(\mathcal{E}_{\mathfrak{m}})$ under totalizations, it is enough to check the following:

1. θ^{op} sends the geometric realization of W to a geometric realization
2. θ^{op} sends free (Y, Y')-bimodules to representable right fibrations.

1. is equivalent to the condition that for every $B \in \mathcal{E}_m$ the composition $\alpha :_Y BMod_{Y'}(\mathcal{D}_m) \xrightarrow{\theta^{op}} \mathcal{P}(\mathcal{E}_m)^{op} \xrightarrow{ev_B} \mathcal{S}^{op}$ sends the geometric realization of W to a geometric realization.

The functor α^{op} is classified by the right fibration $\beta : {}_X BMod_{X'}(\{B\} \times_{\mathcal{E}_m} \mathcal{C}_m) \simeq \{B\} \times_{\mathcal{E}_m} ({}_X BMod_{X'}(\mathcal{C}_m)) \to {}_Y BMod_{Y'}(\mathcal{D}_m)$.

We have a commutative square

$$\begin{array}{ccc} {}_X BMod_{X'}(\{B\} \times_{\mathcal{E}_m} \mathcal{C}_m) & \xrightarrow{V'} & \{B\} \times_{\mathcal{E}_m} \mathcal{C}_m \\ \downarrow \beta & & \downarrow \\ {}_Y BMod_{Y'}(\mathcal{D}_m) & \xrightarrow{V} & \mathcal{D}_m. \end{array}$$

The right fibration $\{B\} \times_{\mathcal{E}_m} \mathcal{C}_m \to \mathcal{D}_m$ reflects split simplicial objects.

So every simplicial object of ${}_X BMod_{X'}(\{B\} \times_{\mathcal{E}_m} \mathcal{C}_m)$ lying over W is V'-split and so admits a geometric realization that is preserved by β. By cor. 5.2.2.37. [18] this implies 1.

2. follows from the fact that the free functor $\mathcal{C}_m \to_X BMod_{X'}(\mathcal{C}_m)$ is a map of left representable right fibrations, i.e. for every $A \in \mathcal{D}_m$ the functor

$$\{A\} \times_{\mathcal{D}_m} \mathcal{C}_m \to \{F(A)\} \times_{({}_Y BMod_{Y'}(\mathcal{D}_m))} ({}_X BMod_{X'}(\mathcal{C}_m))$$

preserves final objects.

Denote $\mathcal{M}_{\mathcal{C}} \to \Delta^1, \mathcal{M}_{\mathcal{D}} \to \Delta^1, \mathcal{M}_{\mathcal{E}} \simeq \mathcal{E}_m \times \Delta^1 \to \Delta^1$ the bicartesian fibrations classifying the free/forgetful adjunctions

$$\mathcal{C}_m \rightleftarrows_X BMod_{X'}(\mathcal{C}_m), \ \mathcal{D}_m \rightleftarrows_Y BMod_{Y'}(\mathcal{D}_m), \ \mathcal{E}_m \simeq BMod_{\mathbb{1}}(\mathcal{E}_m).$$

So we have maps $\mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{D}}, \mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{E}} \simeq \mathcal{E}_m \times \Delta^1$ of bicartesian fibrations over Δ^1.

The induced map $\mathcal{M}_{\mathcal{C}} \to \mathcal{M}_{\mathcal{D}} \times \mathcal{E}_m$ of bicartesian fibrations over Δ^1 induces on the fiber over 0 and 1 the right fibrations $\mathcal{C}_m \to \mathcal{D}_m \times \mathcal{E}_m$ respectively ${}_X BMod_{X'}(\mathcal{C}_m) \to_Y BMod_{Y'}(\mathcal{D}_m) \times \mathcal{E}_m$ and is thus a right fibration.

Let α be the cocartesian section of $\mathcal{M}_{\mathcal{D}} \to \Delta^1$ corresponding to $A \in \mathcal{D}_m$.

Thus the pullback $(\Delta^1 \times \mathcal{E}_m) \times_{(\mathcal{M}_{\mathcal{D}} \times \mathcal{E}_m)} \mathcal{M}_{\mathcal{C}} \to \Delta^1 \times \mathcal{E}_m$ along $\alpha \times \mathcal{E}_m$ is a map of bicartesian fibrations over Δ^1 classifying an adjunction

$$\{A\} \times_{\mathcal{D}_m} \mathcal{C}_m \rightleftarrows \{F(A)\} \times_{({}_Y BMod_{Y'}(\mathcal{D}_m))} ({}_X BMod_{X'}(\mathcal{C}_m))$$

relative to $\mathcal{E}_m$.

This adjunction relative to $\mathcal{E}_m$ induces on the fiber over every $Z \in \mathcal{E}_m$ an adjunction $\{(A, Z)\} \times_{(\mathcal{D}_m \times \mathcal{E}_m)} \mathcal{C}_m \rightleftarrows \{(F(A), Z)\} \times_{({}_Y BMod_{Y'}(\mathcal{D}_m) \times \mathcal{E}_m)} ({}_X BMod_{X'}(\mathcal{C}_m))$ between spaces and is thus an equivalence.

So θ factors through $\mathcal{E}_m$ and we have a canonical equivalence $\theta \circ F^{op} \simeq \phi$ of functors $\mathcal{D}_m^{op} \to \mathcal{E}_m$.

Hence we obtain a canonical equivalence

$$\theta(A) \simeq \theta(\mathrm{colim}_{\Delta^{op}}(W)) \simeq \lim_{\Delta}(\theta^{op} \circ W^{op})$$

and for every $n \in \mathbb{N}$ an equivalence $(\theta \circ W)_n \simeq \theta(F(Z)) \simeq \phi(Z)$.

□

Let $\mathcal{C}$ be a nice stable symmetric monoidal category.

When defining restricted L_∞-algebras by their relation to bialgebras (def. 2.26) we used the definiton of cocommutative coalgebras as the full subcategory $\mathrm{Cocoalg}(\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{F}\mathrm{in}_*, (\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}}$ spanned by the functors over $\mathcal{F}\mathrm{in}_*$ preserving inert morphisms.

This definition provides a symmetric monoidal structure on $\mathrm{Cocoalg}(\mathcal{C})$ such that the forgetful functor $\mathrm{Cocoalg}(\mathcal{C}) \to \mathcal{C}$ is symmetric monoidal and so induces a forgetful functor from bialgebras to associative algebras in $\mathcal{C}$.

On the other hand we used the Koszul-duality (4.2) between spectral Lie algebras in $\mathcal{C}$ and conilpotent cocommutative coalgebras in $\mathcal{C}$ with divided powers that composed with the forgetful functor $\mathrm{Cocoalg}^{\mathrm{dp,conil}}(\mathcal{C}) \to \mathrm{Coalg}_{\mathrm{Cocomm}^{\mathrm{ncu}}}(\mathcal{C})$ leads to an adjunction $\mathrm{Alg}_{\mathrm{Lie}}(\mathcal{C}) \rightleftarrows \mathrm{Coalg}_{\mathrm{Cocomm}^{\mathrm{ncu}}}(\mathcal{C})$.

Consequently to apply Koszul-duality to the theory of restricted L_∞-algebras we need to identify the categories $\mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \simeq \mathrm{Cocoalg}(\mathcal{C})^{\mathrm{ncu}}$ and $\mathrm{Coalg}_{\mathrm{Cocomm}^{\mathrm{ncu}}}(\mathcal{C})$.

In the following we show the equivalent dual statement that there is a canonical equivalence

$$\mathrm{Alg}_{\mathrm{Comm}^{\mathrm{nu}}}(\mathcal{C}) \simeq \mathrm{Calg}(\mathcal{C})^{\mathrm{nu}}$$

over $\mathcal{C}$.

Proposition 4.30. *Let $\mathcal{C}$ be a symmetric monoidal category that admits small colimits.*

1. *There is a canonical equivalence*

$$\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Calg}(\mathcal{C})$$

   over $\mathcal{C}$.

2. *If $\mathcal{C}$ is a preadditive symmetric monoidal category that admits small colimits, there is a canonical equivalence*

$$\mathrm{Alg}_{\mathrm{Comm}^{\mathrm{nu}}}(\mathcal{C}) \simeq \mathrm{Calg}^{\mathrm{nu}}(\mathcal{C})$$

   over $\mathcal{C}$.

Proof. 1:

We may assume that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits.

Otherwise we embed $\mathcal{C}$ symmetric monoidally into $\mathcal{P}(\mathcal{C})$ endowed with Day-convolution and the canonical equivalence

$$\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{P}(\mathcal{C})) \simeq \mathrm{Calg}(\mathcal{P}(\mathcal{C}))$$

over $\mathcal{P}(\mathcal{C})$ restricts to an equivalence

$$\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Calg}(\mathcal{C})$$

over $\mathcal{C}$.

The Hopf operad Comm in $\mathcal{C}$ gives rise to a Hopf monad on $\mathcal{C}$ so that the monadic forgetful functor $\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C}) \to \mathcal{C}$ lifts to a symmetric monoidal functor $\mathrm{Alg}_{\mathrm{Comm}}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$.

Denote S the free functor $\mathrm{Comm} \circ (-) : \mathcal{C} \to \mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$ $X \mapsto \mathrm{Comm} \circ X \simeq \coprod_{n \geq 0} (X^{\otimes n})_{\Sigma_n}$.

For every $X, Y \in \mathcal{C}$ we have a natural equivalence

$$S(X \coprod Y) \simeq \coprod_{n\geq 0}((X \coprod Y)^{\otimes n})_{\Sigma n} \simeq (\coprod_{n\geq 0}(X^{\otimes n})_{\Sigma n}) \otimes (\coprod_{n\geq 0}(Y^{\otimes n})_{\Sigma n})$$

$$\simeq S(X) \otimes S(Y)$$

in $\mathcal{C}$.

For Y the initial object $\varnothing$ of $\mathcal{C}$ this equivalence

$$S(X) \simeq S(X \coprod \varnothing) \simeq S(X) \otimes S(\varnothing) \simeq S(X) \otimes \mathbb{1} \simeq S(X) \tag{19}$$

is the identity.

As next we show that the symmetric monoidal structure on $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$ is cocartesian.

We start by showing that the tensorunit $\mathbb{1}$ of $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$ is initial.

We have to see that the unique morphism $\alpha : S(\varnothing) \to \mathbb{1}$ is an equivalence.

The morphism α factors as $S(\varnothing) \to S(\mathbb{1}) \xrightarrow{\mu} \mathbb{1}$, where the counit μ is induced by the multiplication of $\mathbb{1}$.

So it is enough to see that for every $n \in \mathbb{N}$ the multiplication morphisms $\mathbb{1}^{\otimes n} \to \mathbb{1}$ in $\mathcal{C}$ are equivalences.

The canonical equivalence $\mathrm{Cocoalg}(\mathrm{Cocoalg}(\mathcal{C})) \simeq \mathrm{Cocoalg}(\mathcal{C})$ yields a symmetric monoidal functor $\mathrm{LMod}_{\mathrm{Comm}}(\mathrm{Cocoalg}(\mathcal{C})) \to \mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$.

Thus the tensorunit of $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$ lifts to the tensorunit of $\mathrm{LMod}_{\mathrm{Comm}}(\mathrm{Cocoalg}(\mathcal{C}))$ that lies over the tensorunit of $\mathrm{Cocoalg}(\mathcal{C})$.

So the multiplication morphisms $\mathbb{1}^{\otimes n} \to \mathbb{1}$ in $\mathcal{C}$ lift to endomorphisms of the tensorunit of $\mathrm{Cocoalg}(\mathcal{C})$. But the tensorunit of $\mathrm{Cocoalg}(\mathcal{C})$ is a final object.

Denote $\beta : X \coprod Y \to S(X) \otimes S(Y)$ the morphism in $\mathcal{C}$ that is the morphism

$$X \simeq X \otimes \mathbb{1} \simeq X \otimes S(\varnothing) \to S(X) \otimes S(Y)$$

on the first summand and the morphism

$$Y \simeq \mathbb{1} \otimes Y \simeq S(\varnothing) \otimes Y \to S(X) \otimes S(Y)$$

on the second summand.

By lemma 4.31 and remark 4.32 it is enough to see that for every $X, Y \in \mathcal{C}$ the composition

$$\alpha : S(X \coprod Y) \xrightarrow{S(\beta)} S(S(X) \otimes S(Y)) \xrightarrow{\mu} S(X) \otimes S(Y)$$

adjoint to β is an equivalence, where $\mu : S(S(X) \otimes S(Y)) \to S(X) \otimes S(Y)$ denotes the multiplication of $S(X) \otimes S(Y)$ that factors as

$$S(S(X) \otimes S(Y)) \to S(S(X)) \otimes S(S(Y)) \to S(X) \otimes S(Y).$$

We show that α is the canonical equivalence $S(X \coprod Y) \simeq S(X) \otimes S(Y)$:

By 19 the morphism β factors as $X \coprod Y \to S(X \coprod Y) \simeq S(X) \otimes S(Y)$ so that the morphism $S(\beta) : S(X \coprod Y) \to S(S(X) \otimes S(Y))$ factors as $S(X \coprod Y) \to S(S(X \coprod Y)) \simeq S(S(X) \otimes S(Y))$ and so as

$$S(X \coprod Y) \simeq S(X) \otimes S(Y) \to S(S(X) \otimes S(Y)).$$

So the morphism α factors as $S(X \coprod Y) \simeq$

$$S(X) \otimes S(Y) \to S(S(X) \otimes S(Y)) \to S(S(X)) \otimes S(S(Y)) \to S(X) \otimes S(Y).$$

So by the triangular identities α is the canonical equivalence $S(X \coprod Y) \simeq S(X) \otimes S(Y)$.

Thus the symmetric monoidal structure on $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$ is cocartesian so that the forgetful functor $\mathrm{Calg}(\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})) \to \mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})$ is an equivalence.

Hence the symmetric monoidal functor $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C}) \to \mathcal{C}$ lifts to a forgetful functor $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C}) \simeq \mathrm{Calg}(\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C})) \to \mathrm{Calg}(\mathcal{C})$.

The forgetful functor $\mathrm{Calg}(\mathcal{C}) \to \mathcal{C}$ is monadic with left adjoint Sym.

Thus the forgetful functor $\mathrm{LMod}_{\mathrm{Comm}}(\mathcal{C}) \to \mathrm{Calg}(\mathcal{C})$ is an equivalence if and only if for every $X \in \mathcal{C}$ the morphism $\rho : \mathrm{Sym}(X) \to S(X)$ adjoint to the unit $X \to S(X)$ is an equivalence.

The object $\mathrm{Sym}(X)$ is naturally equivalent in $\mathcal{C}$ to $\coprod_{n \geq 0} (X^{\otimes n})_{\Sigma n}$ such that for every commutative algebra A in $\mathcal{C}$ the counit $\coprod_{n \geq 0} (A^{\otimes n})_{\Sigma n} \simeq \mathrm{Sym}(A) \to A$ as morphism in $\mathcal{C}$ is induced by the multiplication of A.

So we get a natural equivalence $\mathrm{Sym}(X) \simeq S(X)$ in $\mathcal{C}$ and we will show that ρ is homotopic to this equivalence:

The morphism ρ factors as $\mathrm{Sym}(X) \to \mathrm{Sym}(S(X)) \xrightarrow{\mu} S(X)$, where $\mu : \mathrm{Sym}(S(X)) \to S(X)$ denotes the counit.

So ρ factors as $\mathrm{Sym}(X) \simeq S(X) \to S(S(X)) \xrightarrow{\mu'} S(X)$, where $\mu' : S(S(X)) \to S(X)$ is the counit induced by the multiplication of $S(X)$.

By the triangular identities ρ is the canonical equivalence $\mathrm{Sym}(X) \simeq S(X)$.

2: We may assume that the symmetric monoidal structure on $\mathcal{C}$ is compatible with small colimits.

Otherwise denote $\mathcal{C}' := \widehat{\mathcal{P}}^{\Sigma}(\mathcal{C}) \subset \widehat{\mathcal{P}}(\mathcal{C})$ the full subcategory spanned by the functors $\mathcal{C}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ that preserve finite products.

The category $\widehat{\mathcal{P}}^{\Sigma}(\mathcal{C})$ is preadditive.

By the Yoneda-lemma the Yoneda-embedding $\mathcal{C} \to \mathcal{C}' := \mathcal{P}^{\Sigma}(\mathcal{C})$ preserves finite coproducts.

The full subcategory $\widehat{\mathcal{P}}^{\Sigma}(\mathcal{C}) \subset \widehat{\mathcal{P}}(\mathcal{C})$ is a localization compatible with the Day-convolution symmetric monoidal structure on $\widehat{\mathcal{P}}(\mathcal{C})$.

Especially the induced symmetric monoidal structure on $\widehat{\mathcal{P}}^{\Sigma}(\mathcal{C})$ is compatible with small colimits.

The Yoneda-embedding $\mathcal{C} \to \mathcal{P}^{\Sigma}(\mathcal{C})$ is a symmetric monoidal functor and so yields an equivalence

$$\mathrm{LMod}_{\mathrm{Comm}^{\mathrm{nu}}}(\mathcal{C}) \simeq \mathcal{C} \times_{\mathcal{C}'} \mathrm{LMod}_{\mathrm{Comm}^{\mathrm{nu}}}(\mathcal{C}')$$

over $\mathcal{C}$.

So we get an equivalence.

$$\mathrm{LMod}_{\mathrm{Comm}^{\mathrm{nu}}}(\mathcal{C}) \simeq \mathcal{C} \times_{\mathcal{C}'} \mathrm{LMod}_{\mathrm{Comm}^{\mathrm{nu}}}(\mathcal{C}') \simeq \mathcal{C} \times_{\mathcal{C}'} \mathrm{Calg}^{\mathrm{nu}}(\mathcal{C}') \simeq \mathrm{Calg}^{\mathrm{nu}}(\mathcal{C})$$

over $\mathcal{C}$.

So we can assume that $\mathcal{C}$ is compatible with small colimits.

By [18] prop. 5.4.4.8. the forgetful functor $\mathrm{Calg}(\mathcal{C}) \to \mathrm{Calg}^{\mathrm{nu}}(\mathcal{C})$ admits a left adjoint F such that for every $X \in \mathrm{Calg}^{\mathrm{nu}}(\mathcal{C})$ the unit $X \to F(X)$

and the unique morphism $\mathbb{1} \to F(X)$ in $Calg(\mathcal{C})$ induce an equivalence $X \oplus \mathbb{1} \to F(X)$ in $\mathcal{C}$.

The forgetful functor $Calg^{nu}(\mathcal{C}) \to \mathcal{C}$ admits a left adjoint Sym' such that we have a natural equivalence $Sym'(X) \simeq \coprod_{n\geq 1}(X^{\otimes n})_{\Sigma n}$ in $\mathcal{C}$ for $X \in \mathcal{C}$.

Especially we have $Sym'(0) \simeq 0$ so that the category $Calg^{nu}(\mathcal{C})$ admits a zero object 0 that is sent by F to the initial object $\mathbb{1}$ of $Calg(\mathcal{C})$ lying over the tensorunit of $\mathcal{C}$.

So F gives rise to a functor $F' : Calg^{nu}(\mathcal{C}) \simeq Calg^{nu}(\mathcal{C})_{/0} \to Calg(\mathcal{C})_{/\mathbb{1}}$.

By definition $Comm^{nu}$ is the final non-counital Hopf operad in $\mathcal{C}$ and Comm the final Hopf operad in $\mathcal{C}$.

So there is a unique morphism $Comm^{nu} \to Comm$ of Hopf operads in $\mathcal{C}$ that yields a forgetful functor $LMod_{Comm}(\mathcal{C}) \to LMod_{Comm^{nu}}(\mathcal{C})$.

Denote S the free functor

$$Comm \circ (-) : \mathcal{C} \to LMod_{Comm}(\mathcal{C}),\ X \mapsto Comm \circ X \simeq \coprod_{n\geq 0}(X^{\otimes n})_{\Sigma n}$$

and S' the free functor

$$Comm^{nu} \circ (-) : \mathcal{C} \to LMod_{Comm^{nu}}(\mathcal{C}),\ X \mapsto Comm^{nu} \circ X \simeq \coprod_{n\geq 1}(X^{\otimes n})_{\Sigma n}.$$

Especially we have $S(0) \simeq \mathbb{1}$ and $S'(0) \simeq 0$.
Thus the category $LMod_{Comm^{nu}}(\mathcal{C})$ admits a zero object 0 and the category $LMod_{Comm}(\mathcal{C})$ admits an inital object $\mathbb{1}$ lying over the tensorunit of $\mathcal{C}$.

Hence there is a unique morphism $0 \to \mathbb{1}$ in $LMod_{Comm^{nu}}(\mathcal{C})$ that yields a functor

$$\Gamma : LMod_{Comm^{nu}}(\mathcal{C})_{/\mathbb{1}} \to LMod_{Comm^{nu}}(\mathcal{C})_{/0} \simeq LMod_{Comm^{nu}}(\mathcal{C})$$

that takes the augmentation ideal.

By proposition 2.19 we have a canonical equivalence

$$Calg(\mathcal{C}) \simeq LMod_{Comm}(\mathcal{C})$$

over $\mathcal{C}$ that gives rise to an equivalence $Calg(\mathcal{C})_{/\mathbb{1}} \simeq LMod_{Comm}(\mathcal{C})_{/\mathbb{1}}$.

The composition

$$Calg^{nu}(\mathcal{C}) \xrightarrow{F'} Calg(\mathcal{C})_{/\mathbb{1}} \simeq LMod_{Comm}(\mathcal{C})_{/\mathbb{1}} \to$$

$$LMod_{Comm^{nu}}(\mathcal{C})_{/\mathbb{1}} \xrightarrow{\Gamma} LMod_{Comm^{nu}}(\mathcal{C})$$

is a functor over $\mathcal{C}$.

The forgetful functor $Calg^{nu}(\mathcal{C}) \to \mathcal{C}$ is monadic with left adjoint Sym'.

Thus the forgetful functor $Calg^{nu}(\mathcal{C}) \to LMod_{Comm^{nu}}(\mathcal{C})$ is an equivalence if and only if for every $X \in \mathcal{C}$ the morphism $\rho : S'(X) \to Sym'(X)$ adjoint to the unit $X \to Sym'(X)$ is an equivalence.

For every non-unital commutative algebra A in $\mathcal{C}$ the counit $\coprod_{n\geq 1}(A^{\otimes n})_{\Sigma n} \simeq Sym'(A) \to A$ as morphism in $\mathcal{C}$ is induced by the multiplication of A.

So we get a natural equivalence $S'(X) \simeq Sym'(X)$ in $\mathcal{C}$ and we will show that ρ is homotopic to this equivalence:

The morphism ρ factors as $\mathrm{S}'(\mathrm{X}) \to \mathrm{S}'(\mathrm{Sym}'(\mathrm{X})) \xrightarrow{\mu} \mathrm{Sym}'(\mathrm{X})$, where μ denotes the counit induced by the multiplication of $\mathrm{Sym}'(\mathrm{X})$.

So ρ factors as $\mathrm{S}'(\mathrm{X}) \simeq \mathrm{Sym}'(\mathrm{X}) \to \mathrm{Sym}'(\mathrm{Sym}'(\mathrm{X})) \xrightarrow{\mu'} \mathrm{Sym}'(\mathrm{X})$, where μ' is the counit induced by the multiplication of $\mathrm{Sym}'(\mathrm{X})$.

By the triangular identities ρ is the canonical equivalence $\mathrm{S}'(\mathrm{X}) \simeq \mathrm{Sym}'(\mathrm{X})$.

□

To prove proposition 4.30 we used the following lemma:

Lemma 4.31. *Let $\mathcal{C}$ be a symmetric monoidal category that admits finite coproducts.*

Let $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ be a symmetric monoidal and monadic functor with left adjoint F such that the tensorunit of $\mathcal{D}$ is an initial object.

For every $\mathrm{A}, \mathrm{B} \in \mathcal{D}$ we have canonical morphisms $\mathrm{A} \simeq \mathrm{A} \otimes \mathbb{1} \to \mathrm{A} \otimes \mathrm{B}$, $\mathrm{B} \simeq \mathbb{1} \otimes \mathrm{B} \to \mathrm{A} \otimes \mathrm{B}$ in $\mathcal{D}$.

The symmetric monoidal structure on $\mathcal{D}$ is cocartesian if for every $\mathrm{A}, \mathrm{B} \in \mathcal{D}$ that belong to the essential image of F the canonical morphisms $\mathrm{A} \simeq \mathrm{A} \otimes \mathbb{1} \to \mathrm{A} \otimes \mathrm{B}$, $\mathrm{B} \simeq \mathbb{1} \otimes \mathrm{B} \to \mathrm{A} \otimes \mathrm{B}$ in $\mathcal{D}$ exhibit $\mathrm{A} \otimes \mathrm{B}$ as a coproduct of A and B in $\mathcal{D}$.

Remark 4.32. *For every $\mathrm{X}, \mathrm{Y} \in \mathcal{C}$ the canonical morphisms*

$$\mathrm{F}(\mathrm{X}) \to \mathrm{F}(\mathrm{X}) \otimes \mathrm{F}(\mathrm{Y}),\ \mathrm{F}(\mathrm{Y}) \to \mathrm{F}(\mathrm{X}) \otimes \mathrm{F}(\mathrm{Y})$$

in $\mathcal{D}$ define a morphism $\alpha : \mathrm{F}(\mathrm{X} \coprod \mathrm{Y}) \simeq \mathrm{F}(\mathrm{X}) \coprod \mathrm{F}(\mathrm{Y}) \to \mathrm{F}(\mathrm{X}) \otimes \mathrm{F}(\mathrm{Y})$ in $\mathcal{D}$.

Moreover we have a canonical morphism

$$\beta : \mathrm{X} \coprod \mathrm{Y} \to \mathrm{G}(\mathrm{F}(\mathrm{X})) \otimes \mathrm{G}(\mathrm{F}(\mathrm{Y})) \simeq \mathrm{G}(\mathrm{F}(\mathrm{X}) \otimes \mathrm{F}(\mathrm{Y}))$$

in $\mathcal{C}$ that is the morphism

$$\mathrm{X} \simeq \mathrm{X} \otimes \mathbb{1} \simeq \mathrm{X} \otimes \mathrm{G}(\mathrm{F}(\varnothing)) \to \mathrm{G}(\mathrm{F}(\mathrm{X})) \otimes \mathrm{G}(\mathrm{F}(\mathrm{Y}))$$

on the first summand and the morphism

$$\mathrm{Y} \simeq \mathbb{1} \otimes \mathrm{Y} \simeq \mathrm{G}(\mathrm{F}(\varnothing)) \otimes \mathrm{Y} \to \mathrm{G}(\mathrm{F}(\mathrm{X})) \otimes \mathrm{G}(\mathrm{F}(\mathrm{Y}))$$

on the second summand.

The morphism $\alpha : \mathrm{F}(\mathrm{X} \coprod \mathrm{Y}) \to \mathrm{F}(\mathrm{X}) \otimes \mathrm{F}(\mathrm{Y})$ is adjoint to $\beta : \mathrm{X} \coprod \mathrm{Y} \to \mathrm{G}(\mathrm{F}(\mathrm{X}) \otimes \mathrm{F}(\mathrm{Y}))$.

Proof. We write $\mathrm{A} \simeq \mid \bar{\mathrm{A}} \mid$, $\mathrm{B} \simeq \mid \bar{\mathrm{B}} \mid$ for some G-split simplicial objects $\bar{\mathrm{A}}, \bar{\mathrm{B}} : \Delta^{\mathrm{op}} \to \mathcal{D}$ taking values in the essential image of F.

Let $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})$ be endowed with the levelwise symmetric monoidal structure. The tensorunit of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})$ is an initial object as the tensorunit of $\mathcal{D}$ is.

The canonical morphisms $\bar{\mathrm{A}} \simeq \bar{\mathrm{A}} \otimes \mathbb{1} \to \bar{\mathrm{A}} \otimes \bar{\mathrm{B}}$, $\bar{\mathrm{B}} \simeq \mathbb{1} \otimes \bar{\mathrm{B}} \to \bar{\mathrm{A}} \otimes \bar{\mathrm{B}}$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})$ exhibit $\bar{\mathrm{A}} \otimes \bar{\mathrm{B}}$ as a coproduct of $\bar{\mathrm{A}}$ and $\bar{\mathrm{B}}$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})$ as they do after evaluation at every $\mathrm{n} \in \Delta$.

The functor $\otimes : \mathcal{D} \times \mathcal{D} \to \mathcal{D}$ sends the $\mathrm{G} \times \mathrm{G}$-split simplicial object $(\bar{\mathrm{A}}, \bar{\mathrm{B}})$ to the G-split simplicial object $\bar{\mathrm{A}} \otimes \bar{\mathrm{B}} : \Delta^{\mathrm{op}} \to \mathcal{D}$.

Denote $\delta : \mathcal{D} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})$ the diagonal functor.

So the morphism $\bar{A} \otimes \bar{B} \to \delta(A \otimes B)$ in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})$ exhibits $A \otimes B$ as the geometric realization of the simplicial object $\bar{A} \otimes \bar{B}$.

Hence for every $Z \in \mathcal{D}$ the canonical map $\mathcal{D}(A \otimes B, Z) \to \mathcal{D}(A, Z) \times \mathcal{D}(B, Z)$ factors as

$$\mathcal{D}(A \otimes B, Z) \simeq \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})(\bar{A} \otimes \bar{B}, \delta(Z)) \simeq$$

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})(\bar{A}, \delta(Z)) \times \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{D})(\bar{B}, \delta(Z)) \simeq \mathcal{D}(A, Z) \times \mathcal{D}(B, Z).$$

□

4.3 Comparison to simplicial restricted Lie algebras

Given a field K denote $\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$ the category of connective H(K)-module spectra.

By theorem 4.2 there is a forgetful functor

$$\mathrm{Lie}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}).$$

In this section we factor this forgetful functor as

$$\mathrm{Lie}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to (\mathsf{sLie}_{\mathrm{K}}^{\mathrm{res}})_\infty \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}),$$

where $(\mathsf{sLie}_{\mathrm{K}}^{\mathrm{res}})_\infty$ denotes the ∞-category underlying a right induced model structure on the category $\mathsf{sLie}_{\mathrm{K}}^{\mathrm{res}}$ of simplicial restricted Lie algebras over K.

To achieve this factorization we construct for every commutative ring R (and not only a field) the forgetful functor

$$\mathrm{Lie}(\mathrm{Mod}_{\mathrm{H(R)}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}})$$

of theorem 4.2 in a more elementary way (proposition 4.34).

Let $\mathcal{C}$ be a nice symmetric monoidal model category and $\phi : \mathcal{D} \to \mathcal{C}$ a category over $\mathcal{C}$ that admits a model structure with fibrations and weak equivalences the underlying ones of $\mathcal{C}$.

Assume that $\mathsf{Ho}(\mathcal{C})$ is preadditive and the functor $\phi : \mathcal{D} \to \mathcal{C}$ is derived monadic and derived accesible, i.e. $\phi : \mathcal{D} \to \mathcal{C}$ induces a monadic and accesible functor $\mathcal{D}_\infty \to \mathcal{C}_\infty$ on underlying ∞-categories.

We describe what is needed to produce a functor $\mathrm{Lie}(\mathcal{C}_\infty) \to \mathcal{D}_\infty$ compatible with the forgetful functors to $\mathcal{C}_\infty$ (proposition 4.33).

We apply construction 4.33 to the following situations:

1. $\mathcal{C}$ is the category Ch_{R} or $\mathsf{Ch}_{\mathrm{R}}^{\geq 0}$ of (connective) chain complexes over some commutative ring R endowed with the projective model structure and $\mathcal{D}$ is the category $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})$ respectively $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}^{\geq 0})$ of Lie′-algebras for some cofibrant replacement Lie′ of the Lie operad (in the semi-model category of operads in Ch_{R} respectively $\mathsf{Ch}_{\mathrm{R}}^{\geq 0}$).

2. $\mathcal{C}$ is the category $\mathsf{sMod}_{\mathrm{K}}$ of simplicial K-vector spaces for some field K with a model structure right induced from sSet that exists by [2] theorem 5.1. and $\mathcal{D}$ is the category $\mathsf{sLie}_{\mathrm{K}}^{\mathrm{res}}$ of simplicial restricted Lie algebras over K.

To treat the first case we note that there are canonical equivalences

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}}) \simeq \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})_\infty,$$

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}}^{\geq 0}) \simeq \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}^{\geq 0})_\infty$$

over $\mathrm{Mod}_{\mathrm{H(R)}} \simeq (\mathsf{Ch}_{\mathrm{R}})_\infty$ respectively $\mathrm{Mod}_{\mathrm{H(R)}}^{\geq 0} \simeq (\mathsf{Ch}_{\mathrm{R}}^{\geq 0})_\infty$ (remark 4.35) so that the forgetful functors $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}) \to \mathsf{Ch}_{\mathrm{R}}, \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}^{\geq 0}) \to \mathsf{Ch}_{\mathrm{R}}^{\geq 0}$ are derived monadic and derived accessible.

For the second case we use that restricted Lie algebras over K are algebras over some Lawvere theory (remark 4.37) and for every Lawere theory $\mathbb{T}$ the category $\mathsf{sAlg}_{\mathbb{T}}(\mathsf{Set})$ of simplicial $\mathbb{T}$-algebras admits a right induced model structure from sSet ([2] theorem 5.1.) such that the forgetful functor $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet}) \simeq \mathsf{s}\mathrm{Alg}_{\mathbb{T}}(\mathsf{Set}) \to \mathsf{sSet}$ induces a sifted colimits preserving monadic functor on underlying ∞-categories (proposition 4.38).

As the category $\mathrm{Mod}_{\mathrm{K}}$ of K-vector spaces is the category of algebras in Sets over some Lawvere theory, the category $\mathsf{s}\mathrm{Mod}_{\mathrm{K}}$ admits a right induced model structure from sSet such that the forgetful functor $\mathsf{s}\mathrm{Mod}_{\mathrm{K}} \to \mathsf{sSet}$ induces a sifted colimits preserving monadic functor on underlying ∞-categories.

So the model structure on $\mathsf{s}\mathrm{Lie}_{\mathrm{K}}^{\mathrm{res}}$ is right induced from the model structure on $\mathsf{s}\mathrm{Mod}_{\mathrm{K}}$ and the forgetful functor $\mathsf{s}\mathrm{Lie}_{\mathrm{K}}^{\mathrm{res}} \to \mathsf{s}\mathrm{Mod}_{\mathrm{K}}$ induces a sifted colimits preserving monadic functor on underlying ∞-categories.

In the next subsection we collect the results about Lawvere theories needed in this chapter.

We start with explaining the general procedure how to construct functors starting at the category of restricted L_∞-algebras (proposition 4.33) and then apply this procedure to the cases 1. and 2. (proposition 4.34).

For the proof of proposition 4.33 we fix the following notation:

Given a category with weak equivalences $(\mathcal{C}, \mathrm{W})$ denote $\mathcal{C}_\infty$ its underlying ∞-category, i.e. the ∞-categorical localization with respect to $\mathcal{W}$. Especially we use this notation in the case that $(\mathcal{C}, \mathrm{W})$ has the structure of a model category or that $(\mathcal{C}, \mathrm{W})$ is the category of cofibrant objects $\mathcal{M}^{\mathrm{cof}}$ in a model category $\mathcal{M}$ with weak equivalences between cofibrant objects.

For every symmetric monoidal model category $\mathcal{M}$ the symmetric monoidal structure on $\mathcal{M}$ restricts to a symmetric monoidal structure on $\mathcal{M}^{\mathrm{cof}}$ compatible with weak equivalences.

So $(\mathcal{M}^{\mathrm{cof}})_\infty \simeq \mathcal{M}_\infty$ gets a symmetric monoidal structure and the localization functor $\mathcal{M}^{\mathrm{cof}} \to (\mathcal{M}^{\mathrm{cof}})_\infty \simeq \mathcal{M}_\infty$ gets symmetric and so yields functors $\mathrm{Alg}(\mathcal{M}^{\mathrm{cof}}) \to \mathrm{Alg}(\mathcal{M}_\infty)$ and $\mathrm{Bialg}(\mathcal{M}^{\mathrm{cof}}) \to \mathrm{Bialg}(\mathcal{M}_\infty)$.

If we consider $\mathrm{Alg}(\mathcal{M}^{\mathrm{cof}}), \mathrm{Bialg}(\mathcal{M}^{\mathrm{cof}})$ as categories with weak equivalences, whose weak equivalences are the underlying ones of $\mathcal{M}$, we get induced functors $\mathrm{Alg}(\mathcal{M}^{\mathrm{cof}})_\infty \to \mathrm{Alg}(\mathcal{M}_\infty)$ and $\mathrm{Bialg}(\mathcal{M}^{\mathrm{cof}})_\infty \to \mathrm{Bialg}(\mathcal{M}_\infty)$.

Given a model category $\mathcal{M}$, a category $\mathcal{C}$ with weak equivalences and a functor $\phi : \mathcal{M} \to \mathcal{C}$ that preserves weak equivalences between cofibrant objects denote $\mathbb{L}(\phi) : \mathcal{M}_\infty \to \mathcal{C}_\infty$ the functor induced by the functor $\mathcal{M} \xrightarrow{\mathrm{Q}} \mathcal{M}^{\mathrm{cof}} \subset \mathcal{M} \xrightarrow{\phi} \mathcal{C}$ of categories with weak equivalences, where Q denotes a functorial cofibrant replacement for $\mathcal{M}$.

Proposition 4.33. *Let $\mathcal{C}$ be a combinatorial symmetric monoidal model category such that $\mathsf{Ho}(\mathcal{C})$ is preadditive, $\mathcal{D}$ a category over $\mathcal{C}$ that admits a model structure with fibrations and weak equivalences the underlying ones of $\mathcal{C}$ and $\gamma : \mathrm{Alg}(\mathcal{C}) \to \mathcal{D}$ a functor over $\mathcal{C}$ that admits a left adjoint $\psi : \mathcal{D} \to \mathrm{Alg}(\mathcal{C})$.*

Assume that:

- *$\mathcal{C}$ is left proper, the cofibrations of $\mathcal{C}$ are generated by cofibrations between cofibrant objects and $\mathcal{C}$ satisfies the monoid axiom.*
- *$\mathcal{D} \to \mathcal{C}$ is derived monadic and derived accessible, i.e. the functor $\mathcal{D}_\infty \to \mathcal{C}_\infty$ on underlying ∞-categories is monadic and accessible.*

Then every lift $\Phi : \mathcal{D} \to \mathrm{Bialg}(\mathcal{C})$ *of* ψ *gives rise to a functor* $\mathrm{Lie}(\mathcal{C}_\infty) \to \mathcal{D}_\infty$ *over* $\mathcal{C}_\infty$.

Proof. By the first condition the category $\mathrm{Alg}(\mathcal{C})$ admits a right induced model structure from $\mathcal{C}$ such that the forgetful functor $\mathrm{Alg}(\mathcal{C}) \to \mathcal{C}$ preserves cofibrant objects ([18] proposition 4.1.4.3. and its proof).

The canonical symmetric monoidal functor $\mathcal{C}^{\mathrm{cof}} \to (\mathcal{C}^{\mathrm{cof}})_\infty \simeq \mathcal{C}_\infty$ gives rise to a functor $\mathrm{Alg}(\mathcal{C})^{\mathrm{cof}} \subset \mathrm{Alg}(\mathcal{C}^{\mathrm{cof}}) \to \mathrm{Alg}(\mathcal{C}_\infty)$ that induces an equivalence $\mathrm{Alg}(\mathcal{C})_\infty \simeq (\mathrm{Alg}(\mathcal{C})^{\mathrm{cof}})_\infty \to \mathrm{Alg}(\mathcal{C}_\infty)$ by [18] theorem 4.1.4.4.

Moreover the canonical symmetric monoidal functor $\mathcal{C}^{\mathrm{cof}} \to (\mathcal{C}^{\mathrm{cof}})_\infty \simeq \mathcal{C}_\infty$ gives rise to a functor $\mathrm{Bialg}(\mathcal{C}^{\mathrm{cof}})_\infty \to \mathrm{Bialg}(\mathcal{C}_\infty)$.

The adjunction $\psi : \mathcal{D} \to \mathrm{Alg}(\mathcal{C}) : \gamma$ is a Quillen adjunction and thus yields an adjunction $\mathbb{L}(\psi) : \mathcal{D}_\infty \to \mathrm{Alg}(\mathcal{C})_\infty \simeq \mathrm{Alg}(\mathcal{C}_\infty) : \gamma$, where the right adjoint is a functor over $\mathcal{C}_\infty$. So by adjointness $\mathbb{L}(\psi)$ is compatible with the free functors.

Denote F the left adjoint of the monadic functor $\mathcal{D}_\infty \to \mathcal{C}_\infty$.

A lift $\Psi : \mathcal{D} \to \mathrm{Bialg}(\mathcal{C})$ of the left Quillen functor ψ preserves weak equivalences between cofibrant objects and sends cofibrant objects of $\mathcal{D}$ to objects of $\mathrm{Bialg}(\mathcal{C}^{\mathrm{cof}})$. Hence Ψ gives rise to a lift

$$\Psi' : \mathcal{D}_\infty \xrightarrow{\mathbb{L}(\Psi)} \mathrm{Bialg}(\mathcal{C}^{\mathrm{cof}})_\infty \to \mathrm{Bialg}(\mathcal{C}_\infty)$$

of $\mathbb{L}(\psi)$.

As $\mathbb{L}(\psi)$ is compatible with the free functors, the composition $\mathcal{C}_\infty \xrightarrow{\mathrm{F}} \mathcal{D}_\infty \xrightarrow{\Psi'} \mathrm{Bialg}(\mathcal{C}_\infty)$ is the tensoralgebra by the uniqueness of lifts (proposition 3.22).

As $\mathcal{C}$ is a combinatorial symmetric monoidal model category, the underlying ∞-category $\mathcal{C}_\infty$ is a presentably symmetric monoidal ∞-category.

Lifting $\mathbb{L}(\psi)$ the functor $\Psi' : \mathcal{D}_\infty \to \mathrm{Bialg}(\mathcal{C}_\infty)$ preserves small colimits and so admits a right adjoint $\mathfrak{P} : \mathrm{Bialg}(\mathcal{C}_\infty) \to \mathcal{D}_\infty$ that by adjointness lifts the primitives $\mathcal{P} : \mathrm{Bialg}(\mathcal{C}_\infty) \to \mathcal{C}_\infty$ and so by the universal property of $\mathrm{Lie}(\mathcal{C}_\infty)$ (remark 2.27) factors as

$$\mathrm{Bialg}(\mathcal{C}_\infty) \xrightarrow{\bar{\mathcal{P}}} \mathrm{Lie}(\mathcal{C}_\infty) \to \mathcal{D}_\infty$$

for a unique functor $\mathrm{Lie}(\mathcal{C}_\infty) \to \mathcal{D}_\infty$ over $\mathcal{C}_\infty$.

□

Now we apply proposition 4.33 to the cases 1. and 2.:

Proposition 4.34.

1. *For every commutative ring* R *there are forgetful functors*

$$\mathrm{Lie}(\mathrm{Mod}_{\mathrm{H(R)}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}}),$$

$$\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}})$$

over $\mathrm{Mod}_{\mathrm{H(R)}}$ *respectively* $\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}}$.

Moreover these forgetful functors coincide with the forgetful functor of theorem 4.2.

2. In the special case of a field K *the forgetful functor of 1.*

$$\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}})$$

lifts to a forgetful functor

$$\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to (\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}})_{\infty}$$

over $\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}} \simeq (\mathsf{s}\mathrm{Mod}_{\mathrm{K}})_{\infty}$.

Proof. 1: The categories Ch_{R} and $\mathsf{Ch}^{\geq 0}_{\mathrm{R}}$ endowed with the projective model structure form combinatorial symmetric monoidal model categories.

Both model structures are left proper, satisfy the monoid axiom and their cofibrations are generated by cofibrations between cofibrant objects.

Denote Lie' a cofibrant replacement of the Lie operad in Ch_{R} respectively $\mathsf{Ch}^{\geq 0}_{\mathrm{R}}$. By [14] there are model structures on $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})$ and $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}^{\geq 0}_{\mathrm{R}})$ right induced from Ch_{R} respectively $\mathsf{Ch}^{\geq 0}_{\mathrm{R}}$.

We have a map of operads $\mathrm{Lie}' \to \mathrm{Lie} \to \mathrm{Ass}$ in Ch_{R} respectively $\mathsf{Ch}^{\geq 0}_{\mathrm{R}}$ that yields a forgetful functor $\mathrm{Alg}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})$ that is an examples for γ.

We have an enveloping bialgebra functor $\mathcal{U} : \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Bialg}(\mathsf{Ch}_{\mathrm{R}})$ that lifts the enveloping algebra functor $\mathcal{U}' : \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Alg}(\mathsf{Ch}_{\mathrm{R}})$ that is left adjoint to the forgetful functor $\mathrm{Alg}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{Ch}_{\mathrm{R}})$.

The composition

$$\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{Ch}_{\mathrm{R}}) \xrightarrow{\mathcal{U}} \mathrm{Bialg}(\mathsf{Ch}_{\mathrm{R}})$$

of the free functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{Ch}_{\mathrm{R}})$ and $\mathcal{U}$ is an example for Φ.

So 1. follows from 4.33 and remark 4.35.

Moreover the composition

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}}) \simeq \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})_{\infty} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{Ch}_{\mathrm{R}})_{\infty} \xrightarrow{\mathcal{U}} \mathrm{Bialg}(\mathsf{Ch}^{\mathrm{cof}}_{\mathrm{R}})_{\infty}$$
$$\to \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(R)}})$$

is the derived enveloping bialgebra functor of theorem 4.2.

2: The category $\mathsf{s}\mathrm{Mod}_{\mathrm{K}}$ admits a model structure right induced from sSet ([2] theorem 5.1.) that is symmetric monoidal as the model structure on sSet is symmetric monoidal and the free functor $\mathsf{sSet} \to \mathsf{s}\mathrm{Mod}_{\mathrm{K}}$ is symmetric monoidal. Moreover $\mathsf{s}\mathrm{Mod}_{\mathrm{K}}$ is left proper and satisfies the monoid axiom.

We have an enveloping algebra functor $\mathcal{U}' : \mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}} \rightleftarrows \mathsf{s}\mathrm{Alg}(\mathrm{Mod}_{\mathrm{K}})$ left adjoint to the forgetful functor $\mathsf{s}\mathrm{Alg}(\mathrm{Mod}_{\mathrm{K}}) \rightleftarrows \mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}}$ that lifts to the enveloping bialgebra functor $\mathcal{U} : \mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}} \to \mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}})$.

So the assumptions of 4.33 are satisfied and we can apply 4.33 to get a forgetful functor $\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to (\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}})_{\infty}$.

Denote Lie' a cofibrant replacement of the Lie operad in $\mathsf{s}\mathrm{Mod}_{\mathrm{K}}$.

The forgetful functor $\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}})$ gives rise to a forgetful functor

$$\mathsf{s}\mathrm{Lie}^{\mathrm{res}}_{\mathrm{K}} \to \mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}}) \simeq \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}}) \to \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}})$$

that induces a functor

$$(\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}})_\infty \to \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}})_\infty \simeq \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}})$$

on underlying ∞-categories.

We want to see that the composition

$$\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to (\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}})_\infty \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}})$$

is the forgetful functor of 1.

The primitives $\mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}}) \to \mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}})$ factor as

$$\mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}}) \to \mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}} \to \mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}})$$

so that the enveloping bialgebra functor $\mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}}) \to \mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}})$ factors as the free functor $\mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}}) \to \mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}}$ followed by the restricted enveloping bialgebra functor $\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}} \to \mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}})$.

Thus the composition

$$\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}}) \simeq \mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}}) \to \mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}})$$

factors as

$$\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}}) \simeq \mathsf{s}\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{K}}) \to \mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}} \to$$

$$\mathsf{s}\mathrm{Bialg}(\mathrm{Mod}_{\mathrm{K}}).$$

So the functor

$$\alpha : \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}})_\infty \to \mathrm{Bialg}(\mathsf{s}\mathrm{Mod}^{\mathrm{cof}}_{\mathrm{K}})_\infty \to \mathrm{Bialg}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}})$$

factors as the functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}})_\infty \to (\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}})_\infty$ followed by the functor

$$\beta : (\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}})_\infty \to \mathrm{Bialg}(\mathsf{s}\mathrm{Mod}^{\mathrm{cof}}_{\mathrm{K}})_\infty \to \mathrm{Bialg}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}).$$

Thus the right adjoint $\mathrm{Bialg}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}})_\infty$ of α factors as the right adjoint $\mathrm{Bialg}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}}) \to (\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}})_\infty$ of β followed by the forgetful functor $(\mathsf{sLie}^{\mathrm{res}}_{\mathrm{K}})_\infty \to \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{K}})_\infty$.

So the statement follows from the universal property of $\mathrm{Lie}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(K)}})$ (remark 2.27).

□

Remark 4.35. *Let* R *be a commutative ring.*

Denote Lie′ *a cofibrant replacement of the Lie operad in* Ch_{R} *respectively* $\mathsf{Ch}^{\geq 0}_{\mathrm{R}}$ *or* $\mathsf{s}\mathrm{Mod}_{\mathrm{R}}$.

There are canonical equivalences

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}}) \simeq \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})_\infty,$$

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}}) \simeq \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}^{\geq 0}_{\mathrm{R}})_\infty,$$

$$\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}}) \simeq \mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{s}\mathrm{Mod}_{\mathrm{R}})_\infty$$

over $\mathrm{Mod}_{\mathrm{H(R)}} \simeq (\mathsf{Ch}_{\mathrm{R}})_\infty$ *respectively* $\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}} \simeq (\mathsf{Ch}^{\geq 0}_{\mathrm{R}})_\infty$ *and* $\mathrm{Mod}^{\geq 0}_{\mathrm{H(R)}} \simeq (\mathsf{s}\mathrm{Mod}_{\mathrm{R}})_\infty$.

Proof. The canonical functor $\mathsf{Ch}_{\mathrm{R}} \to (\mathsf{Ch}_{\mathrm{R}})_{\infty} \simeq \mathrm{Mod}_{\mathrm{H(R)}}$ is lax symmetric monoidal and sends the operad Lie' to the R-homology of the spectral Lie operad, which is the image of the spectral Lie operad under the symmetric monoidal functor $\mathrm{H(R)} \wedge - : \mathrm{Sp} \to \mathrm{Mod}_{\mathrm{H(R)}}$.

Thus the canonical functor $\mathsf{Ch}_{\mathrm{R}} \to (\mathsf{Ch}_{\mathrm{R}})_{\infty} \simeq \mathrm{Mod}_{\mathrm{H(R)}}$ yields a functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}})$ and so a functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})_{\infty} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}})$ over $(\mathsf{Ch}_{\mathrm{R}})_{\infty} \simeq \mathrm{Mod}_{\mathrm{H(R)}}$.

The forgetful functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}}) \to \mathsf{Ch}_{\mathrm{R}}$ is derived monadic by [14].

As the left adjoint of the functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})_{\infty} \to (\mathsf{Ch}_{\mathrm{R}})_{\infty} \simeq \mathrm{Mod}_{\mathrm{H(R)}}$ is sent to the left adjoint of the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}}) \to \mathrm{Mod}_{\mathrm{H(R)}}$, the functor $\mathrm{Alg}_{\mathrm{Lie}'}(\mathsf{Ch}_{\mathrm{R}})_{\infty} \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(R)}})$ is an equivalence.

The other cases are similar.

□

4.4 Derived monadicity of algebras over a theory

This subsection is devoted to the proofs of proposition 4.38 and lemma 4.39 that show general facts about Lawvere theories.

In the following we present some basic facts about Lawvere theories used in the previous section:

Denote $\mathcal{F}\text{in}$ the category of finite sets.

A Lawvere theory is a pair $(\mathbb{T}, \phi)$ consisting of a small category $\mathbb{T}$ that admits finite products and an essentially surjective functor $\phi : \mathcal{F}\text{in}^{\text{op}} \to \mathbb{T}$ preserving finite products corresponding to an object of $\mathbb{T}$.

A map of Lawvere theories $\mathbb{T} \to \mathbb{T}'$ is a finite products preserving functor $\mathbb{T} \to \mathbb{T}'$ under $\mathcal{F}\text{in}^{\text{op}}$.

Given a Lawvere theory $\mathbb{T}$ and a category $\mathcal{C}$ that admits finite products we call $\text{Alg}_{\mathbb{T}}(\mathcal{C}) \coloneqq \text{Fun}^{\Pi}(\mathbb{T}, \mathcal{C})$ the category of $\mathbb{T}$-algebras in $\mathcal{C}$.

Every adjunction $\text{F} : \mathsf{Set} \rightleftarrows \mathcal{D} : \text{G}$ gives rise to a Lawvere theory $\mathbb{T} \coloneqq \text{F}(\mathcal{F}\text{in})^{\text{op}}$.

We say that $\text{G} : \mathcal{D} \to \mathsf{Set}$ is algebraic if the functor

$$\mathcal{D} \subset \text{Fun}^{\Pi}(\mathcal{D}^{\text{op}}, \mathsf{Set}) \to \text{Fun}^{\Pi}(\mathbb{T}, \mathsf{Set}) = \text{Alg}_{\mathbb{T}}(\mathsf{Set})$$

over Set given by the composition of the Yoneda-embedding with the restriction to $\mathbb{T}$ is an equivalence.

Let $\mathcal{C} \to \mathsf{Set}, \mathcal{D} \to \mathsf{Set}$ be right adjoint functors with associated theories $\mathbb{T}, \mathbb{T}'$.

A right adjoint functor $\mathcal{D} \to \mathcal{C}$ over Set gives rise to a map of theories $\mathbb{T} \to \mathbb{T}'$ as the opposite of its left adjoint functor $\mathcal{C}^{\text{op}} \to \mathcal{D}^{\text{op}}$ under Set^{op} restricts to a finite products preserving functor $\mathbb{T} \to \mathbb{T}'$ under $\mathcal{F}\text{in}^{\text{op}}$.

We have a commutative square

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \text{Alg}_{\mathbb{T}'}(\mathsf{Set}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \longrightarrow & \text{Alg}_{\mathbb{T}}(\mathsf{Set}). \end{array}$$

Remark 4.36.

The full subcategory $\text{Alg}_{\mathbb{T}}(\mathsf{Set}) \subset \text{Fun}(\mathbb{T}, \mathsf{Set})$ *is an accessible localization and is closed under small sifted colimits.*

Thus $\text{Alg}_{\mathbb{T}}(\mathsf{Set})$ *is presentable and the forgetful functor* $\text{Alg}_{\mathbb{T}}(\mathsf{Set}) \to \mathsf{Set}$ *preserves small sifted colimits. So by the theorem of Barr-Beck the forgetful functor* $\text{Alg}_{\mathbb{T}}(\mathsf{Set}) \to \mathsf{Set}$ *is monadic.*

So every algebraic functor $\mathcal{D} \to \mathsf{Set}$ *is monadic and preserves sifted colimits.*

By lemma 4.39 a functor $\mathcal{D} \to \mathsf{Set}$ *is algebraic if and only if it is monadic and preserves filtered colimits. Especially a monadic functor* $\mathcal{D} \to \mathsf{Set}$ *preserves filtered colimits if and only if it preserves sifted colimits.*

So given an algebraic functor $\mathcal{D} \to \mathsf{Set}$ *a monadic functor* $\mathcal{C} \to \mathcal{D}$ *preserves sifted colimits if and only if the composition* $\mathcal{C} \to \mathcal{D} \to \mathsf{Set}$ *is algebraic.*

Moreover we use lemma 4.39 to see that the category of restricted Lie algebras over a field K is algebraic:

Remark 4.37. *For every field* K *the category of restricted Lie algebras over* K *is algebraic.*

Proof. Every K-vector space is the filtered colimit of free finitely generated K-vector spaces. Thus by lemma 4.39 it is enough to check that the category of restricted Lie algebras over K is monadic over $\mathrm{Mod}_{\mathrm{K}}$ and the associated monad preserves filtered colimits.

This follows from a theorem of Fresse ([6] theorem 1.2.5.), according to which restricted Lie algebras over K are divided power Lie algebras in $\mathrm{Mod}_{\mathrm{K}}$.

□

Given a Lawvere theory $\mathbb{T}$ the category $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet}) \simeq \mathrm{sAlg}_{\mathbb{T}}(\mathsf{Set})$ admits a model stucture right induced from sSet by [2] theorem 5.1.

By the next proposition 4.38 the forgetful functor

$$\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet}) \simeq \mathrm{sAlg}_{\mathbb{T}}(\mathsf{Set}) \to \mathsf{sSet}$$

induces a monadic and sifted colimits preserving functor $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet})_\infty \to \mathcal{S}$ on underlying ∞-categories.

Proposition 4.38. *Let* $\mathbb{T}$ *be a Lawvere theory.*

There is a canonical equivalence $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet})_\infty \simeq \mathrm{Alg}_{\mathbb{T}}(\mathcal{S})$ *over* $\mathcal{S}$.

Especially the functor $\mathrm{sAlg}_{\mathbb{T}}(\mathsf{Set})_\infty \to \mathcal{S}$ *is monadic and preserves sifted colimits.*

Proof. By [2] theorem 5.1. the category $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet})$ admits a right induced model structure from sSet.

By [2] theorem 6.4. the projective model structure on $\mathrm{Fun}(\mathbb{T}, \mathsf{sSet})$ admits a left Bousfield localization $\mathrm{Fun}(\mathbb{T}, \mathsf{sSet})^{\mathrm{loc}}$ with local objects the homotopy $\mathbb{T}$-algebras so that the full subcategory inclusion $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet}) \subset \mathrm{Fun}(\mathbb{T}, \mathsf{sSet})^{\mathrm{loc}}$ is a right Quillen equivalence.

So we obtain a fully faithful functor $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet})_\infty \simeq \mathrm{Fun}(\mathbb{T}, \mathsf{sSet})^{\mathrm{loc}}_\infty \subset \mathrm{Fun}(\mathbb{T}, \mathsf{sSet})_\infty$ with essential image the homotopy $\mathbb{T}$-algebras.

By [19] prop. A.3.4.13. we have a canonical equivalence $\mathrm{Fun}(\mathbb{T}, \mathsf{sSet})_\infty \simeq \mathrm{Fun}(\mathbb{T}, \mathcal{S})$ that restricts to an equivalence $\mathrm{Alg}_{\mathbb{T}}(\mathsf{sSet})_\infty \simeq \mathrm{Alg}_{\mathbb{T}}(\mathcal{S})$.

By [9] proposition B.4. the forgetful functor $\mathrm{Alg}_{\mathbb{T}}(\mathcal{S}) \to \mathcal{S}$ is monadic and preserves sifted colimits.

□

Lemma 4.39. *Let* $\mathcal{C} \to \mathsf{Set}$ *be an algebraic functor and* $\mathcal{D} \to \mathcal{C}$ *a monadic functor that preserves filtered colimits.*

Assume that $\mathcal{C}$ *is the only full subcategory of* $\mathcal{C}$ *that contains the essential image of the free functor* $\mathrm{F} : \mathsf{Set} \to \mathcal{C}$ *and is closed under filtered colimits.*

The functor $\mathcal{D} \to \mathcal{C} \to \mathsf{Set}$ *is algebraic.*

Especially the functor $\mathcal{D} \to \mathcal{C}$ *preserves sifted colimits.*

Proof. Let $\mathcal{C} \to \mathsf{Set}, \mathcal{D} \to \mathcal{C} \to \mathsf{Set}$ be the right adjoint functors with associated theories $\mathbb{T}, \mathbb{T}'$.

The adjunction $\mathcal{F} : \mathcal{C} \rightleftarrows \mathcal{D} : \mathcal{G}$ gives rise to a map of theories $\phi : \mathbb{T} \to \mathbb{T}'$ that is the restriction of the functor $\mathcal{F}^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$.

ϕ yields an adjunction $\phi_* : \mathrm{Alg}_{\mathbb{T}}(\mathsf{Set}) \rightleftarrows \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set}) : \phi^*$, where the right adjoint is monadic.

We have a commutative square

$$\begin{array}{ccc} \mathcal{D} & \xrightarrow{\Phi} & \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set}) \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow[\theta]{\simeq} & \mathrm{Alg}_{\mathbb{T}}(\mathsf{Set}), \end{array}$$

where the bottom functor is an equivalence as $\mathcal{C} \to \mathsf{Set}$ is algebraic and where both vertical functors are monadic.

Consequently it is enough to see that the natural transformation

$$\alpha : \phi_* \circ \theta \to \Phi \circ \mathcal{F}$$

adjoint to the natural transformation $\theta \to \theta \circ \mathcal{G} \circ \mathcal{F} \cong \phi^* \circ \Phi \circ \mathcal{F}$ is an isomorphism.

The functor $\mathcal{D} \to \mathcal{C} \to \mathsf{Set}$ preserves filtered colimits so that its left adjoint preserves compact objects. So $\mathbb{T}'^{\mathrm{op}} \subset \mathcal{D}$ consists of compact objects of $\mathcal{D}$.

Thus for every $X \in \mathbb{T}'$ the composition $\mathcal{D}(X, -) : \mathcal{D} \xrightarrow{\Phi} \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set}) \xrightarrow{\mathrm{ev}_X} \mathsf{Set}$ preserves filtered colimits so that the functor $\Phi : \mathcal{D} \to \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})$ preserves filtered colimits.

Consequently by assumption on $\mathcal{C}$ it is enough to see that $\alpha(Y) : \phi_*(\theta(Y)) \to \Phi(\mathcal{F}(Y))$ is an isomorphism for every $Y \in \mathcal{C}$ that is the image of a finite set under the free functor $F : \mathsf{Set} \to \mathcal{C}$.

Let $H \in \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})$ and $X \in \mathcal{F}\mathrm{in}$. Then the induced map

$$\mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})(\Phi(\mathcal{F}(F(X))), H) \to \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})(\phi_*(\theta(F(X))), H)$$

coincides with the map

$$\mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})(\Phi(\mathcal{F}(F(X))), H) \cong \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})(T'(\mathcal{F}(F(X)), -), H) \cong$$

$$\cong H(\mathcal{F}(F(X))) \cong H(\phi(F(X))) \cong \mathrm{Alg}_{\mathbb{T}}(\mathsf{Set})(T(F(X), -), \phi^*(H))$$

$$\cong \mathrm{Alg}_{\mathbb{T}}(\mathsf{Set})(\theta(F(X)), \phi^*(H)) \cong \mathrm{Alg}_{\mathbb{T}'}(\mathsf{Set})(\phi_*(\theta(F(X))), H).$$

□

In the following we give another more elementary proof for the fact that the enveloping bialgebra functor

$$\mathcal{U} : \mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_\infty \simeq \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})$$

is fully faithful when K is a field of char. zero (theorem 4.46).

As the functor $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$ is monadic, this implies that the functor $\bar{\mathcal{P}} : \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to \mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})$ exhibits $\mathrm{Alg}_{\mathrm{Lie}}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})$ as the category of restricted L_∞-algebras in $\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$.

We deduce theorem 4.46 from the theorem of Poincare-Birkhoff-Witt and the fact that for every $\mathrm{X} \in \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$ the canonical morphism $\mathrm{E(X)} \to \mathrm{S(X)}$ in $\mathrm{Cocoalg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})_{\mathbb{1}/}$ is adjoint to an equivalence $\mathrm{X} \to \mathcal{P}'(\mathrm{S(X)})$ (corollary 4.44).

Before we prove corollary 4.44, we remind of the classical situation:

Let K be a field of char. 0 and Y a chain complex of K-vector spaces.

The symmetric algebra $\mathrm{S(Y)} := \bigoplus_{\mathrm{i}\geq 0}(\mathrm{Y}^{\otimes \mathrm{i}})_{\Sigma_{\mathrm{i}}}$ is conilpotent and is the cofree conilpotent coaugmented cocommutative coalgebra on Y:
Denote Γ the functor that takes the cokernel of the coaugmentation.

The canonical map $\Gamma(\mathrm{S(Y)}) = \bigoplus_{\mathrm{i}\geq 1}(\mathrm{Y}^{\otimes \mathrm{i}})_{\Sigma_{\mathrm{i}}} \to \mathrm{Y}$ that is the identity on the first factor and the zero map on every other factor induces for every conilpotent coaugmented cocommutative coalgebra Z in Ch_{K} an isomorphism

$$\mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}})_{\mathbb{1}/}(\mathrm{Z}, \mathrm{S(Y)}) \to \mathsf{Ch}_{\mathrm{K}}(\Gamma(\mathrm{Z}), \Gamma(\mathrm{S(Y)})) \to \mathsf{Ch}_{\mathrm{K}}(\Gamma(\mathrm{Z}), \mathrm{Y}).$$

Especially for Y the co-square-zero extension E(X) on some $\mathrm{X} \in \mathsf{Ch}_{\mathrm{K}}$ the natural map

$$\mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}})_{\mathbb{1}/}(\mathrm{E(X)}, \mathrm{S(Y)}) \to \mathsf{Ch}_{\mathrm{K}}(\mathrm{X}, \Gamma(\mathrm{S(Y)})) \to \mathsf{Ch}_{\mathrm{K}}(\mathrm{X}, \mathrm{Y})$$

is an isomorphism.
So we obtain an isomorphism

$$\mathsf{Ch}_{\mathrm{K}}(\mathrm{X}, \mathrm{Y}) \cong \mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}})_{\mathbb{1}/}(\mathrm{E(X)}, \mathrm{S(Y)}) \cong \mathsf{Ch}_{\mathrm{K}}(\mathrm{X}, \mathcal{P}'(\mathrm{S(Y)}))$$

representing an isomorphism $\mathrm{Y} \to \mathcal{P}'(\mathrm{S(Y)})$ adjoint to the canonical morphism $\mathrm{E(Y)} \to \mathrm{S(Y)}$ in $\mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}})_{\mathbb{1}/}$.

We deduce corollary 4.44 from proposition 4.43, which we prove after some preparations:

Recall that a t-structure on a stable category $\mathcal{C}$ is a pair of full subcategories $\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0}$ such that the following conditons hold, where we set $\mathcal{C}_{\geq \mathrm{n}} := \mathcal{C}_{\geq 0}[\mathrm{n}]$ and $\mathcal{C}_{\leq \mathrm{n}} := \mathcal{C}_{\leq 0}[\mathrm{n}]$ for every $\mathrm{n} \in \mathbb{Z}$:

- For $\mathrm{X} \in \mathcal{C}_{\geq 0}$ and $\mathrm{Y} \in \mathcal{C}_{\leq -1}$ we have $\mathcal{C}(\mathrm{X}, \mathrm{Y}) = 0$.
- We have full subcategory inclusions $\mathcal{C}_{\geq 1} \subset \mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq -1} \subset \mathcal{C}_{\leq 0}$.
- For every $\mathrm{X} \in \mathcal{C}$ there is a fiber sequence $\mathrm{Y} \to \mathrm{X} \to \mathrm{Z}$ with $\mathrm{Y} \in \mathcal{C}_{\geq 0}$ and $\mathrm{Z} \in \mathcal{C}_{\leq -1}$.

We set $\mathcal{C}^{\heartsuit} := \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0}$.

Recall that for every $n \in \mathbb{Z}$ the full subcategory $\mathcal{C}_{\leq n} \subset \mathcal{C}$ is a localization and the full subcategory $\mathcal{C}_{\geq n} \subset \mathcal{C}$ is a colocalization ([18] proposition 1.2.1.5.)

Moreover the localization $\mathcal{C}_{\leq 0} \subset \mathcal{C}$ restricts to a localization $\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\geq 0}$ and the colocalization $\mathcal{C}_{\geq 0} \subset \mathcal{C}$ restricts to a colocalization $\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\leq 0}$ ([18] remark 1.2.1.8.).

Let $\mathcal{C}$ be a stable presentably symmetric monoidal category.

By prop. 3.22 the free commutative algebra functor $\mathcal{C} \to \mathrm{Calg}(\mathcal{C})$ uniquely lifts to a left adjoint functor

$$\mathcal{C} \to \mathrm{Cobialg}(\mathcal{C}) = \mathrm{Cocoalg}(\mathrm{Calg}(\mathcal{C}))$$

that preserves finite products as $\mathrm{Cobialg}(\mathcal{C})$ is preadditive.

Denote S the composition $\mathcal{C} \to \mathrm{Cobialg}(\mathcal{C}) \to \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/}$ and $\Gamma : \mathrm{Cocoalg}(\mathcal{C})_{\mathbb{1}/} \to \mathcal{C}$ the functor that takes the cokernel of the coaugmentation.

For every $Y \in \mathcal{C}$ we have canonical morphisms

$$S(Y) \simeq \bigoplus_{i \geq 0} (Y^{\otimes i})_{\Sigma_i} \to Y, \qquad \Gamma(S(Y)) \simeq \bigoplus_{i \geq 1} (Y^{\otimes i})_{\Sigma_i} \to Y$$

in $\mathcal{C}$ that are the identity on the first factor and the zero morphism on every other factor.

Let $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ be a t-structure on $\mathcal{C}$ such that the symmetric monoidal structure on $\mathcal{C}$ restricts to $\mathcal{C}_{\geq 0}$.

As $\mathcal{C}_{\geq 0}$ is closed under small colimits in $\mathcal{C}$, with Y also $S(Y) \simeq \oplus_{i \geq 0} (Y^{\otimes i})_{\Sigma_i}$ and $\Gamma(S(Y)) \simeq \oplus_{i \geq 1} (Y^{\otimes i})_{\Sigma_i}$ belong to $\mathcal{C}_{\geq 0}$.

As next some remarks about the structure of the (co)free (co)commutative (co)algebra:

Let $\mathcal{C}$ be a symmetric monoidal category such that $\mathcal{C}$ admits small colimits. Denote $V : \mathrm{Calg}(\mathcal{C}) \to \mathcal{C}$ the forgetful functor.

Let $A \in \mathrm{Calg}(\mathcal{C})$ and $\mathsf{f} : X \to V(A)$ a morphism in $\mathcal{C}$.

For every $n \in \mathbb{N}$ the morphism $\mathsf{f} : X \to A$ gives rise to a Σ_n-equivariant morphism $X^{\otimes n} \to A^{\otimes n}$ and the multiplication $A^{\otimes n} \to A$ of A is Σ_n-equivariant (construction 4.40).

For every $Z \in \mathcal{C}$ the functor $- \otimes Z : \mathcal{C} \to \mathcal{C}$ sends the Σ_n-equivariant morphism $X^{\otimes n} \to A^{\otimes n} \to A$ to a Σ_n-equivariant morphism $X^{\otimes n} \otimes Z \to A \otimes Z$ corresponding to a morphism $(X^{\otimes n} \otimes Z)_{\Sigma_n} \to A \otimes Z$ in $\mathcal{C}$.

In the following we will explain that $\mathsf{f} : X \to V(A)$ exhibits A as the free commutative algebra on X in $\mathcal{C}$ if for every $Z \in \mathcal{C}$ the family of morphisms $(X^{\otimes n} \otimes Z)_{\Sigma_n} \to A \otimes Z$ in $\mathcal{C}$ indexed by $n \in \mathbb{N}$ exhibits $A \otimes Z$ as the coproduct of the objects $(X^{\otimes n} \otimes Z)_{\Sigma_n}$.

First we need to explain how we make the morphisms $X^{\otimes n} \to A^{\otimes n}, A^{\otimes n} \to A$ Σ_n-equivariant.

Construction 4.40. *Given an operad $\mathcal{O}^{\otimes}$ denote $\mathcal{O}^{\otimes}_{\mathrm{act}} \subset \mathcal{O}^{\otimes}$ the subcategory spanned by the active morphisms.*

We have a canonical equivalence $\mathrm{Env}(\mathcal{O}) \simeq \mathcal{O}^{\otimes}_{\mathrm{act}}$.

Denote $\mathcal{F}\mathrm{in} \subset \mathsf{Set}$ *the full subcategory spanned by the finite sets and* $\Sigma \subset \mathcal{F}\mathrm{in}$ *the maximal groupoid in* $\mathcal{F}\mathrm{in}$.

The functor $\mathcal{F}\mathrm{in} \to \mathcal{F}\mathrm{in}_* = \mathrm{Comm}^\otimes$ *that adjoins a base point induces an equivalence* $\mathcal{F}\mathrm{in} \simeq \mathrm{Comm}^\otimes_{\mathrm{act}} \simeq \mathrm{Env}(\mathrm{Comm})$, *under which* Σ *corresponds to* $\mathrm{Triv}^\otimes_{\mathrm{act}} \simeq \mathrm{Env}(\mathrm{Triv})$ *so that* Σ *is the intersection* $\mathrm{Triv}^\otimes \cap \mathcal{F}\mathrm{in} \subset \mathcal{F}\mathrm{in}_*$.

The symmetric monoidal category $\mathrm{Env}(\mathrm{Comm})^\otimes$ *is cocartesian so that we have a canonical equivalence* $\mathrm{Env}(\mathrm{Comm})^\otimes \simeq \mathcal{F}\mathrm{in}^\amalg$ *of symmetric monoidal categories, under which* $\mathrm{Env}(\mathrm{Triv})^\otimes \subset \mathrm{Env}(\mathrm{Comm})^\otimes$ *corresponds to a symmetric monoidal subcategory* $\Sigma^\otimes$ *of* $\mathcal{F}\mathrm{in}^\amalg$.

The operad $\mathrm{Triv}^\otimes$ *is the tensorunit of the closed symmetric monoidal structure on* Op_∞ *and so by proposition 6.82* $\mathrm{Env}(\mathrm{Triv})^\otimes \simeq \Sigma^\otimes$ *is the tensorunit of the closed symmetric monoidal structure on* $\mathrm{Cmon}(\mathsf{Cat}_\infty)$.

Especially for every symmetric monoidal category $\mathcal{C}$ *we have canonical equivalences*

$$\mathrm{Fun}^\otimes(\mathcal{F}\mathrm{in}, \mathcal{C}) \simeq \mathrm{Calg}(\mathcal{C}), \quad \mathrm{Fun}^\otimes(\Sigma, \mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{Triv}}(\mathcal{C}) \simeq \mathcal{C},$$

where the last equivalence evaluates at the set with one element.

So every $\mathrm{X} \in \mathcal{C}$ *uniquely extends to a symmetric monoidal functor* $\tilde{\mathrm{X}} : \Sigma \to \mathcal{C}$. *For every* $\mathrm{n} \in \mathbb{N}$ *the restriction* $\tilde{\mathrm{X}}_{|\mathrm{B}(\Sigma_\mathrm{n})} : \mathrm{B}(\Sigma_\mathrm{n}) \subset \Sigma \to \mathcal{C}$ *endows* $\mathrm{X}^{\otimes \mathrm{n}}$ *with a* Σ_n*-action.*

Every $\mathrm{A} \in \mathrm{Calg}(\mathcal{C})$ *uniquely extends to a symmetric monoidal functor* $\bar{\mathrm{A}} : \mathcal{F}\mathrm{in} \to \mathcal{C}$. *The unique natural transformation from the subcategory inclusion* $\Sigma \subset \mathcal{F}\mathrm{in}$ *to the constant functor with value* $\langle 1 \rangle$ *gives rise to a natural transformation* $\phi : \tilde{\mathrm{A}} \simeq \bar{\mathrm{A}}_{|\Sigma} \to \delta(\mathrm{A})$ *of functors* $\Sigma \to \mathcal{C}$, *where* $\delta : \mathcal{C} \to \mathrm{Fun}(\Sigma, \mathcal{C})$ *denotes the diagonal functor.*

So for every $\mathrm{n} \in \mathbb{N}$ *the restriction* $\phi_{|\mathrm{B}(\Sigma_\mathrm{n})}$ *makes the morphism* $\phi(\mathrm{n}) : \mathrm{A}^{\otimes \mathrm{n}} \to \mathrm{A}$ Σ_n*-equivariant, where* $\phi(\mathrm{n}) : \mathrm{A}^{\otimes \mathrm{n}} \to \mathrm{A}$ *is the multiplication of* A.

By construction 4.40 the morphism $\mathsf{f} : \mathrm{X} \to \mathrm{A}$ in $\mathcal{C}$ uniquely extends to a symmetric monoidal natural transformation $\tilde{\mathrm{X}} \to \tilde{\mathrm{A}}$ of symmetric monoidal functors $\Sigma \to \mathcal{C}$. So we get a natural transformation $\tilde{\mathrm{X}} \to \tilde{\mathrm{A}} \xrightarrow{\phi} \delta(\mathrm{A})$ of functors $\Sigma \to \mathcal{C}$ corresponding to a functor $\beta : \Sigma^\triangleright \to \mathcal{C}$ that extends $\tilde{\mathrm{X}}$ and sends the final object of $\Sigma^\triangleright$ to A.

By the following remark 4.41 for every $\mathrm{Z} \in \mathcal{C}$ the following conditions are equivalent:

- The family of morphisms $(\mathrm{X}^{\otimes \mathrm{n}} \otimes \mathrm{Z})_{\Sigma_\mathrm{n}} \to \mathrm{A} \otimes \mathrm{Z}$ in $\mathcal{C}$ indexed by $\mathrm{n} \in \mathbb{N}$ exhibits $\mathrm{A} \otimes \mathrm{Z}$ as the coproduct of the objects $(\mathrm{X}^{\otimes \mathrm{n}} \otimes \mathrm{Z})_{\Sigma_\mathrm{n}}$.
- The composition $\Sigma^\triangleright \xrightarrow{\beta} \mathcal{C} \xrightarrow{-\otimes \mathrm{Z}} \mathcal{C}$ is a colimit diagram.

Remark 4.41. *Let* $\mathrm{H} : \Sigma \to \mathcal{C}$ *be a functor,* $\mathrm{Y} \in \mathcal{C}$ *and* $\psi : \mathrm{H} \to \delta(\mathrm{Y})$ *a natural transformation.*

For every $\mathrm{n} \in \mathbb{N}$ *denote* $\delta_\mathrm{n} : \mathcal{C} \to \mathrm{Fun}(\mathrm{B}(\Sigma_\mathrm{n}), \mathcal{C})$ *the diagonal functor. The morphism* $\psi_{|\mathrm{B}(\Sigma_\mathrm{n})} : \mathrm{H}_\mathrm{n} := \mathrm{H}_{|\mathrm{B}(\Sigma_\mathrm{n})} \to \delta_\mathrm{n}(\mathrm{Y})$ *is adjoint to a morphism* $\phi_\mathrm{n} : (\mathrm{H}_\mathrm{n})_{\Sigma_\mathrm{n}} \to \mathrm{Y}$ *in* $\mathcal{C}$.

The following conditions are equivalent:

1. $\psi : \mathrm{H} \to \delta(\mathrm{Y})$ *exhibits* Y *as the colimit of* H.
2. *The family of morphisms* $\phi_\mathrm{n} : (\mathrm{H}_\mathrm{n})_{\Sigma_\mathrm{n}} \to \mathrm{Y}$ *in* $\mathcal{C}$ *indexed by* $\mathrm{n} \in \mathbb{N}$ *exhibits* Y *as the coproduct of the objects* $(\mathrm{H}_\mathrm{n})_{\Sigma_\mathrm{n}}$.

Proof. We have a canonical equivalence $\coprod_{n\in\mathbb{N}} B(\Sigma_n) \simeq \Sigma$ that yields an equivalence $Fun(\Sigma,\mathcal{C}) \simeq \prod_{n\in\mathbb{N}} Fun(B(\Sigma_n),\mathcal{C})$.

The diagonal functors $\delta:\mathcal{C}\to Fun(\Sigma,\mathcal{C})$ factors as

$$\mathcal{C}\to\prod_{n\in\mathbb{N}}\mathcal{C}\xrightarrow{\prod_{n\in\mathbb{N}}\delta_n}\prod_{n\in\mathbb{N}}Fun(B(\Sigma_n),\mathcal{C})\simeq Fun(\Sigma,\mathcal{C}).$$

So we get a canonical equivalence

$$\mathcal{C}_{H/}\simeq\mathcal{C}\times_{Fun(\Sigma,\mathcal{C})}Fun(\Sigma,\mathcal{C})_{H/}\simeq\mathcal{C}\times_{\prod_{n\in\mathbb{N}}Fun(B(\Sigma_n),\mathcal{C})}\prod_{n\in\mathbb{N}}Fun(B(\Sigma_n),\mathcal{C})_{H_n/}\simeq$$

$$\mathcal{C}\times_{(\prod_{n\in\mathbb{N}}\mathcal{C})}\prod_{n\in\mathbb{N}}(\mathcal{C}\times_{Fun(B(\Sigma_n),\mathcal{C})}Fun(B(\Sigma_n),\mathcal{C})_{H_n/})\simeq\mathcal{C}\times_{(\prod_{n\in\mathbb{N}}\mathcal{C})}\prod_{n\in\mathbb{N}}\mathcal{C}_{(H_n)_{\Sigma_n}/}\simeq$$

$$(\prod_{n\in\mathbb{N}}\mathcal{C})_{((H_n)_{\Sigma_n})_{n\in\mathbb{N}}/}.$$

□

As next we will give an alternative description of the functor $\beta:\Sigma^{\triangleright}\to\mathcal{C}$.

Construction 4.42. *By lemma 6.53 the full subcategory* $\mathcal{C}\subset Env(\mathcal{C})\simeq\mathcal{C}^{\otimes}_{act}$ *is a localization, where a morphism* $X\to Y$ *of* $\mathcal{C}^{\otimes}_{act}$ *with* $Y\in\mathcal{C}$ *is a local equivalence if and only if it is cocartesian with respect to the cocartesian fibration* $\mathcal{C}^{\otimes}\to\mathcal{F}in_*$.

Moreover the localization $\mathcal{C}\subset Env(\mathcal{C})$ *is compatible with the symmetric monoidal structure.*

Denote $L:Env(\mathcal{C})\simeq\mathcal{C}^{\otimes}_{act}\to\mathcal{C}$ *the localization functor.*

So L *lifts to a symmetric monoidal localization functor* $Env(\mathcal{C})^{\otimes}\to\mathcal{C}^{\otimes}$ *that restricts to the identity on* $\mathcal{C}^{\otimes}$ *and is thus adjoint to the identity of* $\mathcal{C}^{\otimes}$.

So given an operad $\mathcal{O}^{\otimes}$ *every* $\mathcal{O}^{\otimes}$*-algebra of* $\mathcal{C}$ *is adjoint to the symmetric monoidal functor* $Env(\mathcal{O})^{\otimes}\to Env(\mathcal{C})^{\otimes}\to\mathcal{C}^{\otimes}$, *whose underlying functor is* $\mathcal{O}^{\otimes}_{act}\to\mathcal{C}^{\otimes}_{act}\xrightarrow{L}\mathcal{C}$.

We will apply this to $\mathcal{O}^{\otimes}=Triv^{\otimes}$ *and* $\mathcal{O}^{\otimes}=Comm^{\otimes}$.

The morphism $f:X\to V(A)$ *uniquely extends to a morphism* $\mathcal{X}\to A_{|Triv^{\otimes}}$ *of* Triv*-algebras in* $\mathcal{C}$ *that restricts to a natural transformation* $\mathcal{X}_{|\Sigma}\to A_{|\Sigma}$ *of functors* $\Sigma\to\mathcal{C}^{\otimes}_{act}$ *over* $\mathcal{F}in$ *that is sent by* L *to the natural transformation* $\tilde{X}\to\tilde{A}$ *of functors* $\Sigma\to\mathcal{C}$.

The unique natural transformation from the subcategory inclusion $\Sigma\subset\mathcal{F}in$ *to the constant functor with value* $\langle 1\rangle$ *gives rise to a natural transformation* $A_{|\Sigma}\to\delta(A)$ *of functors* $\Sigma\to\mathcal{C}^{\otimes}_{act}$ *that is sent by* L *to the natural transformation* $\tilde{A}\simeq\bar{A}_{|\Sigma}\to\delta(A)$ *of functors* $\Sigma\to\mathcal{C}$.

So we obtain a natural transformation $\mathcal{X}_{|\Sigma}\to A_{|\Sigma}\to\delta(A)$ *of functors* $\Sigma\to\mathcal{C}^{\otimes}_{act}$ *corresponding to a functor* $\theta:\Sigma^{\triangleright}\to\mathcal{C}^{\otimes}_{act}$ *that extends* $\mathcal{X}_{|\Sigma}$ *and sends the final object of* $\Sigma^{\triangleright}$ *to* A *that is sent by* L *to the functor* $\beta:\Sigma^{\triangleright}\to\mathcal{C}$.

By [18] proposition 3.1.1.15. and proposition 3.1.1.16. the following conditions are equivalent:

- The composition $\Sigma^{\triangleright}\xrightarrow{\beta\simeq L\circ\theta}\mathcal{C}\xrightarrow{-\otimes Z}\mathcal{C}$ is a colimit diagram.
- The functor $\theta:\Sigma^{\triangleright}\to\mathcal{C}^{\otimes}_{act}$ is an operadic colimit diagram of $\mathcal{C}^{\otimes}\to\mathcal{F}in_*$.

By [18] proposition 3.1.3.2. the morphism $f : X \to V(A)$ exhibits A as the free commutative algebra on X in $\mathcal{C}$ if $\theta : \Sigma^{\triangleright} \to \mathcal{C}^{\otimes}_{act}$ is an operadic colimit diagram of $\mathcal{C}^{\otimes} \to \mathcal{F}in_*$, where we refer to [18] definition 3.1.2. for the definition of operadic colimit.

So we have seen that $f : X \to V(A)$ exhibits A as the free commutative algebra on X in $\mathcal{C}$ if for every $Z \in \mathcal{C}$ the family of morphisms $(X^{\otimes n} \otimes Z)_{\Sigma_n} \to A \otimes Z$ in $\mathcal{C}$ indexed by $n \in \mathbb{N}$ exhibits $A \otimes Z$ as the coproduct of the objects $(X^{\otimes n} \otimes Z)_{\Sigma_n}$.

In the following we will use the dual statement:

Let $\mathcal{C}$ be a symmetric monoidal category such that $\mathcal{C}$ admits small limits. Denote $V : \mathrm{Cocoalg}(\mathcal{C}) \to \mathcal{C}$ the forgetful functor.

Let $A \in \mathrm{Cocoalg}(\mathcal{C})$ and $f : V(A) \to X$ a morphism in $\mathcal{C}$.

For every $n \in \mathbb{N}$ the morphism $f : A \to X$ gives rise to a Σ_n-equivariant morphism $A^{\otimes n} \to X^{\otimes n}$ and the comultiplication $A \to A^{\otimes n}$ of A is Σ_n-equivariant (construction 4.40 applied to $\mathcal{C}^{op}$).
For every $Z \in \mathcal{C}$ the functor $- \otimes Z : \mathcal{C} \to \mathcal{C}$ sends the Σ_n-equivariant morphism $A \to A^{\otimes n} \to X^{\otimes n}$ to a Σ_n-equivariant morphism $A \otimes Z \to X^{\otimes n} \otimes Z$ corresponding to a morphism $A \otimes Z \to (X^{\otimes n} \otimes Z)^{\Sigma_n}$ in $\mathcal{C}$.

So $f : V(A) \to X$ exhibits A as the cofree cocommutative coalgebra on X in $\mathcal{C}$ if for every $Z \in \mathcal{C}$ the family of morphisms $A \otimes Z \to (X^{\otimes n} \otimes Z)^{\Sigma_n}$ in $\mathcal{C}$ indexed by $n \in \mathbb{N}$ exhibits $A \otimes Z$ as the product of the objects $(X^{\otimes n} \otimes Z)^{\Sigma_n}$.

Recall that for all $n \in \mathbb{N}$ and $X \in \mathcal{C}^{\Sigma_n}$ there is norm map $X_{\Sigma_n} \to X^{\Sigma_n}$ (example 6.1.6.22. [18]).

Now we are ready to prove proposition 4.43:

Proposition 4.43. *Let $\mathcal{C}$ be a stable presentably symmetric monoidal category compactly generated by the tensorunit $\mathbb{1}$ of $\mathcal{C}$ and $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$ a t-structure on $\mathcal{C}$ such that the symmetric monoidal structure on $\mathcal{C}$ restricts to $\mathcal{C}_{\geq 0}$.*

1. *Assume that for all $Y, Z \in \mathcal{C}$ and $i \geq 0$ the norm map $(Y^{\otimes i} \otimes Z)_{\Sigma_i} \to (Y^{\otimes i} \otimes Z)^{\Sigma_i}$ is an equivalence and that $\pi_n(Y) := \pi_0(\mathcal{C}(\Sigma^n(\mathbb{1}), Y)) = 0$ if $Y \in \mathcal{C}_{\geq 0}$ and $n < 0$.*

   Then for every $Y \in \mathcal{C}_{\geq 1}$ the canonical morphism $S(Y) \to Y$ in $\mathcal{C}$ exhibits $S(Y)$ as the cofree coaugmented cocommutative coalgebra on Y in the category $\mathcal{C}_{\geq} := \bigcup_{j \in \mathbb{Z}} \mathcal{C}_{\geq j}$, i.e. for every $Z \in \mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}$ the canonical map

   $$\mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}(Z, S(Y)) \to \mathcal{C}_{\geq}(\Gamma(Z), \Gamma(S(Y))) \to \mathcal{C}_{\geq}(\Gamma(Z), Y)$$

   is an equivalence.

2. *Suppose that the symmetric monoidal structure on $\mathcal{C}$ restricts to $\mathcal{C}^{\heartsuit}$ and that $\mathcal{C}^{\heartsuit}$ is closed under countable coproducts in $\mathcal{C}$.*
   Denote $E : \mathcal{C}_{\geq 0} \to \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}$ the co-square-zero extension.
   Then for every $Y \in \mathcal{C}^{\heartsuit}$ and $X \in \mathcal{C}_{\geq 0}$ the canonical map

   $$\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}(E(X), S(Y)) \to \mathcal{C}_{\geq 0}(X, \Gamma(S(Y))) \to \mathcal{C}_{\geq 0}(X, Y)$$

   is an equivalence.

Combining 1. and 2. we get:

3. *Assume that for all* $\mathrm{Y}, \mathrm{Z} \in \mathcal{C}$ *and* $\mathrm{i} \geq 0$ *the norm map* $(\mathrm{Y}^{\otimes \mathrm{i}} \otimes \mathrm{Z})_{\Sigma_\mathrm{i}} \to (\mathrm{Y}^{\otimes \mathrm{i}} \otimes \mathrm{Z})^{\Sigma_\mathrm{i}}$ *is an equivalence and that* $\pi_\mathrm{n}(\mathrm{Y}) := \pi_0(\mathcal{C}(\Sigma^\mathrm{n}(\mathbb{1}), \mathrm{Y})) = 0$ *if* $\mathrm{Y} \in \mathcal{C}_{\geq 0}$ *and* $\mathrm{n} < 0$.

   Suppose that the symmetric monoidal structure on $\mathcal{C}$ *restricts to* $\mathcal{C}^\heartsuit$ *and that* $\mathcal{C}^\heartsuit$ *is closed under countable coproducts in* $\mathcal{C}$.
   Moreover assume that $\mathcal{C}_{\geq 0}$ *is the only full subcategory of* $\mathcal{C}_{\geq 0}$ *that contains* $\mathcal{C}_{\geq 1}$ *and* $\mathcal{C}^\heartsuit$ *and is closed under finite products and retracts.*

   Then for every $\mathrm{X}, \mathrm{Y} \in \mathcal{C}_{\geq 0}$ *the canonical map*

   $$\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}(\mathrm{E}(\mathrm{X}), \mathrm{S}(\mathrm{Y})) \to \mathcal{C}_{\geq 0}(\mathrm{X}, \Gamma(\mathrm{S}(\mathrm{Y}))) \to \mathcal{C}_{\geq 0}(\mathrm{X}, \mathrm{Y})$$

   is an equivalence.
   So we obtain an equivalence

   $$\mathcal{C}(\mathrm{X}, \mathrm{Y}) \simeq \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}(\mathrm{E}(\mathrm{X}), \mathrm{S}(\mathrm{Y})) \simeq \mathcal{C}_{\geq 0}(\mathrm{X}, \mathcal{P}'(\mathrm{S}(\mathrm{Y})))$$

   representing an equivalence $\mathrm{Y} \to \mathcal{P}'(\mathrm{S}(\mathrm{Y}))$ *adjoint to the canonical morphism* $\mathrm{E}(\mathrm{Y}) \to \mathrm{S}(\mathrm{Y})$ *in* $\mathcal{C}$.

Proof. We start by showing how 3. follows from 1. and 2.:

The functor $\mathrm{S} : \mathcal{C}_{\geq 0} \to \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}$ preserves finite products as S factors as the composition $\mathcal{C}_{\geq 0} \to \mathrm{Cobialg}(\mathcal{C}_{\geq 0}) \to \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}$ of finite products preserving functors.

Hence the full subcategory of $\mathcal{C}_{\geq 0}$ spanned by the objects Y such that the canonical map

$$\varphi : \mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}(\mathrm{Z}, \mathrm{S}(\mathrm{Y})) \to \mathcal{C}_{\geq}(\Gamma(\mathrm{Z}), \Gamma(\mathrm{S}(\mathrm{Y}))) \to \mathcal{C}_{\geq}(\Gamma(\mathrm{Z}), \mathrm{Y})$$

is an equivalence for all $\mathrm{Z} \in \mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}$ is closed under finite products and retracts.

So by assumption it is enough to show that φ is an equivalence if $\mathrm{Z} \in \mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}$ and $\mathrm{Y} \in \mathcal{C}^\heartsuit \cup \mathcal{C}_{\geq 1}$.

As next we prove 1.:

First we remark that it is enough to show that for every $\mathrm{Y} \in \mathcal{C}_{\geq 1}$ the canonical morphism $\mathrm{S}(\mathrm{Y}) \to \mathrm{Y}$ in $\mathcal{C}$ exhibits $\mathrm{S}(\mathrm{Y})$ as the cofree cocommutative coalgebra on Y in the category $\mathcal{C}_{\geq}$ (instead of the cofree coaugmented cocommutative coalgebra) i.e. for every $\mathrm{Z} \in \mathrm{Cocoalg}(\mathcal{C}_{\geq})$ the canonical map

$$\mathrm{Cocoalg}(\mathcal{C}_{\geq})(\mathrm{Z}, \mathrm{S}(\mathrm{Y})) \to \mathcal{C}_{\geq}(\mathrm{Z}, \mathrm{S}(\mathrm{Y})) \to \mathcal{C}_{\geq}(\mathrm{Z}, \mathrm{Y})$$

is an equivalence:
For every $\mathrm{Z} \in \mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}$ the canonical map

$$\varphi : \mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}(\mathrm{Z}, \mathrm{S}(\mathrm{Y})) \to \mathcal{C}_{\geq}(\Gamma(\mathrm{Z}), \Gamma(\mathrm{S}(\mathrm{Y}))) \to \mathcal{C}_{\geq}(\Gamma(\mathrm{Z}), \mathrm{Y})$$

factors as

$$\mathrm{Cocoalg}(\mathcal{C}_{\geq})_{\mathbb{1}/}(\mathrm{Z}, \mathrm{S}(\mathrm{Y})) \simeq \{\sigma\} \times_{\mathrm{Cocoalg}(\mathcal{C}_{\geq})(\mathbb{1}, \mathrm{S}(\mathrm{Y}))} \mathrm{Cocoalg}(\mathcal{C}_{\geq})(\mathrm{Z}, \mathrm{S}(\mathrm{Y}))$$

$$\to \{\sigma\} \times_{\mathcal{C}_{\geq}(\mathbb{1}, \mathrm{S}(\mathrm{Y}))} \mathcal{C}_{\geq}(\mathrm{Z}, \mathrm{S}(\mathrm{Y})) \to \{0\} \times_{\mathcal{C}_{\geq}(\mathbb{1}, \mathrm{Y})} \mathcal{C}_{\geq}(\mathrm{Z}, \mathrm{Y}) \simeq \mathcal{C}_{\geq}(\Gamma(\mathrm{Z}), \mathrm{Y}),$$

where $\sigma : \mathbb{1} \to \mathrm{S}(\mathrm{Y})$ is adjoint to the zero morphism $0 : \mathbb{1} \to \mathrm{Y}$ in $\mathcal{C}$.

To show that the canonical morphism $S(Y) \to Y$ in $\mathcal{C}$ exhibits $S(Y)$ as the cofree cocommutative coalgebra on Y in the category $\mathcal{C}_{\geq}$, it is enough to check the following condition:

For every $Z \in \mathcal{C}_{\geq}$ and $i \geq 0$ the morphism $S(Y) \to Y$ and the Σ_i-equivariant comultiplication $S(Y) \to S(Y)^{\otimes i}$ give rise to Σ_i-equivariant morphisms $S(Y) \otimes Z \to S(Y)^{\otimes i} \otimes Z \to Y^{\otimes i} \otimes Z$ in $\mathcal{C}$ that induce a morphism

$$\alpha : S(Y) \otimes Z \to \prod_{i \geq 0} (Y^{\otimes i} \otimes Z)^{\Sigma_i}.$$

Then α is an equivalence.

Set $A_i := (Y^{\otimes i})_{\Sigma_i} \otimes Z \simeq (Y^{\otimes i} \otimes Z)_{\Sigma_i} \simeq (Y^{\otimes i} \otimes Z)^{\Sigma_i}$.
Then α is equivalent to the canonical morphism $\beta : \coprod_{i \geq 0} A_i \to \prod_{i \geq 0} A_i$.
By assumption it is enough to see that for every $k \in \mathbb{Z}$ the morphism

$$\gamma : \pi_k(\coprod_{i \geq 0} A_i) \simeq \bigoplus_{i \geq 0} \pi_k(A_i) \to \prod_{i \geq 0} \pi_k(A_i) \simeq \pi_k(\prod_{i \geq 0} A_i)$$

is an isomorphism.
If $Z \in \mathcal{C}_{\geq j}$ for some $j \in \mathbb{Z}$, the object $A_i \simeq (Y^{\otimes i} \otimes Z)_{\Sigma_i}$ belongs to $\mathcal{C}_{\geq i+j}$.
Hence by assumption for every $k \in \mathbb{Z}$ and every $i > k - j$ we have $\pi_k(A_i) = 0$ so that γ is an isomorphism.

As next we prove 2.:

For every $Y \in \mathcal{C}_{\geq 0}$ the unit $S(Y) \simeq \bigoplus_{i \geq 0} (Y^{\otimes i})_{\Sigma_i} \to \Gamma(S(Y)) \oplus \mathbb{1} \simeq \bigoplus_{i \geq 1} (Y^{\otimes i})_{\Sigma_i} \oplus \mathbb{1}$ is the identity. So $S(Y)$ belongs to $\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}$.

So for every $X, Y \in \mathcal{C}_{\geq 0}$ we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/}(E(X), S(Y)) & \longrightarrow & \mathcal{C}_{\geq 0}(X, \Gamma(S(Y))) & \longrightarrow & \mathcal{C}_{\geq 0}(X, Y) \\
\downarrow \simeq & & \downarrow = & & \downarrow = \\
\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}}(\bar{\Gamma}(E(X)), \bar{\Gamma}(S(Y))) & \longrightarrow & \mathcal{C}_{\geq 0}(X, \Gamma(S(Y))) & \longrightarrow & \mathcal{C}_{\geq 0}(X, Y),
\end{array}$$

where the vertical maps are equivalences.

For every $Y \in \mathcal{C}^{\heartsuit}$ the free commutative algebra $S(Y) \simeq \bigoplus_{i \geq 0} (Y^{\otimes i})_{\Sigma_i}$ and $\Gamma(S(Y)) \simeq \bigoplus_{i \geq 1} (Y^{\otimes i})_{\Sigma_i}$ belong to $\mathcal{C}^{\heartsuit}$:
By assumption for every $i \geq 0$ and $Y \in \mathcal{C}^{\heartsuit}$ the object $Y^{\otimes i}$ belongs to $\mathcal{C}^{\heartsuit}$.
As $\mathcal{C}_{\geq 0}$ is closed in $\mathcal{C}$ under small colimits, $(Y^{\otimes i})^{\Sigma_i} \simeq (Y^{\otimes i})_{\Sigma_i}$ belongs to $\mathcal{C}_{\geq 0}$.
Hence $(Y^{\otimes i})^{\Sigma_i} \to Y^{\otimes i}$ is a limit diagram in $\mathcal{C}_{\geq 0}$ and thus a limit diagram in $\mathcal{C}^{\heartsuit}$. So by assumption $\bigoplus_{i \geq 0} (Y^{\otimes i})_{\Sigma_i}, \bigoplus_{i \geq 1} (Y^{\otimes i})_{\Sigma_i}$ belong to $\mathcal{C}^{\heartsuit}$.

The symmetric monoidal full subcategory inclusion $\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\geq 0}$ admits a left adjoint $\pi_0 : \mathcal{C}_{\geq 0} \to \mathcal{C}^{\heartsuit}$.
So the localization $\pi_0 : \mathcal{C}_{\geq 0} \rightleftarrows \mathcal{C}^{\heartsuit}$ lifts to a localization $\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}} \rightleftarrows \mathrm{Cocoalg}(\mathcal{C}^{\heartsuit})^{\mathrm{ncu}}$.
So for every $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}^{\heartsuit}$ we have a commutative diagram

$$\begin{array}{ccccc}
\mathrm{Cocoalg}(\mathcal{C}^{\heartsuit})^{\mathrm{ncu}}(\bar{\Gamma}(E(\pi_0(X))), \bar{\Gamma}(S(Y))) & \longrightarrow & \mathcal{C}^{\heartsuit}(\pi_0(X), \Gamma(S(Y))) & \longrightarrow & \mathcal{C}^{\heartsuit}(\pi_0(X), Y) \\
\downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
\mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}}(\bar{\Gamma}(E(X)), \bar{\Gamma}(S(Y))) & \longrightarrow & \mathcal{C}_{\geq 0}(X, \Gamma(S(Y))) & \longrightarrow & \mathcal{C}_{\geq 0}(X, Y).
\end{array}$$

Consequently it is enough to see that the top horizontal map

$$\mathrm{Cocoalg}(\mathcal{C}^{\heartsuit})^{\mathrm{ncu}}(\bar{\Gamma}(\mathrm{E}(\pi_0(\mathrm{X}))), \bar{\Gamma}(\mathrm{S}(\mathrm{Y}))) \to \mathcal{C}^{\heartsuit}(\pi_0(\mathrm{X}), \Gamma(\mathrm{S}(\mathrm{Y})))$$
$$\to \mathcal{C}^{\heartsuit}(\pi_0(\mathrm{X}), \mathrm{Y})$$

is an equivalence.

The restriction

$$\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\geq 0} \xrightarrow{\mathrm{E}} \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/} \xrightarrow{\bar{\Gamma}} \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}}$$

factors as the unique section

$$\mathcal{C}^{\heartsuit} \to \mathrm{Cocoalg}(\mathcal{C}^{\heartsuit})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}}$$

by the uniqueness of the co-square-zero extension.

We have shown that for every $\mathrm{Y} \in \mathcal{C}^{\heartsuit}$ the free commutative algebra $\mathrm{S}(\mathrm{Y})$ belongs to $\mathcal{C}^{\heartsuit}$.

So the restriction of the free commutative algebra $\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\geq 0} \to \mathrm{Calg}(\mathcal{C}_{\geq 0})$ factors as the free commutative algebra

$$\mathcal{C}^{\heartsuit} \to \mathrm{Calg}(\mathcal{C}^{\heartsuit}) \subset \mathrm{Calg}(\mathcal{C}_{\geq 0}).$$

Thus by the uniqueness of lifts the restriction

$$\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\geq 0} \xrightarrow{\mathrm{S}} \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})_{\mathbb{1}/} \xrightarrow{\bar{\Gamma}} \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}}$$

factors as $\mathcal{C}^{\heartsuit} \xrightarrow{\mathrm{S}} \mathrm{Cocoalg}(\mathcal{C}^{\heartsuit})_{\mathbb{1}/} \xrightarrow{\bar{\Gamma}} \mathrm{Cocoalg}(\mathcal{C}^{\heartsuit})^{\mathrm{ncu}} \subset \mathrm{Cocoalg}(\mathcal{C}_{\geq 0})^{\mathrm{ncu}}$.

So using that $\mathcal{C}^{\heartsuit}$ is a 1-category we only have to see that every morphism $\mathrm{Z} \to \mathrm{Y}$ in $\mathcal{C}^{\heartsuit}$ uniquely lifts to a map ψ of non-counital coalgebras $\mathrm{Z} \to \bigoplus_{\mathrm{i} \geq 1} (\mathrm{Y}^{\otimes \mathrm{i}})_{\Sigma_{\mathrm{i}}}$ in $\mathcal{C}^{\heartsuit}$, where Z is endowed with 0-comultiplication.

ψ is the composition $\mathrm{Z} \to \mathrm{Y} \to \bigoplus_{\mathrm{i} \geq 1} (\mathrm{Y}^{\otimes \mathrm{i}})_{\Sigma_{\mathrm{i}}}$, where the second map is the canonical map to the first summand that is a map of non-counital coalgebras when Y is endowed with 0-comultiplication.

□

We apply proposition 4.43 to the stable presentably symmetric monoidal category $\mathrm{Mod}_{\mathrm{H(K)}}$ underlying the projective model structure on the category Ch_{K} of chain complexes over some field K of char. 0.

We endow $\mathrm{Mod}_{\mathrm{H(K)}}$ with its natural t-structure so that $\mathrm{Mod}_{\mathrm{H(K)}}^{\heartsuit}$ is the category of K-vector spaces.

Corollary 4.44. *Let* K *be a field of char. 0.*

The canonical morphism $\mathrm{E}(\mathrm{Y}) \to \mathrm{S}(\mathrm{Y})$ *in* $\mathrm{Cocoalg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})_{\mathbb{1}/}$ *is adjoint to an equivalence* $\mathrm{Y} \to \mathcal{P}'(\mathrm{S}(\mathrm{Y}))$ *in* $\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$.

Proof. We check that $\mathrm{Mod}_{\mathrm{H(K)}}$ satisfies the assumptions of proposition 4.43:

The category $\mathrm{Mod}_{\mathrm{H(K)}}$ is compactly generated by its tensorunit and the symmetric monoidal structure on $\mathrm{Mod}_{\mathrm{H(K)}}$ restricts to $\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$.

For every $\mathrm{Y} \in \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$ and $\mathrm{n} < 0$ we have $\pi_{\mathrm{n}}(\mathrm{Y}) = 0$.

The full subcategory $\mathrm{Mod}_{\mathrm{H(K)}}^{\heartsuit}$ is closed under small filtered colimits in $\mathrm{Mod}_{\mathrm{H(K)}}$ and is thus closed under arbitrary small coproducts.

An object of $\mathrm{Mod}_{H(K)}$ belongs to $\mathrm{Mod}_{H(K)}^{\heartsuit}$ if and only if its a small coproduct of tensorunits of $\mathrm{Mod}_{H(K)}$.

So the symmetric monoidal structure on $\mathrm{Mod}_{H(K)}$ restricts to $\mathrm{Mod}_{H(K)}^{\heartsuit}$.

Every chain complex over K is equivalent to its homology considered as a chain complex with zero differentials and is thus a direct sum of a chain complex concentrated in degree zero and a chain complex that vanishes in degree zero.

Especially every object of $\mathrm{Mod}_{H(K)}^{\geq 0}$ is the direct sum of an object in $\mathrm{Mod}_{H(K)}^{\heartsuit}$ and an object in $\mathrm{Mod}_{H(K)}^{\geq 1}$.

□

As next some remarks about the theorem of Poincare-Birkhoff-Witt which we use in the proof of theorem 4.46:

Let K be a field of char. zero, $Y \in \mathsf{Ch}_K^{\geq 0}$ and $X \in \mathrm{Lie}(\mathsf{Ch}_K^{\geq 0})$ lying over $X' \in \mathsf{Ch}_K^{\geq 0}$.

We have a symmetrization map $\gamma : S(Y) \to T(Y)$ in $\mathrm{Cocoalg}(\mathsf{Ch}_K^{\geq 0})_{\mathbb{1}/}$ and a canonical morphism $T(X') \to \mathcal{U}(X)$ in $\mathrm{Bialg}(\mathsf{Ch}_K^{\geq 0})_{\mathbb{1}/}$ that factors as $T(X') \simeq \mathcal{U}(\mathcal{L}(X')) \to \mathcal{U}(X)$.

By the theorem of Poincare-Birkhoff-Witt the composition $S(X') \xrightarrow{\gamma} T(X') \to \mathcal{U}(X)$ is an isomorphism.

The functors

$$T : \mathsf{Ch}_K^{\geq 0} \to \mathrm{Bialg}(\mathsf{Ch}_K^{\geq 0}), \quad S : \mathsf{Ch}_K^{\geq 0} \to \mathrm{Cobialg}(\mathsf{Ch}_K^{\geq 0})$$

factor canonically as $\mathsf{Ch}_K^{\geq 0} \xrightarrow{E} \mathrm{Cocoalg}(\mathsf{Ch}_K^{\geq 0})_{\mathbb{1}/} \xrightarrow{F} \mathrm{Bialg}(\mathsf{Ch}_K^{\geq 0})$ respectively $\mathsf{Ch}_K^{\geq 0} \xrightarrow{E'} \mathrm{Cocoalg}(\mathsf{Ch}_K^{\geq 0})_{\mathbb{1}/} \xrightarrow{F'} \mathrm{Cobialg}(\mathsf{Ch}_K^{\geq 0})$, where the morphisms F, F' denote the free functors.

The canonical morphisms

$$E(Y) = Y \oplus \mathbb{1} \to \bigoplus_{i \geq 0} Y^{\otimes i} = T(Y), E(Y) = Y \oplus \mathbb{1} \to \bigoplus_{i \geq 0} (Y^{\otimes i})_{\Sigma_i} = S(Y)$$

lift to morphisms in $\mathrm{Cocoalg}(\mathsf{Ch}_K^{\geq 0})_{\mathbb{1}/}$ that are the units $E(Y) \to F(E(Y)) \cong T(Y)$ and $E(Y) \to F'(E(Y)) \cong S(Y)$.

The units $E(Y) \to F(E(Y)) \cong T(Y)$ and $E(Y) \to F'(E(Y)) \cong S(Y)$ are sent to morphisms in $\mathrm{Cocoalg}(\mathrm{Mod}_{H(K)}^{\geq 0})_{\mathbb{1}/}$ that are the units of the corresponding adjunctions for $\mathrm{Mod}_{H(K)}^{\geq 0}$ instead of $\mathsf{Ch}_K^{\geq 0}$.

So the commutative triangle

$$\begin{array}{ccc} S(Y) & \xrightarrow{\quad\gamma\quad} & T(Y) \\ & \nwarrow \quad \nearrow & \\ & E(Y) & \end{array}$$

in $\mathsf{Ch}_K^{\geq 0}$ is a commutative triangle in $\mathrm{Cocoalg}(\mathsf{Ch}_K^{\geq 0})_{\mathbb{1}/}$ and is thus sent to a commutative triangle in $\mathrm{Cocoalg}(\mathrm{Mod}_{H(K)}^{\geq 0})_{\mathbb{1}/}$, where the diagonal morphisms are the units.

Remark 4.45. *Let* K *be a field of char. zero.*

The enveloping algebra functor $\mathcal{U}' : \mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0}) \to \mathrm{Alg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})$ *is a left Quillen functor and thus preserves weak equivalences between cofibrant objects.*
The theorem of Poincare-Birkhoff-Witt implies that $\mathcal{U}' : \mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0}) \to \mathrm{Alg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})$ *preserves all weak equivalences:*
The combinatorial model structure on $\mathsf{Ch}_{\mathrm{K}}^{\geq 0}$ *lifts to a right induced model structure on* $\mathrm{Calg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})$ *([24] lemma 5.1.).*
So the free-forgetful adjunction $\mathrm{S} : \mathsf{Ch}_{\mathrm{K}}^{\geq 0} \rightleftarrows \mathrm{Calg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})$ *is a Quillen adjunction so that* S *preserves weak equivalences.*
By the theorem of Poincare-Birkhoff-Witt the functors $\mathcal{U} : \mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0}) \to \mathrm{Bialg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0}) \to \mathsf{Ch}_{\mathrm{K}}^{\geq 0}$ *and* $\mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0}) \to \mathsf{Ch}_{\mathrm{K}}^{\geq 0} \xrightarrow{\mathrm{S}} \mathrm{Calg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0}) \to \mathsf{Ch}_{\mathrm{K}}^{\geq 0}$ *are isomorphic.*

Now we are ready for the proof of theorem 4.46:

Theorem 4.46. *Let* K *be a field of char. 0.*

The enveloping bialgebra functor

$$\mathcal{U} : \mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_{\infty} \to \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})$$

is fully faithful.

Proof. The functors

$$\mathrm{T} : \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0} \to \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}), \quad \mathrm{S} : \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0} \to \mathrm{Cobialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})$$

factor canonically as

$$\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0} \xrightarrow{\mathrm{E}} \mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_{\mathbb{1}/} \xrightarrow{\mathrm{F}} \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}),$$

$$\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0} \xrightarrow{\mathrm{E}'} \mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_{\mathbb{1}/} \xrightarrow{\mathrm{F}'} \mathrm{Cobialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}),$$

where the morphisms F, F' denote the left adjoints of the forgetful functors $\mathrm{V} : \mathrm{Bialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to \mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_{\mathbb{1}/}$ respectively $\mathrm{Cobialg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}) \to \mathrm{Cocoalg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_{\mathbb{1}/}$.
So the unit $\mathrm{id} \to \mathcal{P} \circ \mathrm{T}$ factors as $\mathrm{id} \to \mathcal{P}' \circ \mathrm{E} \to \mathcal{P}' \circ \mathrm{V} \circ \mathrm{F} \circ \mathrm{E} \simeq \mathcal{P} \circ \mathrm{T}$.
Moreover we note that for every $\mathrm{Y} \in \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$ the units

$$\mathrm{E(Y)} \to \mathrm{F(E(Y))} \simeq \mathrm{T(Y)}, \ \mathrm{E(Y)} \to \mathrm{F}'(\mathrm{E(Y)}) \simeq \mathrm{S(Y)}$$

lie over the canonical morphisms $\mathrm{Y} \oplus \mathbb{1} \to \bigoplus_{\mathrm{i} \geq 0} \mathrm{Y}^{\otimes \mathrm{i}}$ and $\mathrm{Y} \oplus \mathbb{1} \to \bigoplus_{\mathrm{i} \geq 0} (\mathrm{Y}^{\otimes \mathrm{i}})_{\Sigma_{\mathrm{i}}}$ in $\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$.
We have a symmetrization map $\gamma : \mathrm{S(Y)} \to \mathrm{T(Y)}$ in $\mathrm{Cocoalg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})_{\mathbb{1}/}$ such that the composition $\mathrm{E(Y)} \to \mathrm{S(Y)} \xrightarrow{\gamma} \mathrm{T(Y)}$ is equivalent to $\mathrm{E(Y)} \to \mathrm{T(Y)}$ in $\mathrm{Cocoalg}(\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0})_{\mathbb{1}/}$.

Let $\mathrm{X} \in \mathrm{Lie}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})_{\infty}$ with underlying object $\mathrm{X}' \in \mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$.
We want to see that the unit $\mathrm{X} \to \bar{\mathcal{P}}(\mathcal{U}(\mathrm{X}))$ is an equivalence or equivalently that its image $\alpha : \mathrm{X}' \to \mathcal{P}(\mathcal{U}(\mathrm{X}))$ in $\mathrm{Mod}_{\mathrm{H(K)}}^{\geq 0}$ is an equivalence.
The counit $\mathcal{L}(\mathrm{X}') \to \mathrm{X}$ yields a morphism $\beta : \mathrm{T}(\mathrm{X}') \simeq \mathcal{U}(\mathcal{L}(\mathrm{X}')) \to \mathcal{U}(\mathrm{X})$ in $\mathrm{Bialg}(\mathsf{Ch}_{\mathrm{K}}^{\geq 0})$ that is adjoint to α so that α factors as

$$\mathrm{X}' \to \mathcal{P}(\mathrm{T}(\mathrm{X}')) \xrightarrow{\mathcal{P}\beta} \mathcal{P}(\mathcal{U}(\mathrm{X})).$$

Thus α factors as

$$X' \to \mathcal{P}'(E(X')) \to \mathcal{P}'(F(E(X'))) \simeq \mathcal{P}'(T(X')) \xrightarrow{\mathcal{P}'\beta} \mathcal{P}'(\mathcal{U}(X))$$

and so as

$$X' \to \mathcal{P}'(E(X')) \to \mathcal{P}'(F'(E(X'))) \simeq \mathcal{P}'(S(X')) \xrightarrow{\mathcal{P}'\gamma} \mathcal{P}'(T(X')) \xrightarrow{\mathcal{P}'\beta} \mathcal{P}'(\mathcal{U}(X)).$$

By the theorem of Poincare-Birkhoff-Witt the composition $S(X') \xrightarrow{\gamma} T(X') \xrightarrow{\beta} \mathcal{U}(X)$ is an equivalence in $\mathrm{Cocoalg}(\mathrm{Mod}^{\geq 0}_{H(K)})_{\mathbb{1}/}$.

By corollary 4.44 the morphism $X' \to \mathcal{P}'(E(X')) \to \mathcal{P}'(F'(E(X'))) \simeq \mathcal{P}'(S(X'))$ is an equivalence, too.

□

5 About the equivalence between monads and monadic functors

Let X be a category. The composition of endofunctors of X defines a monoidal structure on the category of endofunctors of X, whose associative algebras we call monads on X.

Every functor $g : Y \to X$ with left adjoint $f : X \to Y$ gives rise to a monad $T = g \circ f$ on X.

On the other hand given a monad T on X we can form the category of T-algebras $LMod_T(X)$, which comes equipped with a right adjoint forgetful functor $LMod_T(X) \to X$, whose associated monad is T.

Moreover the functor $g : Y \to X$ gives rise to a canonical functor $Y \to LMod_T(X)$ over X that sometimes happens to be an equivalence, in which case we call g a monadic functor.

This way we can turn right adjoint functors to monads and vice versa and obtain a correspondence between monads and monadic functors.

In nature monads often are equipped with extra structure:

Every associative algebra A in a monoidal category X gives rise to a monad $A \otimes - : X \to X$ on X.

If A is a cocommutative bialgebra in a symmetric monoidal category X its associated monad $A \otimes - : X \to X$ is naturally a Hopf monad, i.e. a monad, whose underlying endofunctor on X is oplax symmetric monoidal and whose unit and multiplication are symmetric monoidal natural transformations.

Another source for Hopf monads are Hopf operads: Given a Hopf operad $\mathcal{O}$ in X, i.e. an operad in cocommutative coalgebras in X, its associated monad $\mathcal{O} \circ - : X \to X$ is naturally a Hopf monad.

Given a commutative algebra A in a symmetric monoidal category X its associated monad $A \otimes - : X \to X$ is naturally a lax symmetric monoidal monad, i.e. a monad, whose underlying endofunctor on X is lax symmetric monoidal and whose unit and multiplication are symmetric monoidal natural transformations.

This extra structure on a monad is reflected in its category of algebras:

The category of algebras over a Hopf monad is a symmetric monoidal category such that the forgetful functor from algebras in X to X is symmetric monoidal.

The category of algebras over a lax symmetric monoidal monad is a symmetric monoidal category such that the free functor from X to algebras in X is symmetric monoidal provided that the tensorproduct on X and the endofunctor of the monad commute with geometric realizations.

To treat these examples systematically we develop a theory of monads and monadic functors in an arbitrary 2-category $\mathcal{C}$ and show that there is a similar correspondence between right adjoint morphisms $g : Y \to X$ in $\mathcal{C}$ and monads on X:

We define monads on X to be associative algebras in the monoidal category $[X, X]$ of endomorphisms.

We say that a morphism $g : Y \to X$ in $\mathcal{C}$ is left adjoint to a morphism $f : X \to Y$ of $\mathcal{C}$ if the pair (f, g) satisfies the triangular identities in $\mathcal{C}$.

We call a right adjoint morphism $g : Y \to X$ in $\mathcal{C}$ monadic if for every object Z of $\mathcal{C}$ the induced functor $[Z, Y] \to [Z, X]$ on categories of morphisms is monadic in the usual sense.

Every right adjoint morphisms $g : Y \to X$ of $\mathcal{C}$ gives rise to a monad $T \simeq g \circ f$ on X (prop. 5.31).

To associate a right adjoint morphism $Y \to X$ to a monad T on X that abstracts the category of algebras is more problematic.

Mimicing the essential properties of the category of algebras the morphism $Y \to X$ is monadic and has T as its associated monad. This implies the uniqueness of such a morphism, which we call the Eilenberg-Moore object of T.

In general there is no reason that the monad T admits an Eilenberg-Moore object but we show that in many 2-categories every monad admits an Eilenberg-Moore object.

For example every Hopf monad, which we can identify with a monad in the 2-category of symmetric monoidal categories and oplax symmetric monoidal functors, admits an Eilenberg-Moore object which is preserved by the 2-functor that sends a symmetric monoidal category to its underlying category.

This way the structure of a Hopf monad on a given monad T corresponds to the structure of a symmetric monoidal category on the category of T-algebras such that the forgetful functor is a symmetric monoidal functor. This generalizes theorems about Hopf monads like theorem 7.1. of [22] from 1-categories to ∞-categories.

Other interesting examples of 2-categories with Eilenberg-Moore objects are the following ones:

- the 2-category of operads
- the 2-category of $\mathcal{O}^\otimes$-monoidal categories and oplax $\mathcal{O}^\otimes$-monoidal functors for some operad $\mathcal{O}^\otimes$
- the 2-category of $\mathcal{O}^\otimes$-monoidal categories compatible with geometric realizations and lax $\mathcal{O}^\otimes$-monoidal functors preserving geometric realizations
- the 2-category of left modules over a monoidal category $\mathcal{V}$ and oplax $\mathcal{V}$-linear functors
- the 2-category of left modules over $\mathcal{V}$ compatible with geometric realizations and lax $\mathcal{V}$-linear functors preserving geometric realizations
- the 2-category of double categories

Given a E_{k+1}-monoidal category $\mathcal{D}$ for some natural k every associative algebra A in the monoidal category of E_k-coalgebras in $\mathcal{D}$ gives rise to an oplax E_k-monoidal monad $T := A \otimes - : \mathcal{D} \to \mathcal{D}$ and every E_{k+1}-algebra B in $\mathcal{D}$ gives rise to a lax E_k-monoidal monad $T' := B \otimes - : \mathcal{D} \to \mathcal{D}$.

The second example implies that the category $\mathrm{LMod}_A(\mathcal{D}) \simeq \mathrm{LMod}_T(\mathcal{D})$ is a E_k-monoidal category such that the forgetful functor $\mathrm{LMod}_A(\mathcal{D}) \to \mathcal{D}$ is E_k-monoidal.

The third example implies that the category $\mathrm{LMod}_B(\mathcal{D}) \simeq \mathrm{LMod}_{T'}(\mathcal{D})$ is a E_k-monoidal category such that the free functor $\mathcal{D} \to \mathrm{LMod}_B(\mathcal{D})$ is E_k-monoidal if the E_{k+1}-monoidal category $\mathcal{D}$ is compatible with geometric realizations.

We show in example 5.36 that for every small category S the 2-category of small categories over S admits Eilenberg-Moore objects.

From this we deduce that for every categorical pattern $\mathfrak{P}$ the category of $\mathfrak{P}$-fibered objects admits Eilenberg-Moore objects (prop. 5.47), which includes all the mentioned examples and lots of generalizations of them.

So to reflect structure on a monad T on its category of algebras we view T as a monad in an appropriate 2-category $\mathcal{C}$ equipped with a forgetful functor ϕ to the 2-category of small categories and show that $\mathcal{C}$ admits Eilenberg-Moore objects that are preserved by ϕ.

Given a 2-category $\mathcal{C}$ that admits Eilenberg-Moore objects for every monad we form the category $\mathcal{C}_{/X}$ of morphism with target X and its full subcategory $\mathcal{C}^{R}_{/X}$ of right adjoint morphisms with target X and construct a localization

$$\mathcal{C}^{R}_{/X} \rightleftarrows \mathrm{Alg}(\mathrm{Fun}(X,X))^{op}$$

with local objects the monadic morphisms with target X (theorem 5.62).

The left adjoint sends a morphism $g : Y \to X$ with left adjoint $f : X \to Y$ to its associated monad $g \circ f$ and the right adjoint sends a monad on X to its Eilenberg-Moore object.

Thus the localization restricts to an equivalence

$$(\mathcal{C}_{/X})^{mon} \simeq \mathrm{Alg}([X,X])^{op},$$

where $(\mathcal{C}_{/X})^{mon} \subset \mathcal{C}_{/X}$ denotes the full subcategory spanned by the monadic morphims with target X.

For $\mathcal{C} = \mathsf{Cat}_\infty$ this result is expected by Lurie in [18] remark 4.7.4.8.

Moreover we prove the following global version (theorem 5.73):
We form the full subcategories $\mathrm{Fun}(\Delta^1,\mathcal{C})^{mon} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})^{R} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})$ of monadic morphisms respectively right adjoint morphisms and show that the full subcategory

$$\mathrm{Fun}(\Delta^1,\mathcal{C})^{mon} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})^{R}$$

is a localization relative to $\mathcal{C}$.

Moreover we show that the full subcategory $\mathrm{Fun}(\Delta^1,\mathcal{C})^{mon} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})^{R}$ is a localization of 2-categories if $\mathcal{C}$ is cotensored over Cat_∞.

We use the results over Hopf monads to show that the category of algebras over a Hopf operad in a symmetric monoidal category that admits small colimits carries a canonical symmetric monoidal structure such that the forgetful functor is symmetric monoidal (prop. 5.77).

After reducing to the case that the symmetric monoidal category is compatible with small colimits, we deduce this from the fact that the associated monad of a Hopf operad is naturally a Hopf monad (prop. 5.76).

5.1 Parametrized categories of sections

5.1.1 Parametrized categories of sections

In the first section we study parametrized versions of categories of sections, from which we define parametrized versions of categories of algebras and left modules. Those will serve us as a tool to make constructions involving categories of algebras and modules natural or functorial.

A functor $\psi : \mathrm{T} \to \mathrm{S}$ between small categories gives rise to an adjunction $\psi_* : \mathsf{Cat}_{\infty/\mathrm{T}} \rightleftarrows \mathsf{Cat}_{\infty/\mathrm{S}} : \psi^* = \mathrm{T} \times_{\mathrm{S}} -$.

Being a right adjoint functor $\mathrm{T} \times_{\mathrm{S}} - : \mathsf{Cat}_{\infty/\mathrm{S}} \to \mathsf{Cat}_{\infty/\mathrm{T}}$ preserves finite products and so endows $\mathsf{Cat}_{\infty/\mathrm{T}}$ with a canonical left module structure over $\mathsf{Cat}_{\infty/\mathrm{S}}$.

Let $\phi : \mathcal{C} \to \mathrm{T}$ be a functor. The functor $\xi : \mathsf{Cat}_{\infty/\mathrm{S}} \xrightarrow{\mathrm{T}\times_{\mathrm{S}}-} \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathcal{C}\times_{\mathrm{T}}-} \mathsf{Cat}_{\infty/\mathrm{T}}$ is equivalent to the composition $\mathsf{Cat}_{\infty/\mathrm{S}} \xrightarrow{\mathcal{C}\times_{\mathrm{S}}-} \mathsf{Cat}_{\infty/\mathcal{C}} \xrightarrow{\phi_*} \mathsf{Cat}_{\infty/\mathrm{T}}$.

Hence ξ admits a right adjoint if and only if the functor $\mathcal{C} \times_{\mathrm{S}} - : \mathsf{Cat}_{\infty/\mathrm{S}} \to \mathsf{Cat}_{\infty/\mathcal{C}}$ does. In this case we call the functor $\gamma : \mathcal{C} \to \mathrm{S}$ flat or say that γ exhibits $\mathcal{C}$ as flat over S and write $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, -) : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ for the right adjoint of ξ.

If $\psi : \mathrm{T} \to \mathrm{S}$ is the identity, we write $\mathrm{Map}_{\mathrm{S}}(\mathcal{C}, \mathcal{D})$ for $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})$.

Observation 5.1. *It follows immediately from the definition that flat functors are closed under composition.*

Moreover the opposite functor and the pullback of a flat functor $\mathcal{C} \to \mathrm{S}$ along any functor $\alpha : \mathrm{S}' \to \mathrm{S}$ are flat as we have commutative diagrams

$$\begin{array}{ccc} \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}} & \longrightarrow & \mathsf{Cat}_{\infty/\mathcal{C}^{\mathrm{op}}} \\ {\scriptstyle\simeq}\downarrow{\scriptstyle \mathrm{op}} & & {\scriptstyle\simeq}\downarrow{\scriptstyle \mathrm{op}} \\ \mathsf{Cat}_{\infty/\mathrm{S}} & \longrightarrow & \mathsf{Cat}_{\infty/\mathcal{C}} \end{array} \qquad \begin{array}{ccc} \mathsf{Cat}_{\infty/\mathrm{S}'} & \longrightarrow & \mathsf{Cat}_{\infty/\mathcal{C}'} \\ \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha'_*} \\ \mathsf{Cat}_{\infty/\mathrm{S}} & \longrightarrow & \mathsf{Cat}_{\infty/\mathcal{C}} \end{array}$$

with $\alpha' : \mathcal{C}' := \mathrm{S}' \times_{\mathrm{S}} \mathcal{C} \to \mathcal{C}$ the projection, where α_*, α'_* preserve and reflect small colimits.

By [18] B.3.11. every cocartesian and thus also every cartesian fibration is flat.

Denote $\mathsf{Cat}^{\mathrm{fl}/\mathrm{S}}_{\infty/\mathrm{T}} \subset \mathsf{Cat}_{\infty/\mathrm{T}}$ the full subcategory spanned by the categories over T that are flat over S.

The left action functor $\mathsf{Cat}_{\infty/\mathrm{S}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{T}}$ yields a functor

$$(\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{op}} \times (\mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \to (\mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathsf{Cat}_{\infty/\mathrm{T}}(-,-)} \mathcal{S}$$

adjoint to a functor $(\mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathrm{Fun}((\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{op}}, \mathcal{S})$ that restricts to a functor $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(-,-) : (\mathsf{Cat}^{\mathrm{fl}/\mathrm{S}}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$.

So we get a canonical equivalence

$$\mathsf{Cat}_{\infty/\mathrm{S}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq \mathsf{Cat}_{\infty/\mathrm{T}}(\mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D})$$

natural in small categories $\mathcal{C}, \mathcal{D}$ over T with $\mathcal{C}$ flat over S and a small category $\mathcal{B}$ over S.

Remark 5.2.

Let $\mathrm{T} \to \mathrm{S}, \mathrm{S} \to \mathrm{R}, \mathcal{B} \to \mathrm{S}, \mathcal{C} \to \mathrm{T}, \mathcal{D} \to \mathrm{T}$ *be functors between small categories such that* $\mathcal{C} \to \mathrm{T} \to \mathrm{S}$ *is flat.*

1. *We have a canonical equivalence*

$$\mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{T}}(\mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D})$$

represented by the natural equivalence

$$\mathsf{Cat}_\infty(\mathrm{K}, \mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}))) \simeq \mathsf{Cat}_{\infty/\mathrm{S}}(\mathrm{K} \times \mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq$$

$$\mathsf{Cat}_{\infty/\mathrm{T}}((\mathrm{K} \times \mathcal{B}) \times_{\mathrm{S}} \mathcal{C}, \mathcal{D}) \simeq \mathsf{Cat}_{\infty/\mathrm{T}}(\mathrm{K} \times (\mathcal{B} \times_{\mathrm{S}} \mathcal{C}), \mathcal{D}) \simeq$$
$$\mathsf{Cat}_\infty(\mathrm{K}, \mathrm{Fun}_{\mathrm{T}}(\mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D}))$$

for a small category K.

Generalizing 1. we have a canonical equivalence

$$\mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{T}}(\mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D})$$

over R *represented by the natural equivalence*

$$\mathrm{Fun}_{\mathrm{R}}(\mathrm{K}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}))) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{K} \times_{\mathrm{R}} \mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq$$

$$\mathrm{Fun}_{\mathrm{T}}(\mathrm{K} \times_{\mathrm{R}} \mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{R}}(\mathrm{K}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{T}}(\mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D}))$$

for a small category K *over* R.

2. *We have a canonical equivalence*

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}}_{\mathrm{T}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$$

over S^{op} *represented by the canonical equivalence*

$$\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}}) \simeq \mathsf{Cat}_{\infty/\mathrm{S}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq$$

$$\mathsf{Cat}_{\infty/\mathrm{T}}(\mathcal{B} \times_{\mathrm{S}} \mathcal{C}, \mathcal{D}) \simeq \mathsf{Cat}_{\infty/\mathrm{T}^{\mathrm{op}}}(\mathcal{B}^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} \mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \simeq$$

$$\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}(\mathcal{B}^{\mathrm{op}}, \mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}}_{\mathrm{T}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})).$$

3. *Let* $\mathrm{S}' \to \mathrm{S}$ *be a functor. Set* $\mathrm{T}' := \mathrm{S}' \times_{\mathrm{S}} \mathrm{T}$.

There is a canonical equivalence

$$\mathrm{S}' \times_{\mathrm{S}} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Fun}^{/\mathrm{S}'}_{\mathrm{T}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{D})$$

of categories over S' *represented by the canonical equivalence*

$$\mathrm{Fun}_{\mathrm{S}'}(\mathrm{K}, \mathrm{S}' \times_{\mathrm{S}} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{T}}(\mathcal{C} \times_{\mathrm{S}} \mathrm{K}, \mathcal{D})$$

$$\simeq \mathrm{Fun}_{\mathrm{T}'}(\mathcal{C} \times_{\mathrm{S}} \mathrm{K}, \mathrm{T}' \times_{\mathrm{T}} \mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{T}'}((\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}) \times_{\mathrm{S}'} \mathrm{K}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{D})$$

$$\simeq \mathrm{Fun}_{\mathrm{S}'}(\mathrm{K}, \mathrm{Fun}^{/\mathrm{S}'}_{\mathrm{T}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{D}))$$

natural in a small category K *over* S'.

Especially for every object s *of* S *we have a canonical equivalence*

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})_{\mathsf{s}} \simeq \mathrm{Fun}_{\mathrm{T}_{\mathsf{s}}}(\mathcal{C}_{\mathsf{s}}, \mathcal{D}_{\mathsf{s}}).$$

4. *We have a canonical equivalence*

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{S} \times_{\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathrm{T})} \mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathcal{D})$$

over S *represented by the canonical equivalence*

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{T}}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathcal{D}) \simeq$$

$$\{\phi_*(\mathrm{K}\times_{\mathrm{S}}\mathcal{C})\} \times_{\mathrm{Fun}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathrm{T})} \mathrm{Fun}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathcal{D}) \simeq$$
$$\{\phi_*(\mathrm{K}\times_{\mathrm{S}}\mathcal{C})\} \times_{\mathrm{Fun}_{\mathrm{S}}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathrm{T})} \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathcal{D}) \simeq$$
$$\{\phi_*(\mathrm{K}\times_{\mathrm{S}}\mathcal{C})\} \times_{\mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathrm{T}))} \mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathcal{D}))$$
$$\simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{S}\times_{\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathrm{T})}\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathcal{D}))$$

natural in a small category K *over* S.

5. *Let* $\mathcal{E}\to\mathrm{T},\mathrm{T}\to\mathrm{S}$ *be functors and* $\mathcal{C}\to\mathcal{E},\mathcal{D}\to\mathcal{E}$ *functors over* T.
We have a canonical equivalence

$$\mathrm{S}\times_{\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{E})}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{C},\mathcal{D})$$

over S *given by the composition*

$$\mathrm{S}\times_{\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{E})}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{S}\times_{\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathcal{E})}\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{C},\mathcal{D})$$

of canonical equivalences over S.
More generally given a functor $\mathcal{B}\to\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{E})$ *over* S *we have a canonical equivalence*

$$\mathcal{B}\times_{\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{E})}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{/\mathcal{B}}_{\mathcal{B}\times_{\mathrm{S}}\mathcal{E}}(\mathcal{B}\times_{\mathrm{S}}\mathcal{C},\mathcal{B}\times_{\mathrm{S}}\mathcal{D})$$

over $\mathcal{B}$ *given by the composition*

$$\mathcal{B}\times_{\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{E})}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D}) \simeq \mathcal{B}\times_{(\mathcal{B}\times_{\mathrm{S}}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{E}))}(\mathcal{B}\times_{\mathrm{S}}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D})) \simeq$$

$$\mathcal{B}\times_{\mathrm{Fun}^{/\mathcal{B}}_{\mathcal{B}\times_{\mathrm{S}}\mathrm{T}}(\mathcal{B}\times_{\mathrm{S}}\mathcal{C},\mathcal{B}\times_{\mathrm{S}}\mathcal{E})}\mathrm{Fun}^{/\mathcal{B}}_{\mathcal{B}\times_{\mathrm{S}}\mathrm{T}}(\mathcal{B}\times_{\mathrm{S}}\mathcal{C},\mathcal{B}\times_{\mathrm{S}}\mathcal{D}) \simeq \mathrm{Fun}^{/\mathcal{B}}_{\mathcal{B}\times_{\mathrm{S}}\mathcal{E}}(\mathcal{B}\times_{\mathrm{S}}\mathcal{C},\mathcal{B}\times_{\mathrm{S}}\mathcal{D})$$

of canonical equivalences over $\mathcal{B}$.

6. *Given functors* $\mathcal{C}\to\mathrm{T}',\mathrm{T}'\to\mathrm{T},\mathrm{T}\to\mathrm{S},\mathcal{D}\to\mathrm{T}$ *we have a canonical equivalence*

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D}) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}'}(\mathcal{C},\mathrm{T}'\times_{\mathrm{T}}\mathcal{D})$$

over S *represented by the canonical equivalence*

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C},\mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{T}}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathcal{D}) \simeq$$

$$\mathrm{Fun}_{\mathrm{T}'}(\mathrm{K}\times_{\mathrm{S}}\mathcal{C},\mathrm{T}'\times_{\mathrm{T}}\mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}'}(\mathcal{C},\mathrm{T}'\times_{\mathrm{T}}\mathcal{D}))$$

natural in a small category K *over* S.

Remark 5.3. *Let* R, S, T *be categories and* $\mathrm{T}\to\mathrm{S},\mathrm{R}\to\mathrm{S}$ *and* $\mathrm{X}\to\mathrm{T}\times_{\mathrm{S}}\mathrm{R}$ *be functors. Let* $\mathcal{B}$ *be a category over* T *and* $\mathcal{D}$ *a category over* R.

1. *Assume that the functors* $\mathcal{B} \to \mathrm{S}$ *and* $\mathcal{D} \to \mathrm{R}$ *are flat.*
   There is a canonical equivalence
   $$\mathrm{Map}_{\mathrm{R}}(\mathcal{D}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{X})) \simeq \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{Map}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X}))$$
   of categories over R *represented by the following canonical equivalence natural in a small category* K *over* R, *where we set* $\mathrm{Z} := \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{X})$ *and* $\mathrm{W} := \mathrm{Map}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X})$:
   $$\mathrm{Fun}_{\mathrm{R}}(\mathrm{K}, \mathrm{Map}_{\mathrm{R}}(\mathcal{D}, \mathrm{Z})) \simeq \mathrm{Fun}_{\mathrm{R}}(\mathrm{K}\times_{\mathrm{R}}\mathcal{D}, \mathrm{Z}) \simeq$$
   $$\mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}((\mathrm{K}\times_{\mathrm{R}}\mathcal{D})\times_{\mathrm{R}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}), \mathrm{X}) \simeq$$
   $$\mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}((\mathrm{K}\times_{\mathrm{R}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}))\times_{\mathrm{R}}\mathcal{D}, \mathrm{X}) \simeq \mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}((\mathrm{K}\times_{\mathrm{S}}\mathcal{B})\times_{\mathrm{R}}\mathcal{D}, \mathrm{X}) \simeq$$
   $$\mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}((\mathrm{K}\times_{\mathrm{S}}\mathcal{B})\times_{(\mathrm{T}\times_{\mathrm{S}}\mathrm{R})}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}), \mathrm{X}) \simeq \mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{K}\times_{\mathrm{S}}\mathcal{B}, \mathrm{W}) \simeq$$
   $$\mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{K}\times_{\mathrm{R}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}), \mathrm{W}) \simeq \mathrm{Fun}_{\mathrm{R}}(\mathrm{K}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{W})).$$
2. *Assume that the functors* $\mathcal{B} \to \mathrm{S}$ *and* $\mathcal{D} \to \mathrm{S}$ *are flat.*
   There is a canonical equivalence
   $$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X})) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{R}}(\mathcal{D}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{X}))$$
   over S.
   We have a canonical equivalence natural in a small category L *over* S :
   $$\mathrm{Fun}_{\mathrm{S}}(\mathrm{L}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X}))) \simeq$$
   $$\mathrm{Fun}_{\mathrm{T}}(\mathrm{L}\times_{\mathrm{S}}\mathcal{B}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X})) \simeq \mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}((\mathrm{L}\times_{\mathrm{S}}\mathcal{B})\times_{\mathrm{T}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}), \mathrm{X})$$
   $$\mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}((\mathrm{L}\times_{\mathrm{S}}\mathcal{B})\times_{\mathrm{S}}\mathcal{D}, \mathrm{X})$$
   Changing the roles of R *and* T *and* $\mathcal{D}$ *and* $\mathcal{B}$ *we get a canonical equivalence natural in a small category* L *over* S :
   $$\mathrm{Fun}_{\mathrm{S}}(\mathrm{L}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{R}}(\mathcal{D}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{X}))) \simeq \mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}((\mathrm{L}\times_{\mathrm{S}}\mathcal{D})\times_{\mathrm{S}}\mathcal{B}, \mathrm{X}).$$
   So we get a canonical equivalence
   $$\mathrm{Fun}_{\mathrm{S}}(\mathrm{L}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X}))) \simeq \mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}((\mathrm{L}\times_{\mathrm{S}}\mathcal{B})\times_{\mathrm{S}}\mathcal{D}, \mathrm{X})$$
   $$\mathrm{Fun}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}((\mathrm{L}\times_{\mathrm{S}}\mathcal{D})\times_{\mathrm{S}}\mathcal{B}, \mathrm{X}) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{L}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{R}}(\mathcal{D}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{X})))$$
   natural in a small category L *over* S *that represents a canonical equivalence*
   $$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{B}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathcal{D}, \mathrm{X})) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{R}}(\mathcal{D}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathcal{B}, \mathrm{X}))$$
   over S.
3. *Set* $\mathrm{Y} := \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathrm{R}, \mathrm{X})$.
   For $\mathcal{B} \to \mathrm{T}$ *and* $\mathcal{D} \to \mathrm{R}$ *the identities the canonical equivalence*
   $$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathrm{T}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathrm{R}, \mathrm{X})) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{R}}(\mathrm{R}, \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathrm{T}, \mathrm{X}))$$
   over S *of 2. is adjoint to the functor*
   $$\mathrm{R}\times_{\mathrm{S}}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathrm{T}, \mathrm{Y}) \simeq \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathrm{T}, \mathrm{R}\times_{\mathrm{S}}\mathrm{Y}) \to \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{R}\times_{\mathrm{S}}\mathrm{T}}(\mathrm{R}\times_{\mathrm{S}}\mathrm{T}, \mathrm{X})$$
   over R *induced by the functor*
   $$\mathrm{R}\times_{\mathrm{S}}\mathrm{Y} \simeq \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{R}}(\mathrm{T}\times_{\mathrm{S}}\mathrm{R}, \mathrm{X})\times_{\mathrm{T}}(\mathrm{T}\times_{\mathrm{S}}\mathrm{R}) \to \mathrm{X}$$
   over $\mathrm{T}\times_{\mathrm{S}}\mathrm{R}$.

Remark 5.4. *Let* $\mathrm{T} \to \mathrm{S}$ *and* $\phi : \mathcal{C} \to \mathrm{T}$ *be functors such that the composition* $\mathcal{C} \to \mathrm{T} \to \mathrm{S}$ *is flat and let* $\varphi : \mathcal{D} \to \mathcal{E}$ *a functor over* T.

If the functor $\varphi : \mathcal{D} \to \mathcal{E}$ *is a subcategory inclusion respectively is fully faithful, the induced functor* $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{E})$ *also is.*

Proof. Being right adjoint to the functor $\mathsf{Cat}_{\infty/\mathrm{S}} \xrightarrow{\mathcal{C}\times_{\mathrm{S}}-} \mathsf{Cat}_{\infty/\mathcal{C}} \xrightarrow{\phi_*} \mathsf{Cat}_{\infty/\mathrm{T}}$ the functor $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, -) : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ preserves pullbacks and so monomorphisms. The forgetful functor $\mathsf{Cat}_{\infty/\mathrm{S}} \to \mathsf{Cat}_{\infty}$ preserves and reflects pullbacks and so monomorphisms, where the monorphisms in Cat_{∞} are the subcategory inclusions.

If $\varphi : \mathcal{D} \to \mathcal{E}$ is fully faithful, φ is a subcategory inclusion so that the induced functor $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{E})$ is a subcategory inclusion.

Let $\alpha : \Delta^1 \to \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{E})$ be a morphism of $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{E})$, whose source and target belong to $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \subset \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{E})$ and that lies over a morphism $\mathsf{f} : \mathsf{s} \to \mathrm{t}$ of S.

α corresponds to a functor $\mathrm{F} : \Delta^1 \times_{\mathrm{S}} \mathcal{C} \to \Delta^1 \times_{\mathrm{S}} \mathcal{E}$ over $\Delta^1 \times_{\mathrm{S}} \mathrm{T}$ such that the induced functors $\mathrm{F}_1 : \mathcal{C}_{\mathsf{s}} \to \mathcal{E}_{\mathsf{s}}$ over T_{s} and $\mathrm{F}_2 : \mathcal{C}_{\mathrm{t}} \to \mathcal{E}_{\mathrm{t}}$ over T_{t} factor through $\mathcal{D}_{\mathsf{s}}$ respectively $\mathcal{D}_{\mathrm{t}}$.

As $\mathcal{D}$ is a full subcategory of $\mathcal{E}$, the functor $\mathrm{F} : \Delta^1 \times_{\mathrm{S}} \mathcal{C} \to \Delta^1 \times_{\mathrm{S}} \mathcal{E}$ over $\Delta^1 \times_{\mathrm{S}} \mathrm{T}$ induces a functor $\Delta^1 \times_{\mathrm{S}} \mathcal{C} \to \Delta^1 \times_{\mathrm{S}} \mathcal{D}$ over $\Delta^1 \times_{\mathrm{S}} \mathrm{T}$ corresponding to a morphism $\Delta^1 \to \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})$ of $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})$ that is sent to α.

□

Remark 5.5.

Let $\alpha : \mathrm{T} \to \mathrm{S}, \beta : \mathcal{C} \to \mathrm{T}, \gamma : \mathcal{D} \to \mathrm{T}$ *be functors such that the composition* $\mathcal{C} \to \mathrm{T} \to \mathrm{S}$ *is flat and* $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathrm{S})$ *a full subcategory.*

[18] Theorem B.4.2. implies the following:

Assume that $\alpha : \mathrm{T} \to \mathrm{S}$ *is a cartesian fibration relative to* $\mathcal{E}$.
Denote $\mathcal{E}' \subset \mathrm{Fun}(\Delta^1, \mathrm{T})$ *the full subcategory spanned by the* α*-cartesian morphisms lying over morphism of* $\mathcal{E}$.

If $\beta : \mathcal{C} \to \mathrm{T}$ *is a cartesian fibration relative to* $\mathcal{E}'$ *and* $\gamma : \mathcal{D} \to \mathrm{T}$ *is a cocartesian fibration relative to* $\mathcal{E}'$*, then* $\psi : \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \to \mathrm{S}$ *is a cocartesian fibration relative to* $\mathcal{E}$.

In this case a morphism of $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})$ *lying over a morphism of* $\mathcal{E}$ *is* ψ*-cocartesian if and only if the corresponding functor* $\Delta^1 \times_{\mathrm{S}} \mathcal{C} \to \Delta^1 \times_{\mathrm{S}} \mathcal{D}$ *over* $\Delta^1 \times_{\mathrm{S}} \mathrm{T}$ *sends* β*-cartesian morphisms lying over morphisms of* $\mathcal{E}'$ *to* γ*-cocartesian morphisms.*

For $\mathrm{S} = \Delta^1$ and $\mathcal{E} = \mathrm{Fun}(\Delta^1, \Delta^1)$ we get the following:

The cartesian fibration $\alpha : \mathrm{T} \to \Delta^1$ classifies a functor $\mathrm{G} : \mathrm{T}_1 \to \mathrm{T}_0$. The cartesian fibration $\beta : \mathcal{C} \to \mathrm{T}$ relative to $\mathcal{E}'$ classifies a commutative square of categories corresponding to a functor $\mathcal{C}_1 \to \mathrm{G}^*(\mathcal{C}_0)$ over T_1 and the cocartesian fibration $\gamma : \mathcal{D} \to \mathrm{T}$ relative to $\mathcal{E}'$ classifies a functor $\mathrm{G}^*(\mathcal{D}_0) \to \mathcal{D}_1$ over T_1.

The cocartesian fibration $\psi : \mathrm{Fun}^{/\Delta^1}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \to \Delta^1$ classifies the functor

$$\mathrm{Fun}_{\mathrm{T}_0}(\mathcal{C}_0, \mathcal{D}_0) \to \mathrm{Fun}_{\mathrm{T}_1}(\mathrm{G}^*(\mathcal{C}_0), \mathrm{G}^*(\mathcal{D}_0)) \to \mathrm{Fun}_{\mathrm{T}_1}(\mathcal{C}_1, \mathcal{D}_1).$$

For α the projection $\mathrm{K} \times \mathrm{S} \to \mathrm{S}$ we get the following corollary:

Corollary 5.6. *Let* $\mathcal{C} \to \mathrm{K} \times \mathrm{S}$ *be a map of cartesian fibrations relative to* $\mathcal{E}$ *such that* $\mathcal{C} \to \mathrm{S}$ *is flat and* $\mathcal{D} \to \mathrm{K} \times \mathrm{S}$ *a map of cocartesian fibrations relative to* $\mathcal{E}$.

The functor $\psi : \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{K}\times\mathrm{S}}(\mathcal{C}, \mathcal{D}) \to \mathrm{S}$ *is a cocartesian fibration relative to* $\mathcal{E}$.

In this case a morphism of $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})$ *lying over a morphism of* $\mathcal{E}$ *is* ψ*-cocartesian if and only if the corresponding functor* $\Delta^1 \times_{\mathrm{S}} \mathcal{C} \to \Delta^1 \times_{\mathrm{S}} \mathcal{D}$ *over* $\Delta^1 \times_{\mathrm{S}} \mathrm{T}$ *sends morphisms that are cartesian with respect to* $\mathcal{C} \to \mathrm{S}$ *to morphisms that are cocartesian with respect to* $\mathcal{D} \to \mathrm{S}$.

Especially for K *contractible:*

Let $\mathcal{C} \to \mathrm{S}$ *be a flat functor and cartesian fibration relative to* $\mathcal{E}$ *and* $\mathcal{D} \to \mathrm{S}$ *a cocartesian fibration relative to* $\mathcal{E}$.

The functor $\psi : \mathrm{Map}_{\mathrm{S}}(\mathcal{C}, \mathcal{D}) \to \mathrm{S}$ *is a cocartesian fibration relative to* $\mathcal{E}$.

By the canonical equivalence

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{S} \times_{\mathrm{Map}_{\mathrm{S}}(\mathcal{C},\mathrm{T})} \mathrm{Map}_{\mathrm{S}}(\mathcal{C}, \mathcal{D})$$

over S we get the following corollary:

Corollary 5.7. *Let* $\mathcal{C} \to \mathrm{S}$ *be a flat functor and cartesian fibration relative to* $\mathcal{E}$, $\beta : \mathcal{C} \to \mathrm{T}$ *a functor over* S *that sends morphisms that are cartesian with respect to* $\mathcal{C} \to \mathrm{S}$ *and lie over morphisms of* $\mathcal{E}$ *to morphisms that are cocartesian with respect to* $\mathrm{T} \to \mathrm{S}$ *and* $\gamma : \mathcal{D} \to \mathrm{T}$ *a map of cocartesian fibrations relative to* $\mathcal{E}$.

Then $\psi : \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \to \mathrm{S}$ *is a cocartesian fibration relative to* $\mathcal{E}$.

In this case a morphism of $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})$ *lying over a morphism of* $\mathcal{E}$ *is* ψ*-cocartesian if and only if the corresponding functor* $\Delta^1 \times_{\mathrm{S}} \mathcal{C} \to \Delta^1 \times_{\mathrm{S}} \mathcal{D}$ *over* $\Delta^1 \times_{\mathrm{S}} \mathrm{T}$ *sends morphisms that are cartesian with respect to* $\mathcal{C} \to \mathrm{S}$ *to morphisms that are cocartesian with respect to* $\mathcal{D} \to \mathrm{S}$.

For $\mathrm{S} = \Delta^1$ and $\mathcal{E} = \mathrm{Fun}(\Delta^1, \Delta^1)$ we get the following:

The cocartesian fibration $\mathrm{T} \to \Delta^1$ classifies a functor $\mathrm{F} : \mathrm{T}_0 \to \mathrm{T}_1$.
The functor $\beta : \mathcal{C} \to \mathrm{T}$ classifies a functor $\mathcal{C}_1 \to \mathrm{F}_*(\mathcal{C}_0)$ over T_1 and the map $\gamma : \mathcal{D} \to \mathrm{T}$ of cocartesian fibrations over Δ^1 classifies a commutative square of categories corresponding to a functor $\mathrm{F}_*(\mathcal{D}_0) \to \mathcal{D}_1$ over T_1.
The cocartesian fibration $\psi : \mathrm{Fun}^{/\Delta^1}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}) \to \Delta^1$ classifies the functor

$$\mathrm{Fun}_{\mathrm{T}_0}(\mathcal{C}_0, \mathcal{D}_0) \to \mathrm{Fun}_{\mathrm{T}_1}(\mathrm{F}_*(\mathcal{C}_0), \mathrm{F}_*(\mathcal{D}_0)) \to \mathrm{Fun}_{\mathrm{T}_1}(\mathcal{C}_1, \mathcal{D}_1).$$

5.1.2 Parametrized categories of algebras

Based on parametrized categories of sections we define parametrized categories of algebras in the evident way:

Let S be a category, $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ maps of S-families of operads such that the functor $\mathcal{O}'^{\otimes} \to \mathrm{S}$ is flat.

We define $\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathcal{C}^{\otimes})$ to be the full subcategory spanned by the functors $\mathcal{O}'^{\otimes}_{\mathsf{s}} \to \mathcal{C}^{\otimes}_{\mathsf{s}}$ over $\mathcal{O}^{\otimes}_{\mathsf{s}}$ that preserve inert morphisms for some $\mathsf{s} \in \mathrm{S}$.

So for every $\mathsf{s} \in \mathrm{S}$ the equivalence $\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes)_\mathsf{s} \simeq \mathrm{Fun}_{\mathcal{O}^\otimes_\mathsf{s}}(\mathcal{O}'^\otimes_\mathsf{s}, \mathcal{C}^\otimes_\mathsf{s})$ restricts to an equivalence $\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})_\mathsf{s} \simeq \mathrm{Alg}_{\mathcal{O}'_\mathsf{s}/\mathcal{O}_\mathsf{s}}(\mathcal{C}_\mathsf{s})$.

More generally given a functor $\mathrm{S}' \to \mathrm{S}$ the canonical equivalence

$$\mathrm{S}' \times_\mathrm{S} \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \simeq \mathrm{Fun}^{/\mathrm{S}'}_{\mathrm{S}'\times_\mathrm{S}\mathcal{O}^\otimes}(\mathrm{S}' \times_\mathrm{S} \mathcal{O}'^\otimes, \mathrm{S}' \times_\mathrm{S} \mathcal{C}^\otimes)$$

over S' of remark 5.2 4. restricts to an equivalence

$$\mathrm{S}' \times_\mathrm{S} \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}^{/\mathrm{S}'}_{\mathrm{S}'\times_\mathrm{S}\mathcal{O}'/\mathrm{S}'\times_\mathrm{S}\mathcal{O}}(\mathrm{S}' \times_\mathrm{S} \mathcal{C})$$

over S'.

For every section $\mathrm{S} \to \mathcal{O}'^\otimes$ of the functor $\mathcal{O}'^\otimes \to \mathrm{S}$ lying over some section $\alpha : \mathrm{S} \to \mathcal{O}^\otimes$ of the functor $\mathcal{O}^\otimes \to \mathrm{S}$ we have a forgetful functor

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \to \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{O}^\otimes}(\mathrm{S}, \mathcal{C}^\otimes) \simeq \mathrm{S} \times_{\mathcal{O}^\otimes} \mathcal{C}^\otimes$$

over S, which induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ the forgetful functor $\mathrm{Alg}_{\mathcal{O}'_\mathsf{s}/\mathcal{O}_\mathsf{s}}(\mathcal{C}_\mathsf{s}) \to \{\alpha(\mathsf{s})\} \times_{\mathcal{O}^\otimes_\mathsf{s}} \mathcal{C}^\otimes_\mathsf{s}$.

Given a map of operads $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ we write $\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ for $\mathrm{Alg}^{/\mathrm{S}}_{\mathrm{S}\times\mathcal{O}'/\mathrm{S}\times\mathcal{O}}(\mathcal{C})$ and $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ for $\mathrm{Alg}^{/\mathrm{S}}_{\mathrm{Ass}/\mathrm{Ass}}(\mathcal{C})$.

Remark 5.8. *Given maps $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes, \mathcal{O}^\otimes \to \tilde{\mathcal{O}}^\otimes$ and $\mathcal{C}^\otimes \to \tilde{\mathcal{O}}^\otimes$ of S-families of operads we have a canonical equivalence*

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\tilde{\mathcal{O}}}(\mathcal{C}) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O} \times_{\tilde{\mathcal{O}}} \mathcal{C})$$

over S that is the restriction of the canonical equivalence

$$\mathrm{Fun}^{/\mathrm{S}}_{\tilde{\mathcal{O}}^\otimes}(\mathcal{O}'^\otimes, \mathcal{C}^\otimes) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{O}^\otimes}(\mathcal{O}'^\otimes, \mathcal{O}^\otimes \times_{\tilde{\mathcal{O}}^\otimes} \mathcal{C}^\otimes)$$

over S of remark 5.2 4.

Remark 5.9.

Let $\mathcal{O}'^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$, $\mathcal{C}^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$ be S-families of operads over $\mathcal{O}^\otimes$ for some operad $\mathcal{O}^\otimes$ and $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathrm{S})$ a full subcategory.

If $\mathcal{O}'^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$ is a map of cocartesian fibrations relative to $\mathcal{E}$ and $\mathcal{C}^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$ a map of cartesian fibrations relative to $\mathcal{E}$, the functor $\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathrm{S}\times\mathcal{O}}(\mathcal{C}) \to \mathrm{S}$ is a cartesian fibration relative to $\mathcal{E}$.

If $\mathcal{O}'^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$ is a map of cocartesian fibrations over S classifying a functor $\alpha : \mathrm{S} \to \mathrm{Op}_{\infty/\mathcal{O}^\otimes}$ and $\mathcal{C}^\otimes \to \mathrm{S}\times\mathcal{O}^\otimes$ a map of cartesian fibrations over S classifying a functor $\beta : \mathrm{S}^{\mathrm{op}} \to \mathrm{Op}_{\infty/\mathcal{O}^\otimes}$, by 5.23 the cartesian fibration $\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathrm{S}\times\mathcal{O}}(\mathcal{C}) \to \mathrm{S}$ classifies the functor

$$\mathrm{S}^{\mathrm{op}} \xrightarrow{(\alpha^{\mathrm{op}}, \beta)} (\mathrm{Op}_{\infty/\mathcal{O}^\otimes})^{\mathrm{op}} \times \mathrm{Op}_{\infty/\mathcal{O}^\otimes} \xrightarrow{\mathrm{Alg}_{(-)/\mathcal{O}}(-)} \mathsf{Cat}_\infty.$$

If $\mathcal{C}^\otimes \to \mathrm{S}\times\mathcal{O}^\otimes$ is a $\mathcal{O}^\otimes$-monoidal category over S such that for all $\mathrm{X} \in \mathcal{O}$ the functor $\mathcal{C}_\mathrm{X} \to \mathrm{S}$ is a cartesian fibration relative to $\mathcal{E}$.

Then by corollary 6.43 the functor $\mathcal{C}^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$ is a map of cartesian fibrations relative to $\mathcal{E}$ so that the functor $\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathrm{S}\times\mathcal{O}}(\mathcal{C}) \to \mathrm{S}$ is a cartesian fibration relative to $\mathcal{E}$.

5.1.3 $\mathcal{O}^\otimes$-monoidal categories of sections

For every functor $\mathcal{C} \to \mathrm{T}$ over S such that $\mathcal{C} \to \mathrm{S}$ is flat we have an adjunction $\mathcal{C} \times_{\mathrm{S}} - : \mathsf{Cat}_{\infty/\mathrm{S}} \rightleftarrows \mathsf{Cat}_{\infty/\mathrm{T}} : \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, -)$.

Being a right adjoint functor $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, -) : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ preserves finite products and thus monoid objects.

Let $\mathcal{O}^\otimes$ be an operad and $\mathcal{D} \to \mathrm{T} \times \mathcal{O}^\otimes$ a $\mathcal{O}^\otimes$-monoidal category over T classifying a $\mathcal{O}^\otimes$-monoid ϕ of $\mathsf{Cat}_{\infty/\mathrm{T}}$.

Theorem 5.23 implies that the image of ϕ under the finite products preserving functor $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, -) : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ is classified by the $\mathcal{O}^\otimes$-monoidal category

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes := \mathrm{Fun}^{/\mathrm{S}\times\mathcal{O}^\otimes}_{\mathrm{T}\times\mathcal{O}^\otimes}(\mathcal{C} \times \mathcal{O}^\otimes, \mathcal{D}^\otimes)$$

over S.

This motivates the following definition:

Given functors $\mathcal{C} \to \mathrm{T}$ and $\mathrm{T} \to \mathrm{S}$ over a category R such that the functor $\mathcal{C} \to \mathrm{S}$ is flat, a R-family of operads $\mathcal{O}^\otimes \to \mathrm{R} \times \mathcal{F}\mathrm{in}_*$ and a functor $\mathcal{D}^\otimes \to \mathrm{T} \times_{\mathrm{R}} \mathcal{O}^\otimes$ over R such that for every object r of R the induced functor $\mathcal{D}^\otimes_{\mathrm{r}} \to \mathrm{T}_{\mathrm{r}} \times \mathcal{O}^\otimes_{\mathrm{r}}$ on the fiber over r exhibits $\mathcal{D}^\otimes_{\mathrm{r}}$ as a T_{r}-family of operads over $\mathcal{O}^\otimes_{\mathrm{r}}$ we set

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes := \mathrm{Fun}^{/\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^\otimes}_{\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^\otimes}(\mathcal{C} \times_{\mathrm{R}} \mathcal{O}^\otimes, \mathcal{D}^\otimes).$$

If the functor $\mathrm{T} \to \mathrm{S}$ is the identity, we write $\mathrm{Map}_{\mathrm{S}}(\mathcal{C}, \mathcal{D})^\otimes$ for $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes$.

Given a functor $\mathrm{S}' \to \mathrm{S}$ and a map of R-families of operads $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ we have a canonical equivalence

$$(\mathrm{S}' \times_{\mathrm{R}} \mathcal{O}'^\otimes) \times_{(\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^\otimes)} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes \simeq \mathrm{Fun}^{/\mathrm{S}'}_{\mathrm{S}'\times_{\mathrm{S}}\mathrm{T}}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}, (\mathrm{S}' \times_{\mathrm{R}} \mathcal{O}') \times_{(\mathrm{S}\times_{\mathrm{R}}\mathcal{O})} \mathcal{D})^\otimes$$

by remark 5.2 4. and so canonical equivalences

$$\mathrm{S}' \times_{\mathrm{S}} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes \simeq \mathrm{Fun}^{/\mathrm{S}'}_{\mathrm{S}'\times_{\mathrm{S}}\mathrm{T}}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{D})^\otimes,$$

$$\mathcal{O}'^\otimes \times_{\mathcal{O}^\otimes} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{O}' \times_{\mathcal{O}} \mathcal{D})^\otimes.$$

So especially for every $\mathrm{r} \in \mathrm{R}$ and $\mathrm{X} \in \mathcal{O}_{\mathrm{r}}$ we have canonical equivalences

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes_{\mathrm{r}} \simeq \mathrm{Fun}^{/\mathrm{S}_{\mathrm{r}}}_{\mathrm{T}_{\mathrm{r}}}(\mathcal{C}_{\mathrm{r}}, \mathcal{D}_{\mathrm{r}})^\otimes,$$

$$(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^\otimes_{\mathrm{r}})_{\mathrm{X}} \simeq \mathrm{Fun}^{/\mathrm{S}_{\mathrm{r}}}_{\mathrm{T}_{\mathrm{r}}}(\mathcal{C}_{\mathrm{r}}, (\mathcal{D}_{\mathrm{r}})_{\mathrm{X}}).$$

Remark 5.10. *Theorem B.4.2. [18] implies the following:*

Let T *be a category and* $\mathfrak{P}$ *a categorical pattern on some category* $\mathcal{B}$. *If* $\mathcal{D} \to \mathcal{B} \times \mathrm{T}$ *is a* T*-family of* $\mathfrak{P}$*-fibered objects, the functor*

$$\mathrm{Fun}^{/\mathcal{B}}_{\mathrm{T}\times\mathcal{B}}(\mathrm{T} \times \mathcal{B}, \mathcal{D}) \to \mathcal{B}$$

is $\mathfrak{P}$*-fibered.*

Especially given an operad $\mathcal{O}^\otimes$ *and a* T*-family* $\mathcal{D}^\otimes \to \mathrm{T} \times \mathcal{O}^\otimes$ *of operads over* $\mathcal{O}^\otimes$ *the functor*

$$\mathrm{Fun}_{\mathrm{T}}(\mathrm{T}, \mathcal{D})^\otimes := \mathrm{Fun}^{/\mathcal{O}^\otimes}_{\mathrm{T}\times\mathcal{O}^\otimes}(\mathrm{T} \times \mathcal{O}^\otimes, \mathcal{D}^\otimes) \to \mathcal{O}^\otimes$$

is a map of operads that is a (locally) cocartesian fibration if $\mathcal{D}^\otimes \to \mathrm{T} \times \mathcal{O}^\otimes$ *is a* T*-family of representable* $\mathcal{O}^\otimes$*-operads respectively* $\mathcal{O}^\otimes$*-monoidal categories.*

Remark 5.11. *Given a map* $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ *of* R*-families of operads such that the functor* $\mathcal{O}'^{\otimes} \to \mathrm{R}$ *is flat, remark 5.3 provides a canonical equivalence*

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}(\mathrm{T}\times_{\mathrm{R}}\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes})) \simeq$$

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}(\mathrm{S}\times_{\mathrm{R}}\mathcal{O}'^{\otimes}, \mathrm{Fun}^{/\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}_{\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}(\mathcal{C}\times_{\mathrm{R}}\mathcal{O}^{\otimes}, \mathcal{D}^{\otimes})) =$$
$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}(\mathrm{S}\times_{\mathrm{R}}\mathcal{O}'^{\otimes}, \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^{\otimes})$$

over S.

This equivalence restricts to an equivalence

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathrm{Alg}^{/\mathrm{T}}_{\mathrm{T}\times_{\mathrm{R}}\mathcal{O}'/\mathrm{T}\times_{\mathrm{R}}\mathcal{O}}(\mathcal{D})) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathrm{S}\times_{\mathrm{R}}\mathcal{O}'/\mathrm{S}\times_{\mathrm{R}}\mathcal{O}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}))$$

over S.

Proof. To see this, we can reduce to the case that R and S are contractible according to remark 5.4.

In this case we have to show that the canonical equivalence

$$\mathrm{Fun}_{\mathrm{T}}(\mathcal{C}, \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times\mathcal{O}^{\otimes}}(\mathrm{T}\times\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes})) \simeq \mathrm{Fun}_{\mathcal{O}^{\otimes}}(\mathcal{O}'^{\otimes}, \mathrm{Fun}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^{\otimes})$$

restricts to an equivalence

$$\mathrm{Fun}_{\mathrm{T}}(\mathcal{C}, \mathrm{Alg}^{/\mathrm{T}}_{\mathrm{T}\times\mathcal{O}'/\mathrm{T}\times\mathcal{O}}(\mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})).$$

By remark 5.5 a functor $\mathcal{O}'^{\otimes} \to \mathrm{Fun}_{\mathrm{T}}(\mathcal{C}, \mathcal{D})^{\otimes}$ over $\mathcal{O}^{\otimes}$ belongs to

$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}_{\mathrm{T}}(\mathcal{C}, \mathcal{D}))$ and a functor $\mathcal{C} \to \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times\mathcal{O}^{\otimes}}(\mathrm{T}\times\mathcal{O}'^{\otimes}, \mathcal{D}^{\otimes})$ over T factors through $\mathrm{Alg}^{/\mathrm{T}}_{\mathrm{T}\times\mathcal{O}'/\mathrm{T}\times\mathcal{O}}(\mathcal{D})$ if and only if their corresponding functor $\mathcal{C}\times\mathcal{O}'^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathrm{T}\times\mathcal{O}^{\otimes}$ sends a morphism (f, g) of $\mathcal{C}\times\mathcal{O}'^{\otimes}$ with f an equivalence of $\mathcal{C}$ and g an inert morphism of $\mathcal{O}'^{\otimes}$ to an inert morphism of $\mathcal{D}^{\otimes}$.

□

Moreover by remark 5.3 we have the following compatibility:

Denote φ *the evaluation functor*

$$\mathrm{T}\times_{\mathrm{S}}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathrm{T}, \mathcal{D})^{\otimes} = \mathrm{T}\times_{\mathrm{S}}\mathrm{Fun}^{/\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}_{\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}(\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}, \mathcal{D}^{\otimes}) \simeq$$

$$\mathrm{Fun}^{/\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}_{\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}}(\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}, \mathcal{D}^{\otimes})\times_{(\mathrm{S}\times_{\mathrm{R}}\mathcal{O}^{\otimes})}(\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}) \to \mathcal{D}^{\otimes}$$

over $\mathrm{T}\times_{\mathrm{R}}\mathcal{O}^{\otimes}$.

The composition

$$\mathrm{T}\times_{\mathrm{S}}\times\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathrm{T}, \mathrm{Alg}^{/\mathrm{T}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})) \simeq \mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathrm{T}, \mathcal{D})) \simeq$$

$$\mathrm{Alg}^{/\mathrm{T}}_{\mathcal{O}'/\mathcal{O}}(\mathrm{T}\times_{\mathrm{S}}\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathrm{T}, \mathcal{D})) \xrightarrow{\mathrm{Alg}^{/\mathrm{T}}_{\mathcal{O}'/\mathcal{O}}(\varphi)} \mathrm{Alg}^{/\mathrm{T}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{D})$$

is equivalent to the evaluation functor over T.

5.1.4 Parametrized categories of modules

In this subsection we specialize from parametrized categories of $\mathcal{O}^{\otimes}$-algebras and $\mathcal{O}^{\otimes}$-monoidal categories of sections to parametrized categories of left modules and $\mathrm{LM}^{\otimes}$-monoidal categories of sections by taking $\mathcal{O}^{\otimes} := \mathrm{LM}^{\otimes}$.

We remark that all results given here work for right modules in a similar way:

Let T be a category and $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ a T-family of operads over $\mathrm{LM}^{\otimes}$. Set $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$ and $\mathcal{B} := \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$.

We write $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})$ for $\mathrm{Alg}^{/\mathrm{T}}_{\mathrm{LM}/\mathrm{LM}}(\mathcal{M})$.

For every functor $\mathrm{T}' \to \mathrm{T}$ we have a canonical equivalence

$$\mathrm{T}' \times_{\mathrm{T}} \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) \simeq \mathrm{LMod}^{/\mathrm{T}'}(\mathrm{T}' \times_{\mathrm{T}} \mathcal{B}).$$

We have forgetful functors

$$\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) = \mathrm{Alg}^{/\mathrm{T}}_{\mathrm{LM}/\mathrm{LM}}(\mathcal{M}) \to \mathrm{Alg}^{/\mathrm{T}}_{\mathrm{Ass}/\mathrm{LM}}(\mathcal{M}) \simeq \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C}),$$

$$\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) \subset \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times\mathrm{LM}}(\mathrm{T} \times \mathrm{LM}, \mathcal{M}) \to \mathrm{Fun}^{/\mathrm{T}}_{\mathrm{T}\times\mathrm{LM}}(\mathrm{T} \times \{\mathfrak{m}\}, \mathcal{M}) \simeq \mathcal{B}$$

over T.

Given a section A of $\mathrm{Alg}^{/\mathrm{T}}(\mathcal{C}) \to \mathrm{T}$ we set $\mathrm{LMod}^{/\mathrm{T}}_{\mathrm{A}}(\mathcal{B}) := \mathrm{T} \times_{\mathrm{Alg}^{/\mathrm{T}}(\mathcal{C})} \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})$.

If $\mathcal{C}^{\otimes} = \mathrm{T} \times_{\mathrm{S}} \mathcal{D}^{\otimes}$ for a S-family of operads $\mathcal{D}^{\otimes} \to \mathrm{S} \times \mathrm{Ass}^{\otimes}$ over $\mathrm{Ass}^{\otimes}$ and a functor $\mathrm{T} \to \mathrm{S}$ and A is a section of $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{S}$, we write $\mathrm{LMod}^{/\mathrm{T}}_{\mathrm{A}}(\mathcal{B})$ for $\mathrm{LMod}^{/\mathrm{T}}_{\mathrm{A}'}(\mathcal{B}) \simeq \mathrm{S} \times_{\mathrm{Alg}^{/\mathrm{S}}(\mathcal{D})} \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})$, where A′ denotes the functor $\mathrm{T} \times_{\mathrm{S}} \mathrm{A} : \mathrm{T} \to \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C}) \simeq \mathrm{T} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{D})$ over T.

Remark 5.12. *Let $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathrm{T})$ be a full subcategory.*

If the functor $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ is a map of cartesian fibrations relative to $\mathcal{E}$, by 5.9 the functors

$$\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) \to \mathrm{T}, \ \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C}) \to \mathrm{T}$$

are cartesian fibrations relative to $\mathcal{E}$ and the functor

$$\Phi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) \to \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C})$$

is a map of cartesian fibrations relative to $\mathcal{E}$.

Moreover if $\mathcal{E} = \mathrm{Fun}(\Delta^1, \mathrm{T})$ the forgetful functor $\Phi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) \to \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C})$ is a cartesian fibration, whose cartesian morphisms are those that get cartesian morphisms of $\mathcal{B} \to \mathrm{T}$:

This follows from the fact that Φ induces on the fiber over every $\mathrm{t} \in \mathrm{T}$ the cartesian fibration $\Phi_{\mathrm{t}} : \mathrm{LMod}(\mathcal{B}_{\mathrm{t}}) \to \mathrm{Alg}(\mathcal{C}_{\mathrm{t}})$ whose cartesian morphisms are those that get equivalences in $\mathcal{B}_{\mathrm{t}}$ so that for every morphism $\mathsf{s} \to \mathrm{t}$ of S the induced functor $\mathrm{LMod}(\mathcal{B}_{\mathrm{t}}) \to \mathrm{LMod}(\mathcal{B}_{\mathsf{s}})$ sends Φ_{t}-cartesian morphisms to Φ_{s}-cartesian morphisms.

By corollary 6.43 the functor $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ is a map of cartesian fibrations relative to $\mathcal{E}$ if the functor $\mathcal{M}^{\otimes} \to \mathrm{T}\times\mathrm{LM}^{\otimes}$ is a map of cocartesian fibrations over $\mathrm{LM}^{\otimes}$ and the functors $\mathcal{B} \to \mathrm{T}, \mathcal{C} \to \mathrm{T}$ are cartesian fibrations relative to $\mathcal{E}$.

Let S be a category and $\varphi : \mathcal{B} \to \mathrm{T}$ a map of cartesian fibrations over S. Let $\mathcal{M}^{\otimes}$ be a $\mathrm{LM}^{\otimes}$-monoidal category over T that exhibits the functor

$\mathcal{B} \to \mathrm{T}$ *as a left module over the pullback of a monoidal category* $\mathcal{D}^{\otimes} \to \mathrm{Ass}^{\otimes} \times \mathrm{S}$ *over* S *along the functor* $\mathrm{T} \to \mathrm{S}$.

The forgetful functor $\Phi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{D}) \times_{\mathrm{S}} \mathcal{B}$ *is a map of cartesian fibrations over* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{D})$, *whose cartesian morphisms are those that get cartesian morphisms of* $\mathcal{B} \to \mathrm{S}$ *(lemma 5.13).*

So for every section A *of* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{S}$ *the functor* $\mathrm{LMod}^{/\mathrm{T}}_{\mathrm{A}}(\mathcal{B}) \to \mathrm{S}$ *is a cartesian fibration, whose cartesian morphisms are those that get cartesian morphisms of* $\mathcal{B} \to \mathrm{T}$.

Lemma 5.13. *Let* S *be a category and* $\varphi : \mathcal{D} \to \mathrm{T}$ *a map of cartesian fibrations over* S.

Let $\mathcal{M}^{\otimes}$ *be a* $\mathrm{LM}^{\otimes}$*-monoidal category over* T *that exhibits the functor* $\mathcal{D} \to \mathrm{T}$ *as a left module over the pullback of a monoidal category* $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes} \times \mathrm{S}$ *over* S *along the functor* $\mathrm{T} \to \mathrm{S}$.

The forgetful functor $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}) \times_{\mathrm{S}} \mathcal{D}$ *is a map of cartesian fibrations over* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$.

A morphism of $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D})$ *is cartesian with respect to the cartesian fibration* $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *if and only if its image in* $\mathcal{D}$ *is cartesian with respect to the cartesian fibration* $\mathcal{D} \to \mathrm{S}$.

Proof. Assume first that S is contractible and $\varphi : \mathcal{D} \to \mathrm{T}$ is a cartesian fibration.

In this case remark 5.12 implies that the canonical functor
$\Psi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{T}$ is a cartesian fibration, where a morphism is Ψ-cartesian if and only if its image in $\mathcal{D}$ is φ-cartesian.

Therefore the composition $\Phi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C}) \times \mathrm{T} \to \mathrm{Alg}(\mathcal{C})$ is a cartesian fibration, where a morphism is Φ-cartesian if and only if it is Ψ-cartesian and its image in T is an equivalence, i.e. if and only if its image in $\mathcal{D}$ is an equivalence.

Now let $\varphi : \mathcal{D} \to \mathrm{T}$ be an arbitrary functor but S still be contractible.

In this case we embed the functor $\mathcal{D} \to \mathrm{T}$ into a cartesian fibration:
The subcategory inclusion $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}} \subset \mathsf{Cat}_{\infty/\mathrm{T}}$ admits a left adjoint $\mathcal{E} : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}}$ with the following properties:

1. For every functor $\mathrm{X} \to \mathrm{T}$ the cartesian fibration $\mathcal{E}(\mathrm{X}) \to \mathrm{T}$ is equivalent over T to the functor $\mathrm{X} \times_{\mathrm{Fun}(\{1\},\mathrm{T})} \mathrm{Fun}(\Delta^1, \mathrm{T}) \to \mathrm{Fun}(\Delta^1, \mathrm{T}) \to \mathrm{Fun}(\{0\}, \mathrm{T})$.
2. The unit $\mathrm{X} \to \mathcal{E}(\mathrm{X}) \simeq \mathrm{X} \times_{\mathrm{Fun}(\{1\},\mathrm{T})} \mathrm{Fun}(\Delta^1, \mathrm{T})$ is the pullback of the fully faithful diagonal embedding $\mathrm{T} \to \mathrm{Fun}(\Delta^1, \mathrm{T})$ over $\mathrm{Fun}(\{1\}, \mathrm{T})$ along $\mathrm{X} \to \mathrm{T}$ and is thus itself fully faithful.
3. For every category K and every functor $\mathcal{C} \to \mathrm{T}$ the map $\mathcal{E}(\mathrm{K} \times \mathcal{C}) \to \mathrm{K} \times \mathcal{E}(\mathcal{C})$ of cartesian fibrations over T adjoint to the functor $\mathrm{K} \times \mathcal{C} \to \mathrm{K} \times \mathcal{E}(\mathcal{C})$ over T is an equivalence.

This follows from the following considerations:
Taking the opposite category $\mathsf{Cat}_{\infty/\mathrm{T}} \simeq \mathsf{Cat}_{\infty/\mathrm{T}^{\mathrm{op}}}$ restricts to an equivalence $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}} \simeq \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathrm{T}^{\mathrm{op}}}$.

So it is enough to see that the subcategory inclusion $\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathrm{T}} \subset \mathsf{Cat}_{\infty/\mathrm{T}}$ admits a left adjoint $\mathcal{E}$ with properties 1., 2., 3., where we have to change $\{1\}$ with $\{0\}$.

We have a colocalization $\iota : \mathsf{Cat}_\infty \rightleftarrows \mathrm{Op}_\infty : \gamma$ that induces an equivalence $\iota : \mathsf{Cat}_{\infty/\mathrm{T}} \rightleftarrows \mathrm{Op}_{\infty/\iota(\mathrm{T})} : \gamma$ that restricts to an equivalence $\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathrm{T}} \simeq \mathrm{Op}^{\mathrm{cocart}}_{\infty/\iota(\mathrm{T})}$.

But the subcategory inclusion $\mathrm{Op}^{\mathrm{cocart}}_{\infty/\iota(\mathrm{T})} \subset \mathrm{Op}_{\infty/\iota(\mathrm{T})}$ admits a left adjoint given by the enveloping $\iota(\mathrm{T})$-monoidal category that induces on underlying categories the properties 1., 2., 3., when we change $\{1\}$ with $\{0\}$.

The Cat_∞-left module structure on $\mathsf{Cat}_{\infty/\mathrm{T}}$ induced by the symmetric monoidal functor $-\times \mathrm{T} : \mathsf{Cat}^\times_\infty \to (\mathsf{Cat}_{\infty/\mathrm{T}})^\times$ restricts to a Cat_∞-left module structure on $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}}$ as the functor $-\times\mathrm{T} : \mathsf{Cat}_\infty \to \mathsf{Cat}_{\infty/\mathrm{T}}$ factors through the subcategory $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}} \subset \mathsf{Cat}_{\infty/\mathrm{T}}$.

So the subcategory inclusion $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}} \subset \mathsf{Cat}_{\infty/\mathrm{T}}$ is a Cat_∞-linear functor and so by 3. the adjunction $\mathcal{E} : \mathsf{Cat}_{\infty/\mathrm{T}} \rightleftarrows \mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}}$ is a Cat_∞-linear adjunction.

Thus we get an induced adjunction $\mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/\mathrm{T}}) \rightleftarrows \mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}})$ over the adjunction $\mathcal{E} : \mathsf{Cat}_{\infty/\mathrm{T}} \rightleftarrows \mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}}$ so that the unit $\mathcal{D} \to \mathcal{E}(\mathcal{D})$ lifts to a $\mathcal{C}$-linear functor over T.

So the fully faithful unit $\mathcal{D} \to \mathcal{E}(\mathcal{D})$ induces a full subcategory inclusion $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \subset \mathrm{LMod}^{/\mathrm{T}}(\mathcal{E}(\mathcal{D}))$ over T such that the functor $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C})$ is the restriction of the functor $\psi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{E}(\mathcal{D})) \to \mathrm{Alg}(\mathcal{C})$.

As the lemma holds for the case that $\varphi : \mathcal{D} \to \mathrm{T}$ is a cartesian fibration and S is contractible, the functor $\psi : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{E}(\mathcal{D})) \to \mathrm{Alg}(\mathcal{C})$ is a cartesian fibration, where a morphism is ψ-cartesian if and only if its image in $\mathcal{E}(\mathcal{D})$ is an equivalence.

Consequently every ψ-cartesian morphism has with its target also its source in $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \simeq \mathcal{D} \times_{\mathcal{E}(\mathcal{D})} (\mathrm{LMod}^{/\mathrm{T}}(\mathcal{E}(\mathcal{D}))$ so that the cartesian fibration ψ restricts to a cartesian fibration $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}(\mathcal{C})$ with the same cartesian morphisms.

Now let S be arbitrary.

Let $\mathrm{X} \to \mathrm{Y}$ be a map of cartesian fibrations over S over a cartesian fibration $\mathrm{Z} \to \mathrm{S}$.

Then the functor $\mathrm{X} \to \mathrm{Y}$ is a map of cartesian fibrations over Z if and only if the following two conditions are satisfied:

1. For every object s of S the induced functor $\mathrm{X}_\mathsf{s} \to \mathrm{Y}_\mathsf{s}$ on the fiber over s is a map of cartesian fibrations over Z_s.
2. For every morphism $\mathsf{s}' \to \mathsf{s}$ of S the induced functors $\mathrm{X}_\mathsf{s} \to \mathrm{X}_{\mathsf{s}'}$ and $\mathrm{Y}_\mathsf{s} \to \mathrm{Y}_{\mathsf{s}'}$ on the fiber send $\mathrm{X}_\mathsf{s} \to \mathrm{Z}_\mathsf{s}$-cartesian morphisms to $\mathrm{X}_{\mathsf{s}'} \to \mathrm{Z}_{\mathsf{s}'}$ -cartesian morphisms respectively $\mathrm{Y}_\mathsf{s} \to \mathrm{Z}_\mathsf{s}$-cartesian morphisms to $\mathrm{Y}_{\mathsf{s}'} \to \mathrm{Z}_{\mathsf{s}'}$-cartesian morphisms.

Moreover the functor $\mathrm{X} \to \mathrm{Y}$ is a map of cartesian fibrations over Z that reflects cartesian morphisms over Z if and only if 1. and 2. holds and for every object s of S the induced functor $\mathrm{X}_\mathsf{s} \to \mathrm{Y}_\mathsf{s}$ on the fiber over s reflects cartesian morphisms over Z_s.

By remark 5.12 the functor $\phi(\mathcal{M}^\otimes, \mathcal{C}^\otimes) : \mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})\times_\mathrm{S}\mathcal{D}$ is a map of cartesian fibrations over S over the cartesian fibration $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$.

For every object s of S the induced functor $\phi(\mathcal{M}^\otimes, \mathcal{C}^\otimes)_\mathsf{s} : \mathrm{LMod}^{/\mathrm{T}_\mathsf{s}}(\mathcal{D}_\mathsf{s}) \simeq \mathrm{LMod}^{/\mathrm{T}}(\mathcal{D})_\mathsf{s} \to \mathrm{Alg}(\mathcal{C}_\mathsf{s})\times\mathcal{D}_\mathsf{s}$ on the fiber over s is equivalent to the functor $\phi(\mathcal{M}^\otimes_\mathsf{s}, \mathcal{C}^\otimes_\mathsf{s}) : \mathrm{LMod}^{/\mathrm{T}_\mathsf{s}}(\mathcal{D}_\mathsf{s}) \to \mathrm{Alg}(\mathcal{C}_\mathsf{s}) \times \mathcal{D}_\mathsf{s}$.

As the lemma holds for the case that S is contractible, the functor $\phi(\mathcal{M}^\otimes_\mathsf{s}, \mathcal{C}^\otimes_\mathsf{s}) : \mathrm{LMod}^{/\mathrm{T}_\mathsf{s}}(\mathcal{D}_\mathsf{s}) \to \mathrm{Alg}(\mathcal{C}_\mathsf{s}) \times \mathcal{D}_\mathsf{s}$ is a map of cartesian fibrations over $\mathrm{Alg}(\mathcal{C}_s)$, where a morphism of $\mathrm{LMod}^{/\mathrm{T}_s}(\mathcal{D}_s)$ is cartesian with respect to the cartesian fibration $\mathrm{LMod}^{/\mathrm{T}_s}(\mathcal{D}_s) \to \mathrm{Alg}(\mathcal{C}_s)$ if and only if its image in $\mathcal{D}_s$ is an equivalence.

This implies condition 1. and 2., where we use for condition 2. that the canonical functor $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D}$ is a map of cartesian fibrations over S.

□

Construction 5.14. *Let* $\mathrm{T} \to \mathrm{S}$ *be a functor and* $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{T}$ *a* $\mathrm{LM}^{\otimes}$*-monoidal category over* T *that exhibits a functor* $\mathcal{D} \to \mathrm{T}$ *as a left module over the pullback of a monoidal category* $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes} \times \mathrm{S}$ *over* S *along the functor* $\mathrm{T} \to \mathrm{S}$.

Assume that the functor $\mathcal{D} \to \mathrm{T}$ *is a map of cartesian fibrations over* S *classifying a natural transformation* $\mathrm{H} \to \mathrm{G}$ *of functors* $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}$.

By lemma 5.13 the forgetful functor

$$\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$$

is a map of cartesian fibrations over $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *and so classifies a functor*

$$\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}} \to \mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})$$

over Cat_{∞} *adjoint to a functor*

$$\phi(\mathcal{M}, \mathcal{C}) : \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}} \to \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))$$

over S^{op}.

Denote $\mathrm{X} : \mathrm{S}^{\mathrm{op}} \to \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))$ *the functor over* S^{op} *corresponding to the natural transformation* $\mathrm{H} \to \mathrm{G}$ *of functors* $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}$.

We have a canonical equivalence

$$\mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})) \simeq \mathrm{S}^{\mathrm{op}} \times_{\mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))^{\{1\}}} \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))^{\Delta^1}$$

$$=: \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}}$$

over S^{op} *represented by the canonical equivalence*

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{K}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))) \simeq \mathrm{Fun}_{\mathsf{Cat}_{\infty}}(\mathrm{H}_*(\mathrm{K}), \mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})) \simeq$$

$$\mathrm{Fun}(\mathrm{K}, \mathsf{Cat}_{\infty})_{/\mathrm{H}_*(\mathrm{K})} \simeq (\mathrm{Fun}(\mathrm{K}, \mathsf{Cat}_{\infty})_{/\mathrm{G}_*(\mathrm{K})})_{/\mathrm{H}_*(\mathrm{K})} \simeq$$

$$\mathrm{Fun}_{\mathsf{Cat}_{\infty}}(\mathrm{G}_*(\mathrm{K}), \mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))_{/\mathrm{H}_*(\mathrm{K})} \simeq$$

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{K}, \mathrm{S}^{\mathrm{op}} \times_{\mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))^{\{1\}}} \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))^{\Delta^1})$$

natural in a category K *over* S^{op}.

So we obtain a functor

$$\phi(\mathcal{M}, \mathcal{C}) : \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}} \to \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})) \simeq \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}}$$

over S^{op} *that sends an object* $\mathrm{A} \in \mathrm{Alg}(\mathcal{C}_s)$ *for some* $s \in \mathrm{S}$ *to* $\mathrm{LMod}_{\mathrm{A}}(\mathcal{D}_s) \to \mathcal{D}_s$.

Remark 5.15.

1. *Let* $\varphi : \mathrm{S}' \to \mathrm{S}$ *be a functor. Set* $\mathrm{T}' := \mathrm{S}' \times_{\mathrm{S}} \mathrm{T}$.

   The pullback of the map $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *of cartesian fibrations over* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *along the functor* $\mathrm{Alg}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}) \simeq \mathrm{S}' \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *is canonically equivalent to the map* $\mathrm{LMod}^{/\mathrm{T}'}(\mathrm{T}' \times_{\mathrm{T}} \mathcal{D}) \to (\mathrm{T}' \times_{\mathrm{T}} \mathcal{D}) \times_{\mathrm{S}'} \mathrm{Alg}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C})$ *of cartesian fibrations over* $\mathrm{Alg}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C})$.

   So the functor $\phi(\mathrm{T}' \times_{\mathrm{T}} \mathcal{M}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{C})$:

   $$\mathrm{Alg}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C})^{\mathrm{op}} \to \varphi^*(\mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)))$$

   over $\mathrm{S}'^{\mathrm{op}}$ *is equivalent to the pullback of the functor* $\phi(\mathcal{M}, \mathcal{C})$ *over* S^{op} *along the functor* $\varphi^{\mathrm{op}} : \mathrm{S}'^{\mathrm{op}} \to \mathrm{S}^{\mathrm{op}}$.

2. *Let* $\beta : \mathcal{C}'^{\otimes} \to \mathcal{C}^{\otimes}$ *be a monoidal functor over* S *and* $\mathcal{M}'^{\otimes}$ *the pullback of* $\mathcal{M}^{\otimes}$ *along* $\mathrm{T} \times_{\mathrm{S}} \beta : \mathrm{T} \times_{\mathrm{S}} \mathcal{C}'^{\otimes} \to \mathrm{T} \times_{\mathrm{S}} \mathcal{C}^{\otimes}$.

   The functor $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}')$ *over* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}')$ *is the pullback of the map*

   $$\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$$

   of cartesian fibrations over $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *along the functor* $\mathrm{Alg}^{/\mathrm{S}}(\beta) : \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}') \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$.

   Thus $\phi(\mathcal{M}', \mathcal{C}')$ *is the composition*

   $$\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}')^{\mathrm{op}} \xrightarrow{\mathrm{Alg}^{/\mathrm{S}}(\beta)^{\mathrm{op}}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}} \xrightarrow{\phi(\mathcal{M},\mathcal{C})} \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$

   of functors over S^{op}.

3. *Let* $\mathrm{T} \to \mathrm{S}' \to \mathrm{S}$ *be a factorization of the functor* $\mathrm{T} \to \mathrm{S}$.
   The functor $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D} \times_{\mathrm{S}'} \mathrm{Alg}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}) \simeq \mathcal{D} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *over* $\mathrm{Alg}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{C}) \simeq \mathrm{S}' \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *considered as a functor over* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *is equivalent to the functor* $\mathrm{LMod}^{/\mathrm{T}}(\mathcal{D}) \to \mathcal{D} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$.

4. *Let* $\gamma : \mathrm{S} \to \mathrm{R}$ *be a cartesian fibration.*
   Denote $\mathrm{H}' : \mathrm{R}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *the functor classified by the composition* $\mathcal{D} \to \mathrm{S} \to \mathrm{R}$.

   Denote $\rho : \mathrm{T} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})^{\otimes} \to \mathrm{T} \times_{\mathrm{S}} \mathcal{C}^{\otimes}$ *the pullback of the monoidal counit* $\mathrm{S} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}$ *over* S *along the functor* $\mathrm{T} \to \mathrm{S}$ *and* $\rho^*(\mathcal{M}^{\otimes})$ *the pullback of the* $\mathrm{T} \times_{\mathrm{S}} \mathcal{C}^{\otimes}$*-left module structure on* $\mathcal{D} \to \mathrm{T}$ *along* ρ.
   By 2. the functor

   $$\theta : (\mathrm{S} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})))^{\mathrm{op}} \simeq \mathrm{Alg}^{/\mathrm{S}}(\mathrm{S} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}))^{\mathrm{op}}$$

   $$\xrightarrow{\phi(\rho^*(\mathcal{M}), \mathrm{S} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}))} \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$

   over S^{op} *factors as*

   $$(\mathrm{S} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})))^{\mathrm{op}} \simeq \mathrm{Alg}^{/\mathrm{S}}(\mathrm{S} \times_{\mathrm{R}} \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}))^{\mathrm{op}}$$

   $$\to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}} \xrightarrow{\phi(\mathcal{M},\mathcal{C})} \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)),$$

in other words θ is adjoint to the functor

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}}) \xrightarrow{\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \phi(\mathcal{M},\mathcal{C}))}$$

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)))$$

over R^{op}.

So by lemma 5.16 the composition

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}}) \simeq \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}))^{\mathrm{op}} \simeq \mathrm{Alg}^{/\mathrm{R}}(\mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}))^{\mathrm{op}}$$

$$\xrightarrow{\phi(\rho^*(\mathcal{M}), \mathrm{Fun}^{/\mathrm{R}}_{\mathrm{S}}(\mathrm{S},\mathcal{C}))} \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$

is equivalent to the composition

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})^{\mathrm{op}}) \xrightarrow{\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \phi(\mathcal{M},\mathcal{C}))}$$

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))) \subset \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)).$$

Lemma 5.16. *Let* $\mathrm{Y} \to \mathrm{R}$ *be a functor and* $\gamma : \mathrm{S} \to \mathrm{R}$ *and* $\mathcal{D} \to \mathrm{S}$ *cartesian fibrations .*

Denote $\mathrm{H} : \mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *the functor classified by* $\mathcal{D} \to \mathrm{S}$ *and* $\mathrm{H}' : \mathrm{R}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *the functor classified by the composition* $\mathcal{D} \to \mathrm{S} \to \mathrm{R}$.

Let $\varphi : \mathrm{X} \to \mathcal{D} \times_{\mathrm{S}} (\mathrm{S} \times_{\mathrm{R}} \mathrm{Y}) \simeq \mathcal{D} \times_{\mathrm{R}} \mathrm{Y}$ *be a map of cartesian fibrations over* $\mathrm{S} \times_{\mathrm{R}} \mathrm{Y}$ *that gives rise to a map of cartesian fibrations* φ' *over* Y *via forgetting along the canonical functor* $\mathrm{S} \times_{\mathrm{R}} \mathrm{Y} \to \mathrm{Y}$.

φ *classifies a functor* $\mathrm{H}_*(\mathrm{S}^{\mathrm{op}} \times_{\mathrm{R}^{\mathrm{op}}} \mathrm{Y}^{\mathrm{op}}) \to \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ *over* Cat_∞ *adjoint to a functor* $\alpha : \mathrm{S}^{\mathrm{op}} \times_{\mathrm{R}^{\mathrm{op}}} \mathrm{Y}^{\mathrm{op}} \to \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$ *over* S^{op} *and* φ' *classifies a functor* $\mathrm{H}'_*(\mathrm{Y}^{\mathrm{op}}) \to \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ *adjoint to a functor* $\beta : \mathrm{Y}^{\mathrm{op}} \to \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$ *over* R^{op}.

Then β *factors as the functor*

$$\mathrm{Y}^{\mathrm{op}} \to \mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)))$$

over R^{op} *adjoint to* α *followed by the canonical subcategory inclusion*

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))) \subset \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$

over R^{op}, *which is represented by the subcategory inclusion*

$$\mathrm{Fun}_{\mathrm{R}^{\mathrm{op}}}(\mathrm{K}, \mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)))) \simeq$$

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}} \times_{\mathrm{R}^{\mathrm{op}}} \mathrm{K}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))) \simeq$$
$$\mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{H}_*(\mathrm{S}^{\mathrm{op}} \times_{\mathrm{R}^{\mathrm{op}}} \mathrm{K}), \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) \simeq (\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}\times_{\mathrm{R}}\mathrm{K}^{\mathrm{op}}})_{/\mathcal{D}\times_{\mathrm{S}}(\mathrm{S}\times_{\mathrm{R}}\mathrm{K}^{\mathrm{op}})}$$
$$\simeq (\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}\times_{\mathrm{R}}\mathrm{K}^{\mathrm{op}}})_{/\mathcal{D}\times_{\mathrm{R}}\mathrm{K}^{\mathrm{op}}} \subset {\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{K}^{\mathrm{op}}}}_{/\mathcal{D}\times_{\mathrm{R}}\mathrm{K}^{\mathrm{op}}} \simeq$$
$$\mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{H}'_*(\mathrm{K}), \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) \simeq \mathrm{Fun}_{\mathrm{R}^{\mathrm{op}}}(\mathrm{K}, \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)))$$

natural in a small category K *over* R^{op}.

Remark 5.17. *If* R *is contractible, the canonical subcategory inclusion*

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))) \subset \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) \simeq$$
$$\{\mathcal{D}\} \times_{\mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \simeq \mathsf{Cat}_{\infty/\mathcal{D}}$$

is the composition

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))) \simeq \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{H}_*(\mathrm{S}^{\mathrm{op}}), \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$
$$\simeq (\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}})_{/\mathcal{D}} \subset (\mathsf{Cat}_{\infty/\mathrm{S}})_{/\mathcal{D}} \simeq \mathsf{Cat}_{\infty/\mathcal{D}}.$$

Proof. The assertion of the lemma follows tautologically from the definition of the canonical subcategory inclusion

$$\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))) \subset \mathrm{H}'^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$

over R^{op}.

□

5.1.5 $\mathrm{LM}^{\otimes}$-monoidal categories of sections

Let $\mathcal{D} \to \mathrm{T}, \mathrm{T} \to \mathrm{S}$ be functors such that the composition $\mathcal{D} \to \mathrm{T} \to \mathrm{S}$ is flat.

Let $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ be a T-family of operads over $\mathrm{LM}^{\otimes}$.

Set $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$ and $\mathcal{B} := \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$.

We set

$$\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})^{\otimes} := \mathrm{Fun}_{\mathrm{T} \times \mathrm{LM}^{\otimes}}^{/\mathrm{S} \times \mathrm{LM}^{\otimes}}(\mathcal{D} \times \mathrm{LM}^{\otimes}, \mathcal{M}^{\otimes})$$

and

$$\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{C})^{\otimes} := \mathrm{Fun}_{\mathrm{T} \times \mathrm{Ass}^{\otimes}}^{/\mathrm{S} \times \mathrm{Ass}^{\otimes}}(\mathcal{D} \times \mathrm{Ass}^{\otimes}, \mathcal{C}^{\otimes}).$$

We have canonical equivalences

$$\mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})^{\otimes} \simeq \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{C})^{\otimes}$$

over $\mathrm{S} \times \mathrm{Ass}^{\otimes}$ and

$$\{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})^{\otimes} \simeq \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})$$

over S and for every functor $\mathrm{S}' \to \mathrm{S}$ a canonical equivalence

$$\mathrm{S}' \times_{\mathrm{S}} \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})^{\otimes} \simeq \mathrm{Fun}_{\mathrm{S}' \times_{\mathrm{S}} \mathrm{T}}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{D}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{B})^{\otimes}$$

over $\mathrm{S}' \times \mathrm{LM}^{\otimes}$.

Remark 5.18.
Let $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ be a $\mathrm{LM}^{\otimes}$-monoidal category over T classifying a $\mathrm{LM}^{\otimes}$-monoid ϕ of $\mathsf{Cat}_{\infty/\mathrm{T}}$.

Theorem 5.23 implies that the image of ϕ under the finite products preserving functor $\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}, -) : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ is classified by the $\mathrm{LM}^{\otimes}$-monoidal category

$$\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{T}, \mathcal{B})^{\otimes} = \mathrm{Fun}_{\mathrm{T} \times \mathrm{LM}^{\otimes}}^{/\mathrm{S} \times \mathrm{LM}^{\otimes}}(\mathrm{T} \times \mathrm{LM}^{\otimes}, \mathcal{M}^{\otimes}) \to \mathrm{LM}^{\otimes}$$

over S.

Remark 5.10 specializes to the following:

Given a T-family $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ the functor

$$\mathrm{Fun}_{\mathrm{T}}(\mathrm{T}, \mathcal{B})^{\otimes} = \mathrm{Fun}_{\mathrm{T} \times \mathrm{LM}^{\otimes}}^{/\mathrm{LM}^{\otimes}}(\mathrm{T} \times \mathrm{LM}^{\otimes}, \mathcal{M}^{\otimes}) \to \mathrm{LM}^{\otimes}$$

is a map of operads that is a (locally) cocartesian fibration if $\mathcal{M}^{\otimes} \to \mathrm{T} \times \mathrm{LM}^{\otimes}$ is a T-family of representable $\mathrm{LM}^{\otimes}$-operads respectively $\mathrm{LM}^{\otimes}$-monoidal categories.

Remark 5.19. *1. By remark 5.11 we have a canonical equivalence*

$$\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})) = \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathrm{Alg}_{\mathrm{LM}^{\otimes}/\mathrm{LM}^{\otimes}}^{/\mathrm{T}}(\mathcal{M})) \simeq$$

$$\mathrm{Alg}_{\mathrm{LM}^{\otimes}/\mathrm{LM}^{\otimes}}^{/\mathrm{S}}(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})) = \mathrm{LMod}^{/\mathrm{S}}(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B}))$$

over $\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{B})$, whose pullback along the canonical functor $\mathrm{Ass}^{\otimes} \to \mathrm{LM}^{\otimes}$ is the canonical equivalence

$$\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C})) \simeq \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D}, \mathcal{C}))$$

over S *and such that we have a commutative square*

$$\begin{array}{ccc} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})) & \xrightarrow{\simeq} & \mathrm{LMod}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C})) & \xrightarrow{\simeq} & \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{C})) \end{array} \tag{20}$$

of categories over S.

Let A *be a section of* $\mathrm{Alg}^{/\mathrm{T}}(\mathcal{C}) \to \mathrm{T}$ *and* A′ *the section of* $\mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{C})) \to \mathrm{S}$ *corresponding to the composition* $\mathcal{D} \to \mathrm{T} \xrightarrow{\mathrm{A}} \mathrm{Alg}^{/\mathrm{T}}(\mathcal{C})$ *of functors over* T.

Square 20 induces an equivalence

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{LMod}^{/\mathrm{T}}_{\mathrm{A}}(\mathcal{B})) \simeq \mathrm{LMod}^{/\mathrm{S}}_{\mathrm{A}'}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})).$$

2. *Especially we are interested in the following situation:*

Let $\mathrm{T} \to \mathrm{S}$ *be a functor,* $\mathcal{A}^{\otimes} \to \mathrm{Ass}^{\otimes} \times \mathrm{S}$ *a monoidal category over* S *and* $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{T}$ *a* $\mathrm{LM}^{\otimes}$*-monoidal category over* T *that exhibits a category* $\mathcal{B} \to \mathrm{T}$ *over* T *as a left module over the category* $\mathrm{T} \times_{\mathrm{S}} \mathcal{A}$ *over* T.

We have a canonical diagonal monoidal functor

$$\delta : \mathcal{A}^{\otimes} \simeq \mathrm{Map}_{\mathrm{S}}(\mathrm{S}, \mathcal{A})^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{D}, \mathcal{A})^{\otimes} \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{T} \times_{\mathrm{S}} \mathcal{A})^{\otimes}$$

over S *that induces a functor*

$$\mathrm{Alg}^{/\mathrm{S}}(\delta) : \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Map}_{\mathrm{S}}(\mathcal{D}, \mathcal{A})) \simeq \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{T} \times_{\mathrm{S}} \mathcal{A}))$$
$$\simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{T} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}))$$

over S *that is equivalent over* S *to the diagonal functor*

$$\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{D}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})) \simeq \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{T} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}))$$

over S.
Pulling back the $\mathrm{LM}^{\otimes}$*-monoidal category* $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})^{\otimes}$ *over* S *along* δ *we obtain a* $\mathrm{LM}^{\otimes}$*-monoidal category* $\delta^*(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})^{\otimes})$ *over* S *that exhibits* $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})$ *as a left module over* $\mathcal{A}$.

Square 20 specializes to the commutative square

$$\begin{array}{ccc} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})) & \xrightarrow{\simeq} & \mathrm{LMod}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{S}}(\mathcal{D}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})) & \xrightarrow{\simeq} & \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Map}_{\mathrm{S}}(\mathcal{D}, \mathcal{A})) \end{array} \tag{21}$$

of categories over S.
Pulling back square 21 along the functor $\mathrm{Alg}^{/\mathrm{S}}(\delta) : \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{D}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}))$ *over* S *we obtain a canonical equivalence*

$$\mathrm{Alg}^{/\mathrm{S}}(\delta)^*(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))) \simeq \mathrm{LMod}^{/\mathrm{S}}(\delta^*(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})))$$

over $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}) \times_{\mathrm{S}} \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}, \mathcal{B})$.

3. *We have a canonical equivalence*

$$\mathrm{Alg}^{/\mathrm{S}}(\delta)^*(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))) \simeq$$

$$\mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}^{/\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathcal{D}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}),\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))$$

over $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})$ *represented by the following canonical equivalence natural in every functor* $\alpha:\mathrm{K}\to\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})$ *:*

$$\mathrm{Fun}_{\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathrm{K},\mathrm{Alg}^{/\mathrm{S}}(\delta)^*(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})))) \simeq$$

$$\mathrm{Fun}_{\mathrm{Map}_{\mathrm{S}}(\mathcal{D},\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}))}(\mathrm{Alg}^{/\mathrm{S}}(\delta)_*(\mathrm{K}),\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))) \simeq$$

$$\{\delta\circ\alpha\}\times_{\mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Map}_{\mathrm{S}}(\mathcal{D},\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})))}\mathrm{Fun}_{\mathrm{S}}(\mathrm{K},\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))) \simeq$$

$$\{\alpha\circ\mathrm{p}\}\times_{\mathrm{Fun}_{\mathrm{S}}(\mathcal{D}\times_{\mathrm{S}}\mathrm{K},\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}))}\mathrm{Fun}_{\mathrm{T}}(\mathcal{D}\times_{\mathrm{S}}\mathrm{K},\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})) \simeq$$

$$\mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathcal{D}\times_{\mathrm{S}}\mathrm{K},\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})) \simeq$$

$$\mathrm{Fun}_{\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathrm{K},\mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}^{/\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathcal{D}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}),\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))),$$

where $\mathrm{p}:\mathcal{D}\times_{\mathrm{S}}\mathrm{K}\to\mathrm{K}$ *denotes the canonical functor.*

So we get a canonical equivalence

$$\Psi:\mathrm{LMod}^{/\mathrm{S}}(\delta^*(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathcal{B})))\simeq\mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}^{/\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathcal{D}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}),\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B}))$$

over $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})$ *such that we have a commutative square*

$$\begin{array}{ccc}
\mathrm{LMod}^{/\mathrm{S}}(\delta^*(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathcal{B}))) & \longrightarrow & \mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}^{/\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathcal{D}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}),\mathrm{LMod}^{/\mathrm{T}}(\mathcal{B})) \\
\downarrow & & \downarrow \\
\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})\times_{\mathrm{S}}\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathcal{B}) & \longrightarrow & \mathrm{Fun}_{\mathrm{T}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}^{/\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})}(\mathcal{D}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}),\mathcal{B}\times_{\mathrm{S}}\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A}))
\end{array} \quad (22)$$

of categories over S *with horizontal functors equivalences.*

The pullback of Ψ *along a section* A *of* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{A})\to\mathrm{S}$ *is a canonical equivalence*

$$\mathrm{LMod}_{\mathrm{A}}^{/\mathrm{S}}(\delta^*(\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathcal{B})))\simeq\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{D},\mathrm{LMod}_{\mathrm{A}}^{/\mathrm{T}}(\mathcal{B}))$$

over S.

Let R, S, T be categories and $\mathrm{T}\to\mathrm{R},\alpha:\mathrm{X}\to\mathrm{S}\times\mathrm{T},\beta:\mathrm{Y}\to\mathrm{S}\times\mathrm{T}$ be functors.

If the composition $\mathrm{Y}\to\mathrm{S}\times\mathrm{T}\to\mathrm{S}\times\mathrm{R}$ is a flat functor, there is a functor $\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{Y},\mathrm{X})\to\mathrm{S}\times\mathrm{R}$.

If $\alpha:\mathrm{X}\to\mathrm{S}\times\mathrm{T}$ is a map of cartesian fibrations over S and $\beta:\mathrm{Y}\to\mathrm{S}\times\mathrm{T}$ a map of cocartesian fibrations over S, the functor $\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{Y},\mathrm{X})\to\mathrm{S}\times\mathrm{R}$ is a map of cartesian fibrations over S.

We complete this subsection by showing the following classification result (theorem 5.23):

If $\alpha : X \to S \times T$ classifies a functor $F : S^{op} \to \mathsf{Cat}_{\infty/T}$ and $\beta : Y \to S \times T$ classifies a functor $G : S \to \mathsf{Cat}^{fl/R}_{\infty/T} \subset \mathsf{Cat}_{\infty/T}$, the map

$$\mathrm{Fun}^{/S\times R}_{S\times T}(Y, X) \to S \times R$$

of cartesian fibrations over S classifies the functor

$$S^{op} \xrightarrow{(G^{op}, F)} (\mathsf{Cat}^{fl/R}_{\infty/T})^{op} \times \mathsf{Cat}_{\infty/T} \xrightarrow{\mathrm{Fun}^{/R}_{T}(-,-)} \mathsf{Cat}_{\infty/R}.$$

To prove theorem 5.23 we show the following proposition:

There is a canonical equivalence

$$\mathrm{Cocart} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)$$

of cartesian fibrations over Cat_∞ that restricts to an equivalence

$$\mathcal{L} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathcal{S})$$

of cartesian fibrations over Cat_∞ (proposition 5.20).

Proposition 5.20. *There is a canonical equivalence*

$$\mathrm{Cocart} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)$$

of cartesian fibrations over Cat_∞ *that induces on the fiber over every small category* $\mathcal{C}$ *the canonical equivalence* $\mathsf{Cat}^{cocart}_{\infty/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_\infty)$.

Consequently this equivalence restricts to an equivalence

$$\mathcal{L} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathcal{S})$$

of cartesian fibrations over Cat_∞.

Proof. By Yoneda it is enough to show that for every (large) category S over Cat_∞ there is a bijection between equivalence classes of functors
$S \to \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)$ over Cat_∞ and equivalence classes of functors $S \to \mathrm{Cocart}$ over Cat_∞ such that for every functor $\phi : T \to S$ over Cat_∞ the square

$$\begin{array}{ccc}
\mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)) & \longrightarrow & \mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \mathrm{Cocart}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_{\mathsf{Cat}_\infty}(T, \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)) & \longrightarrow & \mathrm{Fun}_{\mathsf{Cat}_\infty}(T, \mathrm{Cocart})
\end{array}$$

commutes on equivalence classes.
We have a canonical equivalence

$$\widehat{\mathsf{Cat}_\infty}_{/\mathsf{Cat}_\infty}(-, \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)) \simeq \widehat{\mathsf{Cat}_\infty}_{/\mathsf{Cat}_\infty}(- \times_{\mathsf{Cat}_\infty} \mathcal{U}, \mathsf{Cat}_\infty \times \mathsf{Cat}_\infty)$$
$$\simeq \widehat{\mathsf{Cat}_\infty}(- \times_{\mathsf{Cat}_\infty} \mathcal{U}, \mathsf{Cat}_\infty)$$

of functors $(\widehat{\mathsf{Cat}_\infty}_{/\mathsf{Cat}_\infty})^{op} \to \widehat{\mathcal{S}}$.

Consequently it is enough to see that for every functor $\varphi : S \to \mathsf{Cat}_\infty$ there is a bijection between equivalence classes of functors $S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to$

Cat_∞ and equivalence classes of functors $S \to \mathrm{Cocart}$ over Cat_∞ such that for every functor $\phi : T \to S$ over Cat_∞ the square

$$\begin{array}{ccc} \mathrm{Fun}(S \times_{\mathsf{Cat}_\infty} \mathcal{U}, \mathsf{Cat}_\infty) & \longrightarrow & \mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \mathrm{Cocart}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(T \times_{\mathsf{Cat}_\infty} \mathcal{U}, \mathsf{Cat}_\infty) & \longrightarrow & \mathrm{Fun}_{\mathsf{Cat}_\infty}(T, \mathrm{Cocart}) \end{array}$$

commutes on equivalence classes.

Being a right fibration the forgetful functor $\widehat{\mathsf{Cat}_\infty}_{/S} \to \widehat{\mathsf{Cat}_\infty}$ induces an equivalence $(\widehat{\mathsf{Cat}_\infty}_{/S})_{/S \times_{\mathsf{Cat}_\infty} \mathcal{U}} \simeq \widehat{\mathsf{Cat}_\infty}_{/S \times_{\mathsf{Cat}_\infty} \mathcal{U}}$.

The fully faithful map

$$\mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \widehat{\mathrm{Cocart}})^{\simeq} \subset \mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \mathrm{Fun}(\Delta^1, \widehat{\mathsf{Cat}_\infty}))^{\simeq} \simeq$$

$$((\widehat{\mathsf{Cat}}^{\mathrm{cocart}}_{\infty/S})_{/S \times_{\mathsf{Cat}_\infty} \mathcal{U}})^{\simeq} \subset ((\widehat{\mathsf{Cat}_\infty}_{/S})_{/S \times_{\mathsf{Cat}_\infty} \mathcal{U}})^{\simeq} \simeq (\widehat{\mathsf{Cat}_\infty}_{/S \times_{\mathsf{Cat}_\infty} \mathcal{U}})^{\simeq}$$

has essential image the space $(\widehat{\mathsf{Cat}}^{\mathrm{cocart}}_{\infty/S \times_{\mathsf{Cat}_\infty} \mathcal{U}})^{\simeq} \simeq \mathrm{Fun}(S \times_{\mathsf{Cat}_\infty} \mathcal{U}, \widehat{\mathsf{Cat}_\infty})^{\simeq}$:

A functor $Y \to S \times_{\mathsf{Cat}_\infty} \mathcal{U}$ is a cocartesian fibration if and only if it is a map of cocartesian fibrations over S classifying a natural transformation $S \to \mathrm{Fun}(\Delta^1, \widehat{\mathsf{Cat}_\infty})$ of functors $S \to \widehat{\mathsf{Cat}_\infty}$ with target φ that factors through the subcategory $\widehat{\mathrm{Cocart}} \subset \mathrm{Fun}(\Delta^1, \widehat{\mathsf{Cat}_\infty})$.

So we get an equivalence $\mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \widehat{\mathrm{Cocart}})^{\simeq} \simeq \mathrm{Fun}(S \times_{\mathsf{Cat}_\infty} \mathcal{U}, \widehat{\mathsf{Cat}_\infty})^{\simeq}$ that restricts to an equivalence $\mathrm{Fun}_{\mathsf{Cat}_\infty}(S, \mathrm{Cocart})^{\simeq} \simeq \mathrm{Fun}(S \times_{\mathsf{Cat}_\infty} \mathcal{U}, \mathsf{Cat}_\infty)^{\simeq}$.

Given a functor $\phi : T \to S$ over Cat_∞ and a cocartesian fibration $X \to S \times_{\mathsf{Cat}_\infty} \mathcal{U}$ classifying a functor $S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to \mathsf{Cat}_\infty$ the composition $T \times_{\mathsf{Cat}_\infty} \mathcal{U} \to S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to \mathsf{Cat}_\infty$ is classified by the pullback of the cocartesian fibration $X \to S \times_{\mathsf{Cat}_\infty} \mathcal{U}$ along the functor $T \times_{\mathsf{Cat}_\infty} \mathcal{U} \to S \times_{\mathsf{Cat}_\infty} \mathcal{U}$.
Therefore if α denotes the natural transformation of functors $S \to \mathsf{Cat}_\infty$ with target φ corresponding to the functor $S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to \mathsf{Cat}_\infty$ then $\alpha \circ \phi$ is the natural transformation of functors $T \to \mathsf{Cat}_\infty$ with target $\varphi \circ \phi$ corresponding to the composition $T \times_{\mathsf{Cat}_\infty} \mathcal{U} \to S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to \mathsf{Cat}_\infty$.

So the functor $T \to \mathrm{Cocart}$ over Cat_∞ corresponding to the composition $T \times_{\mathsf{Cat}_\infty} \mathcal{U} \to S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to \mathsf{Cat}_\infty$ is the composition $T \xrightarrow{\phi} S \to \mathrm{Cocart}$ of $\phi : T \to S$ and the functor $S \to \mathrm{Cocart}$ over Cat_∞ corresponding to the functor $S \times_{\mathsf{Cat}_\infty} \mathcal{U} \to \mathsf{Cat}_\infty$.

2. follows from the fact that a cocartesian fibration is a left fibration if and only if all its fibers are spaces.

□

Remark 5.21. *Let* T *be a small category.*
There is a canonical equivalence

$$\mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times T)^{\simeq} \simeq \mathsf{Cat}_{\infty/T}$$

of right fibrations over Cat_∞ *represented by the following equivalence*

$$\mathrm{Fun}_{\mathsf{Cat}_\infty}(K, \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times T)^{\simeq}) \simeq \mathrm{Fun}^{\mathrm{cocart}}_K(K \times_{\mathsf{Cat}_\infty} \mathcal{U}, K \times T)^{\simeq}$$

$$\simeq \mathrm{Fun}(K, \mathsf{Cat}_\infty)(\varphi, \delta(T)) \simeq \mathrm{Fun}_{\mathsf{Cat}_\infty}(K, \mathsf{Cat}_{\infty/T})$$

natural in $\varphi : K \to \mathsf{Cat}_\infty$.

So given a cocartesian fibration $\mathrm{X} \to \mathrm{S}$ *classifying a functor* $\mathrm{H} : \mathrm{S} \to \mathsf{Cat}_\infty$ *we have a canonical equivalence*

$$\mathrm{Map}_{\mathrm{S}}(\mathrm{X}, \mathrm{S} \times \mathrm{T})^{\simeq} \simeq \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathsf{Cat}_{\infty/\mathrm{T}}$$

of right fibrations over S.

Thus the right fibration $\mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathrm{T})^{\simeq} \to \mathsf{Cat}_\infty$ *classifies the functor* $\mathsf{Cat}_\infty(-, \mathrm{T}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \mathcal{S}$.

Enlarging the universe we have a canonical equivalence

$$\mathrm{Map}_{\widehat{\mathsf{Cat}_\infty}}(\widehat{\mathcal{U}}, \widehat{\mathsf{Cat}_\infty} \times \mathrm{T})^{\simeq} \simeq \widehat{\mathsf{Cat}_{\infty/\mathrm{T}}}$$

of right fibrations over $\widehat{\mathsf{Cat}_\infty}$, *where* T *is not necessarily small and so especially a canonical equivalence*

$$\mathcal{L}^{\simeq} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathcal{S})^{\simeq} \simeq \mathsf{Cat}_\infty \times_{\widehat{\mathsf{Cat}_\infty}} \widehat{\mathsf{Cat}_{\infty/\mathcal{S}}}$$

of right fibrations over Cat_∞.

Thus the right fibration $\mathcal{L}^{\simeq} \to \mathsf{Cat}_\infty$ *classifies the functor* $\mathrm{Fun}(-, \mathcal{S})^{\simeq} : \mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathcal{S}}$.

By [10] corollary A.31. the cartesian fibration $\mathcal{L} \to \mathsf{Cat}_\infty$ *classifies the functor* $\mathrm{Fun}(-, \mathcal{S}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathcal{S}}$.

So especially the cartesian fibration $\mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathcal{S}) \to \mathsf{Cat}_\infty$ *classifies the functor* $\mathrm{Fun}(-, \mathcal{S}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathcal{S}}$.

Remark 5.22. *By proposition 6.9 we have a canonical fully faithful map* $\mathcal{U} \subset \mathcal{R}$ *of cocartesian fibrations over* Cat_∞ *and by proposition 5.20 we have a canonical equivalence* $\mathcal{L} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathcal{S})$ *over* Cat_∞, *whose pullback along the involution* $(-)^{\mathrm{op}} : \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ *is a canonical equivalence*

$$\mathcal{R} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}^{\mathrm{rev}}, \mathsf{Cat}_\infty \times \mathcal{S})$$

over Cat_∞. *So we obtain a canonical fully faithful map*

$$\chi : \mathcal{U} \subset \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}^{\mathrm{rev}}, \mathsf{Cat}_\infty \times \mathcal{S})$$

of cocartesian fibrations over Cat_∞.

Let $\mathcal{E} \to \mathrm{S}$ *be a cocartesian fibration classifying a functor* $\phi : \mathrm{S} \to \mathsf{Cat}_\infty$.

Pulling back χ *along* $\phi : \mathrm{S} \to \mathsf{Cat}_\infty$ *we get a fully faithful map* $\mathcal{E} \subset \mathcal{P}^{/\mathrm{S}}(\mathcal{E}) := \mathrm{Map}_{\mathrm{S}}(\mathcal{E}^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S})$ *of cocartesian fibrations over* S *adjoint to a functor* $\alpha : \mathcal{E}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{E} \to \mathcal{S}$ *such that for every* $\mathsf{s} \in \mathrm{S}$ *the composition* $(\mathcal{E}_{\mathsf{s}})^{\mathrm{op}} \times \mathcal{E}_{\mathsf{s}} \to \mathcal{E}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{E} \xrightarrow{\alpha} \mathcal{S}$ *is the mapping space functor of* $\mathcal{E}_{\mathsf{s}}$.

We call α *the mapping space functor of* $\mathcal{E} \to \mathrm{S}$ *relative to* S.

Theorem 5.23. *Let* R, S, T *be categories,* $\mathrm{T} \to \mathrm{R}$ *a functor,* $\alpha : \mathrm{X} \to \mathrm{S} \times \mathrm{T}$ *a map of cartesian fibrations over* S *and* $\beta : \mathrm{Y} \to \mathrm{S} \times \mathrm{T}$ *a map of cocartesian fibrations over* S *corresponding to functors* $\mathrm{F} : \mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathrm{T}}$ *respectively* $\mathrm{G} : \mathrm{S} \to \mathsf{Cat}_{\infty/\mathrm{T}}$.

Assume that the composition $\mathrm{X} \to \mathrm{S} \times \mathrm{T} \to \mathrm{S} \times \mathrm{R}$ *is a flat functor so that* F *induces a functor* $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathrm{T}}^{\mathrm{fl}/\mathrm{R}}$.

The map

$$\mathrm{Fun}_{\mathrm{S} \times \mathrm{T}}^{/\mathrm{S} \times \mathrm{R}}(\mathrm{X}, \mathrm{Y}) \to \mathrm{S} \times \mathrm{R}$$

of cocartesian fibrations over S *classifies the functor*

$$\mathrm{S} \xrightarrow{(\mathrm{F}^{\mathrm{op}},\mathrm{G})} (\mathsf{Cat}^{\mathrm{fl}/\mathrm{R}}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathrm{Fun}^{/\mathrm{R}}_{\mathrm{T}}(-,-)} \mathsf{Cat}_{\infty/\mathrm{R}}.$$

Dually, assume that the composition $\mathrm{Y} \to \mathrm{S}\times\mathrm{T} \to \mathrm{S}\times\mathrm{R}$ *is a flat functor so that* G *induces a functor* $\mathrm{S} \to \mathsf{Cat}^{\mathrm{fl}/\mathrm{R}}_{\infty/\mathrm{T}}$.

The map

$$\mathrm{Fun}^{/\mathrm{S}\times\mathrm{R}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{Y},\mathrm{X}) \to \mathrm{S}\times\mathrm{R}$$

of cartesian fibrations over S *classifies the functor*

$$\mathrm{S}^{\mathrm{op}} \xrightarrow{(\mathrm{G}^{\mathrm{op}},\mathrm{F})} (\mathsf{Cat}^{\mathrm{fl}/\mathrm{R}}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathrm{Fun}^{/\mathrm{R}}_{\mathrm{T}}(-,-)} \mathsf{Cat}_{\infty/\mathrm{R}}.$$

Proof. We prove the second statement, the first is dual to the second by the following consideration:

By the second part the map

$$\mathrm{Fun}^{/\mathrm{S}\times\mathrm{R}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{X},\mathrm{Y})^{\mathrm{op}} \simeq \mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}\times\mathrm{R}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}\times\mathrm{T}^{\mathrm{op}}}(\mathrm{X}^{\mathrm{op}},\mathrm{Y}^{\mathrm{op}}) \to \mathrm{S}^{\mathrm{op}}\times\mathrm{R}^{\mathrm{op}}$$

of cartesian fibrations over S^{op} classifies the functor

$$\mathrm{S} \xrightarrow{(\mathrm{F}^{\mathrm{op}},\mathrm{G})} (\mathsf{Cat}^{\mathrm{fl}/\mathrm{R}}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{(-)^{\mathrm{op}}\times(-)^{\mathrm{op}}} (\mathsf{Cat}^{\mathrm{fl}/\mathrm{R}^{\mathrm{op}}}_{\infty/\mathrm{T}^{\mathrm{op}}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}^{\mathrm{op}}}$$

$$\xrightarrow{\mathrm{Fun}^{/\mathrm{R}^{\mathrm{op}}}_{\mathrm{T}^{\mathrm{op}}}(-,-)} \mathsf{Cat}_{\infty/\mathrm{R}^{\mathrm{op}}}$$

being equivalent to the functor $\mathrm{S} \xrightarrow{(\mathrm{F}^{\mathrm{op}},\mathrm{G})} (\mathsf{Cat}^{\mathrm{fl}/\mathrm{R}}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathrm{Fun}^{/\mathrm{R}}_{\mathrm{T}}(-,-)} \mathsf{Cat}_{\infty/\mathrm{R}} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/\mathrm{R}^{\mathrm{op}}}$.

Hence the map $\mathrm{Fun}^{/\mathrm{S}\times\mathrm{R}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{X},\mathrm{Y}) \to \mathrm{S}\times\mathrm{R}$ of cocartesian fibrations over S classifies the functor $\mathrm{S} \xrightarrow{(\mathrm{F}^{\mathrm{op}},\mathrm{G})} (\mathsf{Cat}^{\mathrm{fl}/\mathrm{R}}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathrm{Fun}^{/\mathrm{R}}_{\mathrm{T}}(-,-)} \mathsf{Cat}_{\infty/\mathrm{R}}$.

We will divide the proof into the following reduction steps:

1. The right fibration $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{Y},\mathrm{X})^{\simeq} \to \mathrm{S}$ classifies the functor
$$\mathrm{S}^{\mathrm{op}} \xrightarrow{(\mathrm{G}^{\mathrm{op}},\mathrm{F})} (\mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathsf{Cat}_{\infty/\mathrm{T}}(-,-)} \mathcal{S}.$$
2. T is contractible: the right fibration $\mathrm{Map}_{\mathrm{S}}(\mathrm{Y},\mathrm{X})^{\simeq} \to \mathrm{S}$ classifies the functor $\mathrm{S}^{\mathrm{op}} \xrightarrow{(\mathrm{G}^{\mathrm{op}},\mathrm{F})} \mathsf{Cat}^{\mathrm{op}}_{\infty} \times \mathsf{Cat}_{\infty} \xrightarrow{\mathsf{Cat}_{\infty}(-,-)} \mathcal{S}$.
3. $\mathrm{X} \to \mathrm{S}$ is equivalent over S to $\mathrm{Map}_{\mathrm{S}}(\mathrm{X}'^{\mathrm{rev}}, \mathrm{S}\times\mathcal{S})$ for some bicartesian fibration $\mathrm{X}' \to \mathrm{S}$, where $\mathrm{X}'^{\mathrm{rev}}$ denotes the fiberwise dual of the cocartesian fibration $\mathrm{X}' \to \mathrm{S}$.

1: We reduce the statement to 1:
Denote Ψ the functor $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathrm{R}}$ classified by the map

$$\mathrm{Fun}^{/\mathrm{S}\times\mathrm{R}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{Y},\mathrm{X}) \to \mathrm{S}\times\mathrm{R}$$

of cartesian fibrations over S.

We want to find an equivalence $\Psi \simeq \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{R}}(-,-) \circ (\mathrm{G}^{\mathrm{op}}, \mathrm{F})$ of functors $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathrm{R}}$.

Such an equivalence is represented by an equivalence of functors $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathrm{R}} \subset \mathrm{Fun}((\mathsf{Cat}_{\infty/\mathrm{R}})^{\mathrm{op}}, \mathcal{S})$ adjoint to an equivalence

$$\mathsf{Cat}_{\infty/\mathrm{R}}(\mathcal{B}, \Psi(\mathsf{s})) \simeq \mathsf{Cat}_{\infty/\mathrm{R}}(\mathcal{B}, \mathrm{Fun}_{\mathrm{T}}^{/\mathrm{R}}(\mathrm{G}(\mathsf{s}), \mathrm{F}(\mathsf{s})))$$

$$\simeq \mathsf{Cat}_{\infty/\mathrm{T}}(\mathcal{B} \times_{\mathrm{R}} \mathrm{G}(\mathsf{s}), \mathrm{F}(\mathsf{s}))$$

natural in $\mathcal{B} \in \mathsf{Cat}_{\infty/\mathrm{R}}$ and $\mathsf{s} \in \mathrm{S}$.

In other words we want to see that both functors

$$\alpha : (\mathsf{Cat}_{\infty/\mathrm{R}})^{\mathrm{op}} \times \mathrm{S}^{\mathrm{op}} \xrightarrow{\mathrm{id}\times\Psi} (\mathsf{Cat}_{\infty/\mathrm{R}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{R}} \xrightarrow{\mathsf{Cat}_{\infty/\mathrm{R}}(-,-)} \mathsf{Cat}_{\infty},$$

$$\beta : (\mathsf{Cat}_{\infty/\mathrm{R}})^{\mathrm{op}} \times \mathrm{S}^{\mathrm{op}} \xrightarrow{\mathrm{id}\times(\mathrm{G}^{\mathrm{op}},\mathrm{F})} (\mathsf{Cat}_{\infty/\mathrm{R}})^{\mathrm{op}} \times (\mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{(-\times_{\mathrm{R}}-)\times\mathrm{id}}$$

$$(\mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathsf{Cat}_{\infty/\mathrm{T}}(-,-)} \mathsf{Cat}_{\infty}$$

are equivalent.

Denote $\mathcal{U}_{\mathrm{R}} \to \mathrm{R} \times \mathsf{Cat}_{\infty/\mathrm{R}}$ the map of cocartesian fibrations over $\mathsf{Cat}_{\infty/\mathrm{R}}$ classifying the identity of $\mathsf{Cat}_{\infty/\mathrm{R}}$.

If 1. is shown, the right fibration

$$\mathrm{Fun}_{\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{R}}^{/\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}}(\mathcal{U}_{\mathrm{R}} \times \mathrm{S}, \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{Y},\mathrm{X}))^{\simeq} \to \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{S}$$

classifies the functor α and the right fibration

$$\mathrm{Fun}_{\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{T}}^{/\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}}((\mathcal{U}_{\mathrm{R}} \times \mathrm{S}) \times_{(\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{R})} (\mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{Y}), \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{X})^{\simeq}$$

$$\to \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{S}$$

classifies the functor β.

We have a canonical equivalence

$$\mathrm{Fun}_{\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{R}}^{/\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}}(\mathcal{U}_{\mathrm{R}} \times \mathrm{S}, \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{Y},\mathrm{X})) \simeq$$

$$\mathrm{Fun}_{\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{R}}^{/\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}}(\mathcal{U}_{\mathrm{R}} \times \mathrm{S}, \mathrm{Fun}_{\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{T}}^{/\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{R}}(\mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{Y}, \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{X})) \simeq$$

$$\mathrm{Fun}_{\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{T}}^{/\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}}((\mathcal{U}_{\mathrm{R}} \times \mathrm{S}) \times_{(\mathsf{Cat}_{\infty/\mathrm{R}}\times\mathrm{S}\times\mathrm{R})} (\mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{Y}), \mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{X})$$

over $\mathsf{Cat}_{\infty/\mathrm{R}} \times \mathrm{S}$.

2: As next we reduce to 2:

We have a pullback square

$$\begin{array}{ccc}
\mathsf{Cat}_{\infty/\mathrm{T}}(-,-) \circ (\mathrm{G}^{\mathrm{op}}, \mathrm{F}) & \longrightarrow & \mathsf{Cat}_{\infty}(-,-) \circ (\mathrm{G}^{\mathrm{op}}, \mathrm{F}) \\
\downarrow & & \downarrow \\
* \simeq \mathsf{Cat}_{\infty/\mathrm{T}}(-,-) \circ (\mathrm{G}^{\mathrm{op}}, \mathrm{T}) & \xrightarrow{\phi} & \mathsf{Cat}_{\infty}(-,-) \circ (\mathrm{G}^{\mathrm{op}}, \mathrm{T})
\end{array}$$

of functors $\mathrm{S}^{\mathrm{op}} \to \mathcal{S}$.

The induced map ϕ on mapping spaces is classified by the canonical functor

$$\beta : \mathrm{S} \simeq \mathrm{S} \times_{\mathsf{Cat}_{\infty/\mathrm{T}}} (\mathsf{Cat}_{\infty/\mathrm{T}})_{/\mathrm{T}} \to \mathrm{S} \times_{\mathsf{Cat}_{\infty}} \mathsf{Cat}_{\infty/\mathrm{T}}$$

over S that is adjoint to $\mathrm{G} : \mathrm{S} \to \mathsf{Cat}_{\infty/\mathrm{T}}$.

Thus under the canonical equivalence

$$S \times_{\mathsf{Cat}_\infty} \mathsf{Cat}_{\infty/T} \simeq \mathrm{Map}_S(Y, S \times T)^\simeq$$

over S the functor β corresponds to the section γ of $\mathrm{Map}_S(Y, S \times T)^\simeq \to S$ corresponding to the map $Y \to S \times T$ of cocartesian fibrations over S classifying the functor $G : S \to \mathsf{Cat}_{\infty/T}$.

Hence ϕ is classified by $\gamma : S \to \mathrm{Map}_S(Y, S \times T)^\simeq$.

Consequently if the right fibration $\mathrm{Map}_S(Y, X)^\simeq \to S$ classifies the functor $S^{op} \xrightarrow{(G^{op}, F)} \mathsf{Cat}_\infty^{op} \times \mathsf{Cat}_\infty \xrightarrow{\mathsf{Cat}_\infty(-,-)} \mathcal{S}$ and the canonical map $\mathrm{Map}_S(Y, X)^\simeq \to \mathrm{Map}_S(Y, S \times T)^\simeq$ of right fibrations over S classifies the natural transformation induced by the unique natural transformation $F \to T$ to the constant functor $S^{op} \to \mathsf{Cat}_{\infty/T}$ with image T, the right fibration

$$\mathrm{Fun}^{/S}_{S \times T}(Y, X)^\simeq \simeq S \times_{\mathrm{Map}_S(Y, S \times T)^\simeq} \mathrm{Map}_S(Y, X)^\simeq \to S$$

classifies the functor $S^{op} \xrightarrow{(G^{op}, F)} (\mathsf{Cat}_{\infty/T})^{op} \times \mathsf{Cat}_{\infty/T} \xrightarrow{\mathsf{Cat}_{\infty/T}(-,-)} \mathcal{S}$.

Thus it is enough to verify 2.

3: To do so, we are free to enlarge X in the following way:

Let $Z \to S$ be a cartesian fibration equipped with a fully faithful map $X \to Z$ of cartesian fibrations over S classifying a component-wise fully faithful natural transformation $F \to H$ of functors $S^{op} \to \mathsf{Cat}_\infty$.

If the right fibration $\mathrm{Map}_S(Y, Z)^\simeq \to S$ classifies the functor

$$S^{op} \xrightarrow{(G^{op}, H)} \mathsf{Cat}_\infty^{op} \times \mathsf{Cat}_\infty \xrightarrow{\mathsf{Cat}_\infty(-,-)} \mathcal{S},$$

the right fibration $\mathrm{Map}_S(Y, X)^\simeq \to S$ classifies the functor $S^{op} \xrightarrow{(G^{op}, F)} \mathsf{Cat}_\infty^{op} \times \mathsf{Cat}_\infty \xrightarrow{\mathsf{Cat}_\infty(-,-)} \mathcal{S}$.

This follows from the fact that the fully faithful map $X \to Z$ of cartesian fibrations over S yields a fully faithful map $\mathrm{Map}_S(Y, X)^\simeq \subset \mathrm{Map}_S(Y, Z)^\simeq$ of right fibrations over S, whose essential image coincides with the essential image of the fully faithful map of right fibrations over S that classifies the component-wise fully faithful natural transformation $\mathsf{Cat}_\infty(-,-) \circ (G^{op}, F) \to \mathsf{Cat}_\infty(-,-) \circ (G^{op}, H)$ of functors $S^{op} \to \mathcal{S}$.

By remark 5.22 we have a fully faithful map

$$X^{op} \subset \mathrm{Map}_{S^{op}}((X^{op})^{rev}, S^{op} \times \mathcal{S})$$

of cocartesian fibrations over S^{op}, where $\mathrm{Map}_{S^{op}}((X^{op})^{rev}, S^{op} \times \mathcal{S}) \to S^{op}$ is a bicartesian fibration. Taking the opposite we get a fully faithful map

$$X \subset Z := \mathrm{Map}_{S^{op}}((X^{op})^{rev}, S^{op} \times \mathcal{S})^{op}$$

of cartesian fibrations over S, where $Z \to S$ is a bicartesian fibration.

Consequently we can reduce to the case that $X \to S$ is a bicartesian fibration classifying the functor $F : S^{op} \to \mathsf{Cat}_\infty^R$ and a functor $H : S \xrightarrow{F^{op}} (\mathsf{Cat}_\infty^R)^{op} \simeq \mathsf{Cat}_\infty^L$.

Write $\mathcal{U}^L \to \mathsf{Cat}_\infty^L, \mathcal{R}^L \to \mathsf{Cat}_\infty^L$ for the pullbacks of the cocartesian fibrations $\mathcal{U} \to \mathsf{Cat}_\infty, \mathcal{R} \to \mathsf{Cat}_\infty$ along the subcategory inclusion $\mathsf{Cat}_\infty^L \subset \mathsf{Cat}_\infty$.

The embedding $\mathcal{U} \subset \mathcal{R}$ of cocartesian fibrations over Cat_∞ gives rise to an embedding $\mathcal{U}^L \subset \mathcal{R}^L$ of cocartesian fibrations over Cat_∞^L.

Moreover the cartesian fibration $\mathcal{R}^L \to \mathsf{Cat}_\infty^L$ restricts to a cartesian fibration $\mathcal{U}^L \to \mathsf{Cat}_\infty^L$ with the same cartesian morphisms.

Thus the embedding $\mathcal{U}^L \subset \mathcal{R}^L$ of cocartesian fibrations over Cat_∞^L is also an embedding of cartesian fibrations over Cat_∞^L and so by pulling back along $H : S \to \mathsf{Cat}_\infty^L$ gives rise to an embedding

$$X \simeq S \times_{\mathsf{Cat}_\infty} \mathcal{U} \simeq S \times_{\mathsf{Cat}_\infty^L} \mathcal{U}^L \subset S \times_{\mathsf{Cat}_\infty^L} \mathcal{R}^L \simeq S \times_{\mathsf{Cat}_\infty} \mathcal{R}$$

of cartesian fibrations over S.

By prop. 5.20 we have a canonical equivalence

$$\mathcal{L} \simeq L := \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}, \mathsf{Cat}_\infty \times \mathcal{S})$$

of cartesian fibrations over Cat_∞ and so a canonical equivalence

$$\mathcal{R} \simeq \mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{U}^{\mathrm{rev}}, \mathsf{Cat}_\infty \times \mathcal{S})$$

of cartesian fibrations over Cat_∞ and a canonical equivalence

$$S \times_{\mathsf{Cat}_\infty} \mathcal{R} \simeq \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S})$$

of cartesian fibrations over S.

So we get an embedding $X \subset \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S})$ of cartesian fibrations over S.

By [10] corollary A.31. the cartesian fibration $\mathcal{L} \to \mathsf{Cat}_\infty$ classifies the functor $\mathrm{Fun}(-, \mathcal{S}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathcal{S}}$ so that the cartesian fibration $S \times_{\mathsf{Cat}_\infty} \mathcal{R} \simeq \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S}) \to S$ classifies the functor $\mathrm{Fun}((-)^{\mathrm{op}}, \mathcal{S}) \circ H^{\mathrm{op}} : S^{\mathrm{op}} \to \mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathcal{S}}$.

Consequently it is enough to see that the right fibration

$$\mathrm{Map}_S(Y, \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S}))^{\simeq} \to S$$

classifies the functor

$$S^{\mathrm{op}} \xrightarrow{(G^{\mathrm{op}}, H^{\mathrm{op}})} \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{id} \times \mathrm{Fun}((-)^{\mathrm{op}}, \mathcal{S})} \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty \xrightarrow{\mathrm{Fun}(-,-)^{\simeq}} \mathcal{S}.$$

We have a canonical equivalence

$$\mathrm{Map}_S(Y, \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S})) \simeq \mathrm{Map}_S(Y \times_S X^{\mathrm{rev}}, S \times \mathcal{S})$$

of cartesian fibrations over S that yields an equivalence

$$\mathrm{Map}_S(Y, \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S}))^{\simeq} \simeq \mathrm{Map}_S(Y \times_S X^{\mathrm{rev}}, S \times \mathcal{S})^{\simeq} \simeq S \times_{\widehat{\mathsf{Cat}_\infty}} \widehat{\mathsf{Cat}_{\infty/\mathcal{S}}}$$

of right fibrations over S.

Thus the right fibration $\mathrm{Map}_S(Y, \mathrm{Map}_S(X^{\mathrm{rev}}, S \times \mathcal{S}))^{\simeq} \to S$ classifies the functor

$$S^{\mathrm{op}} \xrightarrow{(G^{\mathrm{op}}, H^{\mathrm{op}})} \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{id} \times (-)^{\mathrm{op}}} \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty^{\mathrm{op}} \xrightarrow{\times} \mathsf{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{Fun}(-, \mathcal{S})^{\simeq}} \mathcal{S}$$

being equivalent to the functor

$$S^{\mathrm{op}} \xrightarrow{(G^{\mathrm{op}}, H^{\mathrm{op}})} \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty^{\mathrm{op}} \xrightarrow{\mathrm{id} \times \mathrm{Fun}((-)^{\mathrm{op}}, \mathcal{S})} \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty \xrightarrow{\mathrm{Fun}(-,-)^{\simeq}} \mathcal{S}.$$

Moreover if $X = S \times T$ for some category T, the equivalent right fibrations

$$\mathrm{Map}_S(Y, S \times T)^{\simeq} \to S, \; S \times_{\mathsf{Cat}_\infty} \mathsf{Cat}_{\infty/T} \to S$$

classify the same functor.

□

Remark 5.24.

- *Let* $\mathcal{O}^\otimes$ *be an operad,* $\mathcal{D}^\otimes \to \mathrm{T} \times \mathcal{O}^\otimes$ *a* $\mathcal{O}^\otimes$*-monoidal category over* T *classified by a* $\mathcal{O}^\otimes$*-monoid* ϕ *of* $\mathsf{Cat}_{\infty/\mathrm{T}}, \mathrm{T} \to \mathrm{R}$ *a functor and* $\mathrm{F}: \mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathrm{T}}^{\mathrm{fl}/\mathrm{R}} \subset \mathsf{Cat}_{\infty/\mathrm{T}}$ *a functor classified by a map* $\mathrm{X} \to \mathrm{S} \times \mathrm{T}$ *of cartesian fibrations over* S.

  The composition $\mathrm{S} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathrm{F}^{\mathrm{op}}\times \mathrm{id}} (\mathsf{Cat}_{\infty/\mathrm{T}}^{\mathrm{fl}/\mathrm{R}})^{\mathrm{op}} \times \mathsf{Cat}_{\infty/\mathrm{T}} \xrightarrow{\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{R}}(-,-)} \mathsf{Cat}_{\infty/\mathrm{R}}$ *is adjoint to a functor*

$$\Psi : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_{\infty/\mathrm{R}}) \simeq (\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}})_{/\mathrm{S}\times\mathrm{R}} \subset \mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{R}}.$$

  As for every $\mathsf{s} \in \mathrm{S}$ *the functor* $\mathrm{Fun}_{\mathrm{T}}^{/\mathrm{R}}(\mathrm{F}(\mathsf{s}), -) : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{R}}$ *preserves finite products, the functor* $\Psi : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{R}}$ *also does and so sends* ϕ *to a* $\mathcal{O}^\otimes$*-monoid* ϕ' *of* $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{R}}$.

  By theorem 5.23 ϕ' *is classified by the* $\mathcal{O}^\otimes$*-monoidal category*

$$\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathcal{D})^\otimes := \mathrm{Fun}_{\mathrm{S}\times\mathrm{T}\times\mathcal{O}^\otimes}^{/\mathrm{S}\times\mathrm{R}\times\mathcal{O}^\otimes}(\mathrm{X} \times \mathcal{O}^\otimes, \mathrm{S} \times \mathcal{D}^\otimes) \to \mathrm{S} \times \mathrm{R} \times \mathcal{O}^\otimes$$

  over $\mathrm{S} \times \mathrm{R}$.

- *Now we specialize to the situation* $\mathcal{O}^\otimes = \mathrm{LM}^\otimes$:
  Let $\mathcal{M}^\otimes \to \mathrm{T} \times \mathrm{LM}^\otimes$ *be a* $\mathrm{LM}^\otimes$*-monoidal category over* T *classifying a* $\mathrm{LM}^\otimes$*-monoid* ϕ *of* $\mathsf{Cat}_{\infty/\mathrm{T}}$ *that exhibits a category* $\mathcal{B}$ *over* T *as a left module over a monoidal category* $\mathcal{C}$ *over* R *with respect to the canonical left module structure on* $\mathsf{Cat}_{\infty/\mathrm{T}}$ *over* $\mathsf{Cat}_{\infty/\mathrm{R}}$.
  Then ϕ' *is classified by the* $\mathrm{LM}^\otimes$*-monoidal category*

$$\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathcal{M})^\otimes = \mathrm{Fun}_{\mathrm{S}\times\mathrm{T}\times\mathrm{LM}^\otimes}^{/\mathrm{S}\times\mathrm{R}\times\mathrm{LM}^\otimes}(\mathrm{X} \times \mathrm{LM}^\otimes, \mathrm{S} \times \mathcal{M}^\otimes) \to \mathrm{S} \times \mathrm{R} \times \mathrm{LM}^\otimes$$

  over $\mathrm{S} \times \mathrm{R}$ *that exhibits the category* $\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathrm{S} \times \mathcal{B})$ *over* $\mathrm{S} \times \mathrm{R}$ *as a left module over the monoidal category*

$$\mathrm{Map}_{\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathcal{C})^\otimes := \mathrm{Map}_{\mathrm{S}\times\mathrm{R}\times\mathrm{Ass}^\otimes}(\mathrm{X} \times \mathrm{Ass}^\otimes, \mathrm{S} \times \mathcal{C}^\otimes) \simeq$$

$$\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathrm{T}\times_{\mathrm{R}}\mathcal{C})^\otimes := \mathrm{Fun}_{\mathrm{S}\times\mathrm{T}\times\mathrm{Ass}^\otimes}^{/\mathrm{S}\times\mathrm{R}\times\mathrm{Ass}^\otimes}(\mathrm{X}\times\mathrm{Ass}^\otimes, \mathrm{S}\times(\mathrm{T}\times_{\mathrm{R}}\mathcal{C}^\otimes)) \to \mathrm{S}\times\mathrm{R}\times\mathrm{Ass}^\otimes$$

  over $\mathrm{S} \times \mathrm{R}$.

  By prop. 6.55 the functor $\mathsf{Cat}_{\infty/\mathrm{T}} \to \mathrm{Fun}((\mathsf{Cat}_{\infty/\mathrm{T}}^{\mathrm{fl}/\mathrm{R}})^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathrm{R}})$ *is lax* $\mathsf{Cat}_{\infty/\mathrm{R}}$*-linear and thus also the functor* $\Psi : \mathsf{Cat}_{\infty/\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{R}}$ *is lax* $\mathsf{Cat}_{\infty/\mathrm{R}}$*-linear and so sends* ϕ *to a canonical left module structure on*

$$\mathcal{B}' := \mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathrm{S} \times \mathcal{B}) \to \mathrm{S} \times \mathrm{R}$$

  over the monoidal category $\mathrm{S}\times\mathcal{C}^\otimes \to \mathrm{S}\times\mathrm{R}\times\mathrm{Ass}^\otimes$ *over* $\mathrm{S}\times\mathrm{R}$ *that is classified by the pullback of the* $\mathrm{LM}^\otimes$*-monoidal category* $\mathrm{Fun}_{\mathrm{S}\times\mathrm{T}}^{/\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathcal{M})^\otimes$ *over* $\mathrm{S} \times \mathrm{R}$ *along the monoidal diagonal functor*

$$\delta : \mathrm{S} \times \mathcal{C}^\otimes \simeq \mathrm{Map}_{\mathrm{S}\times\mathrm{R}}(\mathrm{S} \times \mathrm{R}, \mathcal{C})^\otimes = \mathrm{Map}_{\mathrm{S}\times\mathrm{R}\times\mathrm{Ass}^\otimes}(\mathrm{S} \times \mathrm{R} \times \mathrm{Ass}^\otimes, \mathrm{S} \times \mathcal{C}^\otimes)$$

$$\to \mathrm{Map}_{\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathcal{C})^\otimes = \mathrm{Map}_{\mathrm{S}\times\mathrm{R}\times\mathrm{Ass}^\otimes}(\mathrm{X} \times \mathrm{Ass}^\otimes, \mathrm{S} \times \mathcal{C}^\otimes)$$

  over $\mathrm{S} \times \mathrm{R}$.

  Moreover the induced functor $\mathrm{Alg}^{/\mathrm{S}}(\delta)$ *over* S *is canonically equivalent over* S *to the diagonal functor*

$$\delta' : \mathrm{S}\times\mathrm{Alg}^{/\mathrm{R}}(\mathcal{C}) \simeq \mathrm{Map}_{\mathrm{S}\times\mathrm{R}}(\mathrm{S}\times\mathrm{R}, \mathrm{S}\times\mathrm{Alg}^{/\mathrm{R}}(\mathcal{C})) \to \mathrm{Map}_{\mathrm{S}\times\mathrm{R}}(\mathrm{X}, \mathrm{S}\times\mathrm{Alg}^{/\mathrm{R}}(\mathcal{C}))$$

over $S \times R$.

So by remark 5.19 2. we have a canonical equivalence

$$\mathrm{LMod}^{/S\times R}(\delta^*(\mathcal{B}')) \simeq \delta'^*(\mathrm{Fun}^{/S}_{S\times T}(X, S \times \mathrm{LMod}^{/T}(\mathcal{B})))$$

over $S \times \mathrm{Alg}^{/R}(\mathcal{C})$.

5.2 Endomorphism objects

5.2.1 Basic notions of enriched category theory

We use Lurie's model of enriched categories with some slight modifications:

Let $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ be an operad over $\mathrm{LM}^\otimes$. Set $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ and $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$.

Let X, Y be objects of $\mathcal{D}$ and A an object of $\mathcal{C}$ and let $\alpha \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}, \mathrm{X}; \mathrm{Y})$.

- If (A, α) represents the presheaf $\mathrm{Mul}_{\mathcal{M}}(-, \mathrm{X}; \mathrm{Y}) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$, i.e. if evaluation at α induces an equivalence

  $$\mathcal{C}(\mathrm{B}, \mathrm{A}) \to \mathcal{S}(\mathrm{Mul}_{\mathcal{M}}(\mathrm{A}, \mathrm{X}; \mathrm{Y}), \mathrm{Mul}_{\mathcal{M}}(\mathrm{B}, \mathrm{X}; \mathrm{Y})) \to \mathrm{Mul}_{\mathcal{M}}(\mathrm{B}, \mathrm{X}; \mathrm{Y}),$$

  we say that $\alpha \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}, \mathrm{X}; \mathrm{Y})$ exhibits A as the morphism object of X and Y and write $[\mathrm{X}, \mathrm{Y}]$ for A.
- If $\mathrm{X} = \mathrm{Y}$, we say that $\alpha \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}, \mathrm{X}; \mathrm{X})$ exhibits A as the endomorphism object of X and write $[\mathrm{X}, \mathrm{X}]$ for A.

For every $\mathrm{n} \in \mathbb{N}$ we set $\mathrm{Ass}_{\mathrm{n}} := \mathrm{Mul}_{\mathrm{Ass}}(\underbrace{\mathfrak{a}, ..., \mathfrak{a}}_{\mathrm{n}}; \mathfrak{a})$.

Denote $\sigma \in \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ the unique object. For every $\alpha \in \mathrm{Ass}_{\mathrm{n}}$ for some $\mathrm{n} \in \mathbb{N}$ denote α' the image of α, the identity of $\mathfrak{m}$ and σ under the operadic composition

$$\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m}) \times (\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}; \mathfrak{a}) \times \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{m}; \mathfrak{m})) \to \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m}).$$

We say that $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ exhibits $\mathcal{D}$ as pseudo-enriched in $\mathcal{C}$ if the functor $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$ is a locally cocartesian fibration and the following condition holds:

For every objects $\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}} \in \mathcal{C}$ for some $\mathrm{n} \in \mathbb{N}$ and $\mathrm{X}, \mathrm{Y} \in \mathcal{D}$ and every $\alpha \in \mathrm{Ass}_{\mathrm{n}}$ the canonical map

$$\zeta : \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}), \mathrm{X}; \mathrm{Y}) \simeq$$
$$\{\sigma\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}), \mathrm{X}; \mathrm{Y}) \to$$
$$\{\alpha'\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}, \mathrm{X}; \mathrm{Y})$$

is an equivalence.

We say that an operad $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ over $\mathrm{LM}^\otimes$ exhibits $\mathcal{D}$ as enriched in $\mathcal{C}$ if it exhibits $\mathcal{D}$ as pseudo-enriched in $\mathcal{C}$ and for every objects $\mathrm{X}, \mathrm{Y} \in \mathcal{D}$ there exists a morphism object $[\mathrm{X}, \mathrm{Y}] \in \mathcal{C}$.

Let $\mathcal{M}^\otimes, \mathcal{N}^\otimes$ be operads over $\mathrm{LM}^\otimes$ with $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes \simeq \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$ that exhibit the categories $\{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$, $\{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$ as enriched in $\mathcal{C}$.

We call a map of operads $\mathcal{M}^\otimes \to \mathcal{N}^\otimes$ over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the identity, a $\mathcal{C}$-enriched functor.

Convention 5.25. *We make the following convention for the next sections except the appendix.*

Let $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ *be an operad over* $\mathrm{LM}^\otimes$. *Set* $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ *and* $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$. *Let* X *be an object of* $\mathcal{D}$.

When we say that X *admits an endomorphism object or that an object* $\mathrm{Y} \in \mathcal{C}$ *is the endomorphism object of* X *or that a morphism* $\alpha \in$

$\mathrm{Mul}_{\mathcal{M}}(Y, X; X)$ *exhibits* Y *as the endomorphism object of* X, *we implicitely assume that for every objects* $A_1, ..., A_n \in \mathcal{C}$ *for some* $n \in \mathbb{N}$ *and every* $\alpha \in \mathrm{Ass}_n$ *the canonical map*

$$\zeta : \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(A_1, ..., A_n), X; X) \simeq$$
$$\{\sigma\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a},\mathfrak{m};\mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(A_1, ..., A_n), X; X) \to$$
$$\{\alpha'\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a},...,\mathfrak{a},\mathfrak{m};\mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(A_1, ..., A_n, X; X)$$

is an equivalence.

In many applications we use the following parametrized notion of enrichment:

Let $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times S$ be a locally cocartesian S-family of operads over $\mathrm{LM}^\otimes$. Set $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ and $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$.

We call $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times S$ a locally cocartesian S-family of categories pseudo-enriched respectively enriched in $\mathcal{C}$ if for all $\mathsf{s} \in S$ the induced functor $\mathcal{C}^\otimes_\mathsf{s} \to \mathrm{Ass}^\otimes$ is a locally cocartesian fibration (equivalently if the functor $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes \times S$ is a locally cocartesian fibration) and $\mathcal{M}^\otimes_\mathsf{s}$ exhibits $\mathcal{D}_\mathsf{s}$ as pseudo-enriched respectively enriched in $\mathcal{C}_\mathsf{s}$.

Let $\mathcal{M}^\otimes, \mathcal{N}^\otimes$ be locally cocartesian S-families of categories enriched in $\mathcal{C}$.

We call a map $\mathcal{M}^\otimes \to \mathcal{N}^\otimes$ of locally cocartesian S-families of operads over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the identity, a map of locally cocartesian S-families of $\mathcal{C}$-enriched categories.

Example 5.26. *Let* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ *be a* $\mathrm{LM}^\otimes$*-monoidal category that exhibits a category* $\mathcal{D}$ *as a left module over a monoidal category* $\mathcal{C}$.

The functor $\mathrm{Fun}(\Delta^1, \mathcal{D}) \to \mathrm{Fun}(\{1\}, \mathcal{D})$ *is a left module over* $\mathcal{D} \times \mathcal{C}_{/\mathbb{1}}$ *in* $\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{D}}$ *(remark 6.69) and thus can be promoted to a cocartesian* $\mathcal{D}$*-family of categories pseudo-enriched in* $\mathcal{C}_{/\mathbb{1}}$.

Given a morphism $\mathsf{f} : K \to \mathbb{1}$ *in* $\mathcal{C}$ *and* $g : Y \to X$ *in* $\mathcal{D}$ *we have* $\mathsf{f} \otimes g : K \otimes Y \to \mathbb{1}_{\mathcal{C}} \otimes X \simeq X$.

Given a functor $H : S \to \mathcal{D}$ *this left module structure gives rise to a left module structure on* $\mathrm{Fun}_{\mathcal{D}}(S, \mathrm{Fun}(\Delta^1, \mathcal{D})) \simeq \mathrm{Fun}(S, \mathcal{D})_{/H}$ *over* $\mathrm{Fun}(S, \mathcal{C}_{/\mathbb{1}}) \simeq \mathrm{Fun}(S, \mathcal{C})_{/\mathbb{1}}$, *which is the canonical action.*

Let $\mathcal{M}^\otimes$ be a cocartesian S-family of categories enriched in $\mathcal{C} := \{\mathfrak{a}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$. Set $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$.

By remark 5.22 we have a multi-mapping space functor

$$\mathrm{Mul}_{\mathcal{M}}(-, -; -) : \mathcal{C}^{\mathrm{rev}} \times_S \mathcal{D}^{\mathrm{rev}} \times_S \mathcal{D} \to \mathcal{S}$$

relative to S that is adjoint to a functor $\beta : \mathcal{D}^{\mathrm{rev}} \times_S \mathcal{D} \to \mathrm{Map}_S(\mathcal{C}^{\mathrm{rev}}, \mathcal{S} \times S)$ over S.

As $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times S$ is a cocartesian S-family of categories enriched in $\mathcal{C}$, β induces a functor $\mathcal{D}^{\mathrm{rev}} \times_S \mathcal{D} \to \mathcal{C} \subset \mathrm{Map}_S(\mathcal{C}^{\mathrm{rev}}, \mathcal{S} \times S)$ over S adjoint to a functor $\theta : \mathcal{D} \to \mathrm{Map}_S(\mathcal{D}^{\mathrm{rev}}, \mathcal{C})$ over S.

θ sends an object X of $\mathcal{D}$ lying over some $\mathsf{s} \in S$ to the functor $[-, X]_{\mathcal{D}_\mathsf{s}} : \mathcal{D}^{\mathrm{op}}_\mathsf{s} \to \mathcal{C}_\mathsf{s}$ that sends an object Y of $\mathcal{D}_\mathsf{s}$ to the morphism object $[Y, X]_{\mathcal{D}_\mathsf{s}}$ of Y and X.

In proposition 6.55 we construct a map $\mathcal{M}^\otimes \to \mathrm{Map}_S(\mathcal{D}^{\mathrm{rev}}, \mathcal{C})^\otimes$ of S-families of operads over $\mathrm{LM}^\otimes$, whose underlying functor over S is θ and whose pullback to $\mathrm{Ass}^\otimes$ is the diagonal map $\delta : \mathcal{C}^\otimes \simeq \mathrm{Map}_S(S, \mathcal{C})^\otimes \to \mathrm{Map}_S(\mathcal{D}^{\mathrm{rev}}, \mathcal{C})^\otimes$ of S-families of operads over $\mathrm{Ass}^\otimes$.

For S contractible this guarantees the following:

Let X be an object of $\mathcal{D}$ and $\beta \in \mathrm{Mul}_{\mathcal{M}}(B, X; X)$ an operation that exhibits $B = [X, X]$ as the endomorphism object of X.

Being a map of operads over $LM^{\otimes}$ the functor θ sends the endomorphism $[X, X]$-left module structure on X to a $\delta([X, X])$-left module structure on $[-, X] : \mathcal{D}^{op} \to \mathcal{C}$ corresponding to a lift $\mathcal{D}^{op} \to \mathrm{LMod}_{[X,X]}(\mathcal{C})$ of $[-, X] : \mathcal{D}^{op} \to \mathcal{C}$.

So for every object Y of $\mathcal{D}$ the morphism object $[Y, X]$ is a left module in $\mathcal{C}$ over the endomorphism object $[X, X]$ and for every morphism $Y \to Z$ in $\mathcal{D}$ the induced morphism $[Z, X] \to [Y, X]$ is $[X, X]$-linear.

Let $\mathcal{M}^{\otimes} \to LM^{\otimes}, \mathcal{N}^{\otimes} \to LM^{\otimes}$ be operads over $LM^{\otimes}$ that exhibit categories $\mathcal{D}$ respectively $\mathcal{E}$ as pseudo-enriched in $\mathcal{C} := \{\mathfrak{a}\} \times_{LM^{\otimes}} \mathcal{M}^{\otimes}$ respectively $\mathcal{B} := \{\mathfrak{a}\} \times_{LM^{\otimes}} \mathcal{N}^{\otimes}$ and let $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ a map of operads over $LM^{\otimes}$.

Let X, Y be objects of $\mathcal{D}$ that admit a morphism object $[Y, X] \in \mathcal{C}$ and whose images $F(X), F(Y) \in \mathcal{E}$ admit a morphism object $[F(Y), F(X)] \in \mathcal{B}$.

The map $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ of operads over $LM^{\otimes}$ sends the canonical $[X, X]$-left module structure on $[Y, X]$ in $\mathcal{C}$ to a left module structure on $F([Y, X])$ over $F([X, X])$ in $\mathcal{B}$.

The canonical morphisms

$$F([Y, X]) \to [F(Y), F(X)], \qquad F([X, X]) \to [F(X), F(X)]$$

in $\mathcal{B}$ organize to a morphism of $LM^{\otimes}$-algebras, where $[F(Y), F(X)]$ carries the canonical $[F(X), F(X)]$-left module structure in $\mathcal{B}$.

Especially for $\mathcal{C} = \mathsf{Cat}_{\infty}$ and F a Cat_{∞}-enriched functor we see that the canonical functors

$$[Y, X] \to [F(Y), F(X)], \qquad [X, X] \to [F(X), F(X)]$$

are part of a $LM^{\otimes}$-monoidal functor.

This guarantees the following:

Remark 5.27. *Let $T \in \mathrm{Alg}([X, X])$ be a monad on X and $\phi : Y \to X$ a left module over T in $[Y, X]$.*
Then the morphism $F(\phi) : F(Y) \to F(X)$ in $\mathcal{E}$ is a left module over the monad $F(T) \in \mathrm{Alg}([F(X), F(X)])$.

If T is the endomorphism object of ϕ, the monad $F(T)$ is the endomorphism object of $F(\phi)$ by proposition 5.31.

Let $\mathcal{M}^{\otimes}$ be an operad over $LM^{\otimes}$ that exhibits a category $\mathcal{D}$ as enriched in $\mathcal{C} := \{\mathfrak{a}\} \times_{LM^{\otimes}} \mathcal{M}^{\otimes}$.
Let $\mathcal{B}^{\otimes} \to Ass^{\otimes}$ be a locally cocartesian fibration of operads and $F : \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes}$ a map of locally cocartesian fibrations of operads over $Ass^{\otimes}$, whose underlying functor $\mathcal{B} \to \mathcal{C}$ admits a right adjoint $G : \mathcal{C} \to \mathcal{B}$.

Then by proposition 6.61 combined with lemma 6.64 one can pullback $\mathcal{M}^{\otimes}$ along $F : \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes}$ to obtain an operad $F^*(\mathcal{M})^{\otimes}$ over $LM^{\otimes}$ that exhibits $\mathcal{D}$ as enriched in $\mathcal{B}$.

$F^*(\mathcal{M})^\otimes$ is determined by the condition that for every operad $\mathcal{Q}^\otimes$ over $LM^\otimes$, where we set $\mathcal{A}^\otimes := Ass^\otimes \times_{LM^\otimes} \mathcal{Q}^\otimes$, the commutative square

$$\begin{array}{ccc} Alg_{\mathcal{Q}/LM}(F^*(\mathcal{M})) & \longrightarrow & Alg_{\mathcal{Q}/LM}(\mathcal{M}) \\ \downarrow & & \downarrow \\ Alg_{\mathcal{A}/Ass}(\mathcal{B}) & \longrightarrow & Alg_{\mathcal{A}/Ass}(\mathcal{C}). \end{array} \tag{23}$$

is a pullback square.

The morphism object of two objects $X, Y \in \mathcal{D}$ with respect to $F^*(\mathcal{M})^\otimes$ is given by $G([X, Y]) \in \mathcal{B}$, where $[X, Y] \in \mathcal{C}$ denotes the morphism object of X and Y with respect to $\mathcal{M}^\otimes$.

Now we specialize to the case $\mathcal{C}^\otimes = \mathsf{Cat}_\infty^\times$:

We call a category enriched in $\mathsf{Cat}_\infty^\times$ a 2-category and a $\mathsf{Cat}_\infty^\times$-enriched functor a 2-functor.

We call a (locally) cocartesian S-family of categories enriched in $\mathsf{Cat}_\infty^\times$ a (locally) cocartesian S-family of 2-categories and a map of (locally) cocartesian S-families of $\mathsf{Cat}_\infty^\times$-enriched categories a map of (locally) cocartesian S-families of 2-categories.

We denote the pullback of a 2-category $\mathcal{C}$ along the opposite category involution $(-)^{op} : \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ by $\mathcal{C}_{op}$ so that in $\mathcal{C}_{op}$ the 2-morphisms are reversed.

Given a category S the opposite category involution lifts to a canonical equivalence

$$(\mathsf{Cat}_{\infty/S})_{op} \simeq \mathsf{Cat}_{\infty/S^{op}}$$

of 2-categories as the opposite category involutions $\mathsf{Cat}_{\infty/S^{op}} \simeq \mathsf{Cat}_{\infty/S}$ and $\mathsf{Cat}_\infty \simeq \mathsf{Cat}_\infty$ induce for every operad $\mathcal{Q}^\otimes$ over $LM^\otimes$, where we set $\mathcal{B}^\otimes := Ass^\otimes \times_{LM^\otimes} \mathcal{Q}^\otimes$, a pullback square

$$\begin{array}{ccc} Alg_{\mathcal{Q}/LM}(\mathsf{Cat}_{\infty/S^{op}}) & \xrightarrow{\simeq} & Alg_{\mathcal{Q}/LM}(\mathsf{Cat}_{\infty/S}) \\ \downarrow & & \downarrow \\ Alg_{\mathcal{B}/Ass}(\mathsf{Cat}_\infty) & \xrightarrow{\simeq} & Alg_{\mathcal{B}/Ass}(\mathsf{Cat}_\infty). \end{array}$$

We have a notion of adjunction in any 2-category $\mathcal{C}$:

Let $f : X \to Y$ and $g : Y \to X$ be morphisms of $\mathcal{C}$.

We say that f is left adjoint to g or g is right adjoint to f or that (f, g) is an adjoint pair if there are 2-morphisms $\eta : id_X \to g \circ f$ and $\varepsilon : f \circ g \to id_Y$ such that the triangular identities $(\varepsilon \circ f) \circ (f \circ \eta) = id_f$ and $(g \circ \varepsilon) \circ (\eta \circ g) = id_g$ hold.

5.2.2 Endomorphism objects

Let $\mathcal{M}^\otimes \to LM^\otimes$ be an operad over $LM^\otimes$. Set $\mathcal{C}^\otimes := Ass^\otimes \times_{LM^\otimes} \mathcal{M}^\otimes$ and $\mathcal{D} := \{\mathfrak{m}\} \times_{LM^\otimes} \mathcal{M}^\otimes$. Let X be an object of $\mathcal{D}$.

Denote $\varphi : \Delta^1 \to LM^\otimes$ the morphism of $LM^\otimes$ corresponding to the unique object of $Mul_{LM}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$.

Using φ we form the category $\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes})$ and have canonical functors

$$\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \to \mathcal{M}^{\otimes}_{(\mathfrak{a},\mathfrak{m})} \simeq \mathcal{C} \times \mathcal{D}, \ \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \to \mathcal{M}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{D}$$

evaluating at 0 respectively 1.

We set $\mathcal{C}[\mathrm{X}] \coloneqq \{(\mathrm{X},\mathrm{X})\} \times_{\mathcal{D}\times\mathcal{D}} \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes})$ and have a forgetful functor $\mathcal{C}[\mathrm{X}] \to \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \to \mathcal{C}$ that is a right fibration classifying the functor $\mathrm{Mul}_{\mathcal{M}}(-,\mathrm{X};\mathrm{X}) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ according to lemma 6.70.

So an object of $\mathcal{C}[\mathrm{X}]$ corresponding to a pair (A,α) consisting of an object A of $\mathcal{C}$ and an object α of $\mathrm{Mul}_{\mathcal{M}}(\mathrm{A},\mathrm{X};\mathrm{X})$ is a final object of $\mathcal{C}[\mathrm{X}]$ if and only if for all objects B of $\mathcal{C}$ evaluation at α induces an equivalence

$$\mathcal{C}(\mathrm{B},\mathrm{A}) \to \mathcal{S}(\mathrm{Mul}_{\mathcal{M}}(\mathrm{A},\mathrm{X};\mathrm{X}), \mathrm{Mul}_{\mathcal{M}}(\mathrm{B},\mathrm{X};\mathrm{X})) \to \mathrm{Mul}_{\mathcal{M}}(\mathrm{B},\mathrm{X};\mathrm{X}),$$

i.e. if and only if α exhibits A as the endomorphism object of X.

φ gives rise to a forgetful functor

$$\mathrm{LMod}(\mathcal{D}) \subset \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\mathrm{LM}^{\otimes}, \mathcal{M}^{\otimes}) \to \mathcal{D} \times_{\mathcal{D}\times\mathcal{D}} \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes})$$

over $\mathcal{D}$, where the functor $\mathcal{D} \to \mathcal{D}\times\mathcal{D}$ is the diagonal functor, that induces a forgetful functor

$$\{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{LMod}(\mathcal{D}) \subset \{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\mathrm{LM}^{\otimes}, \mathcal{M}^{\otimes}) \to \mathcal{C}[\mathrm{X}]$$
$$= \{(\mathrm{X},\mathrm{X})\} \times_{\mathcal{D}\times\mathcal{D}} \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}).$$

By proposition 6.50 and convention 5.25 if $\mathcal{C}[\mathrm{X}]$ admits a final object, the final object lifts to a final object of $\{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{LMod}(\mathcal{D})$.

As the forgetful functor $\{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{LMod}(\mathcal{D}) \to \mathcal{C}[\mathrm{X}]$ is conservative, in this case an object of $\{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{LMod}(\mathcal{D})$ is a final object if and only if its image in $\mathcal{C}[\mathrm{X}]$ is.
So by abuse of notation we identify the final object of $\{\mathrm{X}\}\times_{\mathcal{D}}\mathrm{LMod}(\mathcal{D})$ with the final object of $\mathcal{C}[\mathrm{X}]$ if both exist.

Endomorphism objects are functorial in the following way:

Let $\mathrm{F} : \mathcal{M}^{\otimes} \to \mathcal{M}'^{\otimes}$ be a map of operads over $\mathrm{LM}^{\otimes}$.
Set $\mathcal{C}^{\otimes} \coloneqq \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}, \mathcal{C}'^{\otimes} \coloneqq \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}'^{\otimes}, \mathcal{D} \coloneqq \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}, \mathcal{D}' = \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}'^{\otimes}$.
Let X be an object of $\mathcal{D}$ such that X and F(X) admit endomorphism objects $[\mathrm{X},\mathrm{X}]$ respectively $[\mathrm{F}(\mathrm{X}),\mathrm{F}(\mathrm{X})]$.
The map $\mathrm{F} : \mathcal{M}^{\otimes} \to \mathcal{M}'^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ gives rise to a commutative square

$$\begin{array}{ccc} \{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{LMod}(\mathcal{D}) & \longrightarrow & \{\mathrm{F}(\mathrm{X})\} \times_{\mathcal{D}'} \mathrm{LMod}(\mathcal{D}') \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}(\mathcal{C}'). \end{array}$$

The endomorphism objects $[\mathrm{X},\mathrm{X}]$ of X and $[\mathrm{F}(\mathrm{X}),\mathrm{F}(\mathrm{X})]$ of F(X) are by definition the final objects of the categories $\{\mathrm{X}\} \times_{\mathcal{D}} \mathrm{LMod}(\mathcal{D})$ respectively $\{\mathrm{F}(\mathrm{X})\} \times_{\mathcal{D}'} \mathrm{LMod}(\mathcal{D}')$.
Consequently F sends the endomorphism left module structure on X over $[\mathrm{X},\mathrm{X}]$ to a left module structure on F(X) over $\mathrm{F}([\mathrm{X},\mathrm{X}])$ that is

the pullback of the endomorphism left module structure on $F(X)$ over $[F(X), F(X)]$ along a canonical morphism $F([X,X]) \to [F(X), F(X)]$ in $\mathrm{Alg}(\mathcal{C}')$.

5.2.3 Endomorphism objects in families

More coherently we study endomorphism objects relative to S :

Let $\mathcal{M}^\otimes \to S \times \mathrm{LM}^\otimes$ be a S-family of operads over $\mathrm{LM}^\otimes$. Set $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ and $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$. Let X be a section of $\mathcal{D} \to S$.

Denote $\varphi : \Delta^1 \to \mathrm{LM}^\otimes$ the morphism of $\mathrm{LM}^\otimes$ corresponding to the unique object of $\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$.

Using φ we form the category $\mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes)$ and have canonical functors

$$\mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes) \to \mathcal{M}^\otimes_{(\mathfrak{a},\mathfrak{m})} \simeq \mathcal{C} \times_S \mathcal{D}, \; \mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes) \to \mathcal{M}^\otimes_{\mathfrak{m}} \simeq \mathcal{D}$$

over S evaluating at 0 respectively 1.

We set $\mathcal{C}[X]^{/S} := S \times_{\mathcal{D} \times_S \mathcal{D}} \mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes)$ and have a forgetful functor

$$\mathcal{C}[X]^{/S} \to \mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes) \to \mathcal{C}$$

over S that induces on the fiber over $\mathsf{s} \in S$ the right fibration

$$\mathcal{C}_\mathsf{s}[X(\mathsf{s})] = \{(X(\mathsf{s}), X(\mathsf{s}))\} \times_{\mathcal{D}_\mathsf{s} \times \mathcal{D}_\mathsf{s}} \mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes_\mathsf{s}) \to \mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes_\mathsf{s}) \to \mathcal{C}_\mathsf{s}$$

classifying the functor $\mathrm{Mul}_{\mathcal{M}_\mathsf{s}}(-, X(\mathsf{s}); X(\mathsf{s})) : \mathcal{C}_\mathsf{s}^{\mathrm{op}} \to \mathcal{S}$ according to lemma 6.70 and on sections the right fibration

$$\mathrm{Fun}_S(S, \mathcal{C})[X] = \{(X,X)\} \times_{\mathrm{Fun}_S(S,\mathcal{D}) \times \mathrm{Fun}_S(S,\mathcal{D})} \mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathrm{Fun}_S(S, \mathcal{M})^\otimes) \to$$
$$\mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathrm{Fun}_S(S, \mathcal{M})^\otimes) \to \mathrm{Fun}_S(S, \mathcal{C})$$

classifying the functor $\mathrm{Mul}_{\mathrm{Fun}_S(S,\mathcal{D})}(-, X; X) : \mathrm{Fun}_S(S, \mathcal{C})^{\mathrm{op}} \to \mathcal{S}$ according to remark 5.3 2. and lemma 6.70.

φ gives rise to a forgetful functor

$$\mathrm{LMod}^{/S}(\mathcal{D}) \subset \mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \mathrm{LM}^\otimes, \mathcal{M}^\otimes) \to \mathcal{D} \times_{(\mathcal{D} \times_S \mathcal{D})} \mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes)$$

over $\mathcal{D}$, where the functor $\mathcal{D} \to \mathcal{D} \times_S \mathcal{D}$ is the diagonal functor over S, that induces a forgetful functor

$$S \times_{\mathcal{D}} \mathrm{LMod}^{/S}(\mathcal{D}) \subset S \times_{\mathcal{D}} \mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \mathrm{LM}^\otimes, \mathcal{M}^\otimes)$$
$$\to \mathcal{C}[X]^{/S} = S \times_{(\mathcal{D} \times_S \mathcal{D})} \mathrm{Fun}^{/S}_{S \times \mathrm{LM}^\otimes}(S \times \Delta^1, \mathcal{M}^\otimes)$$

over S that induces on the fiber over $\mathsf{s} \in S$ the forgetful functor

$$\{X(\mathsf{s})\} \times_{\mathcal{D}_\mathsf{s}} \mathrm{LMod}(\mathcal{D}_\mathsf{s}) \subset \{X(\mathsf{s})\} \times_{\mathcal{D}_\mathsf{s}} \mathrm{Fun}_{\mathrm{LM}^\otimes}(\mathrm{LM}^\otimes, \mathcal{M}^\otimes_\mathsf{s}) \to \mathcal{C}_\mathsf{s}[X(\mathsf{s})] =$$
$$\{(X(\mathsf{s}), X(\mathsf{s}))\} \times_{\mathcal{D}_\mathsf{s} \times \mathcal{D}_\mathsf{s}} \mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathcal{M}^\otimes_\mathsf{s})$$

and on sections the forgetful functor

$$\{X\} \times_{\mathrm{Fun}_S(S,\mathcal{D})} \mathrm{LMod}(\mathrm{Fun}_S(S, \mathcal{D})) \subset$$
$$\{X\} \times_{\mathrm{Fun}_S(S,\mathcal{D})} \mathrm{Fun}_{\mathrm{LM}^\otimes}(\mathrm{LM}^\otimes, \mathrm{Fun}_S(S, \mathcal{M})^\otimes) \to \mathrm{Fun}_S(S, \mathcal{C})[X]$$
$$= \{(X,X)\} \times_{(\mathrm{Fun}_S(S,\mathcal{D}) \times \mathrm{Fun}_S(S,\mathcal{D}))} \mathrm{Fun}_{\mathrm{LM}^\otimes}(\Delta^1, \mathrm{Fun}_S(S, \mathcal{M})^\otimes).$$

Observation 5.28.

Assume that $\mathcal{M}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ *is a locally cocartesian* S*-family of operads over* $\mathrm{LM}^{\otimes}$ *and* X *is a locally cocartesian section of* $\phi : \mathcal{D} \to \mathrm{S}$ *such that for all* $\mathsf{s} \in \mathrm{S}$ *the image* $\mathrm{X}(\mathsf{s}) \in \mathcal{D}_{\mathsf{s}}$ *admits an endomorphism object, in other words the category* $(\mathcal{C}[\mathrm{X}]^{/\mathrm{S}})_{\mathsf{s}} \simeq \mathcal{C}_{\mathsf{s}}[\mathrm{X}(\mathsf{s})]$ *admits a final object.*

Then by proposition 6.50 for every $\mathsf{s} \in \mathrm{S}$ *the final object of* $\mathcal{C}_{\mathsf{s}}[\mathrm{X}(\mathsf{s})]$ *lifts to a final object of the category* $(\mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}))_{\mathsf{s}} \simeq \{\mathrm{X}(\mathsf{s})\} \times_{\mathcal{D}_{\mathsf{s}}} \mathrm{LMod}(\mathcal{D}_{\mathsf{s}})$.

By remark 5.5 the functor $\mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *over* S *is a map of locally cocartesian fibrations over* S.

So by lemma 5.33 the category

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D})) \simeq \{\mathrm{X}\} \times_{\mathrm{Fun}_{\mathrm{S}}(\mathrm{S},\mathcal{D})} \mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D})) \simeq$$

$$\{\mathrm{X}\} \times_{\mathrm{Fun}_{\mathrm{S}}(\mathrm{S},\mathcal{D})} \mathrm{LMod}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D}))$$

admits a final object Y *such that for every object* $\mathsf{s} \in \mathrm{S}$ *the image* $\mathrm{Y}(\mathsf{s})$ *is the final object of the category* $\{\mathrm{X}(\mathsf{s})\} \times_{\mathcal{D}_{\mathsf{s}}} \mathrm{LMod}(\mathcal{D}_{\mathsf{s}})$.

The functor $\mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$ *over* S *sends* Y *to an object* $[\mathrm{X}, \mathrm{X}]^{/\mathrm{S}}$ *of the category* $\mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}))$.

So Y *exhibits* $[\mathrm{X}, \mathrm{X}]^{/\mathrm{S}}$ *as the endomorphism object of* X *with respect to* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{M})^{\otimes} \to \mathrm{LM}^{\otimes}$.

Observation 5.29.

Let $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ *be a morphism of* S *such that the induced map* $\mathsf{f}_* : \mathcal{M}_{\mathsf{s}}^{\otimes} \to \mathcal{M}_{\mathsf{t}}^{\otimes}$ *of operads over* $\mathrm{LM}^{\otimes}$ *preserves the endomorphism object of* $\mathrm{X}(\mathsf{s})$*, in other words such that the functor* $(\mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}))_{\mathsf{s}} \to (\mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}))_{\mathsf{t}}$ *induced by* f *preserves the final object.*

Then the functor $\mathrm{Y} : \mathrm{S} \to \mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D})$ *over* S *sends* f *to a locally cocartesian morphism of the locally cocartesian fibration* $\mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{S}$ *and thus the composition*

$$[\mathrm{X}, \mathrm{X}]^{/\mathrm{S}} : \mathrm{S} \xrightarrow{\mathrm{Y}} \mathrm{S} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C})$$

sends f *to a locally cocartesian morphism of the locally cocartesian fibration* $\mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$.

Observation 5.28 and 5.29 imply the following:

Let $\psi : \mathrm{T} \to \mathrm{S}$ be a category over S and $\mathrm{X} : \mathrm{T} \to \mathcal{D}$ a functor over S that sends every morphism of T to a locally ϕ-cocartesian morphism corresponding to a cocartesian section of the pullback $\mathrm{T} \times_{\mathrm{S}} \mathcal{D} \to \mathrm{T}$.

Assume that for every object $\mathrm{t} \in \mathrm{T}$ the image $\mathrm{X}(\mathrm{t}) \in \mathcal{D}_{\phi(\mathrm{t})}$ admits an endomorphism object.

Then the category

$$\{\mathrm{X}\} \times_{\mathrm{Fun}_{\mathrm{S}}(\mathrm{T},\mathcal{D})} \mathrm{LMod}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{T}, \mathcal{D})) \simeq \mathrm{Fun}_{\mathrm{T}}(\mathrm{T}, \mathrm{T} \times_{(\mathrm{T} \times_{\mathrm{S}} \mathcal{D})} \mathrm{LMod}^{/\mathrm{T}}(\mathrm{T} \times_{\mathrm{S}} \mathcal{D})) \simeq$$

$$\mathrm{Fun}_{\mathrm{T}}(\mathrm{T}, \mathrm{T} \times_{\mathcal{D}} \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}))$$

admits a final object Y such that for every object $\mathrm{t} \in \mathrm{T}$ the image $\mathrm{Y}(\mathrm{t})$ is the final object of the category $\{\mathrm{X}(\mathrm{t})\} \times_{\mathcal{D}_{\phi(\mathrm{t})}} \mathrm{LMod}(\mathcal{D}_{\phi(\mathrm{t})})$ and that lies over an object $[\mathrm{X}, \mathrm{X}]^{/\mathrm{T}}$ of the category $\mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{T}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{T}, \mathrm{Alg}^{/\mathrm{S}}(\mathcal{C}))$.

In other words Y exhibits $[X,X]^{/T}$ as the endomorphism object of X with respect to $T \times_S \mathcal{D}^\otimes \to LM^\otimes \times T$.

Let $f : s \to t$ be a morphism of T such that the induced map $f_* : \mathcal{D}^\otimes_{\phi(s)} \to \mathcal{D}^\otimes_{\phi(t)}$ of operads over $LM^\otimes$ preserves the endomorphism object of $X(s)$, in other words such that the functor $(T \times_{\mathcal{D}} LMod^{/S}(\mathcal{D}))_s \to (T \times_{\mathcal{D}} LMod^{/S}(\mathcal{D}))_t$ induced by f preserves the final object.

Then the functor $Y : T \to T \times_{\mathcal{D}} LMod^{/S}(\mathcal{D})$ over T sends f to a locally cocartesian morphism of $T \times_{\mathcal{D}} LMod^{/S}(\mathcal{D}) \to T$ so that the composition $[X,X]^{/T} : T \xrightarrow{Y} T \times_{\mathcal{D}} LMod^{/S}(\mathcal{D}) \to T \times_S Alg^{/S}(\mathcal{C})$ sends f to a locally cocartesian morphism of the locally cocartesian fibration $T \times_S Alg^{/S}(\mathcal{C}) \to T$.

Denote $\mathcal{D}^\simeq \subset \mathcal{D}$ the wide subcategory with morphisms those of $\mathcal{D}$ that are cocartesian with respect to the locally cocartesian fibration $\mathcal{D} \to S$ ao that $\mathcal{D}^\simeq \to S$ is a left fibration. Let $\mathcal{E} \subset S$ be a subcategory.

Denote $\mathcal{D}^{univ}_{End} \subset \mathcal{D}_{End} \subset \mathcal{D}^\simeq$ the full subcategories spanned by the objects of $\mathcal{D}$ that admit an endomorphism object respectively that admit an endomorphism object that is preserved by the functors on the fibers of the locally cocartesian fibration $\mathcal{D} \to S$ induced by morphisms of $\mathcal{E}$.

The left fibration $\mathcal{D}^\simeq \to S$ restricts to a left fibration $\mathcal{D}^{univ}_{End} \to S$ relative to $\mathcal{E}$.

For $T = \mathcal{D}_{End} \to S$ and X the canonical inclusion $\mathcal{D}_{End} \subset \mathcal{D}$ the endomorphism object of X is a functor $End : \mathcal{D}_{End} \to Alg^{/S}(\mathcal{C})$ over S.

For $T = \mathcal{D}^{univ}_{End} \to S$ and X the canonical inclusion $\mathcal{D}^{univ}_{End} \subset \mathcal{D}$ the endomorphism object of X is a map $End : \mathcal{D}^{univ}_{End} \to Alg^{/S}(\mathcal{C})$ of locally cocartesian fibrations relative to $\mathcal{E}$.

Remark 5.30.

1. *By lemma 6.15 we have a canonical equivalence over* $Alg^{/S}(\mathcal{C})$ *between the map* $S \times_{\mathcal{D}} LMod^{/S}(\mathcal{D}) \to Alg^{/S}(\mathcal{C})$ *of locally cocartesian fibrations over* S *and the map*

$$Alg^{/S}(\mathcal{C})_{/[X,X]^{/S}} := S \times_{Alg^{/S}(\mathcal{C})^{\{1\}}} Alg^{/S}(\mathcal{C})^{\Delta^1} \to Alg^{/S}(\mathcal{C})^{\{0\}}$$

of cocartesian fibrations over S *that induces on the fiber over* $s \in S$ *the canonical equivalence*

$$\{X(s)\} \times_{\mathcal{D}_s} LMod(\mathcal{D}_s) \simeq Alg(\mathcal{C}_s)_{/[X(s),X(s)]}$$

over $Alg(\mathcal{C}_s)$.

Pulling back this equivalence over $Alg^{/S}(\mathcal{C})$ *along a section of* $Alg^{/S}(\mathcal{C}) \to S$ *we obtain a canonical equivalence*

$$S \times_{(Alg^{/S}(\mathcal{C}) \times_S \mathcal{D})} LMod^{/S}(\mathcal{D}) \simeq S \times_{(Alg^{/S}(\mathcal{C})^{\{0\}} \times_S Alg^{/S}(\mathcal{C})^{\{1\}})} Alg^{/S}(\mathcal{C})^{\Delta^1}$$

over S.

2. *Let* $\psi : T \to S$ *be a category over* S *and* $X : T \to \mathcal{D}_{End}, Y : T \to Alg^{/S}(\mathcal{C})$ *functors over* S.

Applying 1. to the pullback $T \times_S \mathcal{D}^\otimes \to LM^\otimes \times T$ *we obtain a canonical equivalence*

$$T \times_{(Alg^{/S}(\mathcal{C}) \times_S \mathcal{D})} LMod^{/S}(\mathcal{D}) \simeq T \times_{(Alg^{/S}(\mathcal{C})^{\{0\}} \times_S Alg^{/S}(\mathcal{C})^{\{1\}})} Alg^{/S}(\mathcal{C})^{\Delta^1}$$

over T *that induces on the fiber over* $t \in T$ *the canonical equivalence*

$$\{X(t)\} \times_{\mathcal{D}_s} \mathrm{LMod}_{Y(t)}(\mathcal{D}_s) \simeq \mathrm{Alg}(\mathcal{C}_s)(Y(t), [X(t), X(t)]).$$

Especially we obtain a canonical equivalence

$$(\mathrm{Alg}^{/S}(\mathcal{C}) \times_S \mathcal{D}_{\mathrm{End}}) \times_{(\mathrm{Alg}^{/S}(\mathcal{C}) \times_S \mathcal{D})} \mathrm{LMod}^{/S}(\mathcal{D}) \simeq$$

$$(\mathrm{Alg}^{/S}(\mathcal{C}) \times_S \mathcal{D}_{\mathrm{End}}) \times_{(\mathrm{Alg}^{/S}(\mathcal{C})^{\{0\}} \times_S \mathrm{Alg}^{/S}(\mathcal{C})^{\{1\}})} \mathrm{Alg}^{/S}(\mathcal{C})^{\Delta^1}$$

over $\mathrm{Alg}^{/S}(\mathcal{C}) \times_S \mathcal{D}_{\mathrm{End}}$.

5.2.4 Monads as endomorphism objects

Now we use the theory of endomorphism objects to associate a monad to a given right adjoint morphism in a 2-category.

Let $\mathcal{C}$ be a small 2-category and $g : Y \to X$ a morphism of $\mathcal{C}$ that admits a left adjoint.

Let $T \in \mathrm{Alg}([X, X])$ be a monad equipped with a left action on $g : Y \to X$ with respect to the canonical $[X, X]$-left module structure on $[Y, X]$.

We say that the left action map $\mu : T \circ g \to g$ in $[Y, X]$ exhibits T as the monad associated to g if μ exhibits T as the endomorphism object of g with respect to the canonical $[X, X]$-left module structure on $[Y, X]$.

The next proposition tells us that every right adjoint morphism in a 2-category admits an associated monad.

Proposition 5.31. *Let* $\mathcal{C}$ *be a 2-category.*

Let X, Y *be objects of* $\mathcal{C}$ *and* $g : Y \to X$ *a morphism of* $\mathcal{C}$ *that admits a left adjoint* $f : X \to Y$ *in* $\mathcal{C}$.

Denote $\eta : \mathrm{id}_X \to g \circ f$ *the unit and* $\varepsilon : f \circ g \to \mathrm{id}_Y$ *the counit of this adjunction.*

1. *For every morphism* $h : X \to X$ *of* $\mathcal{C}$ *the map*
   $\alpha : [X, X](h, g \circ f) \to [Y, X](h \circ g, g \circ f \circ g) \xrightarrow{[Y,X](h \circ g, g \circ \varepsilon)} [Y, X](h \circ g, g)$
   is an equivalence.

   So $g \circ \varepsilon : g \circ f \circ g \to g$ *exhibits* $g \circ f$ *as the endomorphism object of* $g : Y \to X$ *with respect to the canonical* $[X, X]$*-left module structure on* $[Y, X]$.

2. *Let* $T : X \to X$ *be a morphism of* $\mathcal{C}$ *and* $\varphi : T \circ g \to g$ *a morphism in* $[Y, X]$.

   Denote ψ *the composition* $T \xrightarrow{T \circ \eta} T \circ g \circ f \xrightarrow{\varphi \circ f} g \circ f$ *in* $[X, X]$ *and* γ *the composition*

   $$[X, X](h, T) \to [Y, X](h \circ g, T \circ g) \xrightarrow{[Y,X](h \circ g, \varphi)} [Y, X](h \circ g, g).$$

   The morphism ψ *is an equivalence if and only if for every morphism* $h : X \to X$ *of* $\mathcal{C}$ *the map* γ *is an equivalence.*

   So $\varphi : T \circ g \to g$ *exhibits* T *as the endomorphism object of* $g : Y \to X$ *with respect to the canonical* $[X, X]$*-left module structure on* $[Y, X]$ *if and only if* ψ *is an equivalence.*

Proof. Statement 1. and 2. follow from the following lemma 5.32. □

Lemma 5.32. *Let $\mathcal{C}$ be a 2-category.*

Let X, Y *be objects of* $\mathcal{C}$ *and* $g : Y \to X$ *a morphism of* $\mathcal{C}$ *that admits a left adjoint* $f : X \to Y$ *in* $\mathcal{C}$.

Denote $\eta : \mathrm{id}_X \to g \circ f$ *the unit and* $\varepsilon : f \circ g \to \mathrm{id}_Y$ *the counit of this adjunction.*

1. *For every morphism* $h : X \to X$ *of* $\mathcal{C}$ *the following two maps are inverse to each other:*

   $\alpha : [X,X](h, g \circ f) \to [Y,X](h \circ g, g \circ f \circ g) \xrightarrow{[Y,X](h \circ g, g \circ \varepsilon)} [Y,X](h \circ g, g)$

   $\beta : [Y,X](h \circ g, g) \to [X,X](h \circ g \circ f, g \circ f) \xrightarrow{[X,X](h \circ \eta, g \circ f)} [X,X](h, g \circ f).$

2. *Let* $T : X \to X$ *be a morphism of* $\mathcal{C}$ *and* $\varphi : T \circ g \to g$ *a morphism in* $[Y,X]$.

   Denote ψ *the composition* $T \xrightarrow{T \circ \eta} T \circ g \circ f \xrightarrow{\varphi \circ f} g \circ f$ *in* $[X,X]$.

   Then φ *factors as* $T \circ g \xrightarrow{\psi \circ g} g \circ f \circ g \xrightarrow{g \circ \epsilon} g$.

   Consequently for every morphism $h : X \to X$ *of* $\mathcal{C}$ *the map*

   $$\gamma : [X,X](h, T) \to [Y,X](h \circ g, T \circ g) \xrightarrow{[Y,X](h \circ g, \varphi)} [Y,X](h \circ g, g)$$

   factors as

   $$[X,X](h,T) \xrightarrow{[X,X](h,\psi)} [X,X](h, g \circ f) \xrightarrow{\alpha} [Y,X](h \circ g, g).$$

   Thus ψ *is an equivalence if and only if for every morphism* $h : X \to X$ *of* $\mathcal{C}$ *the map* γ *is an equivalence.*

3. *Let* $g : Y \to X, h : Z \to X$ *be morphisms of* $\mathcal{C}$ *that admit left adjoints* $f : X \to Y$ *respectively* $k : X \to Z$ *and let* $\phi : Y \to Z$ *be a morphism in* $\mathcal{C}$ *over* X.

   Denote ω *the morphism*

   $$h \circ k \to h \circ k \circ g \circ f \simeq h \circ k \circ h \circ \phi \circ f \to h \circ \phi \circ f \simeq g \circ f$$

   in $[X,X]$.

   Then $h \circ k \circ g \xrightarrow{\omega \circ g} g \circ f \circ g \to g$ *is equivalent to the composition*

   $$h \circ k \circ g \simeq h \circ k \circ h \circ \phi \to h \circ \phi \simeq g.$$

Proof. The composition

$$[Y,X](h \circ g, g \circ f \circ g) \xrightarrow{[Y,X](h \circ g, g \circ \varepsilon)} [Y,X](h \circ g, g)$$
$$\to [X,X](h \circ g \circ f, g \circ f)$$

is equivalent to the composition

$$[Y,X](h \circ g, g \circ f \circ g) \to [X,X](h \circ g \circ f, g \circ f \circ g \circ f) \xrightarrow{[X,X](h \circ g \circ f, g \circ \varepsilon \circ f)}$$
$$[X,X](h \circ g \circ f, g \circ f)$$

and the composition

$$[X,X](h\circ g\circ f, g\circ f) \xrightarrow{[X,X](h\circ\eta,g\circ f)} [X,X](h, g\circ f)$$
$$\to [Y,X](h\circ g, g\circ f\circ g)$$

is equivalent to the composition

$$[X,X](h\circ g\circ f, g\circ f) \to [Y,X](h\circ g\circ f\circ g, g\circ f\circ g)$$
$$\xrightarrow{[Y,X](h\circ\eta\circ g,g\circ f\circ g)} [Y,X](h\circ g, g\circ f\circ g).$$

So $\beta\circ\alpha$ is equivalent to

$$[X,X](h, g\circ f) \to [X,X](h\circ g\circ f, g\circ f\circ g\circ f) \xrightarrow{[X,X](h\circ\eta,g\circ f\circ g\circ f)}$$
$$[X,X](h, g\circ f\circ g\circ f) \xrightarrow{[X,X](h,g\circ\varepsilon\circ f)} [X,X](h, g\circ f)$$

and $\alpha\circ\beta$ is equivalent to

$$[Y,X](h\circ g, g) \to [Y,X](h\circ g\circ f\circ g, g\circ f\circ g) \xrightarrow{[Y,X](h\circ g\circ f\circ g,g\circ\varepsilon)}$$
$$[Y,X](h\circ g\circ f\circ g, g) \xrightarrow{[Y,X](h\circ\eta\circ g,g)} [Y,X](h\circ g, g).$$

As

$$[X,X](h, g\circ f) \to [X,X](h\circ g\circ f, g\circ f\circ g\circ f) \xrightarrow{[X,X](h\circ\eta,g\circ f\circ g\circ f)}$$
$$[X,X](h, g\circ f\circ g\circ f)$$

is equivalent to

$$[X,X](h, g\circ f) \xrightarrow{[X,X](h,g\circ f\circ\eta)} [X,X](h, g\circ f\circ g\circ f)$$

and

$$[Y,X](h\circ g, g) \to [Y,X](h\circ g\circ f\circ g, g\circ f\circ g) \xrightarrow{[Y,X](h\circ g\circ f\circ g,g\circ\varepsilon)}$$
$$[Y,X](h\circ g\circ f\circ g, g)$$

is equivalent to

$$[Y,X](h\circ g, g) \xrightarrow{[Y,X](h\circ g\circ\varepsilon,g)} [Y,X](h\circ g\circ f\circ g, g),$$

$\beta\circ\alpha$ is equivalent to

$$[X,X](h, g\circ f) \xrightarrow{[X,X](h,g\circ f\circ\eta)} [X,X](h, g\circ f\circ g\circ f) \xrightarrow{[X,X](h,g\circ\varepsilon\circ f)}$$
$$[X,X](h, g\circ f)$$

and $\alpha\circ\beta$ is equivalent to

$$[Y,X](h\circ g, g) \xrightarrow{[Y,X](h\circ g\circ\varepsilon,g)} [Y,X](h\circ g\circ f\circ g, g) \xrightarrow{[Y,X](h\circ\eta\circ g,g)}$$
$$[Y,X](h\circ g, g).$$

Therefore statement 1. follows from the triangular identities:

The compositions $f \xrightarrow{f\circ\eta} f\circ g\circ f \xrightarrow{\varepsilon\circ f} f$ and $g \xrightarrow{\eta\circ g} g\circ f\circ g \xrightarrow{g\circ\varepsilon} g$ of morphisms of the category $[X,Y]$ respectively $[Y,X]$ are the identities.

It remains to show 2:

The composition $\psi : \mathrm{T} \circ \mathrm{g} \xrightarrow{\mathrm{T}\circ\eta\circ\mathrm{g}} \mathrm{T} \circ \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \xrightarrow{\varphi\circ\mathrm{f}\circ\mathrm{g}} \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \xrightarrow{\mathrm{g}\circ\epsilon} \mathrm{g}$ is equivalent to

$\mathrm{T} \circ \mathrm{g} \xrightarrow{\mathrm{T}\circ\eta\circ\mathrm{g}} \mathrm{T} \circ \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \xrightarrow{\mathrm{T}\circ\mathrm{g}\circ\epsilon} \mathrm{T} \circ \mathrm{g} \xrightarrow{\varphi} \mathrm{g}$ and is thus equivalent to φ due to the triangular identities.

It remains to show 3:
The composition

$$\mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \to \mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \simeq \mathrm{h} \circ \mathrm{k} \circ \mathrm{h} \circ \phi \circ \mathrm{f} \circ \mathrm{g} \to \mathrm{h} \circ \phi \circ \mathrm{f} \circ \mathrm{g} \simeq \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \to \mathrm{g}$$

is equivalent to the composition

$$\mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \to \mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \simeq \mathrm{h} \circ \mathrm{k} \circ \mathrm{h} \circ \phi \circ \mathrm{f} \circ \mathrm{g} \to \mathrm{h} \circ \phi \circ \mathrm{f} \circ \mathrm{g} \to \mathrm{h} \circ \phi \simeq \mathrm{g}$$

and thus equivalent to the composition

$$\mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \to \mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \circ \mathrm{f} \circ \mathrm{g} \to \mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \simeq \mathrm{h} \circ \mathrm{k} \circ \mathrm{h} \circ \phi \to \mathrm{h} \circ \phi \simeq \mathrm{g},$$

which is equivalent to

$$\mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \simeq \mathrm{h} \circ \mathrm{k} \circ \mathrm{h} \circ \phi \to \mathrm{h} \circ \phi \simeq \mathrm{g}$$

by the triangular identities.

□

Proposition 5.33. *Let $\phi : \mathcal{C} \to \mathrm{S}$ be a functor such that for all objects s of S the fiber $\mathcal{C}_\mathsf{s}$ admits a final object $\mathrm{X}(\mathsf{s})$.*

Assume that one of the following conditions is satisfied:

1. *$\phi : \mathcal{C} \to \mathrm{S}$ is a locally cocartesian fibration.*
2. *$\phi : \mathcal{C} \to \mathrm{S}$ is a locally cartesian fibration such that the induced functors on the fibers preserve the final object.*

The category $\mathrm{Fun}_\mathrm{S}(\mathrm{S}, \mathcal{C})$ admits a final object $\alpha : \mathrm{S} \to \mathcal{C}$ such that for every $\mathsf{s} \in \mathrm{S}$ the image $\alpha(\mathsf{s})$ is the final object of $\mathcal{C}_\mathsf{s}$.

Especially a section $\alpha : \mathrm{S} \to \mathcal{C}$ of ϕ is a final object of $\mathrm{Fun}_\mathrm{S}(\mathrm{S}, \mathcal{C})$ if and only if for every $\mathsf{s} \in \mathrm{S}$ the image $\alpha(\mathsf{s})$ is the final object of $\mathcal{C}_\mathsf{s}$.

Proof. Denote $\mathcal{W}$ the full subcategory of Cat_∞ spanned by those categories K with the property that for every functor $\psi : \mathrm{K} \to \mathrm{S}$ the following condition holds:

The category $\mathrm{Fun}_\mathrm{S}(\mathrm{K}, \mathcal{C})$ admits a final object $\alpha : \mathrm{K} \to \mathcal{C}$ such that for every $\mathrm{k} \in \mathrm{K}$ the image $\alpha(\mathrm{k})$ is the final object of $\mathcal{C}_{\psi(\mathrm{k})}$.

We will show that $\mathcal{W} = \mathsf{Cat}_\infty$.

As Cat_∞ is the only full subcategory of Cat_∞ that contains the contractible category and Δ^1 and is closed in Cat_∞ under small colimits, it is enough to see that $\mathcal{W}$ contains the contractible category and Δ^1 and is closed in Cat_∞ under arbitrary coproducts and pushouts.

Tautologically the contractible category belongs to $\mathcal{W}$.

Being right adjoint to the functor $\mathcal{C}^{(-)} : \mathsf{Cat}_\infty \to (\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{op}}$ the functor $\mathrm{Fun}_\mathrm{S}(-, \mathcal{C}) : (\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{op}} \to \mathsf{Cat}_\infty$ sends small colimits to limits.

So the case of coproducts follows from the fact that an object in an arbitrary product of categories is a final object if every component is final in each factor.

Let $X, \mathcal{Y}, \mathcal{Z}$ be objects of $\mathcal{W}$ and $\mathcal{X} \coprod_{\mathcal{Y}} \mathcal{Z} \to S$ a functor. Then $\mathcal{X} \coprod_{\mathcal{Y}} \mathcal{Z} \to S$ is the pushout in $\mathsf{Cat}_{\infty/S}$ of the induced functors $\theta : \mathcal{Y} \to \mathcal{X}$ and $\varsigma : \mathcal{Y} \to \mathcal{Z}$ over S.

So the categories $\mathrm{Fun}_S(\mathcal{X}, \mathcal{C}), \mathrm{Fun}_S(\mathcal{Y}, \mathcal{C}), \mathrm{Fun}_S(\mathcal{Z}, \mathcal{C})$ admit final objects α, β respectively γ that take values in final objects of each fiber.

Hence the unique morphisms $\alpha \circ \theta \to \beta$ and $\gamma \circ \varsigma \to \beta$ in $\mathrm{Fun}_S(\mathcal{Y}, \mathcal{C})$ are equivalences being levelwise equivalences.

Thus the category $\mathrm{Fun}_S(\mathcal{X} \coprod_{\mathcal{Y}} \mathcal{Z}, \mathcal{C}) \simeq \mathrm{Fun}_S(\mathcal{X}, \mathcal{C}) \times_{\mathrm{Fun}_S(\mathcal{Y}, \mathcal{C})} \mathrm{Fun}_S(\mathcal{Z}, \mathcal{C})$ admits a final object that takes values in final objects of each fiber using that every object of the pushout $\mathcal{X} \coprod_{\mathcal{Y}} \mathcal{Z} \to S$ is the image of an object of $\mathcal{X}$ or $\mathcal{Z}$.

It remains to show that Δ^1 belongs to $\mathcal{W}$:

Let $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ be a morphism of S.

By assumption the fibers $\mathcal{C}_\mathsf{s}, \mathcal{C}_\mathsf{t}$ admit final objects $X(\mathsf{s})$ respectively $X(\mathsf{t})$.

If condition 1. holds, there is locally ϕ-cocartesian lift $X(\mathsf{s}) \to \mathsf{f}_*(X(\mathsf{s}))$ of f in $\mathcal{C}$, whose composition with the unique morphism $\mathsf{f}_*(X(\mathsf{s})) \to X(\mathsf{t})$ in $\mathcal{C}_\mathsf{t}$ yields a morphism $\alpha : X(\mathsf{s}) \to \mathsf{f}_*(X(\mathsf{s})) \to X(\mathsf{t})$ in $\mathcal{C}$ lying over f.

If condition 2. holds, there is locally ϕ-cartesian lift $\beta : X(\mathsf{s}) \to X(\mathsf{t})$ of f in $\mathcal{C}$.

Let $F : A \to B, G : X \to Y$ be morphisms of $\mathcal{C}$ lying over f.

We have a canonical equivalence

$$\mathrm{Fun}_S(\Delta^1, \mathcal{C})(F, G) \simeq (\{\mathsf{f}\} \times_{\mathrm{Fun}(\Delta^1, S)} \mathrm{Fun}(\Delta^1, \mathcal{C}))(F, G) \simeq$$

$$\{\mathrm{id}_\mathsf{f}\} \times_{\mathrm{Fun}(\Delta^1, S)(\mathsf{f},\mathsf{f})} \mathrm{Fun}(\Delta^1, \mathcal{C})(F, G) \simeq$$
$$\{\mathrm{id}_\mathsf{f}\} \times_{(S(\mathsf{s},\mathsf{s}) \times_{S(\mathsf{s},\mathsf{t})} S(\mathsf{t},\mathsf{t}))} (\mathcal{C}(A, X) \times_{\mathcal{C}(A,Y)} \mathcal{C}(B, Y)) \simeq$$
$$(\{\mathrm{id}_\mathsf{s}\} \times_{S(\mathsf{s},\mathsf{s})} \mathcal{C}(A, X)) \times_{(\{\mathrm{id}_\mathsf{f}\} \times_{S(\mathsf{s},\mathsf{t})} \mathcal{C}(A,Y))} (\{\mathrm{id}_\mathsf{t}\} \times_{S(\mathsf{t},\mathsf{t})} \mathcal{C}(B, Y)) \simeq$$
$$\mathcal{C}_\mathsf{s}(A, X) \times_{(\{\mathrm{id}_\mathsf{f}\} \times_{S(\mathsf{s},\mathsf{t})} \mathcal{C}(A,Y))} \mathcal{C}_\mathsf{t}(B, Y)$$

$$\simeq \begin{cases} \mathcal{C}_\mathsf{s}(A, X) \times_{\mathcal{C}_\mathsf{t}(\mathsf{f}_*(A),Y)} \mathcal{C}_\mathsf{t}(B, Y) \text{ if 1.holds.} \\ \mathcal{C}_\mathsf{s}(A, X) \times_{\mathcal{C}_\mathsf{t}(A,\mathsf{f}^*(Y))} \mathcal{C}_\mathsf{t}(B, Y) \text{ if 2.holds.} \end{cases}$$

So for $G = \alpha$ or $G = \beta$ we see that α respectively β is the final object of the category $\mathrm{Fun}_S(\Delta^1, \mathcal{C})$.

□

5.3 Eilenberg-Moore objects

In this section we develop the theory of Eilenberg-Moore objects in a given 2-category that abstract the category of algebras over a monad from the 2-category Cat_∞ to an arbitrary 2-category.

To do so we abstract in definition 5.35 the notions monadic functor and monad from Cat_∞ to an arbitrary 2-category.

We show in example 5.36 that for every small category S the 2-category $\mathsf{Cat}_{\infty/\mathrm{S}}$ admits Eilenberg-Moore objects and co-Eilenberg-Moore objects.

From this we deduce in theorem 5.47 that for every categorical pattern $\mathfrak{P}$ on S (see [18] def. B. 0.19) the subcategory of $\mathfrak{P}$-fibered objects of $\mathsf{Cat}_{\infty/\mathrm{S}}$ is closed in $\mathsf{Cat}_{\infty/\mathrm{S}}$ under Eilenberg-Moore objects and coEilenberg-Moore objects.

5.3.1 Eilenberg-Moore objects

Let S be a category and $\mathrm{G}: \mathcal{D} \to \mathcal{C}$ a functor over S that admits a left adjoint relative to S.

By proposition 6.78 the functor G over S admits an endomorphism object $\mathrm{T} \in \mathrm{Fun}_\mathrm{S}(\mathcal{C}, \mathcal{C})$ with respect to the canonical left module structure on $\mathrm{Fun}_\mathrm{S}(\mathcal{D}, \mathcal{C})$ over $\mathrm{Fun}_\mathrm{S}(\mathcal{C}, \mathcal{C})$.

By remark 5.19 for every category $\mathcal{B}$ over S we have a canonical equivalence

$$\theta : \mathrm{LMod}_\mathrm{T}(\mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{C})) \simeq \mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathrm{LMod}_\mathrm{T}^{/\mathrm{S}}(\mathcal{C}))$$

over $\mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{C})$.

For $\mathcal{B} = \mathcal{D}$ the endomorphism left module structure on G over T corresponds to lift $\alpha : \mathcal{D} \to \mathrm{LMod}_\mathrm{T}^{/\mathrm{S}}(\mathcal{C})$ of G.

We say that G is a monadic functor over S or that G exhibits $\mathcal{D}$ as monadic over $\mathcal{C}$ relative to S if α is an equivalence.

If S is contractible, we will drop S. In this case our definition coincides with the usual one.

Remark 5.34. *Let* $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ *be a functor over* S *that admits a left adjoint relative to* S.

Then G *is a monadic functor over* S *if and only if for every category* $\mathcal{B}$ *over* S *the induced functor* $\mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathrm{G}) : \mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{D}) \to \mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{C})$ *is a monadic functor.*

Proof. Let $\alpha : \mathcal{D} \to \mathrm{LMod}_\mathrm{T}^{/\mathrm{S}}(\mathcal{C})$ be the canonical lift of G from above.

By Yoneda α is an equivalence if and only if for every functor $\mathcal{B} \to \mathrm{S}$ the induced functor

$$\beta : \mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{D}) \to \mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathrm{LMod}_\mathrm{T}^{/\mathrm{S}}(\mathcal{C})) \simeq \mathrm{LMod}_\mathrm{T}(\mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{C}))$$

over $\mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{C})$ is an equivalence.

The canonical $\mathrm{Fun}_\mathrm{S}(\mathcal{C}, \mathcal{C})$-left module structure on $\mathcal{C}' := \mathrm{Fun}_\mathrm{S}(\mathcal{B}, \mathcal{C})$ is the pullback of the endomorphism left module structure on $\mathcal{C}'$ over $\mathrm{Fun}(\mathcal{C}', \mathcal{C}')$ along a canonical monoidal functor $\mathrm{Fun}_\mathrm{S}(\mathcal{C}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{C}', \mathcal{C}')$ that sends T to some monad T' on $\mathcal{C}'$. So we obtain a canonical equivalence $\mathrm{LMod}_{\mathrm{T}'}(\mathcal{C}') \simeq \mathrm{LMod}_\mathrm{T}(\mathcal{C}')$ over $\mathcal{C}'$.

By remark 5.27 the 2-functor $\mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, -) : \mathsf{Cat}_{\infty/\mathrm{S}} \to \mathsf{Cat}_{\infty}$ sends the endomorphism left module structure on G over T to the endomorphism left module structure on $\mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathrm{G}) : \mathcal{D}' := \mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathcal{D}) \to \mathcal{C}' = \mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathcal{C})$ over T' with respect to the canonical left module structure on $\mathrm{Fun}(\mathcal{D}', \mathcal{C}')$ over $\mathrm{Fun}(\mathcal{C}', \mathcal{C}')$ corresponding to the functor $\beta : \mathcal{D}' \to \mathrm{LMod}_{\mathrm{T}'}(\mathcal{C}')$ over $\mathcal{C}'$. So β is an equivalence if and only if $\mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathrm{G})$ is monadic.

□

By remark 5.34 the following definition generalizes the notion of monadic functor over S.

Definition 5.35. *(monadic morphism, Eilenberg-Moore object, representable monad)*

Let $\mathcal{C}$ be a 2-category and $\psi : \mathrm{Z} \to \mathrm{X}$ a morphism of $\mathcal{C}$.

- *We say that ψ exhibits Z as monadic over X or call $\psi : \mathrm{Z} \to \mathrm{X}$ a monadic morphism if ψ admits a left adjoint in $\mathcal{C}$ and for every object Y of $\mathcal{C}$ the induced functor $[\mathrm{Y}, \mathrm{Z}] \to [\mathrm{Y}, \mathrm{X}]$ is monadic.*

  Let $\mathrm{T} \in \mathrm{Alg}([\mathrm{X}, \mathrm{X}])$ be a monad.

- *We say that a morphism $\phi : \mathrm{Z} \to \mathrm{X}$ of $\mathcal{C}$ exhibits Z as an Eilenberg-Moore object of T or that $\phi : \mathrm{Z} \to \mathrm{X}$ is an Eilenberg-Moore object of T if ϕ is monadic and there is a left T-module structure on ϕ with respect to the canonical $[\mathrm{X}, \mathrm{X}]$-left module structure on $[\mathrm{Z}, \mathrm{X}]$ that exhibits T as the endomorphism object of ϕ.*

  In this case we say that $\phi : \mathrm{Z} \to \mathrm{X}$ represents the monad T.

- *We call the monad T representable if there is an Eilenberg-Moore object of T.*
- *We say that a right adjoint morphism $\phi : \mathrm{Z} \to \mathrm{X}$ of $\mathcal{C}$ is representable if its associated monad is representable.*

If every monad $\mathrm{T} \in \mathrm{Alg}([\mathrm{X}, \mathrm{X}])$ admits an Eilenberg-Moore object, we say that X admits Eilenberg-Moore objects.
If all objects X of $\mathcal{C}$ admit Eilenberg-Moore objects, we say that $\mathcal{C}$ admits Eilenberg-Moore objects.

If $\phi : \mathrm{Z} \to \mathrm{X}$ is a morphism of $\mathcal{C}$ that admits a left adjoint $\mathsf{f} : \mathrm{X} \to \mathrm{Z}$ and exhibits Z as an Eilenberg-Moore object of T, by proposition 5.31 we have a canonical equivalence $\mathrm{T} \simeq \phi \circ \mathsf{f}$ in $\mathrm{Fun}(\mathrm{X}, \mathrm{X})$.

By proposition 5.31 every right adjoint morphism $\mathrm{Y} \to \mathrm{X}$ admits an endomorphism object with respect to the canonical $[\mathrm{X}, \mathrm{X}]$-left module structure on $[\mathrm{Y}, \mathrm{X}]$. So every monadic morphism $\mathrm{Y} \to \mathrm{X}$ is an Eilenberg-Moore object of some monad T on X.

Tautologically every monadic morphism is representable.

We have the dual notion of comonadic morphism and coEilenberg-Moore object.
Given a 2-category $\mathcal{C}$ we call a morphism $\psi : \mathrm{Z} \to \mathrm{X}$ of $\mathcal{C}$ comonadic if ψ is a monadic morphism in $\mathcal{C}_{\mathrm{op}}$, i.e. if ψ admits a right adjoint and for every object Y of $\mathcal{C}$ the induced functor $[\mathrm{Y}, \mathrm{Z}] \to [\mathrm{Y}, \mathrm{X}]$ is comonadic.

Let $\mathrm{T} \in \mathrm{Alg}([\mathrm{X}, \mathrm{X}]^{\mathrm{op}})$ be a comonad.

We say that a morphism $\phi : \mathrm{Z} \to \mathrm{X}$ of $\mathcal{C}$ exhibits Z as a coEilenberg-Moore object of T or that $\phi : \mathrm{Z} \to \mathrm{X}$ is a coEilenberg-Moore object of T if ϕ is an Eilenberg-Moore object of T in $\mathcal{C}_{\mathrm{op}}$.

Example 5.36. *Let $\mathcal{C} \to \mathrm{S}$ be a functor and $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C}, \mathcal{C}))$ a monad.*

The forgetful functor $\psi : \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathcal{C}$ over S is an Eilenberg-Moore object of T in $\mathsf{Cat}_{\infty/\mathrm{S}}$.

So for every small category S the 2-category $\mathsf{Cat}_{\infty/\mathrm{S}}$ admits Eilenberg-Moore objects.

The opposite category involution $(\mathsf{Cat}_{\infty/\mathrm{S}})_{\mathrm{op}} \simeq \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ lifts to a canonical equivalence of 2-categories.

Thus $(\mathsf{Cat}_{\infty/\mathrm{S}})_{\mathrm{op}} \simeq \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ admits Eilenberg-Moore objects so that $\mathsf{Cat}_{\infty/\mathrm{S}}$ admits coEilenberg-Moore objects.

Given a comonad $\mathrm{L} \in \mathrm{Coalg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C}, \mathcal{C}))$ on a category $\mathcal{C}$ over S its coEilenberg-Moore object is given by the forgetful functor

$$\mathrm{coLMod}_{\mathrm{L}}^{/\mathrm{S}}(\mathcal{C}) \coloneqq \mathrm{LMod}_{\mathrm{L}}^{/\mathrm{S}^{\mathrm{op}}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}$$

over S.

Proof. By proposition 6.78 the functor $\psi : \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathcal{C}$ over S admits a left adjoint relative to S.

By remark 5.19 for every category $\mathcal{B}$ over S we have a canonical equivalence

$$\theta : \mathrm{LMod}_{\mathrm{T}}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}))$$

over $\mathrm{Fun}_{\mathrm{S}}(\mathcal{B}, \mathcal{C})$.

For $\mathcal{B} = \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})$ the identity corresponds under θ to a left T-module structure on $\psi : \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathcal{C}$ that exhibits T as the endomorphism object of ψ by lemma 5.43.

So ψ is a monadic functor over S with associated monad T, in other words ψ is an Eilenberg-Moore object of T in $\mathsf{Cat}_{\infty/\mathrm{S}}$.

□

Every 2-functor $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ that preserves monadic morphisms, also preserves Eilenberg-Moore objects:

Let $\psi : \mathrm{Z} \to \mathrm{X}$ be an Eilenberg-Moore object of some monad $\mathrm{T} \in \mathrm{Alg}([\mathrm{X}, \mathrm{X}])$ on some object X of $\mathcal{D}$.

By remark 5.27 the left T-module structure on ψ gives rise to a G(T)-left module structure on $\mathrm{G}(\psi) : \mathrm{G}(\mathrm{Z}) \to \mathrm{G}(\mathrm{X})$ that exhibits G(T) as the endomorphism object of $\mathrm{G}(\psi)$.

So if $\mathrm{G}(\psi)$ is a monadic morphism, $\mathrm{G}(\psi)$ is an Eilenberg-Moore object of G(T).

Observation 5.37. *Let $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ be a 2-functor that admits a left adjoint* F.

Then $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ preserves monadic morphisms and thus Eilenberg-Moore objects.

Proof. As a 2-functor $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ preserves right adjoint morphisms.

For all $\mathrm{X} \in \mathcal{C}, \mathrm{Y} \in \mathcal{D}$ the induced functor

$$[\mathrm{F}(\mathrm{X}), \mathrm{Y}] \to [\mathrm{G}(\mathrm{F}(\mathrm{X})), \mathrm{G}(\mathrm{Y})] \to [\mathrm{X}, \mathrm{G}(\mathrm{Y})]$$

is an equivalence.

Let $\psi : Z \to X$ be a morphism of $\mathcal{D}$. Then for every object Y of $\mathcal{C}$ the functor $[F(Y), Z] \to [F(Y), X]$ is equivalent to the functor $[Y, G(Z)] \to [Y, G(X)]$ so that with ψ also $G(\psi)$ is monadic.

□

Let $\mathcal{C}$ be a 2-category, $\mathcal{B}$ a subcategory of $\mathcal{C}$ and X an object of $\mathcal{B}$.

Let $\psi : Z \to X$ be an Eilenberg-Moore object in $\mathcal{C}$ of some monad T in $\mathcal{B}$ on X.

Then $\psi : Z \to X$ is an Eilenberg-Moore object of T in $\mathcal{B}$ if and only if ψ is a monadic morphism of $\mathcal{B}$ and by cor. 5.39 if and only if ψ belongs to $\mathcal{B}$ and for all morphisms $\alpha : Y \to Z$ of $\mathcal{C}$ with $\psi \circ \alpha : Y \to Z \to X$ also α belongs to $\mathcal{B}$.

We say that $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects of X if every Eilenberg-Moore object $\psi : Z \to X$ in $\mathcal{C}$ of some monad T in $\mathcal{B}$ on X is an Eilenberg-Moore object of T in $\mathcal{B}$, i.e. is monadic in $\mathcal{B}$.

We say that $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects if $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects of X for all objects X of $\mathcal{B}$.

Thus $\mathcal{B}$ is closed in $\mathcal{C}$ under Eilenberg-Moore objects of X if and only if every Eilenberg-Moore object $\psi : Z \to X$ in $\mathcal{C}$ of some monad T in $\mathcal{B}$ on X belongs to $\mathcal{B}$ such that for every morphism $\alpha : Y \to Z$ of $\mathcal{C}$ with $\psi \circ \alpha : Y \to Z \to X$ also α belongs to $\mathcal{B}$.

This has the following consequence:

Let $\mathcal{A} \subset \mathcal{C}, \mathcal{B} \subset \mathcal{C}$ be subcategories and $X \in \mathcal{A} \cap \mathcal{B} \subset \mathcal{C}$.

If $\mathcal{A}, \mathcal{B}$ are closed in $\mathcal{C}$ under Eilenberg-Moore objects of X, the subcategory $\mathcal{A} \cap \mathcal{B}$ is also closed in $\mathcal{C}$ under Eilenberg-Moore objects of X.

Lemma 5.38. *Suppose we have given a commutative square*

$$\begin{array}{ccc} \mathcal{D}' & \longrightarrow & \mathcal{D} \\ \downarrow{\scriptstyle \psi'} & & \downarrow{\scriptstyle \psi} \\ \mathcal{C}' & \longrightarrow & \mathcal{C}, \end{array} \tag{24}$$

where the horizontal functors are fully faithful and the right vertical functor ψ is monadic and its left adjoint $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ restricts to a functor $\mathcal{C}' \to \mathcal{D}'$.

Then the functor $\psi' : \mathcal{D}' \to \mathcal{C}'$ is monadic if and only if square 24 is a pullback square.

Proof. Denote $\mathrm{Fun}(\mathcal{C}, \mathcal{C})' \subset \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ the full subcategory spanned by the functors $\mathcal{C} \to \mathcal{C}$ that send objects of $\mathcal{C}'$ to objects of $\mathcal{C}'$.

Then the endomorphism left module structure on $\mathcal{C}$ over $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ restricts to a left module structure on $\mathcal{C}'$ over $\mathrm{Fun}(\mathcal{C}, \mathcal{C})'$ that is the pullback of the endomorphism left module structure on $\mathcal{C}'$ over $\mathrm{Fun}(\mathcal{C}', \mathcal{C}')$ along a canonical monoidal functor $\mathrm{Fun}(\mathcal{C}, \mathcal{C})' \to \mathrm{Fun}(\mathcal{C}', \mathcal{C}')$.

As $\psi : \mathcal{D} \to \mathcal{C}$ is monadic, we have a canonical equivalence $\mathcal{D} \simeq \mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ over $\mathcal{C}$ for some monad T on $\mathcal{C}$ with $\mathrm{T} \simeq \psi \circ \mathcal{F}$ in $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$.

As $\mathcal{F}$ restricts to a functor $\mathcal{C}' \to \mathcal{D}'$, the monad T is an associative algebra of $\mathrm{Fun}(\mathcal{C}, \mathcal{C})'$ and so gives rise to a monad T' on $\mathcal{C}'$.

We have a canonical equivalence $\mathrm{LMod}_{\mathrm{T}'}(\mathcal{C}') \simeq \mathcal{C}' \times_{\mathcal{C}} \mathrm{LMod}_{\mathrm{T}}(\mathcal{C}) \simeq \mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$ over $\mathcal{C}'$. So the functor $\mathcal{C}' \times_{\mathcal{C}} \mathcal{D} \to \mathcal{C}'$ is monadic.

Moreover $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ restricts to a functor $\mathcal{F}' : \mathcal{C}' \to \mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$ that restricts to a functor $\mathcal{C}' \to \mathcal{D}'$.

So by theorem 5.62 the canonical functor $\mathcal{D}' \to \mathcal{C}' \times_{\mathcal{C}} \mathcal{D}$ over $\mathcal{C}'$ is an equivalence if and only if ψ' is monadic.

□

Corollary 5.39.

Let $\mathcal{C}$ be a 2-category, $\mathcal{B}$ a subcategory of $\mathcal{C}, \psi : \mathrm{Z} \to \mathrm{X}$ a morphism of $\mathcal{B}$ and T a monad in $\mathcal{B}$ on X.

Assume that $\psi : \mathrm{Z} \to \mathrm{X}$ is a monadic morphism of $\mathcal{C}$ with left adjoint $\mathcal{F} : \mathrm{X} \to \mathrm{Z}$.

By lemma 5.38 the morphism ψ is monadic in $\mathcal{B}$ if and only if $\mathcal{F} : \mathrm{X} \to \mathrm{Z}$ is a morphism of $\mathcal{B}$ and for all $\mathrm{Y} \in \mathcal{B}$ the commutative square

$$\begin{array}{ccc} [\mathrm{Y}, \mathrm{Z}]_{\mathcal{B}} & \longrightarrow & [\mathrm{Y}, \mathrm{Z}]_{\mathcal{C}} \\ \downarrow & & \downarrow \\ [\mathrm{Y}, \mathrm{X}]_{\mathcal{B}} & \longrightarrow & [\mathrm{Y}, \mathrm{X}]_{\mathcal{C}} \end{array}$$

is a pullback square.

In other words the morphism ψ is monadic in $\mathcal{B}$ if and only if for all morphisms $\alpha : \mathrm{Y} \to \mathrm{Z}$ of $\mathcal{C}$ with $\psi \circ \alpha : \mathrm{Y} \to \mathrm{Z} \to \mathrm{X}$ also α belongs to $\mathcal{B}$ and the composition $\psi \circ \mathcal{F} : \mathrm{X} \to \mathrm{Z} \to \mathrm{X}$ is a morphism of $\mathcal{B}$.

Observation 5.40. *Let $\mathcal{C}$ be a 2-category and $\mathcal{B}$ a 2-localization of $\mathcal{C}$.*

Let $\phi : \mathrm{Z} \to \mathrm{X}$ be an Eilenberg-Moore object in $\mathcal{C}$ of a monad $\mathrm{T} \in \mathrm{Alg}([\mathrm{X}, \mathrm{X}])$ on some object X of $\mathcal{B}$.

Then Z belongs to $\mathcal{B}$ so that $\phi : \mathrm{Z} \to \mathrm{X}$ is an Eilenberg-Moore object of T in $\mathcal{B}$.

So if an object X of $\mathcal{B}$ admits Eilenberg-Moore objects in $\mathcal{C}$, it admits Eilenberg-Moore objects in $\mathcal{B}$ and thus with $\mathcal{C}$ also $\mathcal{B}$ admits Eilenberg-Moore objects.

If $\mathcal{B}$ is a 2-localization of $\mathcal{C}$, then $\mathcal{B}_{\mathrm{op}}$ is a 2-localization of $\mathcal{C}_{\mathrm{op}}$.

So every coEilenberg-Moore object in $\mathcal{C}$ of a comonad $\mathrm{T} \in \mathrm{coAlg}([\mathrm{X}, \mathrm{X}])$ on some object X of $\mathcal{B}$ is a coEilenberg-Moore object of T in $\mathcal{B}$.

Proof. Denote $\Phi : \mathfrak{Z} \to \mathfrak{X}$ the map of cocartesian fibrations over $\mathcal{C}^{\mathrm{op}}$ classifying the natural transformation $[-, \phi] : [-, \mathrm{Z}] \to [-, \mathrm{X}]$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

By prop. 5.41 the canonical map $\mathfrak{Z} \to \mathrm{LMod}_{\mathrm{T}}(\mathfrak{X})$ of cocartesian fibrations over $\mathcal{C}^{\mathrm{op}}$ over the cocartesian fibration $\mathfrak{X} \to \mathcal{C}^{\mathrm{op}}$ is an equivalence.

Thus for every morphism $\mathsf{f} : \mathrm{A} \to \mathrm{B}$ of $\mathcal{C}$ the commutative square

$$\begin{array}{ccc} [\mathrm{B}, \mathrm{Z}] & \longrightarrow & [\mathrm{A}, \mathrm{Z}] \\ \downarrow & & \downarrow \\ [\mathrm{B}, \mathrm{X}] & \longrightarrow & [\mathrm{A}, \mathrm{X}] \end{array}$$

is equivalent to the commutative square

$$\begin{array}{ccc} \mathrm{LMod}_{\mathrm{T}}([\mathrm{B},\mathrm{X}]) & \longrightarrow & \mathrm{LMod}_{\mathrm{T}}([\mathrm{A},\mathrm{X}]) \\ \downarrow & & \downarrow \\ [\mathrm{B},\mathrm{X}] & \longrightarrow & [\mathrm{A},\mathrm{X}]. \end{array}$$

So if $\mathsf{f} : \mathrm{A} \to \mathrm{B}$ is a local equivalence of $\mathcal{C}$, the $[\mathrm{X},\mathrm{X}]$-linear functor $[\mathrm{B},\mathrm{X}] \to [\mathrm{A},\mathrm{X}]$ is an equivalence and thus induces an equivalence $\mathrm{LMod}_{\mathrm{T}}([\mathrm{B},\mathrm{X}]) \to \mathrm{LMod}_{\mathrm{T}}([\mathrm{A},\mathrm{X}])$.

Hence the functor $[\mathrm{B},\mathrm{Z}] \to [\mathrm{A},\mathrm{Z}]$ is an equivalence so that Z belongs to $\mathcal{B}$.

□

Proposition 5.41. *Let $\mathcal{C}$ be a 2-category, $\mathrm{T} \in \mathrm{Alg}([\mathrm{X},\mathrm{X}])$ a monad on some object* X *of $\mathcal{C}$ and $\phi : \mathrm{Z} \to \mathrm{X}$ a right adjoint morphism of $\mathcal{C}$.*

Denote $\Phi : \mathfrak{Z} \to \mathfrak{X}$ the map of cocartesian fibrations over $\mathcal{C}^{\mathrm{op}}$ classifying the natural transformation $[-,\phi] : [-,\mathrm{Z}] \to [-,\mathrm{X}]$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}$.

Then $\phi : \mathrm{Z} \to \mathrm{X}$ is an Eilenberg-Moore object of T *if and only if $\mathfrak{Z}$ is equivalent over $\mathfrak{X}$ to $\mathrm{LMod}_{\mathrm{T}}(\mathfrak{X})$.*

Proof. By proposition 6.55 we have a 2-functor $\theta : \mathcal{C} \to \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}}$ that sends the morphism $\phi : \mathrm{Z} \to \mathrm{X}$ to $\Phi : \mathfrak{Z} \to \mathfrak{X}$. By definition of the notion of monadic morphism θ preserves monadic morphisms and thus Eilenberg-Moore objects.

So if $\phi : \mathrm{Z} \to \mathrm{X}$ is an Eilenberg-Moore object of T in $\mathcal{C}$, the map $\Phi : \mathfrak{Z} \to \mathfrak{X}$ of cocartesian fibrations over $\mathcal{C}^{\mathrm{op}}$ is an Eilenberg-Moore object of $\theta(\mathrm{T})$ in $\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}} \subset \mathsf{Cat}_{\infty/\mathcal{C}^{\mathrm{op}}}$.
Thus by 5.36 we have a canonical equivalence $\mathfrak{Z} \simeq \mathrm{LMod}^{/\mathcal{C}^{\mathrm{op}}}_{\mathrm{T}}(\mathfrak{X})$ over $\mathfrak{X}$.

On the other hand if there is an equivalence $\mathrm{LMod}^{/\mathcal{C}^{\mathrm{op}}}_{\mathrm{T}}(\mathfrak{X}) \simeq \mathfrak{Z}$ of cocartesian fibrations over $\mathcal{C}^{\mathrm{op}}$ over the cocartesian fibration $\mathfrak{X} \to \mathcal{C}^{\mathrm{op}}$, the assumptions of lemma 5.42 2. are satisfied so that the morphism $\phi : \mathrm{Z} \to \mathrm{X}$ is an Eilenberg-Moore object of T.

□

Lemma 5.42. *Let $\mathcal{C}$ be a 2-category and $\mathrm{T} \in \mathrm{Alg}([\mathrm{X},\mathrm{X}])$ a monad on some object* X *of $\mathcal{C}$. Let $\phi : \mathrm{Z} \to \mathrm{X}$ be a morphism of $\mathcal{C}$.*

Denote $\mathrm{T}' \in \{\mathrm{T}\} \times_{[\mathrm{X},\mathrm{X}]} \mathrm{LMod}_{\mathrm{T}}([\mathrm{X},\mathrm{X}])$ the left T*-module structure on* T *coming from the associative algebra structure on* T.

1. *Assume that ϕ is endowed with a left* T*-module structure such that for every $\mathrm{Y} \in \mathcal{C}$ the induced left* T*-module structure on $[\mathrm{Y},\phi] : [\mathrm{Y},\mathrm{Z}] \to [\mathrm{Y},\mathrm{X}]$ corresponds to an equivalence $[\mathrm{Y},\mathrm{Z}] \to \mathrm{LMod}_{\mathrm{T}}([\mathrm{Y},\mathrm{X}])$ over $[\mathrm{Y},\mathrm{X}]$.*

   Denote $\mathcal{T} : \mathrm{X} \to \mathrm{Z}$ the image of T' under this equivalence for $\mathrm{Y} = \mathrm{X}$ so that $\mathcal{T} : \mathrm{X} \to \mathrm{Z}$ lifts the functor T *along $\phi : \mathrm{Z} \to \mathrm{X}$ and denote $\eta : \mathrm{id}_{\mathrm{X}} \to \mathrm{T}$ the unit of the monad* T.

   Then $\eta : \mathrm{id}_{\mathrm{X}} \to \mathrm{T} \simeq \phi \circ \mathcal{T}$ exhibits $\mathcal{T} : \mathrm{X} \to \mathrm{Z}$ as the left adjoint of $\phi : \mathrm{Z} \to \mathrm{X}$.

2. *Let* $[\mathrm{X},\mathrm{X}],[\mathrm{Z},\mathrm{X}]$ *be endowed with the canonical left-module structures over* $[\mathrm{X},\mathrm{X}]$.

   Assume that the functor $[\mathcal{T},\mathrm{X}]:[\mathrm{Z},\mathrm{X}]\to[\mathrm{X},\mathrm{X}]$ *is a* $[\mathrm{X},\mathrm{X}]$*-linear functor such that the induced* $\mathcal{C}(\mathrm{X},\mathrm{X})$*-linear functor* $\mathcal{C}(\mathcal{T},\mathrm{X}):\mathcal{C}(\mathrm{Z},\mathrm{X})\to\mathcal{C}(\mathrm{X},\mathrm{X})$ *on maximal subspaces is the canonical* $\mathcal{C}(\mathrm{X},\mathrm{X})$*-linear functor.*

   Suppose that the commutative square

$$\begin{array}{ccc} [\mathrm{Z},\mathrm{Z}] & \xrightarrow{[\mathcal{T},\mathrm{Z}]} & [\mathrm{X},\mathrm{Z}] \\ \downarrow & & \downarrow \\ [\mathrm{Z},\mathrm{X}] & \xrightarrow{[\mathcal{T},\mathrm{X}]} & [\mathrm{X},\mathrm{X}] \end{array}$$

   is equivalent over $[\mathcal{T},\mathrm{X}]:[\mathrm{Z},\mathrm{X}]\to[\mathrm{X},\mathrm{X}]$ *to the commutative square*

$$\begin{array}{ccc} \mathrm{LMod}_{\mathrm{T}}([\mathrm{Z},\mathrm{X}]) & \xrightarrow{\mathrm{LMod}_{\mathrm{T}}([\mathcal{T},\mathrm{X}])} & \mathrm{LMod}_{\mathrm{T}}([\mathrm{X},\mathrm{X}]) \\ \downarrow & & \downarrow \\ [\mathrm{Z},\mathrm{X}] & \xrightarrow{[\mathcal{T},\mathrm{X}]} & [\mathrm{X},\mathrm{X}]. \end{array}$$

   Denote ϕ' *the left* T*-module structure on* $\phi:\mathrm{Z}\to\mathrm{X}$ *that corresponds to the identity of* Z*.*

   Then the left action map $\beta:\mathrm{T}\circ\phi\to\phi$ *of* ϕ' *exhibits* T *as the endomorphism object of* ϕ *with respect to the canonical* $[\mathrm{X},\mathrm{X}]$*-left module structure on* $[\mathrm{Z},\mathrm{X}]$*.*

Proof. 1: We first assume that $\phi:\mathrm{Z}\to\mathrm{X}$ admits a left adjoint $\mathcal{F}$.

Denote $\tilde{\eta}$ the unit of the adjunction $\mathcal{F}:\mathrm{X}\rightleftarrows\mathrm{Z}:\phi$ in $\mathcal{C}$.

The functor $[\mathrm{X},-]:\mathcal{C}\to\mathsf{Cat}_\infty$ is a 2-functor.

Thus the natural transformation $[\mathrm{X},\tilde{\eta}]:\mathrm{id}_{[\mathrm{X},\mathrm{X}]}\to[\mathrm{X},\phi]\circ[\mathrm{X},\mathcal{F}]$ exhibits $[\mathrm{X},\mathcal{F}]:[\mathrm{X},\mathrm{X}]\to[\mathrm{X},\mathrm{Z}]$ as the left adjoint of the forgetful functor $\mathrm{LMod}_{\mathrm{T}}([\mathrm{X},\mathrm{X}])\simeq[\mathrm{X},\mathrm{Z}]\xrightarrow{[\mathrm{X},\phi]}[\mathrm{X},\mathrm{X}]$, i.e. as the free left T-module functor.

As $\mathrm{T}\xrightarrow{\mathrm{T}\circ\eta}\mathrm{T}\circ\mathrm{T}\xrightarrow{\mu}\mathrm{T}$ is the identity, the unit $\eta:\mathrm{id}_{\mathrm{X}}\to\mathrm{T}$ of the monad T exhibits T′ as the free T-module generated by the tensor unit id_{X} of $[\mathrm{X},\mathrm{X}]$.

Thus there is a unique equivalence $\mathcal{T}\to\mathcal{F}\simeq[\mathrm{X},\mathcal{F}](\mathrm{id}_{\mathrm{X}})$ such that $\eta:\mathrm{id}_{\mathrm{X}}\to\phi\circ\mathcal{T}\simeq\phi\circ\mathcal{F}$ is homotopic to $\tilde{\eta}=[\mathrm{X},\tilde{\eta}](\mathrm{id}_{\mathrm{X}}):\mathrm{id}_{\mathrm{X}}\to\phi\circ\mathcal{F}$.

So $\eta:\mathrm{id}_{\mathrm{X}}\to\mathrm{T}\simeq\phi\circ\mathcal{T}$ exhibits $\mathcal{T}:\mathrm{X}\to\mathrm{Z}$ as the left adjoint of $\phi:\mathrm{Z}\to\mathrm{X}$.

Now let ϕ be arbitrary. We will show that ϕ admits a left adjoint.

By lemma 6.78 it is enough to see that for every $\mathrm{Y}\in\mathcal{C}$ the induced natural transformation $[\mathrm{Y},\eta]:\mathrm{id}\to[\mathrm{Y},\phi]\circ[\mathrm{Y},\mathcal{T}]$ of functors $[\mathrm{Y},\mathrm{X}]\to[\mathrm{Y},\mathrm{X}]$ exhibits $[\mathrm{Y},\mathcal{T}]$ as left adjoint to $[\mathrm{Y},\phi]$.

The 2-functor $[\mathrm{Y},-]:\mathcal{C}\to\mathsf{Cat}_\infty$ sends the monad T on X to a monad $[\mathrm{Y},\mathrm{T}]$ on $[\mathrm{Y},\mathrm{X}]$.

The left T-module structure on ϕ gives rise to a left $[\mathrm{Y},\mathrm{T}]$-module structure on $\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\phi]):\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{Z}])\to\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{X}])$.

We have a canonical equivalence

$$\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{Z}]) \simeq \mathrm{Fun}([\mathrm{Y},\mathrm{X}],\mathrm{LMod}_{\mathrm{T}}([\mathrm{Y},\mathrm{X}])) \simeq$$

$$\mathrm{LMod}_{[\mathrm{Y},\mathrm{T}]}(\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{X}]))$$

over $\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{X}])$.

We have a commutative square

$$\begin{array}{ccc} [\mathrm{X},\mathrm{Z}] & \longrightarrow & \mathrm{LMod}_{\mathrm{T}}([\mathrm{X},\mathrm{X}]) \\ \downarrow & & \downarrow \\ \mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{Z}]) & \longrightarrow & \mathrm{LMod}_{[\mathrm{Y},\mathrm{T}]}(\mathrm{Fun}([\mathrm{Y},\mathrm{X}],[\mathrm{Y},\mathrm{X}])). \end{array}$$

The functor $[\mathrm{Y},\phi] : [\mathrm{Y},\mathrm{Z}] \simeq \mathrm{LMod}_{\mathrm{T}}([\mathrm{Y},\mathrm{X}]) \to [\mathrm{Y},\mathrm{X}]$ admits a left adjoint.

So by what we have proved so far, $[\mathrm{Y},\eta] : \mathrm{id} \to [\mathrm{Y},\mathrm{T}] \simeq [\mathrm{Y},\phi] \circ [\mathrm{Y},\mathcal{T}]$ exhibits $[\mathrm{Y},\mathcal{T}] : [\mathrm{Y},\mathrm{X}] \to [\mathrm{Y},\mathrm{Z}]$ as the left adjoint of $[\mathrm{Y},\phi] : [\mathrm{Y},\mathrm{Z}] \to [\mathrm{Y},\mathrm{X}]$.

2.: By 1. the unit $\eta : \mathrm{id}_{\mathrm{X}} \to \mathrm{T} \simeq \phi \circ \mathcal{T}$ of the monad T exhibits $\mathcal{T}$ as the left adjoint of $\phi : \mathrm{Z} \to \mathrm{X}$. Thus by proposition 5.31 we have to see that the composition $\mathrm{T} \xrightarrow{\mathrm{T}\circ\eta} \mathrm{T} \circ \phi \circ \mathcal{T} \xrightarrow{\beta\circ\mathcal{T}} \phi \circ \mathcal{T}$ is an equivalence.

The $[\mathrm{X},\mathrm{X}]$-linear functor $\mathcal{T}^* := [\mathcal{T},\mathrm{X}] : [\mathrm{Z},\mathrm{X}] \to [\mathrm{X},\mathrm{X}]$ yields a functor $\mathrm{LMod}_{\mathrm{T}}([\mathrm{Z},\mathrm{X}]) \to \mathrm{LMod}_{\mathrm{T}}([\mathrm{X},\mathrm{X}])$ that sends the left T-module structure ϕ' on ϕ to the left T-module T'.

So $\mathrm{T} \circ \mathrm{T} \simeq \mathrm{T} \circ \mathcal{T}^*(\phi) \simeq \mathcal{T}^*(\mathrm{T} \circ \phi) \xrightarrow{\beta\circ\mathcal{T}} \mathcal{T}^*(\phi) \simeq \mathrm{T}$ is the multiplication map of the monad T as the canonical equivalence $\mathrm{T} \circ \mathcal{T}^*(\phi) \simeq \mathcal{T}^*(\mathrm{T} \circ \phi)$ is the associativity equivalence of $\mathcal{C}$.

So $\beta \circ \mathcal{T} : \mathrm{T} \circ \phi \circ \mathcal{T} \simeq \mathrm{T} \circ \mathrm{T} \to \phi \circ \mathcal{T} \simeq \mathrm{T}$ is the multiplication map of the monad T.

□

Lemma 5.43. *Let* $\mathrm{X} \to \mathrm{S}$ *be a functor and* $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{X},\mathrm{X}))$ *a monad.*

By remark 5.19 for every category Y *over* S *we have a canonical equivalence*

$$\theta : \mathrm{LMod}_{\mathrm{T}}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{Y},\mathrm{X})) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{Y},\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{X}))$$

over $\mathrm{Fun}_{\mathrm{S}}(\mathrm{Y},\mathrm{X})$.

For $\mathrm{Y} = \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{X})$ *the identity corresponds under* θ *to a left* T*-module structure on the forgetful functor* $\psi : \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{X}) \to \mathrm{X}$ *with respect to the canonical* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{X},\mathrm{X})$*-left module structure on* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{X}),\mathrm{X})$.

This left T*-module structure exhibits* T *as the endomorphism object of* ψ.

Proof. Set $\mathcal{C} := \mathsf{Cat}_{\infty/\mathrm{S}}$. Denote $\mathcal{U}'_{\mathrm{S}} \to \mathcal{C}^{\mathrm{op}} \times \mathrm{S}$ the map of cartesian fibrations over $\mathcal{C}^{\mathrm{op}}$ classifying the identity of $\mathcal{C}$.

We have a 2-functor $\theta : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathsf{Cat}_\infty) \simeq \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}}$ adjoint to the functor $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Cat}_\infty$ that sends ψ to a natural transformation $\mathrm{Fun}_{\mathrm{S}}(-,\phi) : \mathrm{Fun}_{\mathrm{S}}(-,\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{X})) \to \mathrm{Fun}_{\mathrm{S}}(-,\mathrm{X})$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ classified by the map

$$\Psi : \mathfrak{Y} := \mathrm{Fun}^{/\mathcal{C}^{\mathrm{op}}}_{\mathcal{C}^{\mathrm{op}}\times\mathrm{S}}(\mathcal{U}'_{\mathrm{S}}, \mathcal{C}^{\mathrm{op}} \times \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathrm{X})) \to \mathfrak{X} := \mathrm{Fun}^{/\mathcal{C}^{\mathrm{op}}}_{\mathcal{C}^{\mathrm{op}}\times\mathrm{S}}(\mathcal{U}'_{\mathrm{S}}, \mathcal{C}^{\mathrm{op}} \times \mathrm{X})$$

of cocartesian fibrations over $\mathcal{C}^{op}$ by theorem 5.23.

By remark 5.18 we have a $LM^{\otimes}$-monoidal category

$$\mathrm{Fun}^{/\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op} \times X)^{\otimes}$$

over $\mathcal{C}^{op}$, whose pullback along the monoidal diagonal functor

$$\mathcal{C}^{op} \times \mathrm{Fun}_S(X, X)^{\otimes} \to \mathrm{Map}_{\mathcal{C}^{op}}(\mathcal{U}'_S, \mathcal{C}^{op} \times \mathrm{Fun}_S(X, X))^{\otimes}$$

$$\simeq \mathrm{Fun}^{/\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op} \times S \times \mathrm{Fun}_S(X, X))^{\otimes}$$

over $\mathcal{C}^{op}$ classifies a left module structure on $\mathfrak{X} \to \mathcal{C}^{op}$ over $\mathrm{Fun}_S(X, X)$ that is the image of the endomorphism left module structure on $X \to S$ over $\mathrm{Fun}_S(X, X)$ under the 2-functor θ.

By cor. 5.41 $\phi : Y \to X$ is an Eilenberg-Moore object of T if and only if there is an equivalence $\mathrm{LMod}^{/\mathcal{C}^{op}}_T(\mathfrak{X}) \simeq \mathfrak{Y}$ of cocartesian fibrations over $\mathcal{C}^{op}$ over the cocartesian fibration $\mathfrak{X} \to \mathcal{C}^{op}$.

This equivalence is the composition of canonical equivalences

$$\mathrm{LMod}^{/\mathcal{C}^{op}}_T(\mathfrak{X}) = \mathrm{LMod}^{/\mathcal{C}^{op}}_T(\mathrm{Fun}^{\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op} \times X)) \simeq$$

$$\mathrm{Fun}^{/\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathrm{LMod}^{/\mathcal{C}^{op}\times S}_T(\mathcal{C}^{op}\times X)) \simeq \mathrm{Fun}^{/\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op}\times \mathrm{LMod}^{/S}_T(X)) = \mathfrak{Y}$$

over $\mathfrak{X}$ provided by remark 5.19 1.

□

5.3.2 An existence result for 2-categories with Eilenberg-Moore objects

Let S be a category. By example 5.36 the 2-category $\mathsf{Cat}_{\infty/S}$ admits Eilenberg-Moore objects and coEilenberg-Moore objects.

Goal of this subsection is to show that many subcategories of $\mathsf{Cat}_{\infty/S}$ are closed in $\mathsf{Cat}_{\infty/S}$ under Eilenberg-Moore objects and coEilenberg-Moore objects.

We will show that for every categorical pattern $\mathfrak{P}$ on S the subcategory $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S} \subset \mathsf{Cat}_{\infty/S}$ with objects the $\mathfrak{P}$-fibered objects and with morphisms the maps of those admits Eilenberg-Moore objects and coEilenberg-Moore objects which are preserved by the subcategory inclusion $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S} \subset \mathsf{Cat}_{\infty/S}$ (theorem 5.47).

Example 5.44. *Theorem 5.47 will imply that structure on a monad is reflected in structure on its category of algebras and dually structure on a comonad is reflected in structure on its category of coalgebras:*

Let T *be a monad on a category* $\mathcal{C}$ *and denote* $\mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$ *its category of algebras.*

Let L *be a comonad on* $\mathcal{C}$ *and denote* $\mathrm{coLMod}_L(\mathcal{C}) = \mathrm{LMod}_T(\mathcal{C}^{op})^{op} \to \mathcal{C}$ *its category of coalgebras.*

1. *If* $\mathcal{C}$ *carries the structure of an operad and* T *lifts to a map of operads such that the unit and multiplication of* T *are natural transformations of operads, then the forgetful functor* $\mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$ *and its left adjoint lift to maps of operads.*

2. *If $\mathcal{C}$ carries the structure of an operad and* L *lifts to a map of operads such that the counit and comultiplication of* L *are natural transformations of operads, then the forgetful functor* $\mathrm{coLMod}_{\mathrm{L}}(\mathcal{C}) \to \mathcal{C}$ *and its right adjoint lift to maps of operads.*

   Let $\mathcal{V}^{\otimes}$ *be a monoidal category.*

3. *If* $\mathcal{C}$ *carries the structure of a left module over* $\mathcal{V}$ *and* T *lifts to a* $\mathcal{V}$*-linear functor such that the unit and multiplication of* T *are* $\mathcal{V}$*-linear natural transformations, then the forgetful functor* $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C}) \to \mathcal{C}$ *and its left adjoint lift to* $\mathcal{V}$*-linear functors.*
4. *If* $\mathcal{C}$ *carries the structure of a left module over* $\mathcal{V}$ *and* L *lifts to a* $\mathcal{V}$*-linear functor such that the counit and comultiplication of* L *are* $\mathcal{V}$*-linear natural transformations, then the forgetful functor* $\mathrm{coLMod}_{\mathrm{L}}(\mathcal{C}) \to \mathcal{C}$ *and its right adjoint lift to* $\mathcal{V}$*-linear functors.*

5. *If* $\mathcal{C}$ *carries the structure of a symmetric monoidal category and* T *lifts to an oplax symmetric monoidal functor such that the unit and multiplication of* T *are oplax symmetric monoidal natural transformations, then the forgetful functor* $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C}) \to \mathcal{C}$ *lifts to a symmetric monoidal functor.*
6. *If* $\mathcal{C}$ *carries the structure of a symmetric monoidal category and* L *lifts to a lax symmetric monoidal functor such that the unit and multiplication of* L *are lax symmetric monoidal natural transformations, then the forgetful functor* $\mathrm{coLMod}_{\mathrm{L}}(\mathcal{C}) \to \mathcal{C}$ *lifts to a symmetric monoidal functor.*

We start with the following observation:

Observation 5.45. *Let* S *be a small category and* $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathrm{S})$ *a full subcategory.*

1. *The full subcategory of* $\mathsf{Cat}_{\infty/\mathrm{S}}$ *spanned by the (locally) cartesian fibrations relative to* $\mathcal{E}$ *admits Eilenberg-Moore objects, which are preserved by the full subcategory inclusion to* $\mathsf{Cat}_{\infty/\mathrm{S}}$.

   Dually, the full subcategory of $\mathsf{Cat}_{\infty/\mathrm{S}}$ *spanned by the (locally) cocartesian fibrations relative to* $\mathcal{E}$ *admits coEilenberg-Moore objects, which are preserved by the full subcategory inclusion to* $\mathsf{Cat}_{\infty/\mathrm{S}}$.

   Moreover for every (locally) cartesian fibration $\mathcal{C} \to \mathrm{S}$ *relative to* $\mathcal{E}$ *and every monad* $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C}, \mathcal{C}))$ *the functor* $\mathrm{LMod}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}) \to \mathcal{C}$ *is a map of (locally) cartesian fibrations relative to* $\mathcal{E}$.
   Dually for every (locally) cocartesian fibration $\mathcal{C} \to \mathrm{S}$ *relative to* $\mathcal{E}$ *and every comonad* $\mathrm{T} \in \mathrm{coAlg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C}, \mathcal{C}))$ *the functor* $\mathrm{coLMod}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{C}) \to \mathrm{S}$ *is a map of (locally) cocartesian fibrations relative to* $\mathcal{E}$.

2. *The subcategory of* $\mathsf{Cat}_{\infty/\mathrm{S}}$ *with objects the (locally) cartesian fibrations relative to* $\mathcal{E}$ *and with morphisms the maps of (locally) cartesian fibrations relative to* $\mathcal{E}$ *admits Eilenberg-Moore objects and coEilenberg-Moore objects, which are preserved by the subcategory inclusion to* $\mathsf{Cat}_{\infty/\mathrm{S}}$.

   Dually, the subcategory of $\mathsf{Cat}_{\infty/\mathrm{S}}$ *with objects the (locally) cocartesian fibrations relative to* $\mathcal{E}$*and with morphisms the maps of such admits Eilenberg-Moore objects and coEilenberg-Moore objects, which are preserved by the subcategory inclusion to* $\mathsf{Cat}_{\infty/\mathrm{S}}$.

Proof. By the canonical equivalence $(\mathsf{Cat}_{\infty/\mathrm{S}})_{\mathrm{op}} \simeq \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ of 2-categories it is enough to show that the full subcategory of $\mathsf{Cat}_{\infty/\mathrm{S}}$ spanned by the (locally) cartesian fibrations relative to $\mathcal{E}$ and the subcategory of $\mathsf{Cat}_{\infty/\mathrm{S}}$ with objects the (locally) (co)cartesian fibrations relative to $\mathcal{E}$ and with morphisms the maps of such admit Eilenberg-Moore objects, which are preserved by the subcategory inclusions to $\mathsf{Cat}_{\infty/\mathrm{S}}$.

By remark 5.39 2. it is enough to see the following:

1. For every (locally) cartesian fibration $\mathcal{C} \to \mathrm{S}$ relative to $\mathcal{E}$ and every monad $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C}))$ the functor $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is a (locally) cartesian fibration relative to $\mathcal{E}$ and the functor $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathcal{C}$ over S is a map of such.
2. For every (locally) cocartesian fibration $\mathcal{C} \to \mathrm{S}$ relative to $\mathcal{E}$ and every monad $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C}))$, whose underlying endofunctor of $\mathcal{C}$ over S is a map of (locally) cocartesian fibrations relative to $\mathcal{E}$, the functor $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is a (locally) cocartesian fibration relative to $\mathcal{E}$ and the functor $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathcal{C}$ over S preserves and reflects (locally) cocartesian morphisms lying over morpisms of $\mathcal{E}$.

1. follows from remark 5.12.

2: Denote $\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C})' \subset \mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C})$ the full subcategory spanned by the maps of (locally) cocartesian fibrations relative to $\mathcal{E}$.

The restriction of the endomorphism left module structure on $\mathcal{C}$ over $\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C})$ restricts to a left module structure over $\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C})'$ in the subcategory of $\mathsf{Cat}_{\infty/\mathrm{S}}$ of (locally) cocartesian fibrations relative to $\mathcal{E}$ and maps of such.

Thus the left module structure on $\mathcal{C} \to \mathrm{S}$ over $\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C})'$ is classified by a map $\mathcal{M}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ of cocartesian fibrations over $\mathrm{LM}^{\otimes}$ that is a map of (locally) cocartesian fibrations relative to $\mathcal{E}$.

So the functor $\mathrm{LMod}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C})') \times \mathcal{C}$ is a map of (locally) cocartesian fibrations relative to $\mathcal{E}$.

Moreover the map $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathcal{C}$ of (locally) cocartesian fibrations relative to $\mathcal{E}$ induces on the fiber over every object s of S the conservative forgetful functor $\mathrm{LMod}_{\mathrm{T}_{\mathsf{s}}}(\mathcal{C}_{\mathsf{s}}) \to \mathcal{C}_{\mathsf{s}}$.

This implies 2.

□

Observation 5.46. *Let* S *be a small category and* $\mathcal{E} \subset \mathrm{Fun}(\Delta^1,\mathrm{S})$ *a full subcategory.*

Let $\mathcal{B} \subset \mathsf{Cat}_{\infty/\mathrm{S},\mathcal{E}}^{\mathrm{cocart}}$ *be a 2-categorical localization.*

Then $\mathcal{B}$ *is closed in* $\mathsf{Cat}_{\infty/\mathrm{S}}$ *under Eilenberg-Moore objects and coEilenberg-Moore objects.*

We remark that a full subcategory $\mathcal{B} \subset \mathsf{Cat}_{\infty/\mathrm{S},\mathcal{E}}^{\mathrm{cocart}}$ is a 2-categorical localization if it is a localization and for every small category K cotensoring with K restricts to $\mathcal{B}$.

Recall that a categorical pattern $\mathfrak{P}$ on a category S is a triple $(\mathcal{E},\mathcal{F},\mathcal{K})$ consisting of full subcategories $\mathcal{E} \subset \mathrm{Fun}(\Delta^1,\mathrm{S})$, $\mathcal{F} \subset \mathrm{Fun}(\Delta^2,\mathrm{S})$ and a family of functors of the form $\mathrm{K}^{\triangleleft} \to \mathrm{S}$ for some category K such that $\mathcal{E}$ contains all equivalences, $\mathcal{F}$ contains all functors that factor through Δ^1

and every functor $K^{\triangleleft} \to S$ that belongs to $\mathcal{K}$ sends morphisms of $K^{\triangleleft}$ to $\mathcal{E}$ and triangles of $K^{\triangleleft}$ to $\mathcal{F}$ (see [18] B.0.19.).

Given a categorical pattern $\mathfrak{P} = (\mathcal{E}, \mathcal{F}, \mathcal{K})$ on a category S we call a functor $\psi : \mathcal{C} \to S$ fibered with respect to $\mathfrak{P}$ if it satisfies the following conditions:

1. The functor $\psi : \mathcal{C} \to S$ is a locally cocartesian fibration relative to $\mathcal{E}$.
2. Every locally ψ-cocartesian morphism lying over a morphism f of $\mathcal{E}$ is cocartesian with respect to the pullback $\Delta^2 \times_S \mathcal{C} \to \Delta^2$ of ψ along every functor $\Delta^2 \to S$ that belongs to $\mathcal{F}$ and whose restriction to $0 \to 1$ is f.
3. For every functor $K^{\triangleleft} \to S$ of $\mathcal{K}$ the pullback $K^{\triangleleft} \times_S \mathcal{C} \to K^{\triangleleft}$ (that is a cocartesian fibration by 1. and 2.) classifies a functor $K^{\triangleleft} \to \mathsf{Cat}_\infty$ that is a limit diagram.
4. For every functor $K^{\triangleleft} \to S$ of $\mathcal{K}$ and every cocartesian section of the pullback $K^{\triangleleft} \times_S \mathcal{C} \to K^{\triangleleft}$ the composition $K^{\triangleleft} \to K^{\triangleleft} \times_S \mathcal{C} \to \mathcal{C}$ is a ψ-limit diagram.

Denote $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S} \subset \mathsf{Cat}^{\text{loc.cocart}}_{\infty/S,\mathcal{E}}$ the full subcategory spanned by the $\mathfrak{P}$-fibered objects.

The full subcategory $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S} \subset \mathsf{Cat}^{\text{loc.cocart}}_{\infty/S,\mathcal{E}}$ is a localization that can be modeled by a Quillen adjunction as in [18] App. B.

For every small category K cotensoring with K restricts to an endofunctor of $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S}$.

From observation 5.46 we deduce the following proposition:

Proposition 5.47. *Let* S *be a category and* $\mathfrak{P}$ *a categorical pattern on* S.

The subcategory $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S} \subset \mathsf{Cat}_{\infty/S}$ *admits Eilenberg-Moore objects and coEilenberg-Moore objects which are preserved by the subcategory inclusion* $\mathsf{Cat}^{\mathfrak{P}}_{\infty/S} \subset \mathsf{Cat}_{\infty/S}$.

Example 5.48. *Let* $\mathcal{O}^{\otimes}$ *be an operad.*

- *Let* $\mathfrak{P}$ *be the categorial pattern for operads over* $\mathcal{O}^{\otimes}$.
  Then $\mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}$ *is closed in* $\mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ *under Eilenberg-Moore objects and coEilenberg-Moore objects.*

  This implies 1.-4. of example 5.44.

- *Denote* $\mathcal{W}$ *the full subcategory of* $\mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ *spanned by the (locally) cocartesian fibrations over* $\mathcal{O}^{\otimes}$.
  Then $\mathrm{Op}_{\infty/\mathcal{O}^{\otimes}} \cap \mathcal{W} \subset \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ *is the category of* $\mathcal{O}^{\otimes}$*-monoidal categories (respectively representable operads over* $\mathcal{O}^{\otimes}$*) and lax* $\mathcal{O}^{\otimes}$*-monoidal functors.*

  By observation 5.45 $\mathcal{W}$ *is closed in* $\mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ *under coEilenberg-Moore objects. Thus by 5.40 the 2-category* $\mathrm{Op}_{\infty/\mathcal{O}^{\otimes}} \cap \mathcal{W}$ *is closed in* $\mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ *under coEilenberg-Moore objects.*

  This together with observation 5.45 1. implies 6. of example 5.44.

  5. of of example 5.44 follows in the following way:

- *Let* $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *be a* $\mathcal{O}^\otimes$*-monoidal category classifying a* $\mathcal{O}^\otimes$*-monoid* ϕ *of* Cat_∞*.*

  Denote $(\mathcal{C}^\otimes)^{\mathrm{rev}} \to \mathcal{O}^\otimes$ *the fiberwise dual relative to* $\mathcal{O}^\otimes$ *of the cocartesian fibration* $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ *and* $(\mathcal{C}^\otimes)^\vee \simeq ((\mathcal{C}^\otimes)^{\mathrm{rev}})^{\mathrm{op}} \to (\mathcal{O}^\otimes)^{\mathrm{op}}$ *the cartesian fibration classifying* ϕ*.*

  Let $\mathrm{T} \in \mathrm{Coalg}(\mathrm{Fun}_{\mathcal{O}}^{\otimes\,\mathrm{oplax}}(\mathcal{C},\mathcal{C})^{\mathrm{op}})$ *be a comonad in* $\mathrm{Op}_{\infty/\mathcal{O}^\otimes} \cap \mathcal{W}$ *on* $(\mathcal{C}^\otimes)^{\mathrm{rev}} \to \mathcal{O}^\otimes$*, i.e. an oplax* $\mathcal{O}^\otimes$*-monoidal monad on* $\mathcal{C}$*.*
  So the coEilenberg-Moore object

  $$\mathrm{coLMod}_{\mathrm{T}}^{/\mathcal{O}^\otimes}((\mathcal{C}^\otimes)^{\mathrm{rev}}) = \mathrm{LMod}_{\mathrm{T}}^{/(\mathcal{O}^\otimes)^{\mathrm{op}}}(((\mathcal{C}^\otimes)^{\mathrm{rev}})^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{O}^\otimes$$

  of T *in* $\mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$ *is a* $\mathcal{O}^\otimes$*-monoidal category and the forgetful functor* $\mathcal{V} : \mathrm{coLMod}_{\mathrm{T}}^{/\mathcal{O}^\otimes}((\mathcal{C}^\otimes)^{\mathrm{rev}}) \to (\mathcal{C}^\otimes)^{\mathrm{rev}}$ *is a* $\mathcal{O}^\otimes$*-monoidal functor.*
  Thus

  $$\mathrm{LMod}_{\mathrm{T}}^{/(\mathcal{O}^\otimes)^{\mathrm{op}}}((\mathcal{C}^\otimes)^\vee)^\vee \simeq \mathrm{coLMod}_{\mathrm{T}}^{/\mathcal{O}^\otimes}((\mathcal{C}^\otimes)^{\mathrm{rev}})^{\mathrm{rev}} \to \mathcal{O}^\otimes$$

  is a $\mathcal{O}^\otimes$*-monoidal category and the forgetful functor* $\mathcal{V}^{\mathrm{rev}} : \mathrm{LMod}_{\mathrm{T}}^{/(\mathcal{O}^\otimes)^{\mathrm{op}}}((\mathcal{C}^\otimes)^\vee)^\vee \to \mathcal{C}^\otimes$ *is a* $\mathcal{O}^\otimes$*-monoidal functor.*

5.3.3 Kan-extensions in Eilenberg-Moore objects

Let T be a monad on some symmetric monoidal category $\mathcal{C}$ such that T lifts to an oplax symmetric monoidal functor and the unit and multiplication of T are oplax symmetric monoidal natural transformations.

Then by prop. 5.47 the forgetful functor $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C}) \to \mathcal{C}$ lifts to a symmetric monoidal functor.

In this section we will construct another symmetric monoidal structure on $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ with the property that not the forgetful functor $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C}) \to \mathcal{C}$ but the free functor $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ lifts to a symmetric monoidal functor (prop. 5.55):

Let T be a monad on some symmetric monoidal category $\mathcal{C}$ such that T lifts to a lax symmetric monoidal functor and the unit and multiplication of T are symmetric monoidal natural transformations.

Assume that $\mathcal{C}$ admits geometric realizations that are preserved by T and the tensorproduct of $\mathcal{C}$ in each component.

Then the free functor $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ lifts to a symmetric monoidal functor and the tensorproduct of $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ preserves geometric realizations in each component.

Moreover if the tensorproduct of $\mathcal{C}$ preserves small colimits in each component, the tensorproduct of $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ preserves small colimits in each component, too.

We obtain the following examples:

Example 5.49.

1. *Let* $\mathcal{C}$ *be a presentably symmetric monoidal category and* T *a monad on* $\mathcal{C}$ *such that* T *lifts to a lax symmetric monoidal functor and the unit and multiplication of* T *are symmetric monoidal natural transformations.*

Assume that T *is an accessible functor and preserves geometric realizations.*

Then $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ *is a presentably symmetric monoidal category and the free functor* $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ *lifts to a symmetric monoidal functor.*

2. *Let* $\mathcal{C}$ *be a presentably monoidal category and* $\mathcal{M}$ *a presentably left module over* $\mathcal{C}$.

   Let T *be a monad on* $\mathcal{M}$ *such that* T *lifts to a lax* $\mathcal{C}$*-linear functor and the unit and multiplication of* T *are* $\mathcal{C}$*-linear natural transformations.*

   Assume that T *is an accessible functor and preserves geometric realizations.*

   Then $\mathrm{LMod}_{\mathrm{T}}(\mathcal{M})$ *is a presentably left module over* $\mathcal{C}$ *and the free functor* $\mathcal{M} \to \mathrm{LMod}_{\mathrm{T}}(\mathcal{M})$ *is* $\mathcal{C}$*-linear.*

3. *Let* $\mathcal{C}$ *be a monoidal category compatible with geometric realizations and* $\mathcal{D}$ *a left module over* $\mathcal{C}$ *compatible with geometric realizations.*

   Let $\phi : \mathrm{A} \to \mathrm{B}$ *be a map of associative algebras in* $\mathcal{C}$.

   Denote T *the image of* ϕ *under the functor* $\mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{C})) \to \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathrm{Fun}(\mathcal{D}, \mathcal{D}))) \simeq \mathrm{Alg}(\mathrm{Fun}_{\Delta^1}(\mathcal{D} \times \Delta^1, \mathcal{D} \times \Delta^1))$.

   The functor $\mathrm{LMod}_{\mathrm{T}}^{/\Delta^1}(\mathcal{D} \times \Delta^1) \to \Delta^1$ *is a cocartesian fibration classifying the free functor* $\mathrm{B} \otimes_{\mathrm{A}} (-) : \mathrm{LMod}_{\mathrm{A}}(\mathcal{D}) \to \mathrm{LMod}_{\mathrm{B}}(\mathcal{D})$.

This follows from prop. 5.47 and the following prop. 5.55:

Let $\mathcal{C} \to \mathrm{S}$ be a cocartesian fibration that is compatible with geometric realizations and $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C}, \mathcal{C}))$ a monad such that for every object s of S the induced functor $\mathrm{T}_{\mathsf{s}} : \mathcal{C}_{\mathsf{s}} \to \mathcal{C}_{\mathsf{s}}$ on the fiber over s preserves geometric realizations.

Then $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is a cocartesian fibration compatible with geometric realizations and the free functor $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})$ over S is a map of cocartesian fibrations over S.

Moreover if $\mathcal{C} \to \mathrm{S}$ is compatible with small colimits, then $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is compatible with small colimits.

Moreover we can derive the following example:

Example 5.50.
Let $\mathrm{X}^{\otimes} \to \mathrm{E}_{\mathrm{k}}^{\otimes}$ *be an associative monoid in the category of* E_{k}*-operads for some natural* k *and* A *an* $\mathrm{E}_{\mathrm{k}+1}$*-algebra of* X.

Then by prop. 5.47 the category $\mathrm{LMod}_{\mathrm{A}}(\mathrm{X})$ *carries the structure of an* E_{k}*-operad and the forgetful functor* $\mathrm{LMod}_{\mathrm{A}}(\mathrm{X}) \to \mathrm{X}$ *and its left adjoint are maps of* E_{k}*-operads.*

If $\mathrm{X}^{\otimes} \to \mathrm{E}_{\mathrm{k}}^{\otimes}$ *is additionally a* E_{k}*-monoidal category that admits geometric realizations that are preserved by the tensor product of* $\mathrm{X}^{\otimes} \to \mathrm{E}_{\mathrm{k}}^{\otimes}$ *and the functor* $\mathrm{A} \otimes - : \mathrm{X} \to \mathrm{X}$ *induced by the associative monoid structure on* $\mathrm{X}^{\otimes} \to \mathrm{E}_{\mathrm{k}}^{\otimes}$*, then by proposition 5.55 the category* $\mathrm{LMod}_{\mathrm{A}}(\mathrm{X})$ *is a* E_{k}*-monoidal category and the free functor* $\mathrm{X} \to \mathrm{LMod}_{\mathrm{A}}(\mathrm{X})$ *is a* E_{k}*-monoidal functor.*

Moreover if X *admits small colimits that are preserved by the tensor product of* $\mathrm{X}^{\otimes} \to \mathrm{E}_{\mathrm{k}}^{\otimes}$*, then the same holds for* $\mathrm{LMod}_{\mathrm{A}}(\mathrm{X})$.

We deduce prop. 5.55 via cor. 5.59 from the fact that for every monad T on a category $\mathcal{C}$ the category $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ is generated under geometric realizations by the free T-algebras and from cor. 5.54:

Given a 2-category $\mathcal{C}$ the subcategory of $\mathcal{C}$ with objects those that are compatible with geometric realizations and with morphisms those that are compatible with geometric realizations is closed in $\mathcal{C}$ under Eilenberg-Moore objects.

We start with the following definitions:

Let $\mathcal{C}$ be a 2-category, X an object of $\mathcal{C}$ and $\varphi : \mathrm{A} \to \mathrm{B}$ a morphism of $\mathcal{C}$. Let $\mathrm{H} : \mathrm{A} \to \mathrm{X}$ and $\mathrm{H}' : \mathrm{B} \to \mathrm{X}$ be morphisms of $\mathcal{C}$ and $\alpha : \mathrm{H} \to \mathrm{H}' \circ \varphi$ a 2-morphism of $\mathcal{C}$.

- We say that α exhibits H' as the left kan-extension of H along φ and write $\mathrm{lan}_{\varphi}(\mathrm{H})$ for H' if the canonical map $[\mathrm{B}, \mathrm{X}](\mathrm{H}', \mathrm{G}) \to [\mathrm{A}, \mathrm{X}](\mathrm{H}' \circ \varphi, \mathrm{G} \circ \varphi) \to [\mathrm{A}, \mathrm{X}](\mathrm{H}, \mathrm{G} \circ \varphi)$ is an equivalence.
- We say that X admits left kan-extensions along φ if every morphisms $\mathrm{H} : \mathrm{A} \to \mathrm{X}$ admits a left kan-extension $\mathrm{B} \to \mathrm{X}$ along φ.

So X admits left kan-extensions along φ if and only if the functor $[\varphi, \mathrm{X}] : [\mathrm{B}, \mathrm{X}] \to [\mathrm{A}, \mathrm{X}]$ admits a left adjoint $\mathrm{lan}_{\varphi} : [\mathrm{A}, \mathrm{X}] \to [\mathrm{B}, \mathrm{X}]$.

Let $\phi : \mathrm{X} \to \mathrm{Y}$ be a morphism of $\mathcal{C}$. Let $\mathrm{H} : \mathrm{A} \to \mathrm{X}$ and $\mathrm{H}' : \mathrm{B} \to \mathrm{X}$ be morphisms of $\mathcal{C}$ and $\alpha : \mathrm{H} \to \mathrm{H}' \circ \varphi$ a 2-morphism of $\mathcal{C}$ that exhibits H' as the left kan-extension of H along φ.

- We say that $\phi : \mathrm{X} \to \mathrm{Y}$ preserves the left kan-extension of H along φ if $\phi \circ \alpha : \phi \circ \mathrm{H} \to \phi \circ \mathrm{H}' \circ \varphi$ exhibits $\phi \circ \mathrm{H}'$ as the left kan-extension of $\phi \circ \mathrm{H}$ along φ.
- We say that $\phi : \mathrm{X} \to \mathrm{Y}$ preserves left kan-extensions along φ if $\phi : \mathrm{X} \to \mathrm{Y}$ preserves the left kan-extension of every morphism $\mathrm{H} : \mathrm{A} \to \mathrm{X}$ of $\mathcal{C}$ along φ.

Let $\phi' : \mathrm{Y} \to \mathrm{Z}$ a morphism of $\mathcal{C}$. If $\phi : \mathrm{X} \to \mathrm{Y}$ preserves the left kan-extension of H along φ and $\phi' : \mathrm{Y} \to \mathrm{Z}$ preserves the left kan-extension of $\phi \circ \mathrm{H}$ along φ, then $\phi' \circ \phi : \mathrm{X} \to \mathrm{Z}$ preserves the left kan-extension of H along φ. So with $\phi : \mathrm{X} \to \mathrm{Y}$ and $\phi' : \mathrm{Y} \to \mathrm{Z}$ also the composition $\phi' \circ \phi : \mathrm{X} \to \mathrm{Z}$ preserves left kan-extensions along φ.

Let $\mathcal{C}$ be a 2-category, X an object of $\mathcal{C}$ and $\varphi : \mathrm{A} \to \mathrm{B}$ a morphism of $\mathcal{C}$. Let $\mathrm{H} : \mathrm{A} \to \mathrm{X}$ and $\mathrm{H}' : \mathrm{B} \to \mathrm{X}$ be morphisms of $\mathcal{C}$ and $\alpha : \mathrm{H}' \circ \varphi \to \mathrm{H}$ a 2-morphism of $\mathcal{C}$.

We say that α exhibits H' as the right kan-extension of H along φ and write $\mathrm{ran}_{\varphi}(\mathrm{H})$ for H' if α exhibits H' as the left kan-extension of H along φ in $\mathcal{C}_{\mathrm{op}}$.

Proposition 5.51. *Let $\mathcal{C}$ be a 2-category and $\psi : \mathrm{Y} \to \mathrm{X}$ an Eilenberg-Moore object for some monad T on some object X of $\mathcal{C}$.*

Let $\varphi : \mathrm{A} \to \mathrm{B}$ be a morphism of $\mathcal{C}$.

1. *If* X *admits left kan-extensions along* φ *and* $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ *preserves left kan-extensions along* φ*, then* Y *admits left kan-extensions along* φ *that are preserved and reflected by* $\psi: \mathrm{Y} \to \mathrm{X}$.
2. *If* X *admits right kan-extensions along* φ*, then* Y *admits right kan-extensions along* φ *that are preserved and reflected by* $\psi: \mathrm{Y} \to \mathrm{X}$.

The subcategory of $\mathcal{C}$ *with objects those that admit left (right) kan-extensions along* φ *and with morphisms those that preserve left (right) kan-extensions along* φ *is closed in* $\mathcal{C}$ *under Eilenberg-Moore objects and coEilenberg-Moore objects.*

The full subcategory of $\mathcal{C}$ *spanned by the objects of* $\mathcal{C}$ *that admit left (right) kan-extensions along* φ *is closed in* $\mathcal{C}$ *under coEilenberg-Moore objects (Eilenberg-Moore objects).*

Proof. 1.: Denote $[\mathrm{X},\mathrm{X}]'$ the full subcategory of $[\mathrm{X},\mathrm{X}]$ spanned by those morphisms $\mathrm{X} \to \mathrm{X}$ that preserve left kan-extensions along $\varphi: \mathrm{A} \to \mathrm{B}$.

As $[\mathrm{X},\mathrm{X}]'$ is closed under composition in $[\mathrm{X},\mathrm{X}]$, the monoidal structure on $[\mathrm{X},\mathrm{X}]$ restricts to a monoidal structure on $[\mathrm{X},\mathrm{X}]'$.

The functor $[\varphi,\mathrm{X}] : [\mathrm{B},\mathrm{X}] \to [\mathrm{A},\mathrm{X}]$ is $[\mathrm{X},\mathrm{X}]$-linear and thus also $[\mathrm{X},\mathrm{X}]'$-linear after pulling back along the monoidal full subcategory inclusion $[\mathrm{X},\mathrm{X}]' \subset [\mathrm{X},\mathrm{X}]$.

If X admits left kan-extensions along φ, the functor $[\varphi,\mathrm{X}] : [\mathrm{B},\mathrm{X}] \to [\mathrm{A},\mathrm{X}]$ admits a left adjoint $\mathrm{lan}_\varphi : [\mathrm{A},\mathrm{X}] \to [\mathrm{B},\mathrm{X}]$. Denote η the unit of this adjunction and let $\phi : \mathrm{X} \to \mathrm{X}$ a morphism of $\mathcal{C}$ that preserves left kan-extensions along φ.

Then for every morphisms $\mathrm{H} : \mathrm{A} \to \mathrm{X}$ of $\mathcal{C}$ the morphism $\mathrm{lan}_\varphi(\phi \circ \mathrm{H}) \to \phi \circ \mathrm{lan}_\varphi(\mathrm{H})$ in $[\mathrm{B},\mathrm{X}]$ adjoint to the morphism $\phi \circ \eta : \phi \circ \mathrm{H} \to \phi \circ \mathrm{lan}_\varphi(\mathrm{H}) \circ \varphi$ in $[\mathrm{A},\mathrm{X}]$ is an equivalence.

Hence we obtain a $[\mathrm{X},\mathrm{X}]'$-linear adjunction $\mathrm{lan}_\varphi : [\mathrm{A},\mathrm{X}] \rightleftarrows [\mathrm{B},\mathrm{X}] : [\varphi,\mathrm{X}]$.

So given a monad T on X that preserves left kan-extensions along φ, i.e. an associative algebra of $[\mathrm{X},\mathrm{X}]'$ we obtain an adjunction $\mathrm{LMod}_\mathrm{T}([\mathrm{A},\mathrm{X}]) \rightleftarrows \mathrm{LMod}_\mathrm{T}([\mathrm{B},\mathrm{X}])$ and a map of adjunctions from the adjunction

$\mathrm{LMod}_\mathrm{T}([\mathrm{A},\mathrm{X}]) \rightleftarrows \mathrm{LMod}_\mathrm{T}([\mathrm{B},\mathrm{X}])$ to the adjunction $\mathrm{lan}_\varphi : [\mathrm{A},\mathrm{X}] \rightleftarrows [\mathrm{B},\mathrm{X}] : [\varphi,\mathrm{X}]$.

Let $\psi : \mathrm{Y} \to \mathrm{X}$ be an Eilenberg-Moore object for T.

Then by corollary 5.41 the induced functor $[\mathrm{B},\mathrm{Y}] \to [\mathrm{A},\mathrm{Y}]$ is equivalent to the functor $\mathrm{LMod}_\mathrm{T}([\mathrm{B},\mathrm{X}]) \to \mathrm{LMod}_\mathrm{T}([\mathrm{A},\mathrm{X}])$ over the functor $[\mathrm{B},\mathrm{X}] \to [\mathrm{A},\mathrm{X}]$.

So the morphism $\mathrm{Y} \to \mathrm{X}$ yields a map of adjunctions from the adjunction $[\mathrm{A},\mathrm{Y}] \rightleftarrows [\mathrm{B},\mathrm{Y}]$ to the adjunction $[\mathrm{A},\mathrm{X}] \rightleftarrows [\mathrm{B},\mathrm{X}]$.
As the forgetful functors $\mathrm{LMod}_\mathrm{T}([\mathrm{B},\mathrm{X}]) \to [\mathrm{B},\mathrm{X}]$ and $\mathrm{LMod}_\mathrm{T}([\mathrm{A},\mathrm{X}]) \to [\mathrm{A},\mathrm{X}]$ are conservative, wee see that Y admits left kan-extensions along φ that are preserved and reflected by $\psi : \mathrm{Y} \to \mathrm{X}$.

2.: The proof of 2. is similar but easier than 1.

If X admits right kan-extensions along φ, the functor $[\varphi,\mathrm{X}] : [\mathrm{B},\mathrm{X}] \to [\mathrm{A},\mathrm{X}]$ admits a right adjoint $\mathrm{ran}_\varphi : [\mathrm{A},\mathrm{X}] \to [\mathrm{B},\mathrm{X}]$.

Hence we obtain a $[\mathrm{X},\mathrm{X}]$-linear adjunction $[\varphi,\mathrm{X}] : [\mathrm{B},\mathrm{X}] \rightleftarrows [\mathrm{A},\mathrm{X}] : \mathrm{ran}_\varphi$.

So given a monad T on X we obtain an adjunction $\mathrm{LMod}_\mathrm{T}([\mathrm{B},\mathrm{X}]) \rightleftarrows \mathrm{LMod}_\mathrm{T}([\mathrm{A},\mathrm{X}])$ and a map of adjunctions from the adjunction

$\mathrm{LMod}_\mathrm{T}([\mathrm{B},\mathrm{X}]) \rightleftarrows \mathrm{LMod}_\mathrm{T}([\mathrm{A},\mathrm{X}])$ to the adjunction $[\varphi,\mathrm{X}] : [\mathrm{B},\mathrm{X}] \rightleftarrows [\mathrm{A},\mathrm{X}] : \mathrm{ran}_\varphi$.

Let $\psi : \mathrm{Y} \to \mathrm{X}$ be an Eilenberg-Moore object for T.

Then by corollary 5.41 the induced functor $[\mathrm{B},\mathrm{Y}] \to [\mathrm{A},\mathrm{Y}]$ is equivalent to the functor $\mathrm{LMod}_{\mathrm{T}}([\mathrm{B},\mathrm{X}]) \to \mathrm{LMod}_{\mathrm{T}}([\mathrm{A},\mathrm{X}])$ over the functor $[\mathrm{B},\mathrm{X}] \to [\mathrm{A},\mathrm{X}]$.

So the morphism $\mathrm{Y} \to \mathrm{X}$ yields a map of adjunctions from the adjunction $[\mathrm{B},\mathrm{Y}] \rightleftarrows [\mathrm{A},\mathrm{Y}]$ to the adjunction $[\mathrm{B},\mathrm{X}] \rightleftarrows [\mathrm{A},\mathrm{X}]$.
As the forgetful functors $\mathrm{LMod}_{\mathrm{T}}([\mathrm{B},\mathrm{X}]) \to [\mathrm{B},\mathrm{X}]$ and $\mathrm{LMod}_{\mathrm{T}}([\mathrm{A},\mathrm{X}]) \to [\mathrm{A},\mathrm{X}]$ are conservative, wee see that Y admits right kan-extensions along φ that are preserved and reflected by $\psi : \mathrm{Y} \to \mathrm{X}$.

□

For $\mathcal{C} = \mathsf{Cat}_\infty$ proposition 5.51 says the following:

Let T be a monad on a category X and $\varphi : \mathrm{A} \to \mathrm{B}$ a functor.

If X admits left kan-extensions along φ that are preserved by T, then $\mathrm{LMod}_{\mathrm{T}}(\mathrm{X})$ admits left kan-extensions along φ that are preserved and reflected by the forgetful functor $\mathrm{LMod}_{\mathrm{T}}(\mathrm{X}) \to \mathrm{X}$.

If X admits right kan-extensions along φ, then $\mathrm{LMod}_{\mathrm{T}}(\mathrm{X})$ admits right kan-extensions along φ that are preserved and reflected by the forgetful functor $\mathrm{LMod}_{\mathrm{T}}(\mathrm{X}) \to \mathrm{X}$.

In the following we will study some consequences of proposition 5.51.

We begin by giving some further notions:

Let $\mathcal{C}$ be a 2-category, X an object of $\mathcal{C}$ and $\varphi : \mathrm{A} \to \mathrm{B}$ a functor.

- We say that X is compatible with left (right) kan-extensions along φ if for every object Y of $\mathcal{C}$ the category $[\mathrm{Y},\mathrm{X}]$ admits left (right) kan-extensions along φ and for every morphism $\beta : \mathrm{Z} \to \mathrm{Y}$ of $\mathcal{C}$ the functor $[\beta,\mathrm{X}] : [\mathrm{Y},\mathrm{X}] \to [\mathrm{Z},\mathrm{X}]$ preserves left (right) kan-extensions along φ.

- If φ is the full subcategory inclusion $\mathrm{K} \subset \mathrm{K}^{\triangleright}$ for some category K, we say that X is compatible with colimits indexed by K instead of saying that X is compatible with left kan-extensions along φ.

- Dually if φ is the full subcategory inclusion $\mathrm{K} \subset \mathrm{K}^{\triangleleft}$ for some category K, we say that X is compatible with limits indexed by K for saying that X is compatible with right kan-extensions along φ.

  Let X,X' be objects of $\mathcal{C}$ that are compatible with left kan-extensions along $\varphi : \mathrm{A} \to \mathrm{B}$.

- We say that a morphism $\theta : \mathrm{X} \to \mathrm{X}'$ of $\mathcal{C}$ is compatible with left (right) kan-extensions along φ if for every object Y of $\mathcal{C}$ the functor $[\mathrm{Y},\theta] : [\mathrm{Y},\mathrm{X}] \to [\mathrm{Y},\mathrm{X}']$ preserves left (right) kan-extensions along φ.

Observation 5.52. *Let $\mathcal{C}$ be a cotensored left module over Cat_∞.*

Then X *is compatible with left (right) kan-extensions along φ if and only if the morphism $\mathrm{X}^{\varphi} : \mathrm{X}^{\mathrm{B}} \to \mathrm{X}^{\mathrm{A}}$ of $\mathcal{C}$ admits a left (right) adjoint.*

If X,X' are objects of $\mathcal{C}$ that are both compatible with left (right) kan-extensions along $\varphi : \mathrm{A} \to \mathrm{B}$, then a morphism $\theta : \mathrm{X} \to \mathrm{X}'$ of $\mathcal{C}$ is compatible with left (right) kan-extensions along φ if and only if θ induces a map of adjunctions from the adjunction $\mathrm{X}^{\mathrm{A}} \rightleftarrows \mathrm{X}^{\mathrm{B}}$ to the adjunction $\mathrm{X}'^{\mathrm{A}} \rightleftarrows \mathrm{X}'^{\mathrm{B}}$.

Proof. By proposition 6.78 the morphism $X^\varphi : X^B \to X^A$ of $\mathcal{C}$ admits a left adjoint if and only if for every object Y of $\mathcal{C}$ the induced functor $\mathrm{Fun}(\varphi, [Y, X]) : \mathrm{Fun}(B, [Y, X]) \simeq [Y, X^B] \to [Y, X^A] \simeq \mathrm{Fun}(A, [Y, X])$ admits a left adjoint $\mathrm{lan}_\varphi^{[Y,X]}$ and for every morphism $\beta : Z \to Y$ of $\mathcal{C}$ the natural transformation

$$\mathrm{lan}_\varphi^{[Z,X]} \circ \mathrm{Fun}(A, [\beta, X]) \to \mathrm{Fun}(B, [\beta, X]) \circ \mathrm{lan}_\varphi^{[Y,X]}$$

adjoint to $\mathrm{Fun}(A, [\beta, X]) \to \mathrm{Fun}(A, [\beta, X]) \circ \mathrm{Fun}(\varphi, [Y, X]) \circ \mathrm{lan}_\varphi^{[Y,X]} \simeq \mathrm{Fun}(\varphi, [Z, X]) \circ \mathrm{Fun}(B, [\beta, X]) \circ \mathrm{lan}_\varphi^{[Y,X]}$ is an equivalence.

□

Example 5.53. *Let* $\mathcal{C} = \mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *for some small category* S.

Let $\varphi : A \to B$ *be a functor and* $X \to S$ *a cocartesian fibration.*

$X \to S$ *is compatible with left (right) kan-extensions along* φ *if and only if for every* $\mathsf{s} \in S$ *the fiber* X_s *admits left (right) kan-extensions along* φ *and for every morphism* $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ *of* S *the induced functor* $X_\mathsf{s} \to X_\mathsf{t}$ *preserves left (right) kan-extensions along* φ.

Let $X \to S, X' \to S$ *be cocartesian fibrations that are compatible with left (right) kan-extensions along* φ. *A map* $\theta : X \to X'$ *of cocartesian fibrations over* S *is compatible with left (right) kan-extensions along* φ *if and only if for every* $\mathsf{s} \in S$ *the induced functor* $X_\mathsf{s} \to X'_\mathsf{s}$ *on the fiber over* s *preserves left (right) kan-extensions along* φ.

Proof. The map $X^\varphi : X^B \to X^A$ of cocartesian fibrations over S admits a left adjoint in $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$, i.e. a left adjoint relative to S, if and only if for every $\mathsf{s} \in S$ the induced functor $\mathrm{Fun}(\varphi, X_\mathsf{s}) : \mathrm{Fun}(B, X_\mathsf{s}) \to \mathrm{Fun}(A, X_\mathsf{s})$ on the fiber over s admits a left adjoint $\mathrm{lan}_\varphi^{X_\mathsf{s}}$ and for every morphism $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ of S the natural transformation

$$\mathrm{lan}_\varphi^{X_\mathsf{t}} \circ \mathrm{Fun}(A, \mathsf{f}_*) \to \mathrm{Fun}(B, \mathsf{f}_*) \circ \mathrm{lan}_\varphi^{X_\mathsf{s}}$$

adjoint to $\mathrm{Fun}(A, \mathsf{f}_*) \to \mathrm{Fun}(A, \mathsf{f}_*) \circ \mathrm{Fun}(\varphi, X_\mathsf{s}) \circ \mathrm{lan}_\varphi^{X_\mathsf{s}} \simeq \mathrm{Fun}(\varphi, X_\mathsf{t}) \circ \mathrm{Fun}(B, \mathsf{f}_*) \circ \mathrm{lan}_\varphi^{X_\mathsf{s}}$ is an equivalence.

Similarly the map $X^\varphi : X^B \to X^A$ of cocartesian fibrations over S admits a right adjoint in $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$, i.e. a right adjoint relative to S that is a map of cocartesian fibrations over S, if and only if for every $\mathsf{s} \in S$ the induced functor $\mathrm{Fun}(\varphi, X_\mathsf{s}) : \mathrm{Fun}(B, X_\mathsf{s}) \to \mathrm{Fun}(A, X_\mathsf{s})$ on the fiber over s admits a right adjoint $\mathrm{ran}_\varphi^{X_\mathsf{s}}$ and for every morphism $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ of S the natural transformation

$$\mathrm{Fun}(B, \mathsf{f}_*) \circ \mathrm{ran}_\varphi^{X_\mathsf{s}} \to \mathrm{ran}_\varphi^{X_\mathsf{t}} \circ \mathrm{Fun}(A, \mathsf{f}_*)$$

adjoint to $\mathrm{Fun}(\varphi, X_\mathsf{t}) \circ \mathrm{Fun}(B, \mathsf{f}_*) \circ \mathrm{ran}_\varphi^{X_\mathsf{s}} \simeq \mathrm{Fun}(A, \mathsf{f}_*) \circ \mathrm{Fun}(\varphi, X_\mathsf{s}) \circ \mathrm{ran}_\varphi^{X_\mathsf{s}} \to \mathrm{Fun}(A, \mathsf{f}_*)$ is an equivalence.

By remark 5.52 $\theta : X \to X'$ is compatible with left kan-extensions along φ if and only if θ induces a map of adjunctions from the adjunction $X^A \rightleftarrows X^B$ to the adjunction $X'^A \rightleftarrows X'^B$, which is equivalent to the condition that for every $\mathsf{s} \in S$ the induced functor $X_\mathsf{s} \to X'_\mathsf{s}$ on the fiber over s induces a map of adjunctions from the adjunction $\mathrm{Fun}(A, X_\mathsf{s}) \rightleftarrows \mathrm{Fun}(B, X_\mathsf{s})$ to the adjunction $\mathrm{Fun}(A, X'_\mathsf{s}) \rightleftarrows \mathrm{Fun}(B, X'_\mathsf{s})$.

□

Corollary 5.54. *Let $\mathcal{C}$ be a 2-category and $\psi : \mathrm{Y} \to \mathrm{X}$ an Eilenberg-Moore object for some monad* T *on some object* X *of* $\mathcal{C}$. *Let* $\varphi : \mathrm{A} \to \mathrm{B}$ *be a functor.*

1. *If* X *is compatible with left kan-extensions along* φ *and* $\mathrm{T} : \mathrm{X} \to \mathrm{X}$ *is compatible with left kan-extensions along* φ*, then* Y *is compatible with left kan-extensions along* φ *and* $\psi : \mathrm{Y} \to \mathrm{X}$ *is compatible with left kan-extensions along* φ.
2. *If* X *is compatible with right kan-extensions along* φ*, then* Y *is compatible with right kan-extensions along* φ *and* $\psi : \mathrm{Y} \to \mathrm{X}$ *is compatible with right kan-extensions along* φ.

Thus the subcategory of $\mathcal{C}$ *with objects the objects of* $\mathcal{C}$ *that are compatible with left (right) kan-extensions along* φ *and with morphisms the morphisms of* $\mathcal{C}$ *that are compatible with left (right) kan-extensions along* φ *is closed in* $\mathcal{C}$ *under Eilenberg-Moore objects and coEilenberg-Moore objects.*

The full subcategory of $\mathcal{C}$ *spanned by the objects of* $\mathcal{C}$ *that are compatible with left (right) kan-extensions along* φ *is closed in* $\mathcal{C}$ *under coEilenberg-Moore objects (Eilenberg-Moore objects).*

Proposition 5.55. *Let* $\mathcal{C} \to \mathrm{S}$ *be a cocartesian fibration that is compatible with geometric realizations and let* $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathcal{C},\mathcal{C}))$ *be a monad such that for every object* s *of* S *the induced functor* $\mathcal{C}_{\mathsf{s}} \to \mathcal{C}_{\mathsf{s}}$ *on the fiber over* s *preserves geometric realizations.*

Then $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ *is a cocartesian fibration compatible with geometric realizations and the free functor* $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})$ *over* S *is a map of cocartesian fibrations over* S.

Moreover if $\mathcal{C} \to \mathrm{S}$ *is compatible with small colimits, then* $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ *is compatible with small colimits.*

Proof. By corollary 5.54 $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is compatible with geometric realizations.

So by remark 5.52 the functor $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})^{(\Delta^{\mathrm{op}})^{\triangleright}} \to \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})^{\Delta^{\mathrm{op}}}$ over S admits a left adjoint relative to S.

Being a relative left adjoint the free functor $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})$ over S preserves cocartesian morphisms.

For every object s of S the fiber $\mathcal{C}_{\mathsf{s}}$ is the only full subcategory of $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})_{\mathsf{s}} \simeq \mathrm{LMod}_{\mathrm{T}_{\mathsf{s}}}^{/\mathrm{S}}(\mathcal{C}_{\mathsf{s}})$ that contains the free T_{s}-algebras and is closed in $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})_{\mathsf{s}}$ under geometric realizations.

Hence by remark 5.59 $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is a cocartesian fibration and the free functor $\mathcal{C} \to \mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C})$ over S is a map of cocartesian fibrations over S.

By lemma 5.56 $\mathrm{LMod}_{\mathrm{T}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ is compatible with coproducts and is thus compatible with small colimits.

□

Lemma 5.56. *Let* $\mathcal{C}$ *be a category,* T *a monad on* $\mathcal{C}$ *and* I *a set.*
Assume that $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ *admits geometric realizations.*

1. *With* $\mathcal{C}$ *also* $\mathrm{LMod}_{\mathrm{T}}(\mathcal{C})$ *admits coproducts indexed by* I.

2. *Let* $H : LMod_T(\mathcal{C}) \to \mathcal{D}$ *be a functor that preserves geometric realizations.*

   If $LMod_T(\mathcal{C})$ *admits coproducts indexed by* I *and the composition* $H \circ T : \mathcal{C} \to LMod_T(\mathcal{C}) \to \mathcal{D}$ *preserves coproducts indexed by* I, *then* H *preserves coproducts indexed by* I.

Proof. 1:

Denote $\gamma : \mathcal{C} \to Fun(I, \mathcal{C})$ and $\delta : LMod_T(\mathcal{C}) \to Fun(I, LMod_T(\mathcal{C}))$ the diagonal functors.

Denote $\mathcal{W} \subset Fun(I, LMod_T(\mathcal{C}))$ the full subcategory spanned by the families $A = (A_i)_{i \in I}$ in $LMod_T(\mathcal{C})$ that admit a coproduct indexed by I, i.e. that the functor $Fun(I, LMod_T(\mathcal{C}))(A, -) \circ \delta : LMod_T(\mathcal{C}) \to \mathcal{S}$ is corepresentable.

$\mathcal{W}$ is closed under geometric realizations as $LMod_T(\mathcal{C})$ admits geometric realizations.

By [18] prop. 4.7.4.14. every object of $LMod_T(\mathcal{C})$ is the geometric realization of a simplicial object of $LMod_T(\mathcal{C})$ that takes values in the full subcategory of $LMod_T(\mathcal{C})$ spanned by the free T-algebras of $\mathcal{C}$.

Hence it is enough to see that for every family $B = (B_i)_{i \in I}$ in $\mathcal{C}$ the family $A := (T(B_i))_{i \in I}$ in $LMod_T(\mathcal{C})$ belongs to $\mathcal{W}$.

The functor $Fun(I, LMod_T(\mathcal{C}))(A, -) \circ \delta : LMod_T(\mathcal{C}) \to \mathcal{S}$ factors as the forgetful functor $LMod_T(\mathcal{C}) \to \mathcal{C}$ followed by the functor $Fun(I, \mathcal{C})(B, -) \circ \gamma : \mathcal{C} \to \mathcal{S}$.

As $\mathcal{C}$ admits coproducts indexed by I, the functor $Fun(I, \mathcal{C})(B, -) \circ \gamma : \mathcal{C} \to \mathcal{S}$ is corepresentable and thus also its composition with the right adjoint forgetful functor $LMod_T(\mathcal{C}) \to \mathcal{C}$ is corepresentable.

2:

Replacing $H : LMod_T(\mathcal{C}) \to \mathcal{D}$ by the functor $LMod_T(\mathcal{C}) \xrightarrow{H} \mathcal{D} \subset Fun(\mathcal{D}, \mathcal{S})^{op}$ we can assume that $\mathcal{D}$ admits coproducts indexed by I.

As $LMod_T(\mathcal{C})$ and $\mathcal{D}$ admit coproducts indexed by I, the diagonal functors $\delta : LMod_T(\mathcal{C}) \to Fun(I, LMod_T(\mathcal{C}))$ and $\delta' : \mathcal{D} \to Fun(I, \mathcal{D})$ admit left adjoints $\coprod$ respectively $\coprod'$. Denote

$$\alpha : \coprod' \circ Fun(I, H) \to H \circ \coprod$$

the natural transformation of functors $Fun(I, LMod_T(\mathcal{C})) \to \mathcal{D}$ adjoint to the natural transformation $Fun(I, H) \to Fun(I, H) \circ \delta \circ \coprod \simeq \delta' \circ H \circ \coprod$ of functors $Fun(I, LMod_T(\mathcal{C})) \to Fun(I, \mathcal{D})$.

As $LMod_T(\mathcal{C})$ admits geometric realizations, with $H : LMod_T(\mathcal{C})) \to \mathcal{D}$ also $Fun(I, H) : Fun(I, LMod_T(\mathcal{C})) \to Fun(I, \mathcal{D})$ preserves geometric realizations so that source and target of α are geometric realizations preserving functors.

Thus the full subcategory $\mathcal{Q} \subset Fun(I, LMod_T(\mathcal{C}))$ spanned by the objects X such that $\alpha(X)$ is an equivalence is closed under geometric realizations.

Consequently it is enough to see that for every family $B = (B_i)_{i \in I}$ in $\mathcal{C}$ the morphism $\alpha(T(B_1), ..., T(B_n)) : \coprod_{i=1}^n H'(T(B_i)) \to H'(\coprod_{i=1}^n T(B_i))$ is an equivalence.

But the composition $\coprod_{i=1}^n H'(T(B_i)) \to H'(\coprod_{i=1}^n T(B_i)) \simeq H'(T(\coprod_{i=1}^n B_i))$ is the canonical morphism and thus an equivalence as $H \circ T$ preserves coproducts indexed by I.

□

Let $\phi : X \to S$ be a functor and $\mathcal{E} \subset S$ a subcategory.

Denote $\tilde{X} \subset X$ the full subcategory spanned by the objects A lying over some object s of S such that for every morphism $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ of $\mathcal{E}$ there exists a ϕ-cocartesian lift $A \to B$ of f.

Denote $\phi' : \tilde{X} \subset X \xrightarrow{\phi} S$ the restriction.

Observation 5.57. *For every morphism of $\mathcal{E}$ the pullback $\Delta^1 \times_S \tilde{X} \to \Delta^1$ is a cocartesian fibration, whose cocartesian morphisms are ϕ'-cocartesian and the full subcategory inclusion $\tilde{X} \subset X$ sends ϕ'-cocartesian morphisms to ϕ-cocartesian morphisms.*

Proof. Let $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ be a morphism of $\mathcal{E}$ and $A \in \tilde{X}_{\mathsf{s}}$. Then there is a ϕ-cocartesian lift $A \to \mathsf{f}_*(A)$ of f. We will show that $\mathsf{f}_*(A)$ belongs to $\tilde{X}$.

Let $\mathsf{g} : \mathsf{t} \to \mathsf{r}$ be a morphism of $\mathcal{E}$. As A belongs to $\tilde{X}_{\mathsf{s}}$, there is a ϕ-cocartesian lift $A \to (\mathsf{g} \circ \mathsf{f})_*(A)$ of $\mathsf{g} \circ \mathsf{f} : \mathsf{s} \to \mathsf{t} \to \mathsf{r}$.

Using that the morphism $A \to \mathsf{f}_*(A)$ is ϕ-cocartesian, the morphism $A \to (\mathsf{g} \circ \mathsf{f})_*(A)$ factors as the morphism $A \to \mathsf{f}_*(A)$ followed by a lift $\mathsf{f}_*(A) \to (\mathsf{g} \circ \mathsf{f})_*(A)$ of $\mathsf{g} : \mathsf{t} \to \mathsf{r}$.

As the morphisms $A \to \mathsf{f}_*(A)$ and $A \to (\mathsf{g} \circ \mathsf{f})_*(A)$ are ϕ-cocartesian, the morphism $\mathsf{f}_*(A) \to (\mathsf{g} \circ \mathsf{f})_*(A)$ is ϕ-cocartesian, too. Thus $\mathsf{f}_*(A)$ belongs to $\tilde{X}$.

□

Lemma 5.58. *Let $\phi : X \to S$ be a functor, K a category and $\mathcal{E} \subset S$ a subcategory.*

If the diagonal functor $X \to X^K$ over S admits a left adjoint relative to S, then for every object s of S the fiber $\tilde{X}_{\mathsf{s}}$ is closed in X_{s} under colimits indexed by K.

Proof. Let $K^{\triangleright} \to X_{\mathsf{s}}$ be a colimit diagram, whose restriction $H : K \subset K^{\triangleright} \to X_{\mathsf{s}}$ factors through $\tilde{X}_{\mathsf{s}}$. We want to see that colim(H) belongs to $\tilde{X}_{\mathsf{s}}$.

Let $\mathsf{f} : \mathsf{s} \to \mathsf{t}$ be a morphism of $\mathcal{E}$. We have to find a ϕ-cocartesian lift $\mathrm{colim}(H) \to Z$ of f.

Denote $\phi' : \tilde{X} \subset X \xrightarrow{\phi} S$ the restriction and $\psi : \tilde{X}^K \simeq S \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(K, \tilde{X}) \to S$ the cotensor.

For every morphism $\Delta^1 \to \mathcal{E}$ the pullback $\Delta^1 \times_S \tilde{X}$ is a cocartesian fibration, whose cocartesian morphisms are ϕ'-cocartesian.

Thus for every morphism $\Delta^1 \to \mathcal{E}$ the pullback $\Delta^1 \times_S \tilde{X}^K$ is a cocartesian fibration, whose cocartesian morphisms are ψ-cocartesian, i.e. are levelwise ϕ'-cocartesian.

So we get a ψ-cocartesian morphism $\alpha : H \to \mathsf{f}_*(H)$ lying over f.

By assumption the diagonal functor $X \to X^K$ over S admits a left adjoint $\chi : X^K \to X$ relative to S.

χ sends α to a morphism $\beta : \mathrm{colim}(H) \to \mathrm{colim}(\mathsf{f}_*(H))$ of X lying over f.

The morphism β is ϕ-cocartesian as the composition $\tilde{X}^K \subset X^K \xrightarrow{\chi} X$ sends ψ-cocartesian morphisms to ϕ-cocartesian morphisms:

Being a relative left adjoint the functor $\chi : X^K \to X$ over S sends morphisms that are cocartesian with respect to the functor $X^K \to S$ to ϕ-cocartesian morphisms.

The full subcategory inclusion $\tilde{X} \subset X$ sends ϕ'-cocartesian morphisms to ϕ-cocartesian morphisms so that the full subcategory inclusion $\tilde{X}^K \subset X^K$ sends ψ-cocartesian morphisms to levelwise ϕ-cocartesian morphisms, which are especially cocartesian with respect to the functor $X^K \to S$ according to lemma 5.60.

□

Corollary 5.59. *Let $\phi : X \to S, \varphi : Y \to S$ be functors, $\xi : Y \to X$ a functor over S and K a category and $\mathcal{E} \subset S$ a subcategory.*

Assume that $\varphi : Y \to S$ is a cocartesian fibration relative to $\mathcal{E}$ and ξ sends φ-cocartesian morphisms lying over morphisms of $\mathcal{E}$ to ϕ-cocartesian morphisms. (This implies that for every object s of S the fiber $\tilde{X}_\mathsf{s}$ contains the essential image of $\xi_\mathsf{s} : Y_\mathsf{s} \to X_\mathsf{s}$.)

Assume that for every object s of S the fiber X_s is the only full subcategory of X_s that contains the essential image of $\xi_\mathsf{s} : Y_\mathsf{s} \to X_\mathsf{s}$ and is closed in X_s under colimits indexed by K.

If the diagonal functor $X \to X^K$ over S admits a left adjoint relative to S, then $\phi : X \to S$ is a cocartesian fibration relative to $\mathcal{E}$.

Proof. If the diagonal functor $X \to X^K$ over S admits a left adjoint relative to S, by lemma 5.58 $\tilde{X}_\mathsf{s}$ is closed in X_s under colimits indexed by K.

So by assumption we have $\tilde{X}_\mathsf{s} = X_\mathsf{s}$ and so $\tilde{X} = X$.

Thus by observation 5.57 for every morphism of $\mathcal{E}$ the pullback $\Delta^1 \times_S X \to \Delta^1$ is a cocartesian fibration, whose cocartesian morphisms are ϕ-cocartesian.

□

Lemma 5.60. *Let $\phi : X \to S$ be a functor and K a category.*

ϕ induces a functor $\mathrm{Fun}(K, \phi) : \mathrm{Fun}(K, X) \to \mathrm{Fun}(K, S)$.

A morphism τ of $\mathrm{Fun}(K, X)$ is $\mathrm{Fun}(K, \phi)$-cocartesian if it is levelwise ϕ-cocartesian, i.e. for every $k \in K$ the component $\tau(k)$ is ϕ-cocartesian.

Especially we have the following:

Denote $\psi : X^K \simeq S \times_{\mathrm{Fun}(K,S)} \mathrm{Fun}(K, X)$ the cotensor of the category K with the category X over S.

Every levelwise ϕ-cocartesian morphism of X^K is $\mathrm{Fun}(K, \phi)$-cocartesian and thus especially ψ-cocartesian.

Proof. Denote $\mathcal{W} \subset \mathsf{Cat}_\infty$ the full subcategory spanned by those categories K such that every levelwise ϕ-cocartesian morphisms of $\mathrm{Fun}(K, X)$ is $\mathrm{Fun}(K, \phi)$-cocartesian. We want to see that $\mathcal{W} = \mathsf{Cat}_\infty$.

As Cat_∞ is the only full subcategory of Cat_∞ that contains the contractible category and Δ^1 and is closed in Cat_∞ under small colimits, it is enough to check that $\mathcal{W}$ contains the contractible category and Δ^1 and is closed in Cat_∞ under small colimits.

Tautologically the contractible category belongs to $\mathcal{W}$.

To verify that $\mathcal{W}$ is closed in Cat_∞ under small colimits, it is enough to check that $\mathcal{W}$ is closed in Cat_∞ under arbitrary coproducts and pushouts.

As the functor $\mathrm{Fun}(-, X) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \mathsf{Cat}_\infty$ sends small colimits to limits, the case of coproducts follows from the fact that given a family of functors $(\theta_j : Y_j \to Z_j)_{j \in \mathcal{J}}$ a morphism in the product $\prod_{j \in \mathcal{J}} Y_j$ is $\prod_{j \in \mathcal{J}} \theta_j$-cocartesian if for every $j \in \mathcal{J}$ its image in Y_j is θ_j-cocartesian and the case of pushouts follows from the fact that given functors $\alpha : A \to X, \beta : B \to Y, \gamma : C \to$

Z and morphisms $\alpha \to \gamma, \beta \to \gamma$ in $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ a morphism in the pullback $A \times_C B$ is $\alpha \times_\gamma \beta$-cocartesian if its images in A, B, C are α, β respectively γ-cocartesian.

So it remains to show that Δ^1 belongs to $\mathcal{W}$.

We want to see that every levelwise ϕ-cocartesian morphism of $\mathrm{Fun}(\Delta^1, X)$ corresponding to a commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \longrightarrow & D \end{array}$$

in X, whose horizontal morphisms are ϕ-cocartesian, is $\mathrm{Fun}(\Delta^1, \phi)$-cocartesian.

Given a morphism $k : E \to F$ of X the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, X)(h,k) & \longrightarrow & \mathrm{Fun}(\Delta^1, X)(g,k) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\Delta^1, S)(\phi(h), \phi(k)) & \longrightarrow & \mathrm{Fun}(\Delta^1, S)(\phi(g), \phi(k)) \end{array}$$

is equivalent to the commutative square

$$\begin{array}{ccc} X(D,F) \times_{X(B,F)} X(B,E) & \longrightarrow & X(C,F) \times_{X(A,F)} X(A,E) \\ \downarrow & & \downarrow \\ S(\phi(D), \phi(F)) \times_{S(\phi(B),\phi(F))} S(\phi(B), \phi(E)) & \longrightarrow & S(\phi(C), \phi(F)) \times_{S(\phi(A),\phi(F))} S(\phi(A), \phi(E)) \end{array}$$

and is thus a pullback square as the morphisms $A \to B$ and $C \to D$ of X are ϕ-cocartesian and taking pullback preserves pullbacks being a right adjoint.

□

5.4 A localization between monads and right adjoint morphisms

Let S be a category, $\mathcal{C}^{\otimes} \to S \times LM^{\otimes}$ a cocartesian S-family of 2-categories and X a cocartesian section of $\mathcal{C} \to S$.

Denote $(\mathcal{C}^{/S}_{/X})^{mon} \subset (\mathcal{C}^{/S}_{/X})^{rep} \subset (\mathcal{C}^{/S}_{/X})^{R} \subset \mathcal{C}^{/S}_{/X}$ the full subcategories spanned by the morphisms $Y \to X(\mathsf{s})$ in $\mathcal{C}_{\mathsf{s}}$ for some $\mathsf{s} \in S$ that are monadic respectively whose associated monad on $X(\mathsf{s})$ is representable, i.e. admits an Eilenberg-Moore object, respectively admit a left adjoint.

We construct a map $\mathrm{End} : (\mathcal{C}^{/S}_{/X})^{R} \to \mathrm{Alg}^{/S}([X,X]^{/S})^{rev}$ of cocartesian fibrations over S that sends a morphism $g : Y \to X(\mathsf{s})$ for some $\mathsf{s} \in S$ with left adjoint $\mathsf{f} : X(\mathsf{s}) \to Y$ to its associated monad $g \circ \mathsf{f}$ on $X(\mathsf{s})$.

We show that the restriction $\mathrm{End} : (\mathcal{C}^{/S}_{/X})^{rep} \to (\mathrm{Alg}([X,X]^{/S})^{rep})^{rev}$ admits a fully faithful right adjoint Alg relative to S with essential image $(\mathcal{C}^{/S}_{/X})^{mon}$ (theorem 5.62).

Thus the functor End restricts to an equivalence

$$(\mathcal{C}^{/S}_{/X})^{mon} \to (\mathrm{Alg}([X,X]^{/S})^{rep})^{rev}$$

inverse to the functor Alg and the full subcategory $(\mathcal{C}^{/S}_{/X})^{mon} \subset (\mathcal{C}^{/S}_{/X})^{rep}$ is a localization relative to S.

If $\mathcal{C}$ is a subcategory of $\mathsf{Cat}_{\infty/S}$ for some small category S and $X \in \mathcal{C}$, we give a more explicite description of the adjunction $\mathrm{End} : (\mathcal{C}_{/X})^{rep} \to (\mathrm{Alg}([X,X])^{rep})^{op}$.

We show in theorem 5.68 that Alg is the restriction of the functor

$$\mathrm{Alg}(\mathrm{Fun}_S(X,X))^{op} \to ((\mathsf{Cat}_{\infty/S})_{/X})^{R} \subset \mathsf{Cat}_{\infty/X}$$

classified by the map $\mathrm{LMod}^{/S}(X) \to X \times \mathrm{Alg}(\mathrm{Fun}_S(X,X))$ of cartesian fibrations over $\mathrm{Alg}(\mathrm{Fun}_S(X,X))$.

Having this description we are able to give a more coherent version of the adjunction of theorem 5.62 for the case that $\mathcal{C}$ is a subcategory of $\mathsf{Cat}_{\infty/S}$:

We define a category $\mathrm{Alg}([X,X]^{/\mathcal{C}^{op}})^{rep}$ over $\mathcal{C}^{op}$, whose fiber over an object X of $\mathcal{C}$ is the category $\mathrm{Alg}([X,X])^{rep}$ of monads on X that admit an Eilenberg-Moore object in $\mathcal{C}$ that is preserved by the subcategory inclusion $\mathcal{C}_{/X} \subset (\mathsf{Cat}_{\infty/S})_{/X}$.

Denote $\mathrm{Fun}(\Delta^1,\mathcal{C})^{rep} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})$ the full subcategory spanned by the morphisms $Y \to X$, whose associated monad on X admits an Eilenberg-Moore object in $\mathcal{C}$ that is preserved by the subcategory inclusion $\mathcal{C}_{/X} \subset (\mathsf{Cat}_{\infty/S})_{/X}$.

We construct a localization

$$\mathrm{End} : \mathrm{Fun}(\Delta^1,\mathcal{C})^{rep} \rightleftarrows (\mathrm{Alg}([X,X]^{/\mathcal{C}^{op}})^{rep})^{op} : \mathrm{Alg}$$

relative to $\mathcal{C}$ that induces on the fiber over an object X of $\mathcal{C}$ the localization $\mathrm{End} : (\mathcal{C}_{/X})^{rep} \rightleftarrows (\mathrm{Alg}([X,X])^{rep})^{op} : \mathrm{Alg}$ (theorem 5.69), where we use the explicite description of the functor Alg given by theorem 5.68.

So the functor End restricts to an equivalence

$$\mathrm{Fun}(\Delta^1,\mathcal{C})^{mon} \to (\mathrm{Alg}([X,X]^{/\mathcal{C}^{op}})^{rep})^{op}$$

relative to $\mathcal{C}$ and the full subcategory $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{mon}} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{rep}}$ is a localization relative to $\mathcal{C}$.

From this we deduce the statement that for every 2-category $\mathcal{C}$ the full subcategory $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{mon}} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{rep}}$ is a localization relative to $\mathcal{C}$ (theorem 5.73). Moreover we show that this localization can be enhanced to a localization of 2-categories if $\mathcal{C}$ is cotensored over Cat_∞.

So if $\mathcal{C}$ is a 2-category that admits Eilenberg-Moore objects, we obtain a localization $\mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{mon}} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})^{\mathrm{R}}$ from monadic morphisms into right adjoint morphisms.

Construction 5.61.

Let S *be a category and* $\mathcal{C}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ *a cocartesian* S*-family of 2-categories.*

By proposition 6.55 we have a map

$$\theta : \mathcal{C}^\otimes \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathsf{Cat}_\infty)^\otimes$$

of S*-families of operads over* $\mathrm{LM}^\otimes$*, whose pullback to* $\mathrm{Ass}^\otimes$ *is the diagonal map*

$$\mathrm{S} \times \mathsf{Cat}_\infty^\times \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathsf{Cat}_\infty)^\otimes$$

of S*-families of operads over* $\mathrm{Ass}^\otimes$.

θ *induces a* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{S} \times \mathsf{Cat}_\infty)^\times \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty)^\times \simeq (\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}})^\times$*-linear map*

$$\chi : \mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})^\otimes \to \mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathsf{Cat}_\infty))^\otimes \simeq \mathrm{Fun}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathsf{Cat}_\infty)^\times$$

$$\simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{rev}}, \mathsf{Cat}_\infty)^\times \simeq (\mathsf{Cat}_{\infty/\mathcal{C}^{\mathrm{rev}}}^{\mathrm{cocart}})^\times$$

of operads over $\mathrm{LM}^\otimes$.

The composition

$$\mathcal{C} \xrightarrow{\theta} \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathsf{Cat}_\infty) \simeq \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathrm{Cocart} \to \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathcal{L}$$

of functors over S *is equivalent to the Yoneda-embedding*

$$\mathcal{C} \simeq \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathcal{U} \subset \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathcal{R} \simeq \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathcal{L}$$

over S.

Hence the composition $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}) \xrightarrow{\chi} \mathsf{Cat}_{\infty/\mathcal{C}^{\mathrm{rev}}}^{\mathrm{cocart}} \to \mathcal{L}_{\mathcal{C}^{\mathrm{rev}}}$ *is equivalent to the Yoneda-embedding*

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}) \simeq \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}) \subset \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{R}) \simeq \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{L}) \simeq \mathcal{L}_{\mathcal{C}^{\mathrm{rev}}}.$$

Let X *be a cocartesian section of* $\mathcal{C} \to \mathrm{S}$ *and* $\rho : \mathfrak{X} \to \mathcal{C}^{\mathrm{rev}}$ *the cocartesian fibration that classifies the functor* $\mathcal{C}^{\mathrm{rev}} \to \mathsf{Cat}_\infty$ *adjoint to the functor* $\theta \circ \mathrm{X} : \mathcal{C}^{\mathrm{rev}} \to \mathrm{S} \times \mathsf{Cat}_\infty$ *over* S.

So we have a canonical equivalence $(\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rev}} \simeq \mathfrak{X}^\simeq$ *of left fibrations over* $\mathcal{C}^{\mathrm{rev}}$*, where* $\mathfrak{X}^\simeq \subset \mathfrak{X}$ *denotes the subcategory with the same objects and with morphisms the* ρ*-cocartesian morphisms.*

We have a canonical endomorphism left module structure on X *over the cocartesian fibration* $[\mathrm{X}, \mathrm{X}]^{/\mathrm{S}} \to \mathrm{S}$ *with respect to the* $\mathrm{LM}^\otimes$*-operad structure on* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})$ *over* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{S} \times \mathsf{Cat}_\infty) \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty) \simeq \mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}$*, which is sent by* χ *to a left module structure on* $\rho : \mathfrak{X} \to \mathcal{C}^{\mathrm{rev}}$ *over the pullback* $\mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} [\mathrm{X}, \mathrm{X}]^{/\mathrm{S}} \to \mathcal{C}^{\mathrm{rev}}$.

Denote $\mathfrak{X}^{\mathrm{univ}}_{\mathrm{End}} \subset \mathfrak{X}^{\simeq} \simeq (\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rev}}$ *the full subcategory spanned by the morphisms* $\mathrm{g} : \mathrm{Y} \to \mathrm{X}(\mathsf{s})$ *in* $\mathcal{C}_{\mathsf{s}}$ *for some* $\mathsf{s} \in \mathrm{S}$ *that admit an endomorphism object with respect to the canonical* $[\mathrm{X}(\mathsf{s}), \mathrm{X}(\mathsf{s})]$*-left module structure on* $[\mathrm{Y}, \mathrm{X}(\mathsf{s})]$ *that is sent by any morphism* $\phi : \mathsf{s} \to \mathrm{t}$ *of* S *to an endomorphism object of* $\phi_*(\mathrm{g}) : \phi_*(\mathrm{Y}) \to \phi_*(\mathrm{X}_{\mathsf{s}}) \simeq \mathrm{X}(\mathrm{t})$ *with respect to the canonical* $[\mathrm{X}(\mathrm{t}), \mathrm{X}(\mathrm{t})]$*-left module structure on* $[\phi_*(\mathrm{Y}), \mathrm{X}(\mathrm{t})]$.

By proposition 5.31 we have an embedding $((\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{rev}} \subset \mathfrak{X}^{\mathrm{univ}}_{\mathrm{End}}$.

So by 5.2.3 we have a map

$$((\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{rev}} \to \mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})$$

of cocartesian fibrations over S *that is the endomorphism object of the inclusion* $((\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{rev}} \subset \mathfrak{X}$ *with respect to the left module structure on* $\mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(((\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{rev}}, \mathcal{X})$ *over* $\mathrm{Fun}_{\mathcal{C}^{\mathrm{rev}}}(((\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{rev}}, \mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} [\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})$
$\simeq \mathrm{Fun}_{\mathrm{S}}(((\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{rev}}, [\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})$.

Passing to fiberwise duals over S *we get a map*

$$\mathrm{End} : (\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{R}} \to \mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})^{\mathrm{rev}}$$

of cocartesian fibrations over S *that sends a morphism* $\mathrm{g} : \mathrm{Y} \to \mathrm{X}(\mathsf{s})$ *for some* $\mathsf{s} \in \mathrm{S}$ *with left adjoint* $\mathsf{f} : \mathrm{X}(\mathsf{s}) \to \mathrm{Y}$ *to its endomorphism object with respect to the canonical* $[\mathrm{X}(\mathsf{s}), \mathrm{X}(\mathsf{s})]$*-left module structure on* $[\mathrm{Y}, \mathrm{X}(\mathsf{s})]$, *which is given by* $\mathrm{g} \circ \mathsf{f}$ *according to proposition 5.31.*

This functor End *restricts to a functor*

$$\mathrm{End} : (\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rep}} \to (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{rev}}$$

over S.

Now we are ready to state the main theorem:

Theorem 5.62. *Let* S *be a category,* $\mathcal{C}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ *a cocartesian* S*-family of 2-categories and* X *a cocartesian section of* $\mathcal{C} \to \mathrm{S}$.

We have a localization $\mathrm{End} : (\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rep}} \to (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{rev}} : \mathrm{Alg}$ *relative to* S.

For every object $\mathsf{s} \in \mathrm{S}$ *the local objects of* $((\mathcal{C}_{\mathsf{s}})_{/\mathrm{X}(\mathsf{s})})^{\mathrm{rep}}$ *are the monadic morphisms over* $\mathrm{X}(\mathsf{s})$ *so that the restriction*

$$(\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{mon}} \subset (\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rep}} \xrightarrow{\mathrm{End}} (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{rev}}$$

is an equivalence and $(\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{mon}} \subset (\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rep}}$ *is a localization relative to* S.

Let S *be contractible and let* $\mathrm{g} : \mathrm{Y} \to \mathrm{X}, \mathrm{h} : \mathrm{Z} \to \mathrm{X}$ *be morphisms in* $\mathcal{C}$ *that admit left adjoints* $\mathsf{f} : \mathrm{X} \to \mathrm{Y}$ *respectively* $\mathrm{k} : \mathrm{X} \to \mathrm{Z}$.

A morphism $\phi : \mathrm{Y} \to \mathrm{Z}$ *in* $(\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}}$ *is a local equivalence if and only if the morphism*

$$\mathrm{h} \circ \mathrm{k} \to \mathrm{h} \circ \mathrm{k} \circ \mathrm{g} \circ \mathsf{f} \simeq \mathrm{h} \circ \mathrm{k} \circ \mathrm{h} \circ \phi \circ \mathsf{f} \to \mathrm{h} \circ \phi \circ \mathsf{f} \simeq \mathrm{g} \circ \mathsf{f}$$

in $[\mathrm{X}, \mathrm{X}]$ *is an equivalence.*

Especially a morphism $\phi : \mathrm{Y} \to \mathrm{Z}$ *in* $(\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}}$ *with* Z *a local object is a local equivalence if and only if the morphism* $\mathrm{k} \to \mathrm{k} \circ \mathrm{g} \circ \mathsf{f} \simeq \mathrm{k} \circ \mathrm{h} \circ \phi \circ \mathsf{f} \to \phi \circ \mathsf{f}$ *in* $[\mathrm{X}, \mathrm{Z}]$ *is an equivalence.*

Let $\mathrm{g} : \mathrm{Y} \to \mathrm{X}$ *be a right adjoint morphism in* $\mathcal{C}$ *with associated monad* T *that admits an Eilenberg-Moore object* $\psi : \mathrm{Z} \to \mathrm{X}$ *in* $\mathcal{C}$.

We have a canonical equivalence $[\mathrm{Y}, \mathrm{Z}] \simeq \mathrm{LMod}_{\mathrm{T}}([\mathrm{Y}, \mathrm{X}])$ *over* $[\mathrm{Y}, \mathrm{X}]$ *under which the endomorphism left module structure on* $\mathrm{g} : \mathrm{Y} \to \mathrm{X}$ *over* T *corresponds to a lift* $\mathrm{g}' : \mathrm{Y} \to \mathrm{Z}$ *of* $\mathrm{g} : \mathrm{Y} \to \mathrm{X}$.

The morphism $\mathrm{g}' : \mathrm{Y} \to \mathrm{Z}$ *is a local equivalence in* $(\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}}$ *with target a local object.*

Proof. Being a map of cocartesian fibrations over S the functor End : $(\mathcal{C}^{/\mathrm{S}}_{/\mathrm{X}})^{\mathrm{rep}} \to (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{rev}}$ over S admits a fully faithful right adjoint relative to S if and only if for every $\mathsf{s} \in \mathrm{S}$ the induced functor $\mathrm{End}_{\mathsf{s}} : ((\mathcal{C}_{\mathsf{s}})_{/\mathrm{X}(\mathsf{s})})^{\mathrm{rep}} \to (\mathrm{Alg}([\mathrm{X}(\mathsf{s}), \mathrm{X}(\mathsf{s})])^{\mathrm{rep}})^{\mathrm{op}}$ on the fiber over s admits a fully faithful right adjoint.

So we can reduce to the case that S is contractible.

Let $\phi : \mathrm{Z} \to \mathrm{X}$ be a monadic morphism of $\mathcal{C}$ and $\mathrm{T} \simeq \mathrm{End}(\mathrm{Z})$ its endomorphism object with respect to the canonical $[\mathrm{X}, \mathrm{X}]$-left module structure on $[\mathrm{Y}, \mathrm{X}]$.

It is enough to find an equivalence

$$\alpha : \mathcal{C}_{/\mathrm{X}}(-, \mathrm{Z}) \simeq \mathrm{Alg}([\mathrm{X}, \mathrm{X}])^{\mathrm{op}}(\mathrm{End}(-), \mathrm{End}(\mathrm{Z}))$$

of functors $((\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}})^{\mathrm{op}} \to \mathcal{S}$ such that under the induced equivalence

$$\mathcal{C}_{/\mathrm{X}}(\mathrm{Z}, \mathrm{Z}) \simeq \mathrm{Alg}([\mathrm{X}, \mathrm{X}])^{\mathrm{op}}(\mathrm{End}(\mathrm{Z}), \mathrm{End}(\mathrm{Z}))$$

of spaces the identity of Z corresponds to an autoequivalence of End(Z).

The morphism $\phi : \mathrm{Z} \to \mathrm{X}$ induces a natural transformation $[-, \phi] : [-, \mathrm{Z}] \to [-, \mathrm{X}]$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}_{\infty}$ classified by a map $\mathfrak{Z} \to \mathfrak{X}$ of cocartesian fibrations over $\mathcal{C}^{\mathrm{op}}$. By 5.41 there is a canonical equivalence $\mathfrak{Z} \to \mathrm{LMod}_{\mathrm{T}}^{/\mathcal{C}^{\mathrm{op}}}(\mathfrak{X})$ over $\mathfrak{X}$.

By remark 5.30 2. we have a canonical equivalence

$$(\mathrm{Alg}([\mathrm{X}, \mathrm{X}]) \times ((\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{op}}) \times_{(\mathrm{Alg}([\mathrm{X}, \mathrm{X}]) \times \mathfrak{X})} \mathrm{LMod}^{/\mathcal{C}^{\mathrm{op}}}(\mathfrak{X}) \simeq$$

$$(\mathrm{Alg}([\mathrm{X}, \mathrm{X}]) \times ((\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{op}}) \times_{(\mathrm{Alg}([\mathrm{X}, \mathrm{X}])^{\{0\}} \times \mathrm{Alg}([\mathrm{X}, \mathrm{X}])^{\{1\}})} \mathrm{Alg}([\mathrm{X}, \mathrm{X}])^{\Delta^1}$$

over $\mathrm{Alg}([\mathrm{X}, \mathrm{X}]) \times ((\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{op}}$ that gives rise to an equivalence

$$((\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{op}} \times_{\mathfrak{X}} \mathrm{LMod}_{\mathrm{T}}^{/\mathcal{C}^{\mathrm{op}}}(\mathfrak{X}) \simeq ((\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{op}} \times_{\mathrm{Alg}([\mathrm{X}, \mathrm{X}])} \mathrm{Alg}([\mathrm{X}, \mathrm{X}])_{\mathrm{T}/}$$

over $((\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}})^{\mathrm{op}}$.

As $\phi : \mathrm{Z} \to \mathrm{X}$ is monadic, for every $\mathrm{Y} \in \mathcal{C}$ the functor $[\mathrm{Y}, \phi] : [\mathrm{Y}, \mathrm{Z}] \to [\mathrm{Y}, \mathrm{X}]$ is monadic and thus conservative.

Hence the commutative square

$$\begin{array}{ccc} \mathcal{C}_{/\mathrm{Z}} & \longrightarrow & \mathfrak{Z}^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathcal{C}_{/\mathrm{X}} & \longrightarrow & \mathfrak{X}^{\mathrm{op}} \end{array}$$

of cartesian fibrations over $\mathcal{C}$ is a pullback square as it induces on the fiber over every $Y \in \mathcal{C}$ the pullback square

$$\begin{array}{ccc} \mathcal{C}(Y,Z) & \longrightarrow & [Y,Z]^{op} \\ \downarrow & & \downarrow \\ \mathcal{C}(Y,X) & \longrightarrow & [Y,X]^{op}. \end{array}$$

So we get a canonical equivalence

$$(\mathcal{C}_{/X})^R \times_{\mathcal{C}_{/X}} (\mathcal{C}_{/X})_{/Z} \simeq (\mathcal{C}_{/X})^R \times_{\mathcal{C}_{/X}} \mathcal{C}_{/Z} \simeq (\mathcal{C}_{/X})^R \times_{\mathfrak{X}^{op}} \mathfrak{Z}^{op} \simeq$$

$$(\mathcal{C}_{/X})^R \times_{\mathfrak{X}^{op}} \mathrm{LMod}_T^{/\mathcal{C}^{op}}(\mathfrak{X})^{op} \simeq (\mathcal{C}_{/X})^R \times_{\mathrm{Alg}([X,X])^{op}} (\mathrm{Alg}([X,X])_{T/})^{op} \simeq$$

$$(\mathcal{C}_{/X})^R \times_{\mathrm{Alg}([X,X])^{op}} (\mathrm{Alg}([X,X])^{op})_{/T}$$

of right fibrations over $(\mathcal{C}_{/X})^R$ that classifies an equivalence

$$\mathcal{C}_{/X}(-,Z) \simeq \mathrm{Alg}([X,X])^{op}(\mathrm{End}(-),\mathrm{End}(Z))$$

of functors $((\mathcal{C}_{/X})^R)^{op} \to \mathcal{S}$, whose restriction to $((\mathcal{C}_{/X})^{rep})^{op} \subset ((\mathcal{C}_{/X})^R)^{op}$ is the desired equivalence.

Proposition 5.31 guarantees that under the induced equivalence

$$\mathcal{C}_{/X}(Z,Z) \simeq \mathrm{Alg}([X,X])^{op}(\mathrm{End}(Z),\mathrm{End}(Z))$$

the identity of Z corresponds to an autoequivalence of End(Z).

So the functor $\mathrm{End} : (\mathcal{C}_{/X})^{rep} \to (\mathrm{Alg}([X,X])^{rep})^{op}$ admits a fully faithful right adjoint that sends a representable monad T on X to the monadic morphism $Z \to X$ representing the functor $\mathrm{Alg}([X,X])^{op}(\mathrm{End}(-),T) : ((\mathcal{C}_{/X})^{rep})^{op} \to \mathcal{S}$.

So given a monadic morphism $Z \to X$ the right adjoint sends the representable monad End(Z) on X to $Z \to X$.

Hence the local objects of $(\mathcal{C}_{/X})^{rep}$ are exactly the monadic morphisms over X.

The statements about local equivalences follow from lemma 5.32 3.

Let $\mathsf{s} \in S$ and let $g : Y \to X$ be a right adjoint morphism in $\mathcal{C}_\mathsf{s}$ with associated monad T that admits an Eilenberg-Moore object $\psi : Z \to X$ in $\mathcal{C}_\mathsf{s}$.

By definition of α under the equivalence

$$\alpha(Y) : (\mathcal{C}_\mathsf{s})_{/X(\mathsf{s})}(Y,Z) \simeq \mathrm{Alg}([X(\mathsf{s}),X(\mathsf{s})])^{op}(\mathrm{End}(Y),T)$$

the lift $g' : Y \to Z$ of $g : Y \to X$ corresponds to the identity of $T = \mathrm{End}(Y)$ and is thus the unit and so a local equivalence.

□

Remark 5.63. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a 2-functor.*
We have a commutative square

$$\begin{array}{ccc} (\mathcal{C}_{/X})^{rep} & \xrightarrow{\mathrm{End}} & (\mathrm{Alg}([X,X])^{rep})^{op} \\ \downarrow & & \downarrow \\ (\mathcal{D}_{/F(X)})^{rep} & \xrightarrow{\mathrm{End}} & (\mathrm{Alg}([F(X),F(X)])^{rep})^{op}. \end{array}$$

If F *preserves monadic morphisms with target* X*, by remark 5.27* F *preserves the Eilenberg-Moore object of every monad on* X*. In this case the last square induces a commutative square*

$$\begin{array}{ccc} (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}} & \xleftarrow{\mathrm{Alg}} & (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ (\mathcal{D}_{/\mathrm{F}(\mathrm{X})})^{\mathrm{rep}} & \xleftarrow{\mathrm{Alg}} & (\mathrm{Alg}([\mathrm{F}(\mathrm{X}),\mathrm{F}(\mathrm{X})])^{\mathrm{rep}})^{\mathrm{op}}. \end{array}$$

Applying remark 5.63 to the 2-functor $\theta : \mathcal{C} \to \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}}$ that preserves monadic morphisms we obtain a commutative square

$$\begin{array}{ccc} (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} & \xrightarrow{\mathrm{Alg}} & (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}} \\ \downarrow & & \downarrow \\ \mathrm{Alg}([\mathfrak{X},\mathfrak{X}])^{\mathrm{op}} & \xrightarrow{\mathrm{Alg}} & ((\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}})_{/\mathfrak{X}})^{\mathrm{R}} \end{array}$$

with $\mathfrak{X} := \theta(\mathrm{X})$.

As the composition $\mathcal{C} \xrightarrow{\theta} \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}} \xrightarrow{(-)^{\simeq}} \mathcal{L}_{\mathcal{C}^{\mathrm{op}}}$ is the Yoneda-embedding, the composition

$$(\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}} \subset \mathcal{C}_{/\mathrm{X}} \subset \mathcal{L}_{\mathcal{C}^{\mathrm{op}}/(\mathcal{C}_{/\mathrm{X}})^{\mathrm{op}}}$$

is equivalent to the functor

$$(\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} \to \mathrm{Alg}([\mathfrak{X},\mathfrak{X}])^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} ((\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}})_{/\mathfrak{X}})^{\mathrm{R}}$$

$$\subset (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}})_{/\mathfrak{X}} \xrightarrow{(-)^{\simeq}} (\mathcal{L}_{\mathcal{C}^{\mathrm{op}}})_{/(\mathcal{C}_{/\mathrm{X}})^{\mathrm{op}}}.$$

Thus the functor $\mathrm{Alg} : (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} \to (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}}$ is induced by the functor $\mathrm{Alg} : \mathrm{Alg}([\mathfrak{X},\mathfrak{X}])^{\mathrm{op}} \to ((\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}})_{/\mathfrak{X}})^{\mathrm{R}}$.

In the following we will give a more explicite description of the localization

$$\mathrm{End} : ((\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}})_{/\mathfrak{X}})^{\mathrm{R}} \rightleftarrows \mathrm{Alg}([\mathfrak{X},\mathfrak{X}])^{\mathrm{op}} : \mathrm{Alg},$$

i.e. the localization $\mathrm{End} : (\mathcal{D}_{/\mathrm{X}})^{\mathrm{R}}_{\mathrm{rep}} \rightleftarrows \mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{op}}_{\mathrm{rep}} : \mathrm{Alg}$ of theorem 5.62 for $\mathcal{D} = \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{C}^{\mathrm{op}}}$ and $\mathrm{X} = \mathfrak{X} \in \mathcal{D}$.

More generally we will give a more explicite description of the localization

$$\mathrm{End} : (\mathcal{D}_{/\mathrm{X}})^{\mathrm{rep}} \rightleftarrows (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} : \mathrm{Alg}$$

of theorem 5.62 for $\mathcal{D}$ a subcategory of $\mathsf{Cat}_{\infty/\mathrm{S}}$ for some small category S and $\mathrm{X} \in \mathcal{D}$.

To do so, we need some notation:

Let $\mathrm{G} : \mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ be a functor and $\mathcal{C} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) = \mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ a subcategory.

Passing to cotensors over S^{op} we obtain a subcategory inclusion $\mathcal{C}^{\Delta^1} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{\Delta^1} \simeq \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^{\Delta^1})$ over S^{op}.

Denote
$$(\mathcal{C}^{\Delta^1})^{\mathrm{mon}} \subset (\mathcal{C}^{\Delta^1})^{\mathrm{rep}} \subset (\mathcal{C}^{\Delta^1})^{\mathrm{R}} \subset \mathcal{C}^{\Delta^1}$$
the full subcategories spanned by the objects of $\mathrm{Fun}(\Delta^1, \mathcal{C}_\mathsf{s})$ for some $\mathsf{s} \in \mathrm{S}$ corresponding to morphisms in $\mathcal{C}_\mathsf{s}$ that are monadic, whose associated monad admits an Eilenberg-Moore object that is preserved by the subcategory inclusion $\mathcal{C}_\mathsf{s} \subset \mathsf{Cat}_{\infty/\mathrm{G}(\mathsf{s})}$ respectively that admit a left adjoint.

Let X be a section of the functor $\mathcal{C} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) \to \mathrm{S}^{\mathrm{op}}$ corresponding to a natural transformation $\mathrm{H} \to \mathrm{G}$ of functors $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

Set
$$(\mathcal{C}^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}})^{\mathrm{mon}} := \mathrm{S}^{\mathrm{op}} \times_{\mathcal{C}^{\{1\}}} (\mathcal{C}^{\Delta^1})^{\mathrm{mon}}, \quad (\mathcal{C}^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}})^{\mathrm{rep}} := \mathrm{S}^{\mathrm{op}} \times_{\mathcal{C}^{\{1\}}} (\mathcal{C}^{\Delta^1})^{\mathrm{rep}},$$
$$(\mathcal{C}^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}})^{\mathrm{R}} := \mathrm{S}^{\mathrm{op}} \times_{\mathcal{C}^{\{1\}}} (\mathcal{C}^{\Delta^1})^{\mathrm{R}}.$$

Let $\mathcal{D} \to \mathcal{E}$ be a map of cartesian fibrations over S classifying the natural transformation $\mathrm{H} \to \mathrm{G}$ of functors $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

Denote $[\mathrm{X}, \mathrm{X}]^{/\mathrm{S}} \subset \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D})$ the full subcategory spanned by the objects that belong to $[\mathrm{X}(\mathsf{s}), \mathrm{X}(\mathsf{s})]_{\mathcal{C}_\mathsf{s}} \subset \mathrm{Fun}_{\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}, \mathcal{D}_\mathsf{s})$ for some $\mathsf{s} \in \mathrm{S}$.

As for every $\mathsf{s} \in \mathrm{S}$ the monoidal structure on $\mathrm{Fun}_{\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}, \mathcal{D}_\mathsf{s})$ restricts to a monoidal structure on $[\mathrm{X}(\mathsf{s}), \mathrm{X}(\mathsf{s})]_{\mathcal{C}_\mathsf{s}}$, the monoidal structure on $\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D})$ over S restricts to a monoidal structure on $[\mathrm{X}, \mathrm{X}]^{/\mathrm{S}}$ over S.

Denote $\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}} \subset \mathrm{Alg}^{/\mathrm{S}}([\mathrm{X}, \mathrm{X}]^{/\mathrm{S}})$ the full subcategory spanned by the monads on $\mathrm{X}(\mathsf{s})$ for some $\mathsf{s} \in \mathrm{S}$ that admit an Eilenberg-Moore object that is preserved by the subcategory inclusion $\mathcal{C}_\mathsf{s} \subset \mathsf{Cat}_{\infty/\mathcal{E}_\mathsf{s}}$.

Construction 5.64.
The endomorphism $\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D})$*-left module structure on* $\mathcal{D} \to \mathcal{E}$ *gives rise to a forgetful functor*
$$\zeta : \mathrm{LMod}^{/\mathcal{E}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D})) \times_\mathrm{S} \mathcal{D}.$$

By lemma 5.13 the functor ζ *is a map of cartesian fibrations over* $\mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D}))$, *where a morphism of* $\mathrm{LMod}^{/\mathcal{E}}(\mathcal{D})$ *is cartesian with respect to the cartesian fibration* $\mathrm{LMod}^{/\mathcal{E}}(\mathcal{D}) \to \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D}))$ *if and only if its image in* $\mathcal{D}$ *is cartesian with respect to the cartesian fibration* $\mathcal{D} \to \mathrm{S}$.
So ζ *classifies a functor*
$$\xi : \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D}))^{\mathrm{op}} \to \mathrm{H}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) \simeq$$
$$\mathrm{S}^{\mathrm{op}} \times_{\mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{\{1\}}} \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{\Delta^1} = \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}}$$
over S^{op} *that induces on the fiber over* $\mathsf{s} \in \mathrm{S}$ *the functor*
$$\mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}, \mathcal{D}_\mathsf{s}))^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{D}_\mathsf{s}} \simeq (\mathsf{Cat}_{\infty/\mathcal{E}_\mathsf{s}})_{/\mathcal{D}_\mathsf{s}}$$
classified by the map $\mathrm{LMod}^{/\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}) \to \mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}, \mathcal{D}_\mathsf{s})) \times \mathcal{D}_\mathsf{s}$ *of cartesian fibrations over* $\mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}, \mathcal{D}_\mathsf{s}))$.

By example 5.36 for every monad $\mathrm{T} \in \mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s}, \mathcal{D}_\mathsf{s}))$ *the functor* $\mathrm{LMod}^{/\mathcal{E}_\mathsf{s}}_{\mathrm{T}}(\mathcal{D}_\mathsf{s}) \to \mathcal{D}_\mathsf{s}$ *is the Eilenberg-Moore object of* T *in* $\mathsf{Cat}_{\infty/\mathcal{E}_\mathsf{s}}$.

Thus ξ restricts to a functor

$$\mathrm{Alg} : (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{op}} \to (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{mon}} \subset (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}}$$

over S^{op} *that induces on the fiber over* $\mathsf{s} \in \mathrm{S}$ *the functor*

$$\mathrm{Alg} : (\mathrm{Alg}([\mathrm{X}(\mathsf{s}),\mathrm{X}(\mathsf{s})])^{\mathrm{rep}})^{\mathrm{op}} \to (\mathcal{C}_{\mathsf{s}/\mathrm{X}(\mathsf{s})})^{\mathrm{mon}} \subset (\mathsf{Cat}_{\infty/\mathcal{E}_{\mathsf{s}}})_{/\mathcal{D}_{\mathsf{s}}}$$

of theorem 5.68.

Remark 5.65. *By remark 5.15 1. for every functor* $\mathrm{S}' \to \mathrm{S}$ *the pullback*

$$\mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} \mathrm{Alg} : \mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{op}} \to \mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}}$$

is equivalent over $\mathrm{S}'^{\mathrm{op}}$ *to the functor*

$$\mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{op}} \simeq (\mathrm{Alg}^{/\mathrm{S}'}([\mathrm{X}',\mathrm{X}']^{/\mathrm{S}'})^{\mathrm{rep}})^{\mathrm{op}} \xrightarrow{\mathrm{Alg}} (\mathcal{C}'^{/\mathrm{S}'^{\mathrm{op}}}_{/\mathrm{X}'})^{\mathrm{R}}$$
$$\simeq \mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}}$$

over $\mathrm{S}'^{\mathrm{op}}$ *with* $\mathcal{C}' := \mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} \mathcal{C}$ *and* $\mathrm{X}' := \mathrm{S}'^{\mathrm{op}} \times_{\mathrm{S}^{\mathrm{op}}} \mathrm{X}$.

We have a commutative square

$$\begin{array}{ccc} (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{op}} & \xrightarrow{\mathrm{Alg}} & (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{rep}} \\ \downarrow & & \downarrow \\ \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}} & \xrightarrow{\mathrm{Alg}} & ((\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty))_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}} \end{array} \tag{25}$$

of categories over S^{op}.

Construction 5.66.

Let α be a section of the functor

$$(\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty))_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}} \simeq \mathrm{H}^*(\mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty)) \to \mathrm{S}$$

corresponding to a map of cartesian fibrations $\varphi : \mathcal{B} \to \mathcal{D}$ *over* S.

For every $\mathsf{s} \in \mathrm{S}$ *the induced functor* $\varphi_{\mathsf{s}} : \mathcal{B}_{\mathsf{s}} \to \mathcal{D}_{\mathsf{s}}$ *over* $\mathcal{E}_{\mathsf{s}}$ *admits a left adjoint relative to* $\mathcal{E}_{\mathsf{s}}$ *that is a morphism of* $\mathcal{C}_{\mathsf{s}}$ *so that* φ *admits a left adjoint* F *relative to* $\mathcal{E}$ *that is a map of cartesian fibrations over* S.

So by proposition 5.31 φ *admits an endomorphism object* T *with respect to the canonical left module structure on* $\mathrm{Fun}_{\mathcal{E}}(\mathcal{B},\mathcal{D})$ *over* $\mathrm{Fun}_{\mathcal{E}}(\mathcal{D},\mathcal{D})$, *which is given by* $\varphi \circ \mathrm{F}$.

Under the monoidal equivalence $\mathrm{Fun}_{\mathcal{E}}(\mathcal{D},\mathcal{D}) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{S},\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))$ *the monad* T *corresponds to an associative algebra of* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S},\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))$ *corresponding to a functor* $\phi : \mathrm{S} \to \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))$ *over* S *that sends every* $\mathsf{s} \in \mathrm{S}$ *to the morphism* $\varphi_{\mathsf{s}} \circ \mathrm{F}_{\mathsf{s}} : \mathcal{D}_{\mathsf{s}} \to \mathcal{D}_{\mathsf{s}}$ *of* $\mathcal{C}_{\mathsf{s}}$ *that is the endomorphism object of* φ_{s}.
So ϕ *induces a functor* $\mathrm{S} \to \mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}}) \subset \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))$ *over* S.

Given a functor $\beta : \mathcal{W} \to (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}}$ *over* S^{op} *adjoint to a section of* $\mathcal{W} \times_{\mathrm{S}^{\mathrm{op}}} (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}} \simeq ((\mathcal{W} \times_{\mathrm{S}^{\mathrm{op}}} \mathcal{C})_{/\mathcal{W}\times_{\mathrm{S}^{\mathrm{op}}}\mathrm{X}}^{/\mathcal{W}})^{\mathrm{R}} \to \mathcal{W}$ *we get a functor*

$$\mathcal{W}^{\mathrm{op}} \to \mathrm{Alg}^{/\mathcal{W}^{\mathrm{op}}}([\mathcal{W} \times_{\mathrm{S}^{\mathrm{op}}} \mathrm{X}, \mathcal{W} \times_{\mathrm{S}^{\mathrm{op}}} \mathrm{X}]^{/\mathcal{W}^{\mathrm{op}}}) \simeq \mathcal{W}^{\mathrm{op}} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})$$

over $\mathcal{W}^{op}$ *adjoint to a functor* $\psi : \mathcal{W}^{op} \to \mathrm{Alg}^{/S}([X,X]^{/S})$ *over* S.

For β *the identity we obtain a functor*

$$\mathrm{End} : (\mathcal{C}_{/X}^{/S^{op}})^R \to \mathrm{Alg}^{/S}([X,X]^{/S})^{op}$$

over S^{op} *that restricts to a functor* $(\mathcal{C}_{/X}^{/S^{op}})^{rep} \to (\mathrm{Alg}^{/S}([X,X]^{/S})^{rep})^{op}$ *with the same name.*

Remark 5.67. *We have a commutative square*

$$\begin{array}{ccc} (\mathcal{C}_{/X}^{/S^{op}})^R & \xrightarrow{\mathrm{End}} & \mathrm{Alg}^{/S}([X,X]^{/S})^{op} \\ \downarrow & & \downarrow \\ (G^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/X}^{/S^{op}})^R & \xrightarrow{\mathrm{End}} & \mathrm{Alg}^{/S}(\mathrm{Fun}_{\mathcal{E}}^{/S}(\mathcal{D},\mathcal{D}))^{op}, \end{array} \quad (26)$$

where the bottom functor over S^{op} *is* End *for* $\mathcal{C} = G^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$.

By 5.19 3. we have a canonical equivalence

$$\mathrm{LMod}_T(\mathrm{Fun}_{\mathcal{E}}(\mathcal{B},\mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{E}}(\mathcal{B}, \mathrm{LMod}_T^{/\mathcal{E}}(\mathcal{D})),$$

under which the endomorphism T*-left module structure on* φ *corresponds to a lift* $\bar{\varphi} : \mathcal{B} \to \mathrm{LMod}_T^{/\mathcal{E}}(\mathcal{D})$ *of* φ.

As $\mathcal{D} \to S$ *is a cartesian fibration, the functor* $\mathrm{LMod}_T^{/S}(\mathcal{D}) \to S$ *is a cartesian fibration (rem. 5.5), whose cartesian morphisms are those that get cartesian morphisms of* $\mathcal{D} \to S$. *So with* φ *also* $\bar{\varphi}$ *is a map of cartesian fibrations over* S.

We have a canonical equivalence

$$\mathrm{LMod}_T^{/\mathcal{E}}(\mathcal{D}) = S \times_{\mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}}(\mathcal{D},\mathcal{D})) \times S} \mathrm{LMod}^{/\mathcal{E}}(\mathcal{D}) \simeq S \times_{\mathrm{Alg}^{/S}(\mathrm{Fun}_{\mathcal{E}}^{/S}(\mathcal{D},\mathcal{D}))} \mathrm{LMod}^{/\mathcal{E}}(\mathcal{D})$$

over $\mathcal{D}$.

The map $\bar{\varphi} : S \to \mathrm{LMod}_T^{/\mathcal{E}}(\mathcal{D}) \simeq S \times_{\mathrm{Alg}^{/S}(\mathrm{Fun}_{\mathcal{E}}^{/S}(\mathcal{D},\mathcal{D}))} \mathrm{LMod}^{/\mathcal{E}}(\mathcal{D})$ *of cartesian fibrations over* S *over the cartesian fibration* $\mathcal{D} \to S$ *classifies a natural transformation* $\gamma : \alpha \to \xi \circ \phi^{op}$ *of functors*

$$S^{op} \to (S^{op} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/X}^{/S^{op}} \simeq S^{op} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$$

over S^{op} *that sends every* $\mathsf{s} \in S$ *to the functor* $\gamma(\mathsf{s}) : \mathcal{B}_\mathsf{s} \to \mathrm{LMod}_{T_\mathsf{s}}^{/\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s})$ *over* $\mathcal{E}_\mathsf{s}$ *that corresponds to the endomorphism* T_s*-left module structure on the functor* $\varphi_\mathsf{s} : \mathcal{B}_\mathsf{s} \to \mathcal{D}_\mathsf{s}$ *over* $\mathcal{E}_\mathsf{s}$.

With φ_s *also* $\gamma(\mathsf{s})$ *belongs to* $\mathcal{C}_\mathsf{s}$ *so that* γ *induces a natural transformation* $\alpha \to \mathrm{Alg} \circ \phi^{op}$ *of functors* $S^{op} \to (\mathcal{C}_{/X}^{/S^{op}})^R$ *over* S^{op}.

Given a functor $\beta : \mathcal{W} \to (\mathcal{C}_{/X}^{/S^{op}})^R$ *over* S^{op} *adjoint to a section of* $\mathcal{W} \times_{S^{op}} (\mathcal{C}_{/X}^{/S^{op}})^R \simeq ((\mathcal{W} \times_{S^{op}} \mathcal{C})_{/\mathcal{W} \times_{S^{op}} X}^{/\mathcal{W}})^R \to \mathcal{W}$ *we get a natural transformation of functors* $\mathcal{W} \to \mathcal{W} \times_{S^{op}} (\mathcal{C}_{/X}^{/S^{op}})^R \simeq ((\mathcal{W} \times_{S^{op}} \mathcal{C})_{/\mathcal{W} \times_{S^{op}} X}^{/\mathcal{W}})^R$ *over* $\mathcal{W}$ *adjoint to a natural transformation* $\beta \to \mathrm{Alg} \circ \psi^{op}$ *of functors* $\mathcal{W} \to (\mathcal{C}_{/X}^{/S^{op}})^R$ *over* S^{op}.

For β the identity we get a natural transformation $\lambda : \mathrm{id} \to \mathrm{Alg} \circ \mathrm{End}$ of functors $(\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}} \to (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{R}}$ over S^{op} that sends an object $\mathrm{Y} \in ((\mathcal{C}_\mathsf{s})_{/\mathrm{X}(\mathsf{s})})^{\mathrm{R}} \subset ((\mathsf{Cat}_{\infty/\mathcal{E}_\mathsf{s}})_{/\mathcal{D}_\mathsf{s}})^{\mathrm{R}}$ for some $\mathsf{s} \in \mathrm{S}$ to the the functor $\mathrm{Y} \to \mathrm{LMod}_{\mathrm{End}(\mathrm{Y})}^{/\mathcal{E}_\mathsf{s}}(\mathcal{D}_\mathsf{s})$ over $\mathcal{D}_\mathsf{s}$ that corresponds to the endomorphism $\mathrm{End}(\mathrm{Y})$-left module structure on the right adjoint functor $\mathrm{Y} \to \mathcal{D}_\mathsf{s}$ over $\mathcal{E}_\mathsf{s}$.

Proposition 5.68. *Let* S *be a small category,* $\mathcal{C} \subset \mathsf{Cat}_{\infty/\mathrm{S}}$ *a subcategory and* X *an object of* $\mathcal{C}$.

The functor $\mathrm{Alg} : (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} \to (\mathcal{C}_{/\mathrm{X}})^{\mathrm{mon}} \subset (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}}$ *of construction 5.64 1. is right adjoint to the functor*

$$\mathrm{End} : (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}} \to (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}}$$

of theorem 5.62.

Especially the functor Alg *is fully faithful.*

So if every monad on X *admits an Eilenberg-Moore object that is preserved by the subcategory inclusion* $\mathcal{C} \subset \mathsf{Cat}_{\infty/\mathrm{S}}$, *the functor*

$$\mathrm{Alg} : \mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{op}} \to (\mathcal{C}_{/\mathrm{X}})^{\mathrm{mon}} \subset (\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}}$$

of construction 5.64 1. is a fully faithful right adjoint of the functor $\mathrm{End} : (\mathcal{C}_{/\mathrm{X}})^{\mathrm{R}} \to \mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{op}}$ *of theorem 5.62.*

Proof. We first observe that we can reduce to the case $\mathcal{C} = \mathsf{Cat}_{\infty/\mathrm{S}}$:

We have commutative squares

$$\begin{array}{ccc} (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} & \xrightarrow{\mathrm{Alg}} & (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}} \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{X},\mathrm{X}))^{\mathrm{op}} & \xrightarrow{\mathrm{Alg}} & ((\mathsf{Cat}_{\infty/\mathrm{S}})_{/\mathrm{X}})^{\mathrm{R}}. \end{array}$$

and

$$\begin{array}{ccc} (\mathcal{C}_{/\mathrm{X}})^{\mathrm{rep}} & \xrightarrow{\mathrm{End}} & (\mathrm{Alg}([\mathrm{X},\mathrm{X}])^{\mathrm{rep}})^{\mathrm{op}} \\ \downarrow & & \downarrow \\ ((\mathsf{Cat}_{\infty/\mathrm{S}})_{/\mathrm{X}})^{\mathrm{R}} & \xrightarrow{\mathrm{End}} & \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{X},\mathrm{X}))^{\mathrm{op}}, \end{array}$$

where the vertical functors are subcategory inclusions.

For every representable monad $\mathrm{T} \in \mathrm{Alg}([\mathrm{X},\mathrm{X}]) \subset \mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{X},\mathrm{X}))$ and morphism $\psi : \mathrm{Y} \to \mathrm{X}$ of $\mathcal{C} \subset \mathsf{Cat}_{\infty/\mathrm{S}}$ that admits a left adjoint in $\mathcal{C}$ the canonical map

$$\mathrm{Alg}(\mathrm{Fun}_{\mathrm{S}}(\mathrm{X},\mathrm{X}))^{\mathrm{op}}(\mathrm{End}(\psi),\mathrm{T}) \simeq (\mathsf{Cat}_{\infty/\mathrm{S}})_{/\mathrm{X}}(\psi,\mathrm{Alg}(\mathrm{T}))$$

is canonically equivalent to the map

$$\mathrm{Alg}([\mathrm{X},\mathrm{X}])(\mathrm{End}(\psi),\mathrm{T})^{\mathrm{op}} \simeq \mathcal{C}_{/\mathrm{X}}(\psi,\mathrm{Alg}(\mathrm{T})),$$

where by remark 5.39 the full subcategory inclusion $\mathcal{C}_{/\mathrm{X}}(\psi,\mathrm{Alg}(\mathrm{T})) \subset$

$(\mathsf{Cat}_{\infty/\mathrm{S}})_{/\mathrm{X}}(\psi,\mathrm{Alg}(\mathrm{T}))$ is an equivalence as $\mathrm{Alg}(\mathrm{T})$ is an Eilenberg-Moore object for T that is preserved by the subcategory inclusion $\mathcal{C} \subset \mathsf{Cat}_{\infty/\mathrm{S}}$.

As $\mathsf{Cat}_{\infty/S}$ admits Eilenberg-Moore objects, for $\mathcal{C} = \mathsf{Cat}_{\infty/S}$ we have to show that the functor $\mathrm{Alg} : \mathrm{Alg}(\mathrm{Fun}_S(X,X))^{op} \to ((\mathsf{Cat}_{\infty/S})_{/X})^R$ is right adjoint to the functor $((\mathsf{Cat}_{\infty/S})_{/X})^R \to \mathrm{Alg}(\mathrm{Fun}_S(X,X))^{op}$.

To show this, we will construct an equivalence

$$\mathrm{Alg}(\mathrm{Fun}_S(X,X))^{op}(\mathrm{End}(\psi), T) \simeq (\mathsf{Cat}_{\infty/S})_{/X}(\psi, \mathrm{Alg}(T))$$
$$\simeq \mathsf{Cat}_{\infty/X}(\psi, \mathrm{Alg}(T))$$

natural in every monad $T \in \mathrm{Alg}(\mathrm{Fun}_S(X,X))$ and functor $\psi : Y \to X$ over S that admits a left adjoint relative to S.

Let $X \to S$ be endowed with the canonical endomorphism left module structure over $\mathrm{Fun}_S(X,X)$.

Denote $\mathcal{W} \to ((\mathcal{C}_{/X})^R)^{op} \times X$ the map of cartesian fibrations over $((\mathcal{C}_{/X})^R)^{op}$ classifying the subcategory inclusion $(\mathcal{C}_{/X})^R \subset \mathsf{Cat}_{\infty/X}$ and set $\mathcal{B} := \mathrm{Alg}(\mathrm{Fun}_S(X,X))$.

As the functor $\mathcal{W} \to ((\mathcal{C}_{/X})^R)^{op} \times X$ is a map of cartesian fibrations over $((\mathcal{C}_{/X})^R)^{op}$, the functor

$$\Psi : \mathrm{Fun}^{/\mathcal{B}\times((\mathcal{C}_{/X})^R)^{op}}_{\mathcal{B}\times((\mathcal{C}_{/X})^R)^{op}\times X}(\mathcal{B}\times\mathcal{W}, ((\mathcal{C}_{/X})^R)^{op}\times \mathrm{LMod}^{/S}(X)) \to \mathcal{B}\times((\mathcal{C}_{/X})^R)^{op}$$

is a map of cocartesian fibrations over $((\mathcal{C}_{/X})^R)^{op}$.

Ψ induces on the fiber over a functor $\psi : Y \to X$ over S that admits a left adjoint relative to S the functor $\varphi : \mathrm{Fun}^{/\mathcal{B}}_{\mathcal{B}\times X}(\mathcal{B}\times Y, \mathrm{LMod}^{/S}(X)) \to \mathcal{B}$.

φ is a cartesian fibration by remark 5.5 3. and the fact that the functor $\mathrm{LMod}^{/S}(X) \to \mathrm{Alg}(\mathrm{Fun}_S(X,X)) \times X$ is a map of cartesian fibrations over $\mathrm{Alg}(\mathrm{Fun}_S(X,X))$ due to remark 5.12.

By proposition 5.23 Ψ classifies the functor

$$((\mathcal{C}_{/X})^R)^{op} \to \mathsf{Cat}^{cart}_{\infty/\mathcal{B}} \simeq \mathrm{Fun}(\mathcal{B}^{op}, \mathsf{Cat}_\infty)$$

adjoint to the functor $((\mathcal{C}_{/X})^R)^{op} \times \mathcal{B}^{op} \subset (\mathsf{Cat}_{\infty/X})^{op} \times \mathcal{B}^{op} \xrightarrow{\mathrm{id}\times\mathrm{Alg}}$

$$(\mathsf{Cat}_{\infty/X})^{op} \times \mathsf{Cat}_{\infty/X} \xrightarrow{\mathrm{Fun}_X(-,-)} \mathsf{Cat}_\infty.$$

The functor Φ :

$$((\mathcal{C}_{/X})^R)^{op}\times_{\mathrm{Fun}(\{1\},\mathrm{Alg}(\mathcal{B}))}\mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{B})) \to \mathrm{Fun}(\{0\}, \mathrm{Alg}(\mathcal{B}))\times((\mathcal{C}_{/X})^R)^{op}$$

is a map of cocartesian fibrations over $((\mathcal{C}_{/X})^R)^{op}$ that induces on the fiber over a functor $\psi : Y \to X$ over S that admits a left adjoint relative to S the right fibration $\mathrm{Alg}(\mathrm{Fun}_S(X,X))_{/\mathrm{End}(\psi)} \to \mathrm{Alg}(\mathrm{Fun}_S(X,X))$.

By proposition 6.9 Φ classifies the functor $((\mathcal{C}_{/X})^R)^{op} \to \mathcal{R}_{\mathrm{Alg}(\mathcal{B})} \subset \mathsf{Cat}^{cart}_{\infty/\mathrm{Alg}(\mathcal{B})}$ adjoint to the functor

$$((\mathcal{C}_{/X})^R)^{op}\times\mathrm{Alg}(\mathcal{B})^{op} \xrightarrow{\mathrm{End}^{op}\times\mathrm{id}} \mathrm{Alg}(\mathcal{B})\times\mathrm{Alg}(\mathcal{B})^{op} \xrightarrow{\mathrm{Alg}(\mathcal{B})^{op}(-,-)} \mathcal{S} \subset \mathsf{Cat}_\infty.$$

Consequently we have to construct an equivalence

$$\zeta : \mathrm{Fun}^{/\mathcal{B}\times((\mathcal{C}_{/X})^R)^{op}}_{\mathcal{B}\times((\mathcal{C}_{/X})^R)^{op}\times X}(\mathcal{B}\times\mathcal{U}'_X, ((\mathcal{C}_{/X})^R)^{op}\times \mathrm{LMod}^{/S}(X))$$
$$\simeq ((\mathcal{C}_{/X})^R)^{op} \times_{\mathrm{Fun}(\{1\},\mathrm{Alg}(\mathcal{B}))} \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{B}))$$

over $\mathrm{Alg}(\mathcal{B}) \times ((\mathcal{C}_{/X})^R)^{op}$.

Denote $\mathcal{U}'_S \to (\mathsf{Cat}_{\infty/S})^{op} \times X$ the map of cartesian fibrations over $(\mathsf{Cat}_{\infty/S})^{op}$ classifying the identity of $\mathsf{Cat}_{\infty/S}$.

By remark 5.18 the endomorphism left module structure on $X \to S$ over $\mathrm{Fun}_S(X, X)$ gives rise to a $\mathrm{LM}^{\otimes}$-monoidal category $\mathrm{Fun}^{\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op}\times X)^{\otimes}$ over $\mathcal{C}^{op}$, whose pullback along the monoidal diagonal functor

$$\mathcal{C}^{op} \times \mathrm{Fun}_S(X, X)^{\otimes} \to \mathrm{Map}_{\mathcal{C}^{op}}(\mathcal{U}'_S, \mathcal{C}^{op} \times \mathrm{Fun}_S(X, X))^{\otimes}$$

$$\simeq \mathrm{Fun}^{/\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op} \times S \times \mathrm{Fun}_S(X, X))^{\otimes}$$

over $\mathcal{C}^{op}$ exhibits $\mathfrak{X}' := \mathrm{Fun}^{/\mathcal{C}^{op}}_{\mathcal{C}^{op}\times S}(\mathcal{U}'_S, \mathcal{C}^{op}\times X)$ as a left module over $\mathrm{Fun}_S(X, X)$.

The endomorphism left module structure on $X \to S$ over $\mathrm{Fun}_S(X, X)$ gives rise to a canonical left module structure over $\mathrm{Fun}_S(X, X)$ on the cocartesian fibration $\mathfrak{X} \to (\mathsf{Cat}_{\infty/S})^{op}$ classifying the functor

$$\mathrm{Fun}_S(-, X) : (\mathsf{Cat}_{\infty/S})^{op} \to \mathsf{Cat}_{\infty}.$$

By remark 5.24 we have a canonical $\mathrm{Fun}_S(X, X)$-linear equivalence $\mathfrak{X} \simeq \mathfrak{X}'$ of cocartesian fibrations over $(\mathsf{Cat}_{\infty/S})^{op}$.

By remark 5.19 3. we have a canonical equivalence

$$\mathrm{LMod}^{/\mathcal{C}^{op}}(\mathfrak{X}) \simeq \mathrm{LMod}^{/\mathcal{C}^{op}}(\delta^*(\mathfrak{X}')) \simeq$$

$$\mathcal{N} := \mathrm{Fun}^{/\mathcal{B}\times\mathcal{C}^{op}}_{\mathcal{B}\times\mathcal{C}^{op}\times S}(\mathcal{B} \times \mathcal{U}'_S, \mathcal{C}^{op} \times \mathrm{LMod}^{/S}(X))$$

over

$$\mathcal{B} \times \mathfrak{X} \simeq \mathcal{B} \times \mathfrak{X}' \simeq \mathrm{Fun}^{/\mathcal{B}\times\mathcal{C}^{op}}_{\mathcal{B}\times\mathcal{C}^{op}\times S}(\mathcal{B} \times \mathcal{U}'_S, \mathcal{C}^{op} \times \mathcal{B} \times X).$$

By remark 5.2 7. we have a canonical equivalence

$$((\mathcal{C}_{/X})^R)^{op} \times_{\mathfrak{X}'} \mathcal{N} \simeq (\mathrm{Alg}(\mathcal{B}) \times ((\mathcal{C}_{/X})^R)^{op}) \times_{(\mathrm{Alg}(\mathcal{B})\times\mathfrak{X}')} \mathcal{N} \simeq$$

$$(\mathrm{Alg}(\mathcal{B}) \times ((\mathcal{C}_{/X})^R)^{op}) \times_{\mathrm{Fun}^{/\mathrm{Alg}(\mathcal{B})\times\mathcal{C}^{op}}_{\mathrm{Alg}(\mathcal{B})\times\mathcal{C}^{op}\times S}(\mathrm{Alg}(\mathcal{B})\times\mathcal{U}'_S, \mathcal{C}^{op}\times\mathrm{Alg}(\mathcal{B})\times X)} \mathcal{N} \simeq$$

$$\mathrm{Fun}^{/\mathrm{Alg}(\mathcal{B})\times((\mathcal{C}_{/X})^R)^{op}}_{\mathrm{Alg}(\mathcal{B})\times((\mathcal{C}_{/X})^R)^{op}\times X}(\mathrm{Alg}(\mathcal{B}) \times \mathcal{U}'_X, ((\mathcal{C}_{/X})^R)^{op} \times \mathrm{LMod}^{/S}(X))$$

over $\mathrm{Alg}(\mathcal{B}) \times ((\mathcal{C}_{/X})^R)^{op}$.

By 5.30 we have a canonical equivalence

$$((\mathcal{C}_{/X})^R)^{op} \times_{\mathfrak{X}} \mathrm{LMod}^{/\mathcal{C}^{op}}(\mathfrak{X}) \simeq ((\mathcal{C}_{/X})^R)^{op} \times_{\mathrm{Fun}(\{1\},\mathrm{Alg}(\mathcal{B}))} \mathrm{Fun}(\Delta^1, \mathrm{Alg}(\mathcal{B}))$$

over $\mathrm{Alg}(\mathcal{B}) \times ((\mathcal{C}_{/X})^R)^{op}$.

So we obtain the desired equivalence ζ over $\mathrm{Alg}(\mathcal{B}) \times ((\mathcal{C}_{/X})^R)^{op}$.

□

Let S be a category, $G : S^{op} \to \mathsf{Cat}_{\infty}$ a functor, $\mathcal{C} \subset G^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty}))$ a subcategory and X a section of the functor $\mathcal{C} \subset G^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})) \to S^{op}$.

In the following we will see that the functor

$$\mathrm{Alg} : (\mathrm{Alg}^{/S}([X, X]^{/S})^{rep})^{op} \to (\mathcal{C}^{/S^{op}}_{/X})^{rep}$$

of construction 5.64 1. is a fully faitful right adjoint relative to S^{op} of the functor $\mathrm{End} : (\mathcal{C}^{/S^{op}}_{/X})^{rep} \to (\mathrm{Alg}^{/S}([X, X]^{/S})^{rep})^{op}$ constructed in 5.64 2.

By theorem 5.68 this localization

$$\mathrm{End} : (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{rep}} \rightleftarrows (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{op}} : \mathrm{Alg}$$

relative to S^{op} induces on the fiber over every object $\mathsf{s} \in \mathrm{S}$ the localization of theorem 5.62 applied to $\mathcal{C}_{\mathsf{s}}$ and $\mathrm{X}(\mathsf{s})$.

But different to the situation of theorem 5.62 we don't need to assume X to be a cocartesian section.

This flexibility is essential to prove corollary 5.70.

Theorem 5.69. *Let* S *be a category,* $\mathrm{G} : \mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *a functor,* $\mathcal{C} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$ *a subcategory and* X *a section of the functor* $\mathcal{C} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)) \to \mathrm{S}^{\mathrm{op}}$.

We have a localization

$$\mathrm{End} : (\mathcal{C}_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{rep}} \rightleftarrows (\mathrm{Alg}^{/\mathrm{S}}([\mathrm{X},\mathrm{X}]^{/\mathrm{S}})^{\mathrm{rep}})^{\mathrm{op}} : \mathrm{Alg}$$

relative to S^{op} *constructed in 5.64.*

Proof. Let $\mathcal{D} \to \mathcal{E}$ be the map of cartesian fibrations over S classifying the natural transformation $\mathrm{H} \to \mathrm{G}$ of functors $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ corresponding to the functor $\mathrm{X} : \mathrm{S}^{\mathrm{op}} \to \mathcal{C} \subset \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$ over S^{op}.

In view of the commutative squares 25 and 26 we can reduce to the case that $\mathcal{C} = \mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$.

We first show that the functor

$$\mathrm{Alg} : \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}} \to (\mathrm{G}^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{mon}}$$

is an equivalence.

This is equivalent to the condition that for every functor $\alpha : \mathrm{S}' \to \mathrm{S}$ the induced functor

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, \mathrm{Alg}) : \mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}}) \to$$

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, ((\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{mon}})$$

is an equivalence.

By remark 5.15 1. this functor $\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, \mathrm{Alg})$ factors as

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}}) \simeq$$

$$\mathrm{Fun}_{\mathrm{S}'^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}'}(\mathrm{Fun}_{\mathrm{S}' \times_{\mathrm{S}} \mathcal{E}}^{/\mathrm{S}'}(\mathrm{S}' \times_{\mathrm{S}} \mathcal{D}, \mathrm{S}' \times_{\mathrm{S}} \mathcal{D}))^{\mathrm{op}}) \xrightarrow{\mathrm{Fun}_{\mathrm{S}'^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, \mathrm{Alg})}$$

$$\mathrm{Fun}_{\mathrm{S}'^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, ((\mathrm{S}'^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/\mathrm{X} \circ \alpha}^{/\mathrm{S}'^{\mathrm{op}}})^{\mathrm{mon}}) \simeq$$

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}'^{\mathrm{op}}, ((\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}})^{\mathrm{mon}}).$$

So we can reduce to the case that $\alpha : \mathrm{S}' \to \mathrm{S}$ is the identity.

By remark 5.15 3. the functor

$$\mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}} \simeq \mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}_{\mathcal{E}}^{/\mathrm{S}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}})$$

$$\xrightarrow{\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Alg})} \mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, (\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))_{/\mathrm{X}}^{/\mathrm{S}^{\mathrm{op}}}) \simeq$$

$$(\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cart}})_{/\mathcal{D}} \subset \mathsf{Cat}_{\infty/\mathcal{D}}$$

is equivalent to the functor $\mathrm{Alg} : \mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}}(\mathcal{D},\mathcal{D}))^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{D}}$.

By theorem 5.68 this functor induces an equivalence

$$\mathrm{Alg} : \mathrm{Alg}(\mathrm{Fun}_{\mathcal{E}}(\mathcal{D}, \mathcal{D}))^{\mathrm{op}} \to ((\mathsf{Cat}_{\infty/\mathcal{E}})_{/\mathcal{D}})^{\mathrm{mon}}.$$

Consequently it is enough to see that the subcategory inclusion

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, (\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}}) \simeq$$

$$(\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}})_{/\mathcal{D}} \subset (\mathsf{Cat}_{\infty/\mathcal{E}})_{/\mathcal{D}}$$

restricts to a subcategory inclusion

$$\mathrm{Fun}_{\mathrm{S}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}, ((\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}})^{\mathrm{mon}}) \subset ((\mathsf{Cat}_{\infty/\mathcal{E}})_{/\mathcal{D}})^{\mathrm{mon}}.$$

Let $\varphi : \mathcal{B} \to \mathcal{D}$ be a map of cartesian fibrations over S over the cartesian fibration $\mathcal{E} \to \mathrm{S}$ that induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ a functor $\varphi_{\mathsf{s}} : \mathcal{B}_{\mathsf{s}} \to \mathcal{D}_{\mathsf{s}}$ over $\mathcal{E}_{\mathsf{s}}$ that admits a left adjoint relative to $\mathcal{E}_{\mathsf{s}}$.

Being a map of cartesian fibrations over S the functor $\varphi : \mathcal{B} \to \mathcal{D}$ admits a left adjoint relative to $\mathcal{E}$ and so admits an associated monad T in $\mathsf{Cat}_{\infty/\mathcal{E}}$.

The T-left module structure on φ corresponds to a functor $\beta : \mathcal{B} \to \mathrm{LMod}^{/\mathcal{E}}_{\mathrm{T}}(\mathcal{D})$ over $\mathcal{D}$ that induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ the functor $\mathcal{B}_{\mathsf{s}} \to \mathrm{LMod}^{/\mathcal{E}_{\mathsf{s}}}_{\mathrm{T}_{\mathsf{s}}}(\mathcal{D}_{\mathsf{s}})$ over $\mathcal{D}_{\mathsf{s}}$ corresponding to the endomorphism T_{s}-left module structure on φ_{s}.

As $\mathcal{D} \to \mathrm{S}$ is cartesian fibration, by remark 5.5 the functor $\mathrm{LMod}^{/\mathrm{S}}_{\mathrm{T}}(\mathcal{D}) \to \mathrm{S}$ is a cartesian fibration, whose cartesian morphisms are those that get cartesian morphisms of $\mathcal{D} \to \mathrm{S}$. So with φ also β is a map of cartesian fibrations over S.

Hence φ is monadic in $\mathsf{Cat}_{\infty/\mathcal{E}}$ if and only if for every $\mathsf{s} \in \mathrm{S}$ the functor $\varphi_{\mathsf{s}} : \mathcal{E}_{\mathsf{s}} \to \mathcal{D}_{\mathsf{s}}$ is monadic in $\mathsf{Cat}_{\infty/\mathcal{E}_{\mathsf{s}}}$.

In this case a morphism of $\mathcal{B}$ is cartesian with respect to $\mathcal{B} \to \mathrm{S}$ if and only if its image in $\mathcal{D}$ is cartesian with respect to $\mathcal{D} \to \mathrm{S}$.

So we have seen that

$$\mathrm{Alg} : \mathrm{Alg}^{/\mathrm{S}}(\mathrm{Fun}^{/\mathrm{S}}_{\mathcal{E}}(\mathcal{D}, \mathcal{D}))^{\mathrm{op}} \to ((\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}})^{\mathrm{R}}$$

is fully faithful.

By construction 5.64 3. we have a natural transformation $\lambda : \mathrm{id} \to \mathrm{Alg} \circ \mathrm{End}$ of endofunctors of $((\mathrm{S}^{\mathrm{op}} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))^{/\mathrm{S}^{\mathrm{op}}}_{/\mathrm{X}})^{\mathrm{R}}$ over S^{op} that sends an object $\mathrm{Y} \in ((\mathsf{Cat}_{\infty/\mathcal{E}_{\mathsf{s}}})_{/\mathcal{D}_{\mathsf{s}}})^{\mathrm{R}}$ for some $\mathsf{s} \in \mathrm{S}$ to the the functor $\lambda(\mathrm{Y}) : \mathrm{Y} \to \mathrm{LMod}^{/\mathcal{E}_{\mathsf{s}}}_{\mathrm{T}}(\mathcal{D}_{\mathsf{s}})$ over $\mathcal{E}_{\mathsf{s}}$ that corresponds to the endomorphism T-left module structure on the right adjoint functor $\mathrm{Y} \to \mathcal{D}_{\mathsf{s}}$ over $\mathcal{E}_{\mathsf{s}}$ with associated monad T.

We will show that $\lambda : \mathrm{id} \to \mathrm{Alg} \circ \mathrm{End}$ exhibits End as left adjoint to Alg relative to S^{op}.

As Alg is fully faithful, it is enough to see that $\mathrm{End} \circ \lambda : \mathrm{End} \to \mathrm{End} \circ \mathrm{Alg} \circ \mathrm{End}$ and $\lambda \circ \mathrm{Alg} : \mathrm{Alg} \to \mathrm{Alg} \circ \mathrm{End} \circ \mathrm{Alg}$ are equivalences or equivalently that for every $\mathsf{s} \in \mathrm{S}$ the induced natural transformations $\mathrm{End}_{\mathsf{s}} \circ \lambda_{\mathsf{s}} : \mathrm{End}_{\mathsf{s}} \to \mathrm{End}_{\mathsf{s}} \circ \mathrm{Alg}_{\mathsf{s}} \circ \mathrm{End}_{\mathsf{s}}$ and $\lambda_{\mathsf{s}} \circ \mathrm{Alg}_{\mathsf{s}} : \mathrm{Alg}_{\mathsf{s}} \to \mathrm{Alg}_{\mathsf{s}} \circ \mathrm{End}_{\mathsf{s}} \circ \mathrm{Alg}_{\mathsf{s}}$ on the fiber over s are equivalences.

So it is enough to see that for every $\mathsf{s} \in \mathrm{S}$ the natural transformation

$\lambda_{\mathsf{s}} : \mathrm{id} \to \mathrm{Alg}_{\mathsf{s}} \circ \mathrm{End}_{\mathsf{s}}$ exhibits $\mathrm{End}_{\mathsf{s}}$ as left adjoint to the fully faithful functor $\mathrm{Alg}_{\mathsf{s}}$, i.e. that for every $\mathrm{Y} \in ((\mathsf{Cat}_{\infty/\mathcal{E}_{\mathsf{s}}})_{/\mathcal{D}_{\mathsf{s}}})^{\mathrm{R}}$ the functor $\lambda(\mathrm{Y})$:

$Y \to \mathrm{LMod}_T^{/\mathcal{E}_s}(\mathcal{D}_s)$ over $\mathcal{D}_s$ induces for every $\mathcal{E} \in ((\mathsf{Cat}_{\infty/\mathcal{E}_s})_{/\mathcal{D}_s})^{\mathrm{mon}}$ an equivalence $\mathsf{Cat}_{\infty/\mathcal{D}_s}(\mathrm{LMod}_T^{/\mathcal{E}_s}(\mathcal{D}_s), \mathcal{E}) \to \mathsf{Cat}_{\infty/\mathcal{D}_s}(Y, \mathcal{E})$.

By theorem 5.62 the full subcategory $((\mathsf{Cat}_{\infty/\mathcal{E}_s})_{/\mathcal{D}_s})^{\mathrm{mon}} \subset ((\mathsf{Cat}_{\infty/\mathcal{E}_s})_{/\mathcal{D}_s})^{\mathrm{R}}$ is a localization and $\lambda(Y) : Y \to \mathrm{LMod}_T^{/\mathcal{E}_s}(\mathcal{D}_s)$ is a local equivalence.

□

Let S be a category, $G : S^{\mathrm{op}} \to \mathsf{Cat}_\infty$ a functor and $\mathcal{C} \subset G^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$ a subcategory.

Set $\mathcal{C}' := \mathcal{C} \times_{S^{\mathrm{op}}} \mathcal{C} \subset \mathcal{C} \times_{\mathsf{Cat}_\infty} \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ and denote U the section of $\mathcal{C}' \to \mathcal{C}$ adjoint to the identity of $\mathcal{C}$.

Then we have a canonical equivalence $\mathcal{C}'^{/\mathcal{C}}_{/U} \simeq \mathcal{C}^{\Delta^1}$ over $\mathcal{C}^{\{1\}}$.

So we obtain the following corollary:

Corollary 5.70. *Let* S *be a category,* $G : S^{\mathrm{op}} \to \mathsf{Cat}_\infty$ *a functor and* $\mathcal{C} \subset G^*(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$ *a subcategory.*

We have a localization

$$\mathrm{End} : (\mathcal{C}^{\Delta^1})^{\mathrm{rep}} \rightleftarrows (\mathrm{Alg}^{/\mathcal{C}^{\mathrm{op}}}([U,U]^{/\mathcal{C}^{\mathrm{op}}})^{\mathrm{rep}})^{\mathrm{op}} : \mathrm{Alg}$$

relative to $\mathcal{C}$ *with local objects those of* $(\mathcal{C}^{\Delta^1})^{\mathrm{mon}}$.
So the restriction

$$(\mathcal{C}^{\Delta^1})^{\mathrm{mon}} \subset (\mathcal{C}^{\Delta^1})^{\mathrm{rep}} \xrightarrow{\mathrm{End}} (\mathrm{Alg}^{/\mathcal{C}^{\mathrm{op}}}([U,U]^{/\mathcal{C}^{\mathrm{op}}})^{\mathrm{rep}})^{\mathrm{op}}$$

is an equivalence and the full subcategory $(\mathcal{C}^{\Delta^1})^{\mathrm{mon}} \subset (\mathcal{C}^{\Delta^1})^{\mathrm{rep}}$ *is a localization relative to* $\mathcal{C}$.

Lemma 5.71. *Suppose we have given a commutative square*

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \mathcal{C}' \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow{\psi} & \mathcal{D}' \end{array} \qquad (27)$$

of categories and let $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}, \mathcal{A}' \subset \mathcal{B}' \subset \mathcal{C}'$ *full subcategories with* $\varphi(\mathcal{A}) \subset \mathcal{A}', \varphi(\mathcal{B}) \subset \mathcal{B}'$.

Assume that the functor $\mathcal{C} \to \varphi(\mathcal{C})$ *induced by* φ *admits a left inverse.*
Assume that the full subcategory inclusion $\mathcal{A}' \subset \mathcal{B}'$ *admits a left adjoint relative to* $\mathcal{D}'$ *and for every object* X *of* $\mathcal{D}$ *the full subcategory inclusion* $\mathcal{A}_X \subset \mathcal{B}_X$ *admits a left adjoint and the induced functor* $\mathcal{C}_X \to \mathcal{C}'_{\psi(X)}$ *preserves local equivalences.*

1. *The full subcategory inclusion* $\mathcal{A} \subset \mathcal{B}$ *admits a left adjoint relative to* $\mathcal{D}$.

2. *Assume that* $\mathcal{C}$ *is a* $\mathcal{V}$*-enriched category that is cotensored over* $\mathcal{V}$ *such that for every* $K \in \mathcal{V}$ *cotensoring with* K *restricts to an endofunctor of* $\mathcal{A}$.

   The embedding $\mathcal{A} \subset \mathcal{B}$ *admits a* $\mathcal{V}$*-enriched left adjoint.*

Proof. Let Y be an object of $\mathcal{B}$ lying over some object X of $\mathcal{D}$.

For 1. it is enough to find a morphism $Y \to Z$ of $\mathcal{B}_X$ with $Z \in \mathcal{A}$ such that for every object A of $\mathcal{A}$ the induced map $\mathcal{C}(Z,A) \to \mathcal{C}(Y,A)$ is an equivalence, for 2. it is enough to find a morphism $Y \to Z$ of $\mathcal{B}_X$ with $Z \in \mathcal{A}$ such that for every object A of $\mathcal{A}$ the induced morphism $[Z,A] \to [Y,A]$ is an equivalence.

As the full subcategory inclusion $\mathcal{A}_X \subset \mathcal{B}_X$ admits a left adjoint, we find a local equivalence $\mathsf{f} : Y \to Z$ of $\mathcal{B}_X$ with $Z \in \mathcal{A}$. Set $X' := \psi(X)$.

By assumption the image $\varphi(\mathsf{f}) : \varphi(Y) \to \varphi(Z)$ is a local equivalence with respect to the localization $\mathcal{A}' \subset \mathcal{B}'$.

So for every object A of $\mathcal{A}$ the induced map

$$\mathcal{C}'(\varphi(Z), \varphi(A)) \to \mathcal{C}'(\varphi(Y), \varphi(A))$$

is an equivalence.

As the functor $\mathcal{C} \to \varphi(\mathcal{C})$ induced by φ admits a left inverse, we have a commutative square

$$\begin{array}{ccccc}
\mathcal{C}(Z,A) & \longrightarrow & \mathcal{C}'(\varphi(Z),\varphi(A)) & \longrightarrow & \mathcal{C}(Z,A) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}(Y,A) & \longrightarrow & \mathcal{C}'(\varphi(Y),\varphi(A)) & \longrightarrow & \mathcal{C}(Y,A)
\end{array}$$

of spaces, where the compositions $\mathcal{C}(Z,A) \to \mathcal{C}'(\varphi(Z),\varphi(A)) \to \mathcal{C}(Z,A)$ and $\mathcal{C}(Y,A) \to \mathcal{C}'(\varphi(Y),\varphi(A)) \to \mathcal{C}(Y,A)$ are the identity.

So with the map $\mathcal{C}'(\varphi(Z),\varphi(A)) \to \mathcal{C}'(\varphi(Y),\varphi(A))$ also the map $\mathcal{C}(Z,A) \to \mathcal{C}(Y,A)$ is an equivalence. This shows 1.

2: By 1. for every $A \in \mathcal{A}$ and $K \in \mathcal{V}$ the induced map

$$\mathcal{C}(Z, A^K) \to \mathcal{C}(Y, A^K)$$

is an equivalence so that the equivalent map $\mathcal{V}(K,[Z,A]) \to \mathcal{V}(K,[Y,A])$ is an equivalence, too.

So by Yoneda the morphism $[Z,A] \to [Y,A]$ is an equivalence.

□

Observation 5.72. *Let* S *be a category and* $\mathcal{C} \to S$ *a cocartesian* S*-family of 2-categories.*

By proposition 6.55 we have a functor $\theta : \mathcal{C} \to \mathrm{Map}_S(\mathcal{C}^{\mathrm{rev}}, \mathsf{Cat}_\infty) \simeq S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart}$ *over* S *that sends an object* X *of* $\mathcal{C}$ *lying over some* $\mathsf{s} \in S$ *to the cocartesian fibration over* $\mathcal{C}_{\mathsf{s}}^{\mathrm{op}}$ *classifying the functor* $[-,X] : \mathcal{C}_{\mathsf{s}}^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

The functor $\theta' : \mathcal{C} \to \theta(\mathcal{C})$ *over* S *induced by* θ *admits a left inverse over* S.

Proof. The composition

$$\theta : \mathcal{C} \to S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart} \xrightarrow{(-)^{\simeq}} S \times_{\mathsf{Cat}_\infty} \mathcal{L} \simeq \mathcal{P}^{/S}(\mathcal{C})$$

over S is the Yoneda-embedding relative to S.

Thus the functor $(-)^{\simeq} : S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart} \to S \times_{\mathsf{Cat}_\infty} \mathcal{L} \simeq \mathcal{P}^{/S}(\mathcal{C})$ over S restricts to a functor $\theta(\mathcal{C}) \to \mathcal{C} \subset \mathcal{P}^{/S}(\mathcal{C})$ over S and the composition $\mathcal{C} \xrightarrow{\theta'} \theta(\mathcal{C}) \to \mathcal{C}$ is the identity.

□

Theorem 5.73. *Let* S *be a category and* $\mathcal{C} \to S$ *a cocartesian* S*-family of 2-categories.*

The embedding $(\mathcal{C}^{\Delta^1})^{mon} \subset (\mathcal{C}^{\Delta^1})^{rep}$ *of categories admits a left adjoint relative to* $\mathcal{C}$.

Let S *be contractible so that embedding* $\mathrm{Fun}(\Delta^1, \mathcal{C})^{mon} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})^{rep}$ *admits a left adjoint relative to* $\mathcal{C}$.
If $\mathcal{C}$ *is cotensored over* Cat_∞*, the embedding* $\mathrm{Fun}(\Delta^1, \mathcal{C})^{mon} \subset \mathrm{Fun}(\Delta^1, \mathcal{C})^{rep}$ *admits a 2-categorical left adjoint.*

Proof. We apply lemma 5.71:

The functor $\theta : \mathcal{C} \to S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart}$ over S induces a commutative square

$$\begin{array}{ccc} \mathcal{C}^{\Delta^1} & \xrightarrow{\theta^{\Delta^1}} & S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart}^{\Delta^1} \\ \downarrow & & \downarrow \\ \mathcal{C}^{\{1\}} & \xrightarrow{\theta} & S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart}^{\{1\}} \end{array}$$

of categories over S.

θ restricts to functors

$$(\mathcal{C}^{\Delta^1})^{R} \to S \times_{\mathsf{Cat}_\infty} (\mathrm{Cocart}^{\Delta^1})^{R}, \ (\mathcal{C}^{\Delta^1})^{mon} \to S \times_{\mathsf{Cat}_\infty} (\mathrm{Cocart}^{\Delta^1})^{mon}$$

over S.

By observation 5.72 the functor $\theta' : \mathcal{C} \to \theta(\mathcal{C})$ over S induced by θ admits a left inverse over S. Thus the functor ${\theta'}^{\Delta^1} : \mathcal{C}^{\Delta^1} \to \theta(\mathcal{C})^{\Delta^1}$ over S also does and so, as we have an embedding $\theta^{\Delta^1}(\mathcal{C}^{\Delta^1}) \subset \theta(\mathcal{C})^{\Delta^1}$, the functor $\mathcal{C}^{\Delta^1} \to \theta^{\Delta^1}(\mathcal{C}^{\Delta^1})$ over S induced by $\theta^{\Delta^1} : \mathcal{C}^{\Delta^1} \to S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart}^{\Delta^1}$ admits a left inverse over S.

By theorem 5.62 for every object X of $\mathcal{C}$ lying over some object s of S the full subcategory inclusions

$$((\mathcal{C}_{\mathsf{s}})_{/X})^{mon} \subset ((\mathcal{C}_{\mathsf{s}})_{/X})^{R}, \ ((\mathsf{Cat}^{cocart}_{\infty/\mathcal{C}_{\mathsf{s}}^{op}})_{/\theta_{\mathsf{s}}(X)})^{mon} \subset ((\mathsf{Cat}^{cocart}_{\infty/\mathcal{C}_{\mathsf{s}}^{op}})_{/\theta_{\mathsf{s}}(X)})^{R}$$

admit left adjoints and the canonical 2-functor $(\mathcal{C}_{\mathsf{s}})_{/X} \to (\mathsf{Cat}^{cocart}_{\infty/\mathcal{C}_{\mathsf{s}}^{op}})_{/\theta_{\mathsf{s}}(X)}$ preserves local equivalences being a 2-functor.

By corollary 5.70 the embedding $S \times_{\mathsf{Cat}_\infty} (\mathrm{Cocart}^{\Delta^1})^{mon} \subset S \times_{\mathsf{Cat}_\infty} (\mathrm{Cocart}^{\Delta^1})^{R}$ admits a left adjoint relative to $S \times_{\mathsf{Cat}_\infty} \mathrm{Cocart}$.

So all requirements are satisfied to apply lemma 5.71 1.

Let S be contractible. With $\mathcal{C}$ also the 2-category $\mathrm{Fun}(\Delta^1, \mathcal{C})$ is cotensored over Cat_∞ with levelwise cotensor.

For every monadic functor $\mathcal{A} \to \mathcal{B}$ and every category $\mathcal{W}$ the induced functor $\mathrm{Fun}(\mathcal{W}, \mathcal{A}) \to \mathrm{Fun}(\mathcal{W}, \mathcal{B})$ is monadic. Thus given a monadic morphism $\mathsf{f} : Y \to X$ of $\mathcal{C}$, an object $Z \in \mathcal{C}$ and a small category K the functor $\mathrm{Fun}(K, [Z, Y]) \to \mathrm{Fun}(K, [Z, X])$ is monadic so that the equivalent functor $[Z, Y^K] \to [Z, X^K]$ is monadic, too. Hence also the morphism $Y^K \to X^K$ is monadic.

So 2. follows from lemma 5.71 2.

□

Construction 5.74. *Let* $\mathcal{C}$ *be a* E_2*-monoidal category compatible with geometric realizations.*

The forgetful functor $\rho : \mathrm{RMod}(\mathcal{C}) \to \mathrm{Alg}(\mathcal{C})$ *lifts to a cocartesian fibration* $\mathrm{RMod}(\mathcal{C})^\otimes \to \mathrm{Alg}(\mathcal{C})^\otimes$ *of monoidal categories.*

Given a functor $B : S \to \mathrm{Alg}(\mathcal{C})$ *denote* $\alpha : \mathrm{LM}^\otimes \times S \to \mathrm{Alg}(\mathcal{C})^\otimes$ *the functor over* $\mathrm{Ass}^\otimes$ *adjoint to the composition*

$$S \to \mathrm{Alg}(\mathcal{C}) \simeq \mathrm{LMod}_{\mathbb{1}}(\mathrm{Alg}(\mathcal{C})) \to \mathrm{LMod}(\mathrm{Alg}(\mathcal{C})).$$

The pullback

$$\mathcal{X}^\otimes := (\mathrm{LM}^\otimes \times S) \times_{\mathrm{Alg}(\mathcal{C})^\otimes} \mathrm{RMod}(\mathcal{C})^\otimes \to \mathrm{LM}^\otimes \times S$$

along α *is a cocartesian* S*-family of* $\mathrm{LM}^\otimes$*-monoidal categories with* $\{\mathfrak{m}\}\times_{\mathrm{LM}^\otimes} \mathcal{X}^\otimes \simeq S \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{RMod}(\mathcal{C}) \to S$ *and* $\mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{X}^\otimes \simeq \mathcal{C}^\otimes \times S \to \mathrm{Ass}^\otimes \times S$.

Thus $\mathcal{X}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{Alg}(\mathcal{C})$ *corresponds to a* $\mathrm{LM}^\otimes$*-monoid of* $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *that exhibits the cocartesian fibration* $S \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{RMod}(\mathcal{C}) \to S$ *as a left module over* $\mathcal{C} \times S$ *in* $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *and so especially as a cocartesian* S*-family of* $\mathcal{C}$*-enriched categories.*

The left module structure on $S \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{RMod}(\mathcal{C}) \to S$ *over* $\mathcal{C} \times S$ *in* $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *induces on sections a left module structure on*

$$\mathrm{Fun}_S(S, S\times_{\mathrm{Alg}(\mathcal{C})}\mathrm{RMod}(\mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{Alg}(\mathcal{C})}(S, \mathrm{RMod}(\mathcal{C})) \simeq \mathrm{RMod}_B(\mathrm{Fun}(S,\mathcal{C}))$$

over $\mathrm{Fun}(S,\mathcal{C})$ *encoded by the* $\mathrm{LM}^\otimes$*-monoidal category*

$$\mathrm{Fun}^{/\mathrm{LM}^\otimes}_{\mathrm{LM}^\otimes\times S}(\mathrm{LM}^\otimes \times S, \mathcal{X}^\otimes) \to \mathrm{LM}^\otimes.$$

This left module structure is the canonical left module structure on $\mathrm{RMod}_B(\mathrm{Fun}(S,\mathcal{C}))$ *over* $\mathrm{Fun}(S,\mathcal{C})$ *encoded by the* $\mathrm{LM}^\otimes$*-monoidal category*

$$\mathrm{LM}^\otimes \times_{\mathrm{Alg}(\mathrm{Fun}(S,\mathcal{C}))^\otimes} \mathrm{RMod}(\mathrm{Fun}(S,\mathcal{C}))^\otimes$$

as we have a canonical equivalence

$$\mathrm{Fun}^{/\mathrm{LM}^\otimes}_{\mathrm{LM}^\otimes\times S}(\mathrm{LM}^\otimes\times S, \mathcal{X}^\otimes) = \mathrm{Fun}^{/\mathrm{LM}^\otimes}_{\mathrm{LM}^\otimes\times S}(\mathrm{LM}^\otimes\times S, (\mathrm{LM}^\otimes\times S)\times_{\mathrm{Alg}(\mathcal{C})^\otimes}\mathrm{RMod}(\mathcal{C})^\otimes)$$

$$\simeq \mathrm{LM}^\otimes \times_{\mathrm{Fun}(S,\mathrm{Alg}(\mathcal{C}))^\otimes} \mathrm{Fun}(S, \mathrm{RMod}(\mathcal{C}))^\otimes \simeq$$

$$\mathrm{LM}^\otimes \times_{\mathrm{Alg}(\mathrm{Fun}(S,\mathcal{C}))^\otimes} \mathrm{RMod}(\mathrm{Fun}(S,\mathcal{C}))^\otimes$$

of $\mathrm{LM}^\otimes$*-monoidal categories.*

The first equivalence is represented by the following equivalence natural in every functor $K \to \mathrm{LM}^\otimes$*:*

$$\mathrm{Fun}_{\mathrm{LM}^\otimes}(K, \mathrm{Fun}^{/\mathrm{LM}^\otimes}_{\mathrm{LM}^\otimes\times S}(\mathrm{LM}^\otimes \times S, (\mathrm{LM}^\otimes \times S) \times_{\mathrm{Alg}(\mathcal{C})^\otimes} \mathrm{RMod}(\mathcal{C})^\otimes)) \simeq$$

$$\mathrm{Fun}_{\mathrm{LM}^\otimes\times S}(K \times S, (\mathrm{LM}^\otimes \times S) \times_{\mathrm{Alg}(\mathcal{C})^\otimes} \mathrm{RMod}(\mathcal{C})^\otimes) \simeq$$

$$\mathrm{Fun}_{\mathrm{Alg}(\mathcal{C})^\otimes}(K \times S, \mathrm{RMod}(\mathcal{C})^\otimes) \simeq \mathrm{Fun}_{\mathrm{Fun}(S,\mathrm{Alg}(\mathcal{C}))^\otimes}(K, \mathrm{Fun}(S, \mathrm{RMod}(\mathcal{C}))^\otimes)$$

$$\simeq \mathrm{Fun}_{\mathrm{LM}^\otimes}(K, \mathrm{LM}^\otimes \times_{\mathrm{Fun}(S,\mathrm{Alg}(\mathcal{C}))^\otimes} \mathrm{Fun}(S, \mathrm{RMod}(\mathcal{C}))^\otimes)$$

Let A *be an associative algebra in* $\mathrm{Fun}(S,\mathcal{C})$ *and* X *a* (A, B)*-bimodule in* $\mathrm{Fun}(S,\mathcal{C})$ *corresponding to a functor* $S \to \mathrm{BMod}(\mathcal{C})$.

Assume that the composition $S \to \mathrm{BMod}(\mathcal{C}) \to \mathrm{RMod}(\mathcal{C})$ *is a cocartesian section of* $S \times_{\mathrm{Alg}(\mathcal{C})} \mathrm{RMod}(\mathcal{C}) \to S$*, i.e. that for every morphism* $\mathsf{s} \to \mathrm{t}$ *of* S *the canonical* B_t*-linear map* $X_\mathsf{s} \otimes_{B_\mathsf{s}} B_t \to X_t$ *is an equivalence.*

Suppose that for every $\mathsf{s} \in S$ *the image* $X(\mathsf{s}) \in \mathrm{RMod}_{B(\mathsf{s})}(\mathcal{C})$ *admits an endomorphism object* $[X(\mathsf{s}), X(\mathsf{s})] \in \mathcal{C}$*.*

Then $X \in \mathrm{RMod}_B(\mathrm{Fun}(S, \mathcal{C}))$ *admits an endomorphism object* $[X, X]^{/S}$ *with respect to the canonical left module structure on* $\mathrm{RMod}_B(\mathrm{Fun}(S, \mathcal{C}))$ *over* $\mathrm{Fun}(S, \mathcal{C})$*.*

The (A, B)*-bimodule structure on* X *corresponds to a left* A*-module structure on the right* B*-module* X *with respect to the canonical left module structure on* $\mathrm{RMod}_B(\mathrm{Fun}(S, \mathcal{C}))$ *over* $\mathrm{Fun}(S, \mathcal{C})$ *and so in turn corresponds to a map of associative algebras* $\beta : A \to [X, X]^{/S}$ *in* $\mathrm{Fun}(S, \mathcal{C})$*.*

So we end up with a map $A \to [X, X]^{/S}$ *in* $\mathrm{Alg}(\mathrm{Fun}(S, \mathcal{C}))$*.*

If X *is the* (B, B)*-bimodule structure on* B *that comes from the associative algebra structure on* B*, the map* β *is an equivalence.*

Example 5.75. *We apply construction 5.74 to* $\mathcal{C} = \mathsf{Cat}_\infty$*:*

Let $\psi : S \to \mathrm{Alg}(\mathsf{Cat}_\infty)$ *be a functor corresponding to an associative monoid* $\mathcal{B} \to S$ *in* $\mathrm{Fun}(S, \mathsf{Cat}_\infty) \simeq \mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$*.*

Construction 5.74 asserts that the pullback

$$S \times_{\mathrm{Alg}(\mathsf{Cat}_\infty)} \mathrm{RMod}(\mathsf{Cat}_\infty) \to S$$

of $\mathrm{RMod}(\mathsf{Cat}_\infty) \to \mathrm{Alg}(\mathsf{Cat}_\infty)$ *along* ψ *has the structure of a cocartesian* S*-family of 2-categories.*

Let $\mathcal{A} \to S$ *be a further associative monoid in* $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *and* $\mathcal{M} \to S$ *a* $(\mathcal{A}, \mathcal{B})$*-bimodule in* $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *corresponding to a functor* $\varphi : S \to \mathrm{BMod}(\mathsf{Cat}_\infty)$*.*

Assume that the composition $X : S \to \mathrm{BMod}(\mathsf{Cat}_\infty) \to \mathrm{RMod}(\mathsf{Cat}_\infty)$ *is a cocartesian section of* $S \times_{\mathrm{Alg}(\mathsf{Cat}_\infty)} \mathrm{RMod}(\mathsf{Cat}_\infty) \to S$*, i.e. that for every morphism* $\mathsf{s} \to \mathrm{t}$ *of* S *the canonical* $\mathcal{B}_t$*-linear functor* $\mathcal{M}_\mathsf{s} \otimes_{\mathcal{B}_\mathsf{s}} \mathcal{B}_t \to \mathcal{M}_t$ *is an equivalence.*

Then by construction 5.74 applied to $\mathcal{C} = \mathsf{Cat}_\infty$ *we have a map* $\beta : \mathcal{A} \to [\mathcal{M}, \mathcal{M}]^{/S}$ *of associative monoids in* $\mathsf{Cat}_{\infty/S}^{\mathrm{cocart}}$ *that yields a map* $\mathrm{Alg}^{/S}(\mathcal{A}) \to \mathrm{Alg}^{/S}([\mathcal{M}, \mathcal{M}]^{/S})$ *of cocartesian fibrations over* S*.*

If $\mathcal{M}$ *is the* $(\mathcal{B}, \mathcal{B})$*-bimodule structure on* $\mathcal{B} \to S$ *that comes from the associative monoid structure on* $\mathcal{B} \to S$*, the map* β *is an equivalence.*

Theorem 5.62 applied to the cocartesian S*-family of 2-categories* $S \times_{\mathrm{Alg}(\mathsf{Cat}_\infty)} \mathrm{RMod}(\mathsf{Cat}_\infty) \to S$ *and its cocartesian section* X *asserts that we have a localization*

$$\mathrm{End} : ((S \times_{\mathrm{Alg}(\mathsf{Cat}_\infty)} \mathrm{RMod}(\mathsf{Cat}_\infty))^{/S}_{/X})^{\mathrm{R}} \to \mathrm{Alg}^{/S}([\mathcal{M}, \mathcal{M}]^{/S})^{\mathrm{rev}} : \mathrm{Alg}$$

relative to S*.*

So we get a functor

$$\mathrm{Alg}^{/S}(\mathcal{A})^{\mathrm{rev}} \to \mathrm{Alg}^{/S}([\mathcal{M}, \mathcal{M}]^{/S})^{\mathrm{rev}} \subset ((S \times_{\mathrm{Alg}(\mathsf{Cat}_\infty)} \mathrm{RMod}(\mathsf{Cat}_\infty))^{/S}_{/X})^{\mathrm{R}}$$

over S *that induces on the fiber over* $\mathsf{s} \in S$ *a functor*

$$\mathrm{Alg}(\mathcal{A}_\mathsf{s})^{\mathrm{op}} \to \mathrm{Alg}([\mathcal{M}_\mathsf{s}, \mathcal{M}_\mathsf{s}])^{\mathrm{op}} \subset (\mathrm{RMod}_{\mathcal{B}_\mathsf{s}}(\mathsf{Cat}_\infty)_{/\mathcal{M}_\mathsf{s}})^{\mathrm{R}}$$

that sends an associative algebra A *of* $\mathcal{A}_s$ *to the forgetful functor* $\mathrm{LMod}_A(\mathcal{M}_s) \simeq \mathrm{LMod}_T(\mathcal{M}_s) \to \mathcal{M}_s$*, where* $T := A \otimes -$ *is the* $\mathcal{B}_s$*-linear monad on* $\mathcal{M}_s$ *associated to* A.

We have a canonical equivalence $\mathcal{B}^\otimes \simeq (S \times \mathrm{Ass}^\otimes) \times_{\mathsf{Cat}_\infty} \mathcal{U}$ *over* $S \times \mathrm{Ass}^\otimes$.

So we get a canonical equivalence $\mathrm{Alg}^{/S}(\mathcal{B}) \simeq S \times_{\mathrm{Mon}(\mathsf{Cat}_\infty)} \mathrm{Mon}(\mathcal{U})$ *over* S *that is the restriction of the canonical equivalence* $\mathrm{Fun}^{/S}_{\mathrm{Ass}^\otimes \times S}(\mathrm{Ass}^\otimes \times S, \mathcal{B}) \simeq S \times_{\mathrm{Fun}(\mathrm{Ass}^\otimes, \mathsf{Cat}_\infty)} \mathrm{Fun}(\mathrm{Ass}^\otimes, \mathcal{U})$ *over* S *represented by the canonical equivalence*

$$\mathrm{Fun}_S(K, \mathrm{Fun}^{/S}_{\mathrm{Ass}^\otimes \times S}(\mathrm{Ass}^\otimes \times S, \mathcal{B}^\otimes)) \simeq \mathrm{Fun}_{S \times \mathrm{Ass}^\otimes}(K \times \mathrm{Ass}^\otimes, (S \times \mathrm{Ass}^\otimes) \times_{\mathsf{Cat}_\infty} \mathcal{U})$$

$$\simeq \mathrm{Fun}_{\mathsf{Cat}_\infty}(K \times \mathrm{Ass}^\otimes, \mathcal{U}) \simeq \mathrm{Fun}_{\mathrm{Fun}(\mathrm{Ass}^\otimes, \mathsf{Cat}_\infty)}(K, \mathrm{Fun}(\mathrm{Ass}^\otimes, \mathcal{U})) \simeq$$

$$\mathrm{Fun}_S(K, S \times_{\mathrm{Fun}(\mathrm{Ass}^\otimes, \mathsf{Cat}_\infty)} \mathrm{Fun}(\mathrm{Ass}^\otimes, \mathcal{U}))$$

natural in every functor $K \to S$.

Let $\mathcal{M}$ *be the* $(\mathcal{B}, \mathcal{B})$*-bimodule structure on* $\mathcal{B} \to S$ *that comes from the associative monoid structure on* $\mathcal{B} \to S$ *and* $\psi : S \to \mathrm{Alg}(\mathsf{Cat}_\infty)$ *the identity.*

Set

- $\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}} := \mathrm{RMod}(\mathsf{Cat}_\infty)^{/S}_{/X}$
- $(\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}})^{R} := (\mathrm{RMod}(\mathsf{Cat}_\infty)^{/S}_{/X})^{R}$
- $(\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}})^{\mathrm{mon}} := (\mathrm{RMod}(\mathsf{Cat}_\infty)^{/S}_{/X})^{\mathrm{mon}}$

So we have for every monoidal category $\mathcal{C}$ *canonical equivalences*

$$\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}}_{\mathcal{C}} \simeq \mathrm{RMod}_{\mathcal{C}}(\mathsf{Cat}_\infty)_{/\mathcal{C}},$$

$$(\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}})^{R}_{\mathcal{C}} \simeq (\mathrm{RMod}_{\mathcal{C}}(\mathsf{Cat}_\infty)_{/\mathcal{C}})^{R},$$
$$(\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}})^{\mathrm{mon}}_{\mathcal{C}} \simeq (\mathrm{RMod}_{\mathcal{C}}(\mathsf{Cat}_\infty)_{/\mathcal{C}})^{\mathrm{mon}}.$$

So we get a localization

$$\mathrm{End} : (\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}})^{R} \rightleftarrows \mathrm{Alg}(\mathcal{U})^{\mathrm{rev}} : \mathrm{Alg}$$

relative to $\mathrm{Alg}(\mathsf{Cat}_\infty)$ *that induces on the fiber over every monoidal category* $\mathcal{C}$ *a localization*

$$\mathrm{End} : (\mathrm{RMod}_{\mathcal{C}}(\mathsf{Cat}_\infty)_{/\mathcal{C}})^{R} \rightleftarrows \mathrm{Alg}(\mathcal{C})^{\mathrm{op}} : \mathrm{Alg}.$$

The right adjoint Alg *sends a pair* $(\mathcal{C}, A)$ *consisting of a monoidal category* $\mathcal{C}$ *and an associative algebra* A *of* $\mathcal{C}$ *(corresponding to the* $\mathcal{C}$*-linear monad* $T := A \otimes -$ *on* $\mathcal{C}$ *associated to* A*) to a right* $\mathcal{C}$*-linear functor lifting the forgetful functor* $\mathrm{LMod}_A(\mathcal{C}) \simeq \mathrm{LMod}_T(\mathcal{C}) \to \mathcal{C}$.

Especially we get an equivalence

$$(\mathrm{RMod}(\mathsf{Cat}_\infty)^{\mathrm{aug}})^{\mathrm{mon}} \simeq \mathrm{Alg}(\mathcal{U})^{\mathrm{rev}}$$

over S *that induces on the fiber over every monoidal category* $\mathcal{C}$ *an equivalence*

$$(\mathrm{RMod}_{\mathcal{C}}(\mathsf{Cat}_\infty)_{/\mathcal{C}})^{\mathrm{mon}} \simeq \mathrm{Alg}(\mathcal{C})^{\mathrm{op}}.$$

5.5 From Hopf operads to Hopf monads

In this subsection we will show that every Hopf operad gives rise to a Hopf monad.

So by example 5.44 the category of algebras over every Hopf operad gets a canonical symmetric monoidal structure.

We start by defining Hopf operads in a symmetric monoidal category $\mathcal{C}$ compatible with small colimits.

To do so, we first need to define the composition product on symmetric sequences in $\mathcal{C}$.

Denote $\Sigma \simeq \coprod_{n\geq 0} B(\Sigma_n)$ the groupoid of finite sets and bijections.

The category $\mathcal{C}^\Sigma := \mathrm{Fun}(\Sigma, \mathcal{C}) \simeq \prod_{n\geq 0} \mathrm{Fun}(B(\Sigma_n), \mathcal{C})$ admits a symmetric monoidal structure compatible with small colimits given by Day-convolution.

We have a fully faithful symmetric monoidal functor $\mathcal{C} \to \mathcal{C}^\Sigma$ left adjoint to evaluation at 0 that considers an object of $\mathcal{C}$ as a symmetric sequence concentrated in degree zero.

We have a monoidal structure on $\mathcal{C}^\Sigma$ called the composition product corresponding to composition under the canonical equivalence

$$\mathcal{C}^\Sigma \simeq \mathrm{Fun}^\otimes(\Sigma, \mathcal{C}^\Sigma) \simeq \mathrm{Fun}^{\otimes,\mathrm{coc}}(\mathcal{S}^\Sigma, \mathcal{C}^\Sigma) \simeq \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{C}/}(\mathcal{C}^\Sigma, \mathcal{C}^\Sigma).$$

For every $X, Y \in \mathcal{C}^\Sigma$ we have $X \circ Y = \coprod_{k\geq 0} X_k \otimes_{\Sigma_k} Y^{\otimes k}$, where we embed $\mathcal{C}$ into $\mathcal{C}^\Sigma$, and so for every $n \geq 0$ we have a canonical equivalence

$$(X \circ Y)_n \simeq \coprod_{k\geq 0}\Big(\coprod_{n_1+\ldots+n_k=n} \Sigma_n \times_{(\Sigma_0\times\Sigma_{n_1}\times\ldots\times\Sigma_{n_k})} (X_k \otimes (\bigotimes_{1\leq j\leq k} Y_{n_j}))\Big)_{\Sigma_k}.$$

The composition product on $\mathcal{C}^\Sigma$ makes $\mathcal{C}^\Sigma$ to a left module over itself. This left module structure on $\mathcal{C}^\Sigma$ over itself restricts to a left module structure on $\mathcal{C}$ over $\mathcal{C}^\Sigma$ that sends $(X, Y) \in \mathcal{C}^\Sigma \times \mathcal{C}$ to $\coprod_{k\geq 0} X_k \otimes_{\Sigma_k} Y^{\otimes k}$.

This left module structure on $\mathcal{C}$ over $\mathcal{C}^\Sigma$ corresponds to a monoidal functor $T : \mathcal{C}^\Sigma \to \mathrm{Fun}(\mathcal{C}, \mathcal{C})$ that sends every associative algebra respectively coassociative coalgebra in $\mathcal{C}^\Sigma$ to a monad respectively comonad on $\mathcal{C}$.

For every symmetric monoidal functor $\mathcal{B} \to \mathcal{C}$ that preserves small colimits between symmetric monoidal categories compatible with small colimits the co-base-change functor $\mathrm{Calg}(\mathsf{Cat}^{\mathrm{coc}}_\infty)_{\mathcal{B}/} \to \mathrm{Calg}(\mathsf{Cat}^{\mathrm{coc}}_\infty))_{\mathcal{C}/}$ is naturally a 2-functor and so yields a monoidal functor $\mathcal{B}^\Sigma \simeq \mathrm{Fun}^{\otimes,\mathrm{L}}_{\mathcal{B}/}(\mathcal{B}^\Sigma, \mathcal{B}^\Sigma) \to \mathcal{C}^\Sigma \simeq \mathrm{Fun}^{\otimes,\mathrm{L}}_{\mathcal{C}/}(\mathcal{C}^\Sigma, \mathcal{C}^\Sigma)$.

Especially the symmetric monoidal small colimits preserving functor $\mathrm{Cocoalg}(\mathcal{C}) \to \mathcal{C}$ between symmetric monoidal categories compatible with small colimits yields a monoidal functor $\mathrm{Cocoalg}(\mathcal{C})^\Sigma \to \mathcal{C}^\Sigma$.

We call associative algebras in $\mathcal{C}^\Sigma$ with respect to the composition product operads in $\mathcal{C}$ and define the category of $\mathcal{O}$-algebras in $\mathcal{C}$ by
$\mathrm{LMod}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{T_{\mathcal{O}}}(\mathcal{C})$.

We refer to operads in $\mathrm{Cocoalg}(\mathcal{C})$ as Hopf operads in $\mathcal{C}$ so that every Hopf operad in $\mathcal{C}$ has an underlying operad in $\mathcal{C}$.

Now we are able to state the main proposition:

Proposition 5.76. *There is a monoidal functor*

$$\mathrm{Cocoalg}(\mathcal{C})^{\Sigma} \to \mathrm{Fun}^{\otimes,\mathrm{oplax}}(\mathcal{C},\mathcal{C})$$

that fits into a commutative square

$$\begin{array}{ccc} \mathrm{Cocoalg}(\mathcal{C})^{\Sigma} & \longrightarrow & \mathrm{Fun}^{\otimes,\mathrm{oplax}}(\mathcal{C},\mathcal{C}) \\ \downarrow & & \downarrow \\ \mathcal{C}^{\Sigma} & \longrightarrow & \mathrm{Fun}(\mathcal{C},\mathcal{C}) \end{array}$$

of monoidal categories.

This commutative square of monoidal categories yields a commutative square

$$\begin{array}{ccc} \mathrm{Alg}(\mathrm{Cocoalg}(\mathcal{C})^{\Sigma}) & \longrightarrow & \mathrm{Alg}(\mathrm{Fun}^{\otimes,\mathrm{oplax}}(\mathcal{C},\mathcal{C})) \\ \downarrow & & \downarrow \\ \mathrm{Alg}(\mathcal{C}^{\Sigma}) & \longrightarrow & \mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})). \end{array}$$

In other words the associated monad of a Hopf operad in $\mathcal{C}$ is an oplax symmetric monoidal monad.

So example 5.44 implies that category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathrm{T}_{\mathcal{H}}}(\mathcal{C})$ of $\mathcal{H}$-algebras in $\mathcal{C}$ carries a canonical symmetric monoidal structure such that the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ gets symmetric monoidal.

More generally if $\mathcal{C}$ admits small colimits but the symmetric monoidal structure on $\mathcal{C}$ is not compatible with small colimits, by constr. 5.80 we only have a representable operad $(\mathcal{C}^{\Sigma})^{\otimes} \to \mathrm{Ass}^{\otimes}$ over $\mathrm{Ass}^{\otimes}$ that is the symmetric monoidal category encoding the composition product if $\mathcal{C}$ is compatible with small colimits.

Moreover for every $\mathcal{O}_1, ..., \mathcal{O}_n \in \mathcal{C}^{\Sigma}$ for some $n \geq 1$ and $X \in \mathcal{C}$ the composition $\mathcal{O}_1 \circ ... \circ \mathcal{O}_n \circ X$ belongs to $\mathcal{C}$.

So the representable operad $\mathrm{LM}^{\otimes} \times_{\mathrm{Ass}^{\otimes}} (\mathcal{C}^{\Sigma})^{\otimes} \to \mathrm{LM}^{\otimes}$ over $\mathrm{LM}^{\otimes}$ restricts to a representable operad over $\mathrm{LM}^{\otimes}$ with fiber over $\mathfrak{a}$ the category $\mathcal{C}^{\Sigma}$ and with fiber over $\mathfrak{m}$ the category $\mathcal{C}$.

We define operads in $\mathcal{C}$ as associative algebras in $(\mathcal{C}^{\Sigma})^{\otimes}$.

With $\mathcal{C}$ also $\mathrm{Cocoalg}(\mathcal{C})$ admits small colimits and we define Hopf operads in $\mathcal{C}$ as operads in $\mathrm{Cocoalg}(\mathcal{C})$ in this more general sense.

Given an operad $\mathcal{O}$ in $\mathcal{C}$ we define the category of $\mathcal{O}$-algebras in $\mathcal{C}$ by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) := \mathrm{LMod}_{\mathcal{O}}(\mathcal{C})$.

So we get the following proposition:

Proposition 5.77. *Let $\mathcal{C}$ be a symmetric monoidal category that admits small colimits and $\mathcal{H}$ a Hopf operad in $\mathcal{C}$.*

Then the category $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C})$ of $\mathcal{H}$-algebras in $\mathcal{C}$ carries a canonical symmetric monoidal structure such that the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ gets symmetric monoidal.

Proof. By the functoriality of constr. 5.80 the symmetric monoidal Yoneda-embedding $\mathcal{C} \subset \mathcal{C}' := \mathcal{P}(\mathcal{C})$ yields a canonical equivalence $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \simeq \mathcal{C} \times_{\mathcal{C}'} \mathrm{Alg}_{\mathcal{H}}(\mathcal{C}')$ over $\mathcal{C}$ that makes the forgetful functor $\mathrm{Alg}_{\mathcal{H}}(\mathcal{C}) \to \mathcal{C}$ symmetric monoidal as $\mathcal{C}' := \mathcal{P}(\mathcal{C})$ is compatible with small colimits.

□

We deduce proposition 5.76 respectively its fibered version 5.84 from proposition 5.83, which relates the composition product on a category of sections to the monoidal structure on endofunctors given by composition.

To prove proposition 5.83, we need to make the composition product on $\mathcal{C}^{\Sigma}$ functorial in $\mathcal{C}$.

We first construct a functor $\Psi : \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \to \mathrm{Mon}(\widehat{\mathsf{Cat}_{\infty}})$ that sends $\mathcal{C}$ to $\mathcal{C}^{\Sigma} \simeq \mathrm{Fun}^{\otimes,\mathrm{L}}_{\mathcal{C}/}(\mathcal{C}^{\Sigma}, \mathcal{C}^{\Sigma})$.

In a second step we extend Ψ to a functor

$$\bar{\Psi} : \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}} \to \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}},$$

where $\mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}} \subset \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})$ denotes the full subcategory spanned by the symmetric monoidal categories, whose underlying category admits small colimits.

$\bar{\Psi}$ takes values in the full subcategory of $\widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}}$ spanned by the representable operads over $\mathrm{Ass}^{\otimes}$.

As we work with cocartesian S-families of symmetric monoidal categories for some category S, we make the following definition:

Let $\mathcal{D}^{\otimes} \to \mathrm{S} \times \mathcal{F}\mathrm{in}_{*}$ be a cocartesian S-family of symmetric monoidal categories classifying a functor $\mathrm{S} \to \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}})$.

Denote

$$(\mathcal{D}^{\Sigma})^{\otimes} \to \mathrm{S} \times \mathrm{Ass}^{\otimes}$$

the cocartesian S-family of monoidal categories classifying the functor $\mathrm{S} \to \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \xrightarrow{\Psi} \mathrm{Mon}(\widehat{\mathsf{Cat}_{\infty}})$.

We start with constructing the functor $\Psi : \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \to \mathrm{Mon}(\widehat{\mathsf{Cat}_{\infty}})$ that sends $\mathcal{C}$ to $\mathcal{C}^{\Sigma} \simeq \mathrm{Fun}^{\otimes,\mathrm{L}}_{\mathcal{C}/}(\mathcal{C}^{\Sigma}, \mathcal{C}^{\Sigma})$.

Construction 5.78.

The finite products preserving functor $\mathcal{F}\mathrm{in}_{*} \times - : \widehat{\mathsf{Cat}_{\infty}} \to \widehat{\mathsf{Cat}}^{\mathrm{cocart}}_{\infty/\mathcal{F}\mathrm{in}_{*}}$ *makes* $\widehat{\mathsf{Cat}}^{\mathrm{cocart}}_{\infty/\mathcal{F}\mathrm{in}_{*}}$ *to a closed left module over* $\widehat{\mathsf{Cat}_{\infty}}$ *and so to a 2-category.*

For $\mathrm{X}, \mathrm{Y} \in \widehat{\mathsf{Cat}}^{\mathrm{cocart}}_{\infty/\mathcal{F}\mathrm{in}_{*}}$ *the morphism object is given by* $\mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{F}\mathrm{in}_{*}}(\mathrm{X}, \mathrm{Y})$. *Thus also the subcategories*

$$\mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \subset \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}}) \subset \widehat{\mathsf{Cat}}^{\mathrm{cocart}}_{\infty/\mathcal{F}\mathrm{in}_{*}}$$

get 2-categories. For $\mathrm{X}, \mathrm{Y} \in \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}})$ *the morphism object is given by* $\mathrm{Fun}^{\otimes,\mathrm{coc}}(\mathrm{X}, \mathrm{Y}) \subset \mathrm{Fun}^{\mathrm{cocart}}_{\mathcal{F}\mathrm{in}_{*}}(\mathrm{X}, \mathrm{Y})$.

So by 6.69 4. the cartesian fibration

$$\zeta : \mathrm{Fun}(\Delta^{1}, \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}})) \to \mathrm{Fun}(\{0\}, \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}))$$

lifts to a cartesian $\mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}})$*-family of 2-categories.*

For every morphism $\mathcal{C} \to \mathcal{D}$ *of* $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})$ *the induced functor*

$$\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{D}/} \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{C}/}$$

admits a left adjoint $\mathcal{D} \otimes_{\mathcal{C}} - : \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{C}/} \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{D}/}$.

For every $\mathrm{X} \in \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{C}/}$ *and* $\mathrm{Y} \in \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{D}/}$ *the functor*

$$\mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{D}/}(\mathcal{D} \otimes_{\mathcal{C}} \mathrm{X}, \mathrm{Y}) \to \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{C}/}(\mathcal{D} \otimes_{\mathcal{C}} \mathrm{X}, \mathrm{Y}) \to \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{C}/}(\mathrm{X}, \mathrm{Y})$$

is an equivalence as for every $\mathrm{K} \in \widehat{\mathsf{Cat}_\infty}$ *the induced functor*

$$\widehat{\mathsf{Cat}_\infty}(\mathrm{K}, \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{D}/}(\mathcal{D} \otimes_{\mathcal{C}} \mathrm{X}, \mathrm{Y})) \to \widehat{\mathsf{Cat}_\infty}(\mathrm{K}, \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{C}/}(\mathcal{D} \otimes_{\mathcal{C}} \mathrm{X}, \mathrm{Y}))$$

$$\to \widehat{\mathsf{Cat}_\infty}(\mathrm{K}, \mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{C}/}(\mathrm{X}, \mathrm{Y}))$$

is equivalent to the equivalence

$$\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{D}/}(\mathcal{D} \otimes_{\mathcal{C}} \mathrm{X}, \mathrm{Y}^{\mathrm{K}}) \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{C}/}(\mathcal{D} \otimes_{\mathcal{C}} \mathrm{X}, \mathrm{Y}^{\mathrm{K}})$$

$$\to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{C}/}(\mathrm{X}, \mathrm{Y}^{\mathrm{K}}).$$

Hence by cor. 6.74 the 2-functor $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{D}/} \to \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})_{\mathcal{C}/}$ *admits a 2-categorical left adjoint.*

Thus the cartesian $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})$*-family of 2-categories*

$$\zeta : \mathrm{Fun}(\Delta^1, \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})) \to \mathrm{Fun}(\{0\}, \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}}))$$

is a bicartesian $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})$*-family of 2-categories.*

The unique small colimits preserving symmetric monoidal functor $\mathcal{S} \to \mathcal{P}(\Sigma)$ *yields a natural transformation* $\mathrm{id} \simeq \mathcal{S} \otimes - \to \mathrm{Fun}(\Sigma, -) \simeq \mathcal{P}(\Sigma) \otimes -$ *of endofunctors of* $\mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}})$ *corresponding to a cocartesian section of* ζ.

So by 5.28 there is a functor $\Psi : \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}}) \to \mathrm{Mon}(\widehat{\mathsf{Cat}_\infty})$ *that sends* $\mathcal{C}$ *to* $\mathcal{C}^\Sigma \simeq \mathrm{Fun}^{\otimes,\mathrm{L}}_{\mathcal{C}/}(\mathcal{C}^\Sigma, \mathcal{C}^\Sigma)$ *and fits into a commutative square*

$$\begin{array}{ccc} \mathrm{Calg}(\mathsf{Cat}_\infty^{\mathrm{coc}}) & \xrightarrow{\Psi} & \mathrm{Mon}(\widehat{\mathsf{Cat}_\infty}) \\ \downarrow & & \downarrow \\ \mathsf{Cat}_\infty^{\mathrm{coc}} & \xrightarrow{\mathrm{Fun}(\Sigma,-)} & \widehat{\mathsf{Cat}_\infty}. \end{array}$$

As next we extend Ψ to a functor $\mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \to \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$ that takes values in the full subcategory of $\widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$ spanned by the representable operads over $\mathrm{Ass}^\otimes$.

To do so, we make the following definitions:

Let $\mathcal{K} \subset \mathsf{Cat}_\infty$ be a full subcategory and $\mathrm{S} \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \subset \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})$ a functor corresponding to a cocartesian S-family $\mathcal{D}^\otimes \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ of symmetric monoidal categories.

Denote

- $\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ the cocartesian S-family of symmetric monoidal categories corresponding to the functor

$$\mathrm{S} \to \mathrm{Calg}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \subset \mathrm{Calg}(\widehat{\mathsf{Cat}_\infty}) \xrightarrow{\widehat{\mathcal{P}}} \mathrm{Calg}(\widehat{\widehat{\mathsf{Cat}_\infty}}^{\mathrm{coc}}) \subset \mathrm{Calg}(\widehat{\widehat{\mathsf{Cat}_\infty}}).$$

- $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes$ the full subfamily of operads spanned by the functors $\mathcal{D}_\mathsf{s}^{\mathrm{op}} \to \widehat{\mathcal{S}}$ that preserve $\mathcal{K}$-indexed limits for some $\mathsf{s} \in \mathrm{S}$ and $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes$ the full subfamily of operads such that for every $\mathsf{s} \in \mathrm{S}$ the full subcategory $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s}$ is the smallest full subcategory of $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s}$ that contains $\mathcal{D}_\mathsf{s}$ and is closed under small colimits.

If $\mathcal{K}$ is empty, we write $\mathcal{P}^{/\mathrm{S}}(\mathcal{D})^\otimes$ for $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes$.
If $\mathcal{K} = \mathsf{Cat}_\infty$, we have $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes = \mathcal{D}^\otimes$.
If S is contractible, we drop S in the notation.

We have a Yoneda-embedding map $\mathcal{D}^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes$ of cocartesian S-families of symmetric monoidal categories that induces an embedding $\mathcal{D}^\otimes \subset \mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes$ of S-families of operads.

For every $\mathsf{s} \in \mathrm{S}$ the full subcategory $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s} \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})_\mathsf{s}$ is a localization. The left adjoint $\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})_\mathsf{s} \to \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s}$ restricts to a functor $\mathcal{P}^{/\mathrm{S}}(\mathcal{D})_\mathsf{s} \to \mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s}$.

Especially we see that the full subcategory $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})_\mathsf{s} \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})_\mathsf{s}$ is a localization so that also the full subcategory $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})_\mathsf{s} \subset \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s}$ is a localization. The localization $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s} \rightleftarrows \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})_\mathsf{s}$ restricts to a localization $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s} \rightleftarrows \mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})_\mathsf{s} = \mathcal{D}_\mathsf{s}$.

If the cocartesian fibration $\mathcal{D}^\otimes \to \mathrm{S}\times\mathcal{F}\mathrm{in}_*$ is compatible with $\mathcal{K}$-indexed colimits, this fiberwise localization is compatible with the cocartesian fibration $\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$. In this case the restriction $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ is a cocartesian fibration and the full subcategory inclusion $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes$ admits a left adjoint relative to $\mathrm{S} \times \mathcal{F}\mathrm{in}_*$.

This implies that the cocartesian fibration $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ is compatible with colimits and so restricts to a cocartesian fibration $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ compatible with small colimits with the same cocartesian morphisms.

So there is a cocartesian S-family $(\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma)^\otimes \to \mathrm{S} \times \mathrm{Ass}^\otimes$ of monoidal categories with underlying cocartesian fibration $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma \to \mathrm{S}$.

Remark 5.79. *The embedding*

$$\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes$$

of cocartesian S*-families of symmetric monoidal categories yields an embedding*

$$(\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma)^\otimes \subset (\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma)^\otimes \subset (\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\Sigma)^\otimes$$

of S*-families of operads over* $\mathrm{Ass}^\otimes$:

Being a map of cocartesian fibrations over $\mathrm{S} \times \mathcal{F}\mathrm{in}_*$ *the left adjoint* $\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\otimes \to \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\otimes$ *relative to* $\mathrm{S} \times \mathcal{F}\mathrm{in}_*$ *gives rise to a map*

$$(\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\Sigma)^\otimes \to (\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma)^\otimes$$

of cocartesian S*-families of monoidal categories that induces on the fiber over every* $\mathsf{s} \in \mathrm{S}$ *and* $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*$ *the left adjoint of the full subcategory inclusion* $\mathrm{Fun}(\Sigma, \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_\mathsf{s}) \subset \mathrm{Fun}(\Sigma, \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})_\mathsf{s})$ *and thus admits a fully faithful right adjoint* $(\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma)^\otimes \subset (\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\Sigma)^\otimes$ *relative to* $\mathrm{S} \times \mathrm{Ass}^\otimes$ *lifting the canonical embedding* $\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^\Sigma \subset \widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^\Sigma$.

The embedding $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^{\otimes} \subset \widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^{\otimes}$ of cocartesian S*-families of symmetric monoidal categories yields an embedding $(\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^{\Sigma})^{\otimes} \subset (\widehat{\mathcal{P}}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^{\Sigma})^{\otimes}$ of cocartesian* S*-families of monoidal categories.*

Construction 5.80. *Denote*

$$(\mathcal{D}^{\Sigma})^{\otimes} \subset (\mathcal{P}^{/\mathrm{S}}(\mathcal{D})^{\Sigma})^{\otimes}$$

the full subfamily spanned by the objects of $\mathcal{D}^{\Sigma} \subset \mathcal{P}^{/\mathrm{S}}(\mathcal{D})^{\Sigma}$.

As the embedding $\mathcal{D}^{\Sigma} \subset \mathcal{P}^{/\mathrm{S}}(\mathcal{D})^{\Sigma}$ is a map of cocartesian fibrations over S*, the restriction $(\mathcal{D}^{\Sigma})^{\otimes} \subset (\mathcal{P}^{/\mathrm{S}}(\mathcal{D})^{\Sigma})^{\otimes} \to \mathrm{S} \times \mathrm{Ass}^{\otimes}$ is a cocartesian* S*-family of operads over $\mathrm{Ass}^{\otimes}$ corresponding to a functor $\mathrm{S} \to \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}}$ and the embedding $(\mathcal{D}^{\Sigma})^{\otimes} \subset (\mathcal{P}^{/\mathrm{S}}(\mathcal{D})^{\Sigma})^{\otimes}$ is a map of such.*

For $\mathrm{S} \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}}$ the identity we denote the resulting functor $\mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}} \to \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}}$ by $\bar{\Psi}$.

If the cocartesian S*-family $\mathcal{D}^{\otimes} \to \mathrm{S} \times \mathcal{F}\mathrm{in}_{*}$ of symmetric monoidal categories classifies a functor $\mathrm{S} \to \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \subset \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})$, the embedding $\mathcal{D}^{\otimes} \subset \mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_{\infty}}(\mathcal{D})^{\otimes}$ of cocartesian* S*-families of symmetric monoidal categories classifies a natural transformation of functors $\mathrm{S} \to \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \subset \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})$ and so gives rise to an embedding $(\mathcal{D}^{\Sigma})^{\otimes} \subset (\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_{\infty}}(\mathcal{D})^{\Sigma})^{\otimes}$ of cocartesian* S*-families of operads over $\mathrm{Ass}^{\otimes}$.*
Thus by remark 5.79 we have an embedding $(\mathcal{D}^{\Sigma})^{\otimes} \subset (\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_{\infty}}(\mathcal{D})^{\Sigma})^{\otimes} \subset (\widehat{\mathcal{P}}^{/\mathrm{S}}(\mathcal{D})^{\Sigma})^{\otimes}$ of S*-families of operads over $\mathrm{Ass}^{\otimes}$.*

So the functor $\bar{\Psi} : \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}} \to \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}}$ extends the functor $\Psi : \mathrm{Calg}(\mathsf{Cat}_{\infty}^{\mathrm{coc}}) \to \mathrm{Alg}(\widehat{\mathsf{Cat}_{\infty}})$ and fits into a commutative square

$$\begin{array}{ccc}
\mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}} & \xrightarrow{\bar{\Psi}} & \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}} \\
\downarrow & & \downarrow \\
\widehat{\mathsf{Cat}_{\infty}} & \xrightarrow{\mathrm{Fun}(\Sigma,-)} & \widehat{\mathsf{Cat}_{\infty}}.
\end{array}$$

For every $\mathsf{s} \in \mathrm{S}$ the localization $\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_{\mathsf{s}} \rightleftarrows \mathcal{D}_{\mathsf{s}}$ yields a localization $\mathrm{Fun}(\Sigma, \mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})_{\mathsf{s}}) \rightleftarrows \mathrm{Fun}(\Sigma, \mathcal{D}_{\mathsf{s}})$.
So the restriction $(\mathcal{D}^{\Sigma})^{\otimes} \subset (\mathcal{P}^{/\mathrm{S}}_{\mathcal{K}}(\mathcal{D})^{\Sigma})^{\otimes} \to \mathrm{S} \times \mathrm{Ass}^{\otimes}$ is a locally cocartesian fibration.

Hence the functor $\bar{\Psi} : \mathrm{Cmon}(\widehat{\mathsf{Cat}_{\infty}})^{\mathrm{coc}} \to \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^{\otimes}}$ takes values in the full subcategory spanned by the representable operads over $\mathrm{Ass}^{\otimes}$.

Thus a symmetric monoidal functor $\phi : \mathcal{B} \to \mathcal{C}$ between symmetric monoidal categories that admit small colimits gives rise to a map $(\mathcal{B}^{\Sigma})^{\otimes} \to (\mathcal{C}^{\Sigma})^{\otimes}$ of representable operads over $\mathrm{Ass}^{\otimes}$ that is an embedding of planar operads if ϕ is fully faithful.

Remark 5.81. *If ϕ preserves small colimits, the lax monoidal functor $(\mathcal{B}^{\Sigma})^{\otimes} \to (\mathcal{C}^{\Sigma})^{\otimes}$ is monoidal.*

Proof. The symmetric monoidal functor ϕ extends to a symmetric monoidal small colimits preserving functor $\phi' : \mathcal{B}' := \mathcal{P}(\mathcal{B}) \to \mathcal{C}' := \mathcal{P}(\mathcal{C})$ along symmetric monoidal embeddings $\mathcal{B} \subset \mathcal{B}', \mathcal{C} \subset \mathcal{C}'$ that admit left adjoints $\mathrm{L} : \mathcal{B}' \to \mathcal{B}, \mathrm{L}' : \mathcal{C}' \to \mathcal{C}$.

The symmetric monoidal embeddings $\mathcal{B} \subset \mathcal{B}', \mathcal{C} \subset \mathcal{C}'$ lift to lax monoidal embeddings $(\mathcal{B}^\Sigma)^\otimes \subset (\mathcal{B}'^\Sigma)^\otimes, (\mathcal{C}^\Sigma)^\otimes \subset (\mathcal{C}'^\Sigma)^\otimes$ on composition products.

The symmetric monoidal functors ϕ, ϕ' lift to lax monoidal functor $(\mathcal{B}^\Sigma)^\otimes \to (\mathcal{C}^\Sigma)^\otimes$ respectively a monoidal functor $(\mathcal{B}'^\Sigma)^\otimes \to (\mathcal{C}'^\Sigma)^\otimes$ on composition products such that the monoidal functor $(\mathcal{B}'^\Sigma)^\otimes \to (\mathcal{C}'^\Sigma)^\otimes$ restricts to the lax monoidal functor $(\mathcal{B}^\Sigma)^\otimes \to (\mathcal{C}^\Sigma)^\otimes$.

To see that the lax monoidal functor $(\mathcal{B}^\Sigma)^\otimes \to (\mathcal{C}^\Sigma)^\otimes$ is already monoidal, it is enough to check that ϕ' and so $\phi'^\Sigma : \mathcal{B}'^\Sigma \to \mathcal{C}'^\Sigma$ preserve local equivalences.

We show that for every $X \in \mathcal{B}'$ the canonical morphism $L'(\phi'(X)) \to \phi(L(X))$ is an equivalence.

This is true for every $X \in \mathcal{B}$ and so also true for every $X \in \mathcal{B}'$ as ϕ preserves small colimits and $\mathcal{B}'$ is the only full subcategory of itself that contains $\mathcal{B}$ and is closed under small colimits.

□

As next we prepare the proof of proposition 5.76.

We need the following definitions and observations:

- Let $\mathcal{D}^\otimes \to S \times \mathcal{F}in_*$ be a cocartesian S-family of symmetric monoidal categories classifying a functor $S \to \mathrm{Calg}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}}$.
  Denote
  $$(\mathcal{D}^\Sigma)^\otimes \to S \times \mathrm{Ass}^\otimes$$
  the cocartesian S-family of representable operads over $\mathrm{Ass}^\otimes$ classifying the functor $S \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \xrightarrow{\bar{\Psi}} \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$.

- Similarly let $\mathcal{D}^\otimes \to S \times \mathcal{F}in_*$ be a cartesian S-family of symmetric monoidal categories classifying a functor $S^{\mathrm{op}} \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}}$ (and a commutative monoid in $\widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/S}$ on the cartesian fibration $\mathcal{D} \to S$).
  Then we write
  $$(\mathcal{D}^\Sigma)^\otimes \to S \times \mathrm{Ass}^\otimes$$
  for the cartesian S-family of representable operads over $\mathrm{Ass}^\otimes$ classifying the functor $S^{\mathrm{op}} \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \xrightarrow{\bar{\Psi}} \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$.

If for every $\mathsf{s} \in S$ the induced symmetric monoidal structure on the fiber $\mathcal{D}_\mathsf{s}$ is compatible with small colimits, the functor $S^{\mathrm{op}} \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \xrightarrow{\bar{\Psi}} \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$ takes values in the full subcategory spanned by the monoidal categories and so by cor. 6.43 the cartesian S-family $(\mathcal{D}^\Sigma)^\otimes \to S \times \mathrm{Ass}^\otimes$ of representable operads over $\mathrm{Ass}^\otimes$ is a map of cocartesian fibrations over $\mathrm{Ass}^\otimes$ classifying an associative monoid in $\widehat{\mathsf{Cat}}_{\infty/S}$ on the cartesian fibration $\mathcal{D}^\Sigma \to S$.

If for every morphism $\mathsf{s} \to \mathrm{t}$ in S the induced functor $\mathcal{D}_\mathsf{s} \to \mathcal{D}_\mathrm{t}$ preserves small colimits, the functor $S^{\mathrm{op}} \to \mathrm{Cmon}(\widehat{\mathsf{Cat}_\infty})^{\mathrm{coc}} \xrightarrow{\bar{\Psi}} \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$ factors as $S^{\mathrm{op}} \to \mathrm{Calg}(\mathsf{Cat}^{\mathrm{coc}}_\infty) \xrightarrow{\Psi} \mathrm{Mon}(\widehat{\mathsf{Cat}_\infty}) \subset \widehat{\mathrm{Op}}_{\infty/\mathrm{Ass}^\otimes}$ so that the associative monoid structure on $\mathcal{D}^\Sigma \to S$ in $\widehat{\mathsf{Cat}}_{\infty/S}$ is an associative monoid in $\widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/S}$.

Moreover given commutative monoids $\mathcal{C} \to S, \mathcal{D} \to S$ in $\widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/S}$ a map of commutative monoids $\mathcal{C} \to \mathcal{D}$ in $\widehat{\mathsf{Cat}}_{\infty/S}$ that induces on the fiber over every $\mathsf{s} \in S$ a small colimits preserving symmetric monoidal functor between symmetric monoidal categories compatible with small colimits yields a map of associative monoids $\mathcal{C}^{\Sigma} \to \mathcal{D}^{\Sigma}$ in $\widehat{\mathsf{Cat}}_{\infty/\mathrm{S}}$.

- Let $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathrm{T})$ be a full subcategory.
  Given cartesian fibrations $\mathcal{B} \to \mathrm{S} \times \mathrm{T}, \mathcal{D} \to \mathrm{S} \times \mathrm{T}$ denote

  $$\mathrm{Fun}^{/\mathrm{S},\mathcal{E}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{B}, \mathcal{D}) \subset \mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{B}, \mathcal{D})$$

  the full subcategory spanned by the maps $\mathcal{B}_{\mathsf{s}} \to \mathcal{D}_{\mathsf{s}}$ of cartesian fibrations relative to $\mathcal{E}$ for some $\mathsf{s} \in \mathrm{S}$.

  For $\mathcal{E} = \mathrm{Fun}(\Delta^1, \mathrm{T})$ we write $\mathrm{Fun}^{/\mathrm{S},\mathrm{cart}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{B}, \mathcal{D})$ for $\mathrm{Fun}^{/\mathrm{S},\mathcal{E}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{B}, \mathcal{D})$.
  For S contractible we write $\mathrm{Fun}^{\mathcal{E}}_{\mathrm{T}}(\mathcal{B}, \mathcal{D})$ for $\mathrm{Fun}^{/\mathrm{S},\mathcal{E}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{B}, \mathcal{D})$ and $\mathrm{Fun}^{\mathrm{cart}}_{\mathrm{T}}(\mathcal{B}, \mathcal{D})$ for $\mathrm{Fun}^{/\mathrm{S},\mathrm{cart}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{B}, \mathcal{D})$.

  Example 5.82.
  For $\mathrm{T} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}, \mathcal{E} \subset \mathrm{Fun}(\Delta^1, \mathcal{F}\mathrm{in}_*^{\mathrm{op}})$ *the full subcategory spanned by the inert morphisms and cocartesian* S*-families* $\mathcal{B}^{\otimes} \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*, \mathcal{D}^{\otimes} \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ *of symmetric monoidal categories we have*

  $$\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\otimes,\mathrm{oplax}}(\mathcal{B}, \mathcal{D}) := \mathrm{Fun}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{B}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})^{\mathrm{op}} \simeq$$

  $$\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\mathcal{E}}_{\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_*^{\mathrm{op}}}(((\mathcal{B}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}}, ((\mathcal{D}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}}).$$

  For $\mathcal{B}^{\otimes} \to \mathrm{S} \times \mathcal{F}\mathrm{in}_*$ *the identity we set*

  $$\mathrm{Cocoalg}^{/\mathrm{S}^{\mathrm{op}}}(\mathcal{D}) := \mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\otimes,\mathrm{oplax}}(\mathrm{S}, \mathcal{D}).$$

  By 5.6 the functor $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, \mathcal{D}) \to \mathrm{S}$ is a cartesian fibration that restricts to a cartesian fibration $\mathrm{Fun}^{/\mathrm{S},\mathcal{E}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, \mathcal{D}) \to \mathrm{S}$ with the same cartesian morphisms.

  The endomorphism associative monoid structure on $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{D}, \mathcal{D})$ in $\mathsf{Cat}_{\infty/\mathrm{S}}$ restricts to $\mathrm{Fun}^{/\mathrm{S},\mathcal{E}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{D}, \mathcal{D})$ so that the endomorphism associative monoid structure on $\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_*^{\mathrm{op}}}(((\mathcal{D}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}}, ((\mathcal{D}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})$ in $\mathsf{Cat}_{\infty/\mathrm{S}}$ restricts to $\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\otimes,\mathrm{oplax}}(\mathcal{D}, \mathcal{D})$.

- By 5.6 the finite products preserving functor

  $$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, -) : \mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}} \to \mathsf{Cat}_{\infty/\mathrm{S}}$$

  restricts to a finite products preserving functor $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}\times\mathrm{T}} \to \mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}}$ that sends a commutative monoid $\mathcal{C} \to \mathrm{S}\times\mathrm{T}$ in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}\times\mathrm{T}}$ to a commutative monoid $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, \mathcal{C})$ in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}}$.

  If for every $\mathsf{s} \in \mathrm{S}, \mathrm{t} \in \mathrm{T}$ the induced symmetric monoidal category $\mathcal{C}_{\mathsf{s},\mathrm{t}}$ is compatible with small colimits, the commutative monoid $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, \mathcal{C})$ in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}}$ induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ a symmetric monoidal category $\mathrm{Fun}_{\mathrm{T}}(\mathrm{T}, \mathcal{C}_{\mathsf{s}})$ compatible with small colimits.
  So the commutative monoid $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, \mathcal{C})$ in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{S}}$ gives rise to an associative monoid $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T}, \mathcal{C})^{\Sigma}$ in $\mathsf{Cat}_{\infty/\mathrm{S}}$.

Proof. For every $t \in T$ we have a natural transformation $\mathrm{Fun}_T(T,-) \to \mathrm{Fun}_T(\{t\},-)$ of finite products preserving functors that yields a symmetric monoidal small colimits preserving functor $\mathrm{Fun}_T(T,\mathcal{C}_s) \to \mathrm{Fun}_T(\{t\},\mathcal{C}_s) \simeq \mathcal{C}_{s,t}$ between cocomplete categories using that $\mathcal{C}_s \to T$ is a cartesian fibration.

□

The commutative monoid structure on $\mathrm{Fun}^{/S}_{S\times T}(S\times T,\mathcal{C}) \to S$ in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/S}$ restricts to $\mathrm{Fun}^{/S,\mathcal{E}}_{S\times T}(S\times T,\mathcal{C}) \to S$ such that for every $s \in S$ the fiber $\mathrm{Fun}^{/S,\mathcal{E}}_{S\times T}(S\times T,\mathcal{C})_s \subset \mathrm{Fun}^{/S}_{S\times T}(S\times T,\mathcal{C})_s$ is closed under small colimits.
Thus the composition product on $\mathrm{Fun}^{/S}_{S\times T}(S\times T,\mathcal{C})^{\Sigma} \to S$ restricts to $\mathrm{Fun}^{/S,\mathcal{E}}_{S\times T}(S\times T,\mathcal{C})^{\Sigma} \to S$.

Proof. To see this, we may reduce to the case that S is contractible.

If for every $t \in T$ the category $\mathcal{C}_t$ admits small colimits, the full subcategory $\mathrm{Fun}^{\mathcal{E}}_T(T,\mathcal{C}) \subset \mathrm{Fun}_T(T,\mathcal{C})$ is closed under small colimits, which follows from the case $\mathcal{E} = T = \Delta^1$.
For $\mathcal{D} \to S\times T$ the cartesian fibration corresponding to the identity of $S^{\mathrm{op}} = \mathsf{Cat}^{\mathrm{cart}}_{\infty/T}$ the embedding $\mathrm{Fun}^{/S,\mathcal{E}}_{S\times T}(S\times T,\mathcal{D}) \subset \mathrm{Fun}^{/S}_{S\times T}(S\times T,\mathcal{D})$ of cartesian fibrations over S classifies a natural transformation

$$\beta : \mathrm{Fun}^{\mathcal{E}}_T(T,-) \to \mathrm{Fun}_T(T,-)$$

of finite products preserving functors $\mathsf{Cat}^{\mathrm{cart}}_{\infty/T} \to \mathsf{Cat}_\infty$ that sends $\mathcal{C} \to T$ to a fully faithful symmetric monoidal functor

$$\mathrm{Fun}^{\mathcal{E}}_T(T,\mathcal{C}) \to \mathrm{Fun}_T(T,\mathcal{C}).$$

□

Now we are able to state the main proposition, from which we dedude proposition 5.84 as a corollary.

Proposition 5.83. *Let* S, T *be categories and* $\mathcal{C} \to S\times T$ *a commutative monoid in* $\mathsf{Cat}^{\mathrm{cart}}_{\infty/S\times T}$ *such that for every* $s \in S, t \in T$ *the induced symmetric monoidal category* $\mathcal{C}_{s,t}$ *is compatible with small colimits.*

There is a map

$$\mathrm{Fun}^{/S}_{S\times T}(S\times T,\mathcal{C})^{\Sigma} \to \mathrm{Fun}^{/S}_{S\times T}(\mathcal{C},\mathcal{C})$$

of associative monoids in $\mathsf{Cat}_{\infty/S}$ *that sends a symmetric sequence* A *in* $\mathrm{Fun}_T(T,\mathcal{C}_s)$ *and an object* $X \in \mathcal{C}_{s,t}$ *for some* $s \in S, t \in T$ *to the object* $A(t) \circ X \in \mathcal{C}_{s,t}$.

Let $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, T)$ *be a full subcategory.*

If for every $s \in S$ *and morphism* $f : t \to t'$ *in* T *that belongs to* $\mathcal{E}$ *the induced functor* $\mathcal{C}_{s,t} \to \mathcal{C}_{s,t'}$ *preserves small colimits, this map restricts to a map*

$$\mathrm{Fun}^{/S,\mathcal{E}}_{S\times T}(S\times T,\mathcal{C})^{\Sigma} \to \mathrm{Fun}^{/S,\mathcal{E}}_{S\times T}(\mathcal{C},\mathcal{C})$$

of associative monoids in $\mathsf{Cat}_{\infty/S}$.

Proof. The counit transformation $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T},-)\times\mathrm{T}\to\mathrm{id}$ of finite products preserving functors $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}\to\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ yields a map $\alpha:\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T},\mathcal{C})\times\mathrm{T}\to\mathcal{C}$ of commutative monoids in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$.

α induces on the fiber over every $\mathsf{s}\in\mathrm{S},\mathrm{t}\in\mathrm{T}$ the small colimits preserving functor $\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C}_{\mathsf{s}})\to\mathrm{Fun}_{\mathrm{T}}(\{\mathrm{t}\},\mathcal{C}_{\mathsf{s}})\simeq\mathcal{C}_{\mathsf{s},\mathrm{t}}$.

So α yields a map

$$\phi:\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T},\mathcal{C})^{\Sigma}\times\mathrm{T}\to\mathcal{C}^{\Sigma}$$

of associative monoids in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$.

The evaluation map $\mathcal{C}^{\Sigma}\to\mathcal{C}^{\{0\}}$ of cartesian fibrations over $\mathrm{S}\times\mathrm{T}$ induces on the fiber over every $\mathsf{s}\in\mathrm{S},\mathrm{t}\in\mathrm{T}$ the evaluation functor $\mathcal{C}^{\Sigma}_{\mathsf{s},\mathrm{t}}\to\mathcal{C}^{\{0\}}_{\mathsf{s},\mathrm{t}}$ right adjoint to the fully faithful functor that considers an object of $\mathcal{C}_{\mathsf{s},\mathrm{t}}$ as a symmetric sequence concentrated in degree 0.

Being a map of cartesian fibrations over $\mathrm{S}\times\mathrm{T}$ the functor $\mathcal{C}^{\Sigma}\to\mathcal{C}^{\{0\}}$ admits a fully faithful left adjoint $\mathcal{C}\to\mathcal{C}^{\Sigma}$ relative to $\mathrm{S}\times\mathrm{T}$.

The associative monoid structure on $\mathcal{C}^{\Sigma}\to\mathrm{S}\times\mathrm{T}$ in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ endows $\mathcal{C}^{\Sigma}\to\mathrm{S}\times\mathrm{T}$ with a left module structure over itself in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ that restricts to a left module structure on $\mathcal{C}\to\mathrm{S}\times\mathrm{T}$ over $\mathcal{C}^{\Sigma}\to\mathrm{S}\times\mathrm{T}$ as for every $\mathsf{s}\in\mathrm{S},\mathrm{t}\in\mathrm{T}$ the induced left module structure on $\mathcal{C}^{\Sigma}_{\mathsf{s},\mathrm{t}}$ over itself restricts to a left module structure on $\mathcal{C}_{\mathsf{s},\mathrm{t}}$ over $\mathcal{C}^{\Sigma}_{\mathsf{s},\mathrm{t}}$.

Pulling back along ϕ we get a left module structure on $\mathcal{C}\to\mathrm{S}\times\mathrm{T}$ over $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T},\mathcal{C})^{\Sigma}$ with respect to the canonical $\mathsf{Cat}_{\infty/\mathrm{S}}$-left module structure on $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ corresponding to a map of associative monoids

$$\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathrm{S}\times\mathrm{T},\mathcal{C})^{\Sigma}\to\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathrm{T}}(\mathcal{C},\mathcal{C})$$

in $\mathsf{Cat}_{\infty/\mathrm{S}}$ that sends a symmetric sequence A in $\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C}_{\mathsf{s}})$ and an object $\mathrm{X}\in\mathcal{C}_{\mathsf{s},\mathrm{t}}$ for some $\mathsf{s}\in\mathrm{S},\mathrm{t}\in\mathrm{T}$ to the object $\mathrm{A}(\mathrm{t})\circ\mathrm{X}\in\mathcal{C}_{\mathsf{s},\mathrm{t}}$.

2. To show 2. we can assume that S is contractible.

For every functor $\mathrm{T}'\to\mathrm{T}$ we have a commutative square

$$\begin{array}{ccc}
\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C})^{\Sigma} & \longrightarrow & \mathrm{Fun}_{\mathrm{T}}(\mathcal{C},\mathcal{C}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_{\mathrm{T}'}(\mathrm{T}',\mathrm{T}'\times_{\mathrm{T}}\mathcal{C})^{\Sigma} & \longrightarrow & \mathrm{Fun}_{\mathrm{T}'}(\mathrm{T}'\times_{\mathrm{T}}\mathcal{C},\mathrm{T}'\times_{\mathrm{T}}\mathcal{C})
\end{array}$$

of monoidal categories.

So to prove 2. we can reduce to the case $\mathcal{E}=\mathrm{Fun}(\Delta^1,\mathrm{T})$.

If for every morphism $\mathsf{f}:\mathsf{s}\to\mathsf{t}$ in T the induced functor $\mathcal{C}_{\mathsf{s}}\to\mathcal{C}_{\mathrm{t}}$ preserves small colimits, the associative monoid $\mathcal{C}^{\Sigma}\to\mathrm{T}$ in $\mathsf{Cat}_{\infty/\mathrm{T}}$ is an associative monoid in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}}$.

Moreover the embedding $\mathcal{C}\subset\mathcal{C}^{\Sigma}$ is a map of cartesian fibrations over T so that $\mathcal{C}\to\mathrm{T}$ is a left module over $\mathcal{C}^{\Sigma}\to\mathrm{T}$ in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathrm{T}}$.

The functor $\alpha:\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C})\times\mathrm{T}\to\mathcal{C}$ over T restricts to a map $\mathrm{Fun}^{\mathrm{cart}}_{\mathrm{T}}(\mathrm{T},\mathcal{C})\times\mathrm{T}\to\mathcal{C}$ of cartesian fibrations over T.

Thus the restriction

$$\mathrm{Fun}^{\mathrm{cart}}_{\mathrm{T}}(\mathrm{T},\mathcal{C})^{\Sigma}\times\mathrm{T}\subset\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C})^{\Sigma}\times\mathrm{T}\xrightarrow{\phi}\mathcal{C}^{\Sigma}$$

is a map of cartesian fibrations over T.

The composition product on $\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C})^{\Sigma}$ restricts to $\mathrm{Fun}_{\mathrm{T}}^{\mathrm{cart}}(\mathrm{T},\mathcal{C})^{\Sigma}$.

So the left module structure on $\mathcal{C}\to\mathrm{T}$ over $\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C})^{\Sigma}$ with respect to the canonical Cat_{∞}-left module structure on $\mathsf{Cat}_{\infty/\mathrm{T}}$ restricts to a left module structure on $\mathcal{C}\to\mathrm{T}$ over $\mathrm{Fun}_{\mathrm{T}}^{\mathrm{cart}}(\mathrm{T},\mathcal{C})^{\Sigma}$ with respect to the canonical Cat_{∞}-left module structure on $\mathsf{Cat}_{\infty/\mathrm{T}}^{\mathrm{cart}}$.

So the monoidal functor $\mathrm{Fun}_{\mathrm{T}}(\mathrm{T},\mathcal{C})^{\Sigma}\times\mathrm{T}\to\mathrm{Fun}_{\mathrm{T}}(\mathcal{C},\mathcal{C})$ restricts to a monoidal functor $\mathrm{Fun}_{\mathrm{T}}^{\mathrm{cart}}(\mathrm{T},\mathcal{C})^{\Sigma}\times\mathrm{T}\to\mathrm{Fun}_{\mathrm{T}}^{\mathrm{cart}}(\mathcal{C},\mathcal{C})$.

□

Let $\mathcal{C}\to\mathrm{S}$ be a commutative monoid in $\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}$ corresponding to a cocartesian S-family $\mathcal{C}^{\otimes}\to\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}$ of symmetric monoidal categories.

The category $\mathrm{Cmon}(\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}})$ is preadditive so that the forgetful functor $\mathrm{Cmon}(\mathrm{Cmon}(\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}))\to\mathrm{Cmon}(\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}})$ is an equivalence.

Thus the cocartesian S-family $\mathcal{C}^{\otimes}\to\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}$ of symmetric monoidal categories lifts canonically to a commutative monoid in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}}^{\mathrm{cocart}}$ that induces on the fiber over every $\mathsf{s}\in\mathrm{S},\langle\mathrm{n}\rangle\in\mathcal{F}\mathrm{in}_{*}$ the canonical symmetric monoidal structure on $\mathcal{C}^{\otimes}_{\mathsf{s},\langle\mathrm{n}\rangle}\simeq\mathcal{C}_{\mathsf{s}}^{\times\mathrm{n}}$.

So if for every $\mathsf{s}\in\mathrm{S}$ the induced symmetric monoidal category $\mathcal{C}_{\mathsf{s}}$ is compatible with small colimits, the commutative monoid structure on $\mathcal{C}^{\otimes}\to\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}$ in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}}^{\mathrm{cocart}}$ induces on the fiber over every $\mathsf{s}\in\mathrm{S},\langle\mathrm{n}\rangle\in\mathcal{F}\mathrm{in}_{*}$ a symmetric monoidal category compatible with small colimits.

Hence the cartesian fibration $\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}}(\mathrm{S}\times\mathcal{F}\mathrm{in}_{*},(\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}}\simeq$ $\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_{*}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_{*}^{\mathrm{op}},((\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})\to\mathrm{S}^{\mathrm{op}}$ carries a canonical structure of a commutative monoid in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}^{\mathrm{cart}}$ that restricts to

$$\mathrm{Cocoalg}^{/\mathrm{S}^{\mathrm{op}}}(\mathcal{C})\simeq\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\mathcal{E}}_{\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_{*}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_{*}^{\mathrm{op}},((\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})$$

and thus the composition product on $(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}}(\mathrm{S}\times\mathcal{F}\mathrm{in}_{*},(\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})^{\Sigma}$ in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ restricts to an associative monoid structure on $\mathrm{Cocoalg}^{/\mathrm{S}^{\mathrm{op}}}(\mathcal{C})^{\Sigma}$ in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$.

Proposition 5.84. *Let* S *be a category and* $\mathcal{C}\to\mathrm{S}$ *a commutative monoid in* $\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}$ *such that for every* $\mathsf{s}\in\mathrm{S}$ *the induced symmetric monoidal category* $\mathcal{C}_{\mathsf{s}}$ *is compatible with small colimits.*

There is a map of associative monoids

$$\mathrm{Cocoalg}^{/\mathrm{S}^{\mathrm{op}}}(\mathcal{C})^{\Sigma}\to\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\otimes,\mathrm{oplax}}(\mathcal{C},\mathcal{C})$$

in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$.

Proof. The cocartesian S-family $\mathcal{C}^{\otimes}\to\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}$ of symmetric monoidal categories corresponding to the commutative monoid $\mathcal{C}\to\mathrm{S}$ in $\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}$ lifts canonically to a commutative monoid in $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathcal{F}\mathrm{in}_{*}}^{\mathrm{cocart}}$ that induces on the fiber over every $\mathsf{s}\in\mathrm{S},\langle\mathrm{n}\rangle\in\mathcal{F}\mathrm{in}_{*}$ the symmetric monoidal structure on $\mathcal{C}^{\otimes}_{\mathsf{s},\langle\mathrm{n}\rangle}\simeq\mathcal{C}_{\mathsf{s}}^{\times\mathrm{n}}$ that is compatible with small colimits.

So by prop. 5.83 we have a map

$$\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_{*}^{\mathrm{op}}}(\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_{*}^{\mathrm{op}},((\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})^{\Sigma}\to$$

$$\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}}}_{\mathrm{S}^{\mathrm{op}}\times\mathcal{F}\mathrm{in}_*^{\mathrm{op}}}(((\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}},((\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})$$

of associative monoids in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ canonically equivalent to a map

$$(\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathcal{F}\mathrm{in}_*}(\mathrm{S}\times\mathcal{F}\mathrm{in}_*,(\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}})^{\Sigma}\to\mathrm{Fun}^{/\mathrm{S}}_{\mathrm{S}\times\mathcal{F}\mathrm{in}_*}((\mathcal{C}^{\otimes})^{\mathrm{rev}},(\mathcal{C}^{\otimes})^{\mathrm{rev}})^{\mathrm{op}}$$

of associative monoids in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$.

This map restricts to the desired map of associative monoids

$$\mathrm{Cocoalg}^{/\mathrm{S}^{\mathrm{op}}}(\mathcal{C})^{\Sigma}\to\mathrm{Fun}^{/\mathrm{S}^{\mathrm{op}},\otimes,\mathrm{oplax}}(\mathcal{C},\mathcal{C})$$

in $\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ as for every $\mathsf{s}\in\mathrm{S}$ and inert morphism $\langle\mathrm{n}\rangle\to\langle\mathrm{m}\rangle$ the induced functor $\mathcal{C}^{\otimes}_{\mathsf{s},\langle\mathrm{n}\rangle}\to\mathcal{C}^{\otimes}_{\mathsf{s},\langle\mathrm{m}\rangle}$ preserves small colimits.

□

6 Appendix

6.1 Appendix A : About the universal properties of the Day-convolution

Given an operad $\mathcal{O}^{\otimes}$, a small $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes}$ and a $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{D}^{\otimes}$ compatible with small colimits we define a $\mathcal{O}^{\otimes}$-monoidal category $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ compatible with small colimits and satisfying $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}_{\mathrm{X}} \simeq \mathrm{Fun}(\mathcal{C}_{\mathrm{X}},\mathcal{D}_{\mathrm{X}})$ for every $\mathrm{X} \in \mathcal{O}$, which we call the Day-convolution $\mathcal{O}$-monoidal structure.

Denote $\widehat{\mathcal{R}} \subset \mathrm{Fun}(\Delta^1, \widehat{\mathsf{Cat}_\infty})$ the full subcategory spanned by the right fibrations that is closed under finite products.

Given a $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{B}^{\otimes}$ corresponding to a $\mathcal{O}^{\otimes}$-algebra $\phi: \mathcal{O}^{\otimes} \to \widehat{\mathsf{Cat}_\infty}^{\times}$ denote $\widehat{\mathcal{P}}(\mathcal{B})^{\otimes}$ the pullback of the symmetric monoidal functor $\widehat{\mathcal{R}}^{\times} \subset \mathrm{Fun}(\Delta^1, \widehat{\mathsf{Cat}_\infty})^{\times} \to \mathrm{Fun}(\{1\}, \widehat{\mathsf{Cat}_\infty})^{\times}$ along ϕ.

We define $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\otimes}$ to be the full suboperad spanned by the objects of

$$\widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\otimes}_{\mathrm{X}} \simeq \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{op}}_{\mathrm{X}} \times \mathcal{D}_{\mathrm{X}}) \simeq \mathrm{Fun}(\mathcal{C}_{\mathrm{X}}, \widehat{\mathcal{P}}(\mathcal{D}_{\mathrm{X}}))$$

for some $\mathrm{X} \in \mathcal{O}$ that belong to $\mathrm{Fun}(\mathcal{C}_{\mathrm{X}},\mathcal{D}_{\mathrm{X}}) \subset \mathrm{Fun}(\mathcal{C}_{\mathrm{X}}, \widehat{\mathcal{P}}(\mathcal{D}_{\mathrm{X}}))$.

We prove that the restriction $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$ is a cocartesian fibration of operads (prop. 6.4).

We show that $\mathcal{O}^{\otimes}$-algebras in the Day-convolution $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes}$ are lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ (proposition 6.26).

This proves by the way that for $\mathcal{O}^{\otimes}$ the commutative operad our Day-convolution coincides with Glasman's Day-convolution [11].

We show that there is a canonical $\mathcal{O}^{\otimes}$-monoidal equivalence $\mathrm{Fun}(\mathcal{C},\mathcal{D})^{\otimes} \simeq \mathcal{P}(\mathcal{C}^{\mathrm{rev}})^{\otimes} \otimes \mathcal{D}^{\otimes}$ (prop. 6.23). This implies that our Day-convolution coincides with Lurie's Day-convolution [18].

We use this description of the Day-convolution and its universal properties in section 3. to prove theorem 3.21.

Moreover we use the characterization of $\mathcal{O}$-algebras in the Day-convolution to deduce the following result (prop. 6.35):

Given $\mathcal{O}^{\otimes}$-monoidal categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ taking the right adjoint defines an equivalence between the category of left adjoint oplax $\mathcal{O}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and the category of right adjoint lax $\mathcal{O}$-monoidal functors $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.

6.1.1 Construction of the Day convolution

The evaluation at the target functor $\varrho : \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)$ is a bicartesian fibration as Cat_∞ admits pullbacks.

Denote $\mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ the full subcategory spanned by the right fibrations.

As right fibrations are stable under pullback, ϱ restricts to a cartesian fibration $\mathrm{q} : \mathcal{R} \to \mathsf{Cat}_\infty$ with the same cartesian morphisms classifying the functor $\mathrm{Fun}((-)^{\mathrm{op}}, \mathcal{S}) : \mathsf{Cat}_\infty^{\mathrm{op}} \to \widehat{\mathsf{Cat}_\infty^{\mathrm{R}}} \subset \widehat{\mathsf{Cat}_\infty}$.

Hence q is also a cocartesian fibration classifying a functor $\mathcal{P} : \mathsf{Cat}_\infty \to \widehat{\mathsf{Cat}_\infty^{\mathrm{L}}} \subset \widehat{\mathsf{Cat}_\infty}$ that induces a functor $\mathcal{P} : \mathsf{Cat}_\infty \to \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}$.

The evaluation at the target functor ϱ preserves finite products and so induces a symmetric monoidal functor $\varrho^\times : \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \to \mathsf{Cat}_\infty^\times$ on cartesian structures that is equivalent to the symmetric monoidal functor $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \simeq (\mathsf{Cat}_\infty^\times)^{\Delta^1} \to (\mathsf{Cat}_\infty^\times)^{\{1\}}$ by the uniqueness of the cartesian structure.

Corollary 6.14 guarantees that $\varrho^\times$ is a cocartesian fibration of symmetric monoidal categories compatible with small colimits.

The full subcategory $\mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ is closed under finite products so that we get a symmetric monoidal functor $\mathrm{q}^\times : \mathcal{R}^\times \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \xrightarrow{\varrho^\times} \mathsf{Cat}_\infty^\times$.

The next proposition 6.1 tells us that $\mathrm{q}^\times : \mathcal{R}^\times \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \xrightarrow{\varrho^\times} \mathsf{Cat}_\infty^\times$ is a localization of $\varrho^\times : \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \to \mathsf{Cat}_\infty^\times$ relative to $\mathsf{Cat}_\infty^\times$.

This implies that $\mathrm{q}^\times : \mathcal{R}^\times \to \mathsf{Cat}_\infty^\times$ is a cocartesian fibration of symmetric monoidal categories compatible with small colimits classifying a lax symmetric monoidal functor $\mathsf{Cat}_\infty^\times \to \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}^\otimes \subset \widehat{\mathsf{Cat}_\infty}^\times$ lifting $\mathcal{P}$ that sends a small $\mathcal{O}^\otimes$-monoidal category $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ to the $\mathcal{O}^\otimes$-monoidal category

$$\mathcal{P}(\mathcal{D})^\otimes := \mathcal{O}^\otimes \times_{\mathsf{Cat}_\infty^\times} \mathcal{R}^\times \to \mathcal{O}^\otimes.$$

Denote $\mathcal{U} \subset \mathcal{R}$ the full subcategory spanned by the representable right fibrations. By theorem 6.12 the restriction $\mathcal{U} \subset \mathcal{R} \to \mathsf{Cat}_\infty$ classifies the identity of Cat_∞. As $\mathcal{U}$ is closed under finite products in $\mathcal{R}$, the embedding $\mathcal{U} \subset \mathcal{R}$ of cocartesian fibrations over Cat_∞ induces a symmetric monoidal embedding $\mathcal{U}^\times \subset \mathcal{R}^\times$ that is a map of cocartesian fibrations over $\mathsf{Cat}_\infty^\times$.

By corollary 6.12 the cocartesian fibration $\mathcal{U}^\times \to \mathsf{Cat}_\infty^\times$ corresponds to the identity of $\mathsf{Cat}_\infty^\times$ and the map $\mathcal{U}^\times \subset \mathcal{R}^\times$ of cocartesian fibrations over $\mathsf{Cat}_\infty^\times$ corresponds to a symmetric monoidal natural transformation from the symmetric monoidal embedding $\mathsf{Cat}_\infty^\times \subset \widehat{\mathsf{Cat}_\infty}^\times$ to the lax symmetric monoidal functor $\mathsf{Cat}_\infty^\times \to \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}^\otimes \subset \widehat{\mathsf{Cat}_\infty}^\times$ lifting $\mathcal{P}$.

So we get a $\mathcal{O}^\otimes$-monoidal Yoneda-embedding

$$\mathcal{D}^\otimes \simeq \mathcal{O}^\otimes \times_{\mathsf{Cat}_\infty^\times} \mathcal{U}^\times \subset \mathcal{P}(\mathcal{D})^\otimes = \mathcal{O}^\otimes \times_{\mathsf{Cat}_\infty^\times} \mathcal{R}^\times.$$

Thus lemma 6.7 1. implies that the lax symmetric monoidal functor $\mathsf{Cat}_\infty^\times \to \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}^\otimes$ lifting $\mathcal{P}$ is symmetric monoidal.

Proposition 6.1. *The restriction* $\mathrm{q}^\times : \mathcal{R}^\times \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \xrightarrow{\varrho^\times} \mathsf{Cat}_\infty^\times$ *is a localization of* $\varrho^\times : \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)^\times \to \mathsf{Cat}_\infty^\times$ *relative to* $\mathsf{Cat}_\infty^\times$.

Proof. Recall the notion of a factorization system:

Given a category $\mathcal{C}$ we call a pair (L,R) consisting of full subcategories $\mathrm{L},\mathrm{R} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})$ a factorization system on $\mathcal{C}$ if the following conditions are satisfied:

1. Factorization: Every morphism in $\mathcal{C}$ admits a factorization $\mathrm{X} \to \mathrm{Y} \to \mathrm{Z}$, where the morphism $\mathrm{X} \to \mathrm{Y}$ belongs to L and the morphism $\mathrm{Y} \to \mathrm{Z}$ belongs to R.
2. Retracts: $\mathrm{L},\mathrm{R} \subset \mathrm{Fun}(\Delta^1,\mathcal{C})$ are closed under retracts.
3. Liftings: For every morphism $\mathrm{A} \to \mathrm{B}$ of L and every morphism $\mathrm{X} \to \mathrm{Y}$ of R the induced square

$$\begin{array}{ccc} \mathcal{C}(\mathrm{B},\mathrm{X}) & \longrightarrow & \mathcal{C}(\mathrm{A},\mathrm{X}) \\ \downarrow & & \downarrow \\ \mathcal{C}(\mathrm{B},\mathrm{Y}) & \longrightarrow & \mathcal{C}(\mathrm{A},\mathrm{Y}) \end{array}$$

   is a pullback square.

Denote $\mathrm{Cof} \subset \mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty)$ the full subcategory spanned by the cofinal functors.

The full subcategories $\mathrm{Cof},\mathcal{R} \subset \mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty)$ define a factorization system $(\mathrm{Cof},\mathcal{R})$ on the category Cat_∞.

1. follows from the fact that there is a strict factorization system on sSet, where the left class are the right anodyne maps and the right class are the right fibrations and that every right anodyne functor is cofinal.

2. follows from the following descriptions of right fibrations and cofinal functors:

A functor $\mathcal{A} \to \mathcal{B}$ is a right fibration if and only if the commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1,\mathcal{A}) & \longrightarrow & \mathrm{Fun}(\Delta^1,\mathcal{B}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\{1\},\mathcal{A}) & \longrightarrow & \mathrm{Fun}(\{1\},\mathcal{B}) \end{array}$$

is a pullback square ([19] cor. 2.1.2.10.).

A functor $\mathcal{A} \to \mathcal{B}$ is cofinal if and only if for every $\mathrm{X} \in \mathcal{B}$ the pullback $\mathcal{A} \times_{\mathcal{B}} \mathcal{B}_{\mathrm{X}/}$ is weakly contractible ([19] theorem 4.1.3.1.).

These descriptions also imply that the full subcategories $\mathrm{Cof},\mathcal{R} \subset \mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty)$ are closed under finite products.

3. follows from [19] cor. 2.1.2.9. and the fact that every cofinal functor factors as a right anodyne functor followed by an equivalence according to [19] cor. 4.1.1.12..

Remark 6.18 implies that the restriction $\mathrm{q}^\times : \mathcal{R}^\times \subset \mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty)^\times \xrightarrow{\varrho^\times} \mathsf{Cat}_\infty^\times$ is a localization of $\varrho^\times : \mathrm{Fun}(\Delta^1,\mathsf{Cat}_\infty)^\times \to \mathsf{Cat}_\infty^\times$ compatible with the cocartesian fibration.

□

As next we consider localizations of presheaf categories.

Let $\mathcal{C}^\otimes,\mathcal{D}^\otimes$ be $\mathcal{O}^\otimes$-monoidal categories for some operad $\mathcal{O}^\otimes$.

Let $\mathcal{B}^\otimes \subset \widehat{\mathcal{P}}(\mathcal{D})^\otimes$ be a full $\mathcal{O}^\otimes$-monoidal subcategory.

We write

$$\mathrm{Fun}(\mathcal{C},\mathcal{B})^\otimes \subset \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^\otimes$$

for the full subcategory spanned by the objects of $\widehat{\mathcal{P}}(\mathcal{C}_X^{\mathrm{op}} \times \mathcal{D}_X) \simeq \mathrm{Fun}(\mathcal{C}_X, \widehat{\mathcal{P}}(\mathcal{D}_X))$ for some $X \in \mathcal{O}$ that belong to $\mathrm{Fun}(\mathcal{C}_X, \mathcal{B}_X) \subset \mathrm{Fun}(\mathcal{C}_X, \widehat{\mathcal{P}}(\mathcal{D}_X))$.

Observation 6.2. *Let $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ be $\mathcal{O}^\otimes$-monoidal categories for some operad $\mathcal{O}^\otimes$.*

Let $\mathcal{B}^\otimes \subset \widehat{\mathcal{P}}(\mathcal{D})^\otimes$ be a $\mathcal{O}^\otimes$-monoidal localization and $\mathcal{A}^\otimes \subset \mathcal{B}^\otimes$ a full $\mathcal{O}^\otimes$-monoidal subcategory.

1. *Then $\mathrm{Fun}(\mathcal{C},\mathcal{B})^\otimes \subset \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^\otimes$ is a $\mathcal{O}^\otimes$-monoidal localization.*

2. *Assume that for every operation $h \in \mathrm{Mul}_{\mathcal{O}}(X_1,...,X_n,Y)$ for some objects $X_1,...,X_n,Y \in \mathcal{O}$ with $n \in \mathbb{N}$ left kan extension*

$$\mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_Y) \to \mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y)$$

along the functor $\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n} \to \mathcal{C}_Y$ induced by h restricts to a functor $\mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{A}_Y) \to \mathrm{Fun}(\mathcal{C}_Y, \mathcal{A}_Y)$.

Then the restriction $\mathrm{Fun}(\mathcal{C},\mathcal{A})^\otimes \subset \mathrm{Fun}(\mathcal{C},\mathcal{B})^\otimes \to \mathcal{O}^\otimes$ is a $\mathcal{O}^\otimes$-monoidal category and the embedding $\mathrm{Fun}(\mathcal{C},\mathcal{A})^\otimes \subset \mathrm{Fun}(\mathcal{C},\mathcal{B})^\otimes$ is a $\mathcal{O}^\otimes$-monoidal functor.

Proof. 1: For every $X \in \mathcal{O}$ the localization $\mathcal{B}_X \subset \widehat{\mathcal{P}}(\mathcal{D}_X)$ yields a localization $\mathrm{Fun}(\mathcal{C}_X, \mathcal{B}_X) \subset \mathrm{Fun}(\mathcal{C}_X, \widehat{\mathcal{P}}(\mathcal{D}_X))$.

Given an operation $h \in \mathrm{Mul}_{\mathcal{O}}(X_1,...,X_n,Y)$ for some objects $X_1,...,X_n,Y \in \mathcal{O}$ with $n \in \mathbb{N}$ the induced functor

$$\mathcal{P}(\mathcal{C}_{X_1}^{\mathrm{op}} \times \mathcal{D}_{X_1}) \times ... \times \mathcal{P}(\mathcal{C}_{X_n}^{\mathrm{op}} \times \mathcal{D}_{X_n}) \to \mathcal{P}(\mathcal{C}_Y^{\mathrm{op}} \times \mathcal{D}_Y)$$

factors as

$$\mathcal{P}(\mathcal{C}_{X_1}^{\mathrm{op}} \times \mathcal{D}_{X_1}) \times ... \times \mathcal{P}(\mathcal{C}_{X_n}^{\mathrm{op}} \times \mathcal{D}_{X_n}) \to \mathcal{P}((\mathcal{C}_{X_1}^{\mathrm{op}} \times \mathcal{D}_{X_1}) \times ... \times (\mathcal{C}_{X_n}^{\mathrm{op}} \times \mathcal{D}_{X_n})) \simeq$$

$$\mathcal{P}((\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n})^{\mathrm{op}} \times (\mathcal{D}_{X_1} \times .... \times \mathcal{D}_{X_n})) \to \mathcal{P}(\mathcal{C}_Y^{\mathrm{op}} \times \mathcal{D}_Y)$$

and thus as the functor α :

$$\mathcal{P}(\mathcal{C}_{X_1}^{\mathrm{op}} \times \mathcal{D}_{X_1}) \times ... \times \mathcal{P}(\mathcal{C}_{X_n}^{\mathrm{op}} \times \mathcal{D}_{X_n}) \simeq \mathrm{Fun}(\mathcal{C}_{X_1}, \mathcal{P}(\mathcal{D}_{X_1})) \times ... \mathrm{Fun}(\mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_{X_n})) \to$$

$$\mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_{X_1}) \times ... \times \mathcal{P}(\mathcal{D}_{X_n})) \to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_{X_1} \times ... \times \mathcal{D}_{X_n}))$$

$$\to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_Y))$$

followed by the functor

$$\beta : \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_Y)) \to \mathrm{Fun}(\mathcal{C}_Y, \mathcal{P}(\mathcal{D}_Y)) \simeq \mathcal{P}(\mathcal{C}_Y^{\mathrm{op}} \times \mathcal{D}_Y)$$

by lemma 6.6.

The functor α preserves local equivalences as the functor $\mathcal{P}(\mathcal{D}_{X_1}) \times ... \times \mathcal{P}(\mathcal{D}_{X_n}) \to \mathcal{P}(\mathcal{D}_{X_1} \times ... \times \mathcal{D}_{X_n}) \to \mathcal{P}(\mathcal{D}_Y)$ induced by h does by our assumption that $\mathcal{B}^\otimes \subset \widehat{\mathcal{P}}(\mathcal{D})^\otimes$ is a $\mathcal{O}^\otimes$-monoidal localization.

Denote $L_Y : \mathcal{P}(\mathcal{D}_Y) \to \mathcal{B}_Y$ the left adjoint of the full subcategory inclusion $\mathcal{B}_Y \subset \mathcal{P}(\mathcal{D}_Y)$.

We have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y) & \longrightarrow & \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_Y) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}_Y, \mathcal{P}(\mathcal{D}_Y)) & \longrightarrow & \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_Y)) \end{array}$$

of right adjoints that yields a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_Y)) & \xrightarrow{\beta} & \mathrm{Fun}(\mathcal{C}_Y, \mathcal{P}(\mathcal{D}_Y)) \\ \downarrow{\scriptstyle \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, L_Y)} & & \downarrow{\scriptstyle \mathrm{Fun}(\mathcal{C}_Y, L_Y)} \\ \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_Y) & \xrightarrow{\beta'} & \mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y) \end{array}$$

of left adjoints, where the functor $\mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y) \to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_Y)$ admits a left adjoint β' because $\mathcal{B}_Y$ admits large colimits as a localization of a category with large colimits. Hence β preserves local equivalences.

2: 1. implies that the restriction $\mathrm{Fun}(\mathcal{C}, \mathcal{B})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$ is a $\mathcal{O}^{\otimes}$-monoidal category.

$$\mathrm{Fun}(\mathcal{C}_{X_1}, \mathcal{B}_{X_1}) \times ... \times \mathrm{Fun}(\mathcal{C}_{X_n}, \mathcal{B}_{X_n}) \to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_{X_1} \times ... \times \mathcal{B}_{X_n})$$

For every operation $h \in \mathrm{Mul}_{\mathcal{O}}(X_1, ..., X_n, Y)$ for some objects $X_1, ..., X_n, Y \in \mathcal{O}$ with $n \in \mathbb{N}$ the induced functor

$$\mathrm{Fun}(\mathcal{C}_{X_1}, \mathcal{B}_{X_1}) \times ... \times \mathrm{Fun}(\mathcal{C}_{X_n}, \mathcal{B}_{X_n}) \to \mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y)$$

factors as

$$\mathrm{Fun}(\mathcal{C}_{X_1}, \mathcal{B}_{X_1}) \times ... \times \mathrm{Fun}(\mathcal{C}_{X_n}, \mathcal{B}_{X_n}) \subset \mathrm{Fun}(\mathcal{C}_{X_1}, \mathcal{P}(\mathcal{D}_{X_1})) \times ... \times \mathrm{Fun}(\mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_{X_n})) \to$$

$$\mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_{X_1}) \times ... \times \mathcal{P}(\mathcal{D}_{X_n})) \to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_{X_1} \times ... \times \mathcal{D}_{X_n}))$$
$$\to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_Y))$$

followed by the functor

$$\mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{P}(\mathcal{D}_Y)) \to \mathrm{Fun}(\mathcal{C}_Y, \mathcal{P}(\mathcal{D}_Y)) \xrightarrow{\mathrm{Fun}(\mathcal{C}_Y, L_Y)} \mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y)$$

and thus factors as

$$\mathrm{Fun}(\mathcal{C}_{X_1}, \mathcal{B}_{X_1}) \times ... \times \mathrm{Fun}(\mathcal{C}_{X_n}, \mathcal{B}_{X_n}) \to \mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_{X_1} \times ... \times \mathcal{B}_{X_n}) \to$$

$$\mathrm{Fun}(\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n}, \mathcal{B}_Y) \xrightarrow{\beta'} \mathrm{Fun}(\mathcal{C}_Y, \mathcal{B}_Y)$$

using that $\mathcal{B}^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{D})^{\otimes}$ is a $\mathcal{O}^{\otimes}$-monoidal localization.

□

As next we turn to generalized presheaf categories:

Given full subcategories $\mathcal{K}, \mathcal{K}' \subset \mathsf{Cat}_\infty$ and a small category $\mathcal{C}$ denote

$$\mathcal{P}_{\mathcal{K}}(\mathcal{C}) \subset \mathcal{P}(\mathcal{C})$$

the full subcategory spanned by the functors $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ that preserve limits indexed by categories that belong to $\mathcal{K}$.

The Yoneda-embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ induces an embedding $\mathcal{C} \to \mathcal{P}_{\mathcal{K}}(\mathcal{C})$ as every representable functor preserves small limits.

Moreover by the Yoneda-lemma the Yoneda-embedding $\mathcal{C} \to \mathcal{P}_{\mathcal{K}}(\mathcal{C})$ preserves colimits indexed by categories that belong to $\mathcal{K}$.

Denote
$$\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$$
the smallest full subcategory of $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$ that contains the representable presheaves and closed under colimits indexed by categories that belong to $\mathcal{K}'$.

The Yoneda-embedding $\mathcal{C} \to \mathcal{P}(\mathcal{C})$ factors as an embedding $\mathcal{C} \to \mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ that preserves colimits indexed by categories that belong to $\mathcal{K}$ followed by an embedding $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C}) \subset \mathcal{P}_{\mathcal{K}}(\mathcal{C})$ that preserves colimits indexed by categories that belong to $\mathcal{K}'$.

Remark 6.3.

1. *If $\mathcal{K} \subset \mathsf{Cat}_\infty$, the category $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})$ is the localization of $\widehat{\mathcal{P}}(\mathcal{C})$ with respect to* $\mathrm{S}_{\mathcal{K}}^{\mathcal{C}} := \{\mathrm{colim}(\mathsf{y} \circ \mathrm{H}) \to \mathsf{y}(\mathrm{colim}(\mathrm{H})) \mid \mathrm{H} : \mathrm{J} \to \mathcal{C}$ *a functor with* $\mathrm{J} \in \mathcal{K}$ *that admits a colimit*$\}$
2. *If $\mathcal{K}' \subset \mathsf{Cat}_\infty$ and $\mathcal{C} \in \widehat{\mathsf{Cat}_\infty}$ the category $\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ belongs to $\widehat{\mathsf{Cat}_\infty}$.*
3. *Let $\mathcal{K} \subset \mathcal{K}'$. For arbitrary categories $\mathcal{C} \in \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}(\mathcal{K})}$ and $\mathcal{D} \in \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}(\mathcal{K}')$ the functor*
$$\mathrm{Fun}^{\mathrm{coc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\mathrm{coc},\mathcal{K}}(\mathcal{C}, \mathcal{D})$$
induced by composition with the Yoneda embedding $\mathcal{C} \subset \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ is an equivalence.
4. *Let $\mathcal{K} \subset \mathsf{Cat}_\infty, \mathcal{K}' = \mathsf{Cat}_\infty$ and $\mathcal{C} \in \mathsf{Cat}_\infty(\mathcal{K})$.*
   Then $\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C})$ is the smallest full subcategory of $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{C})$ that contains the representable presheaves and closed under colimits indexed by small categories and thus coincides with $\mathcal{P}_{\mathcal{K}}(\mathcal{C})$.
   Especially we have the following:
   If $\mathcal{C}$ is a small category, $\mathcal{K} = \varnothing$ and $\mathcal{K}' = \mathsf{Cat}_\infty$, we have $\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{C}) = \mathcal{P}(\mathcal{C})$.

Let $\mathcal{K}, \mathcal{K}' \subset \mathsf{Cat}_\infty$ be full subcategories.
Let $\mathcal{O}^\otimes$ be an operad and $\mathcal{D}^\otimes$ a small $\mathcal{O}^\otimes$-monoidal category.

We write $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D})^\otimes \subset \mathcal{P}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \mathcal{P}(\mathcal{D})^\otimes$ for the full suboperads spanned by the objects of $\mathcal{P}(\mathcal{D}_\mathrm{X})$ that belong to $\mathcal{P}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_\mathrm{X})$ respectively $\mathcal{P}_{\mathcal{K}}(\mathcal{D}_\mathrm{X})$ for some $\mathrm{X} \in \mathcal{O}$.

By prop. 6.5 for every $\mathcal{O}^\otimes$-monoidal category $\mathcal{D}^\otimes$ compatible with colimits indexed by categories that belong to $\mathcal{K}$ the full suboperad $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}(\mathcal{D})^\otimes$ is a $\mathcal{O}^\otimes$-monoidal localization so that $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D})^\otimes$ is a $\mathcal{O}^\otimes$-monoidal category compatible with large colimits.

As we have a $\mathcal{O}^\otimes$-monoidal Yoneda-embedding $\mathcal{D}^\otimes \subset \mathcal{P}(\mathcal{D})^\otimes$, the suboperad $\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D})^\otimes \subset \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D})^\otimes$ is a full $\mathcal{O}^\otimes$-monoidal subcategory.

So by observation 6.2 1. for every $\mathcal{O}^\otimes$-monoidal category $\mathcal{C}^\otimes$ the full suboperad $\mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D}))^\otimes \subset \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^\otimes$ is a $\mathcal{O}^\otimes$-monoidal localization and thus especially a $\mathcal{O}^\otimes$-monoidal category compatible with large colimits.

Given a full $\mathcal{O}^{\otimes}$-monoidal subcategory $\mathcal{B}^{\otimes} \subset \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D})^{\otimes}$ such that for every operation $h \in \mathrm{Mul}_{\mathcal{O}}(X_1, ..., X_n, Y)$ for some objects $X_1, ..., X_n, Y \in \mathcal{O}$ with $n \in \mathbb{N}$ the category $\mathcal{B}_Y$ admits all left kan extensions along the functor $\mathcal{C}_{X_1} \times ... \times \mathcal{C}_{X_n} \to \mathcal{C}_Y$ induced by h that are preserved by the embedding $\mathcal{B}_Y \subset \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D}_Y)$ by 6.2 2. the full suboperad $\mathrm{Fun}(\mathcal{C}, \mathcal{B})^{\otimes} \subset \mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D}))^{\otimes}$ is a full $\mathcal{O}^{\otimes}$-monoidal subcategory.

Choosing $\mathcal{K} = \mathsf{Cat}_{\infty}$ and $\mathcal{B}^{\otimes} = \mathcal{D}^{\otimes} \subset \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D})^{\otimes}$ we find that $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \subset \mathrm{Fun}(\mathcal{C}, \widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D}))^{\otimes}$ is a full $\mathcal{O}^{\otimes}$-monoidal subcategory if $\mathcal{C}^{\otimes}$ is a small $\mathcal{O}^{\otimes}$-monoidal category and $\mathcal{D}^{\otimes}$ is compatible with small colimits.

This leads to the following proposition:

Proposition 6.4. *Let $\mathcal{O}^{\otimes}$ be an operad, $\mathcal{C}^{\otimes}$ a small $\mathcal{O}^{\otimes}$-monoidal category and $\mathcal{D}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category compatible with small colimits.*

Then $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$ *is a $\mathcal{O}^{\otimes}$-monoidal category compatible with small colimits.*

We start with prop. 6.5:

Proposition 6.5. *Let $\mathcal{K} \subset \mathsf{Cat}_{\infty}$ be a full subcategory.*

Let $\mathcal{O}^{\otimes}$ be an operad and $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category compatible with colimits indexed by categories that belong to $\mathcal{K}$.

$\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D})^{\otimes} \subset \widehat{\mathcal{P}}(\mathcal{D})^{\otimes}$ is a $\mathcal{O}^{\otimes}$-monoidal localization.

Proof. By remark 6.3 1. for every $X \in \mathcal{O}$ the full subcategory $\widehat{\mathcal{P}}_{\mathcal{K}}(\mathcal{D}_X) \subset \widehat{\mathcal{P}}(\mathcal{D}_X)$ is a localization.

It is enough to check the following conditions:

1. For every morphism $G : \mathcal{D} \to \mathcal{D}'$ of $\mathsf{Cat}_{\infty}^{\mathrm{coc}}(\mathcal{K})$ the induced functor $\widehat{\mathcal{P}}(G) : \widehat{\mathcal{P}}(\mathcal{D}) \to \widehat{\mathcal{P}}(\mathcal{D}')$ preserves local equivalences.
2. For every natural $n \in \mathbb{N}$ and arbitrary categories $\mathcal{D}_1, ..., \mathcal{D}_n \in \mathsf{Cat}_{\infty}^{\mathrm{coc}}(\mathcal{K})$ the canonical functor

$$\alpha : \widehat{\mathcal{P}}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}(\mathcal{D}_n) \to \widehat{\mathcal{P}}(\mathcal{D}_1 \times ... \times \mathcal{D}_n) \to \widehat{\mathcal{P}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$$

preserves local equivalences.

1: The set of morphisms of $\widehat{\mathcal{P}}(\mathcal{D})$ that are sent to local equivalences by $\widehat{\mathcal{P}}(G)$ is strongly saturated because $\widehat{\mathcal{P}}(G)$ preserves small colimits. As the set of local equivalences of $\widehat{\mathcal{P}}(\mathcal{D})$ is the smallest strongly saturated set of $\widehat{\mathcal{P}}(\mathcal{D})$ that contains $S_{\mathcal{K}}^{\mathcal{D}}$, it is enough to show that $\widehat{\mathcal{P}}(G)$ sends morphisms of $S_{\mathcal{K}}^{\mathcal{D}}$ to local equivalences.

Let a functor $H : J \to \mathcal{D}$ with $J \in \mathcal{K}$ be given and denote $\mathsf{y}_{\mathcal{D}} : \mathcal{D} \to \widehat{\mathcal{P}}(\mathcal{D})$ and $\mathsf{y}_{\mathcal{D}'} : \mathcal{D}' \to \widehat{\mathcal{P}}(\mathcal{D}')$ the corresponding Yoneda-embeddings.
The natural transformation $\widehat{\mathcal{P}}(G)(\mathrm{colim}(\mathsf{y}_{\mathcal{D}} \circ H)) \to \widehat{\mathcal{P}}(G)(\mathsf{y}_{\mathcal{D}}(\mathrm{colim}(H)))$ factors as

$$\widehat{\mathcal{P}}(G)(\mathrm{colim}(\mathsf{y}_{\mathcal{D}} \circ H)) \simeq \mathrm{colim}(\widehat{\mathcal{P}}(G) \circ \mathsf{y}_{\mathcal{D}} \circ H) \simeq \mathrm{colim}(\mathsf{y}_{\mathcal{D}'} \circ G \circ H)$$

$$\xrightarrow{\varepsilon} \mathsf{y}_{\mathcal{D}'}(\mathrm{colim}(G \circ H)) \simeq \mathsf{y}_{\mathcal{D}'}(G(\mathrm{colim}(H))) \simeq \widehat{\mathcal{P}}(G)(\mathsf{y}_{\mathcal{D}}(\mathrm{colim}(H))),$$

where we use that G and $\widehat{\mathcal{P}}(G)$ preserve colimits indexed by categories that belong to $\mathcal{K}$.

As $\varepsilon : \mathrm{colim}(\mathsf{y}_{\mathcal{D}'} \circ G \circ H) \to \mathsf{y}_{\mathcal{D}'}(\mathrm{colim}(G \circ H))$ belongs to $S_{\mathcal{K}}^{\mathcal{D}'}$, the functor $\widehat{\mathcal{P}}(G)(\mathrm{colim}(\mathsf{y}_{\mathcal{D}} \circ H)) \to \widehat{\mathcal{P}}(G)(\mathsf{y}_{\mathcal{D}}(\mathrm{colim}(H)))$ is a local equivalence in $\widehat{\mathcal{P}}(\mathcal{D}')$.

2: The functor α preserves small colimits in each component because the cocartesian fibration of symmetric monoidal categories $q^\times : \mathcal{R}^\times \to \mathsf{Cat}_\infty^\times$ is compatible with small colimits.

This implies that for every natural $i \in \{1, ..., n\}$ and arbitrary presheaves $F_j \in \widehat{\mathcal{P}}(\mathcal{D}_j)$ for $j \in \{1, ..., n\} \setminus \{i\}$ the set of morphisms in $\widehat{\mathcal{P}}(\mathcal{D}_i)$ that are sent to local equivalences under $\alpha(F_1, ..., F_{i-1}, -, F_{i+1}, ..., F_n) : \widehat{\mathcal{P}}(\mathcal{D}_i) \to \widehat{\mathcal{P}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ is strongly saturated.

As the set of local equivalences of $\widehat{\mathcal{P}}(\mathcal{D}_i)$ is the smallest strongly saturated set of $\widehat{\mathcal{P}}(\mathcal{D}_i)$ that contains $S_{\mathcal{K}}^{\mathcal{D}_i}$, it is enough to show that for every natural $i \in \{1, ..., n\}$ and arbitrary presheaves $F_j \in \widehat{\mathcal{P}}(\mathcal{D}_j)$ for $j \in \{1, ..., n\} \setminus \{i\}$ the functor $\alpha(F_1, ..., F_{i-1}, -, F_{i+1}, ..., F_n) : \widehat{\mathcal{P}}(\mathcal{D}_i) \to \widehat{\mathcal{P}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ sends morphims of $S_{\mathcal{K}}^{\mathcal{D}_i}$ to local equivalences.

Using again that α preserves small colimits in each component and that the category of presheaves is generated by the representable presheaves under small colimits, it is enough to check the last condition for the case that all presheaves $F_j \in \widehat{\mathcal{P}}(\mathcal{D}_j)$ for $j \in \{1, ..., n\} \setminus \{i\}$ are representable.

That one may reduce to the case of representable presheaves follows by induction from the fact that for every $i, j \in \{1, ..., n\}$ the full subcategory of $\widehat{\mathcal{P}}(\mathcal{D}_j)$ spanned by the objects X with the following property is closed under small colimits:

For all presheaves $F_k \in \widehat{\mathcal{P}}(\mathcal{D}_k)$ with $k < j$ and $k \neq i$ and all representable presheaves $F_k \in \widehat{\mathcal{P}}(\mathcal{D}_k)$ with $k > j$ and $k \neq i$ the functor

$\alpha(F_1, ..., F_{j-1}, X, F_{j+1}, ..., F_{i-1}, -, F_{i+1}, ..., F_n) : \widehat{\mathcal{P}}(\mathcal{D}_i) \to \widehat{\mathcal{P}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ sends morphims of $S_{\mathcal{K}}^{\mathcal{D}_i}$ to local equivalences.

So let a natural $i \in \{1, ..., n\}$, objects $X_j \in \mathcal{D}_j$ for $j \in \{1, ..., n\} \setminus \{i\}$ and a functor $H : J \to \mathcal{D}_i$ with $J \in \mathcal{K}$ be given and denote $\mathsf{y}_{\mathcal{D}_k} : \mathcal{D}_k \to \widehat{\mathcal{P}}(\mathcal{D}_k)$ the Yoneda-embedding of $\mathcal{D}_k$ for $k \in \{1, ..., n\}$.

We have to see that

$$\phi : \alpha(\mathsf{y}_{\mathcal{D}_1}(X_1), ..., \mathsf{y}_{\mathcal{D}_{i-1}}(X_{i-1}), \mathrm{colim}(\mathsf{y}_{\mathcal{D}_i} \circ H), \mathsf{y}_{\mathcal{D}_{i+1}}(X_{i+1}), ..., \mathsf{y}_{\mathcal{D}_n}(X_n)) \to$$

$$\alpha(\mathsf{y}_{\mathcal{D}_1}(X_1), ..., \mathsf{y}_{\mathcal{D}_{i-1}}(X_{i-1}), \mathsf{y}_{\mathcal{D}_i}(\mathrm{colim}(H)), \mathsf{y}_{\mathcal{D}_{i+1}}(X_{i+1}), ..., \mathsf{y}_{\mathcal{D}_n}(X_n))$$

is a local equivalence in $\widehat{\mathcal{P}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$.

Denote $\beta : \mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n$ the canonical functor.

Corollary 6.12 implies that we have a canonical equivalence

$$\alpha \circ (\mathsf{y}_{\mathcal{D}_1} \times ... \times \mathsf{y}_{\mathcal{D}_n}) \simeq \mathsf{y}_{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n} \circ \beta,$$

via which we can factor ϕ as

$$\alpha(\mathsf{y}_{\mathcal{D}_1}(X_1), ..., \mathsf{y}_{\mathcal{D}_{i-1}}(X_{i-1}), \mathrm{colim}(\mathsf{y}_{\mathcal{D}_i} \circ H), \mathsf{y}_{\mathcal{D}_{i+1}}(X_{i+1}), ..., \mathsf{y}_{\mathcal{D}_n}(X_n)) \simeq$$

$$\mathrm{colim}(\alpha(\mathsf{y}_{\mathcal{D}_1}(X_1), ..., \mathsf{y}_{\mathcal{D}_{i-1}}(X_{i-1}), \mathsf{y}_{\mathcal{D}_i} \circ H(-), \mathsf{y}_{\mathcal{D}_{i+1}}(X_{i+1}), ..., \mathsf{y}_{\mathcal{D}_n}(X_n))) \simeq$$

$$\mathrm{colim}(\mathsf{y}_{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n} \circ \beta(X_1, ..., X_{i-1}, H(-), X_{i+1}, ..., X_n)) \xrightarrow{\psi}$$

$$\mathsf{y}_{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n}(\mathrm{colim}(\beta(X_1, ..., X_{i-1}, H(-), X_{i+1}, ..., X_n)) \simeq$$

$$\mathsf{y}_{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n}(\beta(X_1, ..., X_{i-1}, \mathrm{colim}(H), X_{i+1}, ..., X_n)) \simeq$$

$$\alpha(\mathsf{y}_{\mathcal{D}_1}(X_1), ..., \mathsf{y}_{\mathcal{D}_{i-1}}(X_{i-1}), \mathsf{y}_{\mathcal{D}_i}(\mathrm{colim}(H)), \mathsf{y}_{\mathcal{D}_{i+1}}(X_{i+1}), ..., \mathsf{y}_{\mathcal{D}_n}(X_n)).$$

By definition

$$\psi : \mathrm{colim}(\mathsf{y}_{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n} \circ \beta(X_1, ..., X_{i-1}, H(-), X_{i+1}, ..., X_n)) \to$$

$$\mathsf{y}_{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n}(\mathrm{colim}(\beta(X_1, ..., X_{i-1}, H(-), X_{i+1}, ..., X_n)))$$

belongs to $S_{\mathcal{K}}^{\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n}$ so that ϕ is a local equivalence in $\widehat{\mathcal{P}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$.

□

Lemma 6.6.

1. *Let* $F : \mathcal{C} \to \mathcal{C}'$ *and* $G : \mathcal{D} \to \mathcal{D}'$ *be functors between small categories. The functor* $\mathcal{P}(\mathcal{C}^{op} \times \mathcal{D}) \to \mathcal{P}(\mathcal{C}'^{op} \times \mathcal{D}')$ *induced by* F *and* G *is equivalent to the composition*

$$\mathcal{P}(\mathcal{C}^{op}\times\mathcal{D}) \simeq \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D})) \to \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}')) \xrightarrow{\varphi} \mathrm{Fun}(\mathcal{C}',\mathcal{P}(\mathcal{D}')) \simeq \mathcal{P}(\mathcal{C}'^{op}\times\mathcal{D}'),$$

where φ *denotes a left adjoint of the functor* $\mathrm{Fun}(\mathcal{C}',\mathcal{P}(\mathcal{D}')) \to \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}'))$ *given by composition with* $F : \mathcal{C} \to \mathcal{C}'$.

2. *Let* $n \in \mathbb{N}$ *be a natural and* $\mathcal{C}_1, ..., \mathcal{C}_n, \mathcal{D}_1, ..., \mathcal{D}_n$ *be small categories. Denote*

$$\theta^{\mathcal{D}_1,...,\mathcal{D}_n}_{\mathcal{C}_1,...,\mathcal{C}_n} : \mathrm{Fun}(\mathcal{C}_1,\mathcal{D}_1)\times...\times\mathrm{Fun}(\mathcal{C}_n,\mathcal{D}_n) \to \mathrm{Fun}(\mathcal{C}_1\times...\times\mathcal{C}_n, \mathcal{D}_1\times...\times\mathcal{D}_n)$$

the functor adjoint to the functor

$$(\mathcal{C}_1 \times ... \times \mathcal{C}_n) \times (\mathrm{Fun}(\mathcal{C}_1,\mathcal{D}_1) \times ... \times \mathrm{Fun}(\mathcal{C}_n,\mathcal{D}_n)) \simeq$$
$$(\mathcal{C}_1 \times \mathrm{Fun}(\mathcal{C}_1,\mathcal{D}_1)) \times ... \times (\mathcal{C}_n \times \mathrm{Fun}(\mathcal{C}_n,\mathcal{D}_n)) \to \mathcal{D}_1 \times ... \times \mathcal{D}_n$$

induced by the evaluation functors $\mathcal{C}_i \times \mathrm{Fun}(\mathcal{C}_i,\mathcal{D}_i) \to \mathcal{D}_i$ *for* $i \in \{1, ..., n\}$.

The canonical functor

$$\mathcal{P}((\mathcal{C}_1)^{op}\times\mathcal{D}_1)\times...\times\mathcal{P}((\mathcal{C}_n)^{op}\times\mathcal{D}_n) \to \mathcal{P}(((\mathcal{C}_1)^{op}\times\mathcal{D}_1)\times...\times((\mathcal{C}_n)^{op}\times\mathcal{D}_n)) \simeq$$

$$\mathcal{P}(((\mathcal{C}_1)^{op}\times...\times(\mathcal{C}_n)^{op})\times(\mathcal{D}_1\times...\times\mathcal{D}_n)) \simeq \mathcal{P}((\mathcal{C}_1\times...\times\mathcal{C}_n)^{op}\times(\mathcal{D}_1\times...\times\mathcal{D}_n))$$

is equivalent to the composition

$$\mathcal{P}((\mathcal{C}_1)^{op}\times\mathcal{D}_1)\times...\times\mathcal{P}((\mathcal{C}_n)^{op}\times\mathcal{D}_n) \simeq \mathrm{Fun}(\mathcal{C}_1,\mathcal{P}(\mathcal{D}_1))\times...\times\mathrm{Fun}(\mathcal{C}_n,\mathcal{P}(\mathcal{D}_n))$$

$$\xrightarrow{\theta^{\mathcal{P}(\mathcal{D}_1),...,\mathcal{P}(\mathcal{D}_n)}_{\mathcal{C}_1,...,\mathcal{C}_n}} \mathrm{Fun}(\mathcal{C}_1 \times ... \times \mathcal{C}_n, \mathcal{P}(\mathcal{D}_1) \times ... \times \mathcal{P}(\mathcal{D}_n)) \to$$

$$\mathrm{Fun}(\mathcal{C}_1\times...\times\mathcal{C}_n, \mathcal{P}(\mathcal{D}_1\times...\times\mathcal{D}_n)) \simeq \mathcal{P}((\mathcal{C}_1\times...\times\mathcal{C}_n)^{op}\times(\mathcal{D}_1\times...\times\mathcal{D}_n)).$$

Proof. We start by proving 1.

The functors $F : \mathcal{C} \to \mathcal{C}'$ and $G : \mathcal{D} \to \mathcal{D}'$ yield a commutative square

$$\begin{array}{ccc} \mathcal{P}(\mathcal{C}'^{op} \times \mathcal{D}') & \xrightarrow{\simeq} & \mathrm{Fun}(\mathcal{C}',\mathcal{P}(\mathcal{D}')) \\ \downarrow & & \downarrow \\ & & \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}')) \\ & & \downarrow \\ \mathcal{P}(\mathcal{C}^{op} \times \mathcal{D}) & \xrightarrow{\simeq} & \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D})), \end{array} \tag{28}$$

where the vertical functors are given by composition.

If we set $G : \mathcal{D} \to \mathcal{D}'$ to be the identity of $\mathcal{D}'$, we see that the functor $\mathrm{Fun}(\mathcal{C}',\mathcal{P}(\mathcal{D}')) \to \mathrm{Fun}(\mathcal{C},\mathcal{P}(\mathcal{D}'))$ given by composition with $F : \mathcal{C} \to \mathcal{C}'$

admits a left adjoint φ . This shows the existence of φ in the assertion of 1. Turning to left adjoints square 28 gives rise to a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D})) & \xrightarrow{\simeq} & \mathcal{P}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}')) & & \\ \downarrow \varphi & & \\ \mathrm{Fun}(\mathcal{C}', \mathcal{P}(\mathcal{D}')) & \xrightarrow{\simeq} & \mathcal{P}(\mathcal{C}'^{\mathrm{op}} \times \mathcal{D}'), \end{array}$$

that provides the desired equivalence.

2: Denote $\beta_{\mathcal{B}_1,...,\mathcal{B}_n} : \mathcal{P}(\mathcal{B}_1) \times ... \times \mathcal{P}(\mathcal{B}_n) \to \mathcal{P}(\mathcal{B}_1 \times ... \times \mathcal{B}_n)$ the functor adjoint to the functor

$$(\mathcal{B}_1 \times ... \times \mathcal{B}_n)^{\mathrm{op}} \times (\mathcal{P}(\mathcal{B}_1) \times ... \times \mathcal{P}(\mathcal{B}_n)) \simeq ((\mathcal{B}_1)^{\mathrm{op}} \times ... \times (\mathcal{B}_n)^{\mathrm{op}}) \times (\mathcal{P}(\mathcal{B}_1) \times ... \times \mathcal{P}(\mathcal{B}_n)) \simeq$$

$$((\mathcal{B}_1)^{\mathrm{op}} \times \mathcal{P}(\mathcal{B}_1)) \times ... \times ((\mathcal{B}_n)^{\mathrm{op}} \times \mathcal{P}(\mathcal{B}_n)) \to \mathcal{S}^{\times n} \to \mathcal{S}$$

induced by the evaluation functors $(\mathcal{B}_i)^{\mathrm{op}} \times \mathcal{P}(\mathcal{B}_i) \to \mathcal{S}$ for $i \in \{1, ..., n\}$ and the functor $\mathcal{S}^{\times n} \to \mathcal{S}$ determined by the cartesian structure on $\mathcal{S}$.

By lemma 6.7 for 2. it is enough to see that the canonical functor

$$\psi : \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \xrightarrow{\beta_{(\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1, ..., (\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n}}$$

$$\mathcal{P}(((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times ((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n)) \simeq \mathcal{P}(((\mathcal{C}_1)^{\mathrm{op}} \times ... \times (\mathcal{C}_n)^{\mathrm{op}}) \times (\mathcal{D}_1 \times ... \times \mathcal{D}_n))$$
$$\simeq \mathcal{P}((\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{D}_1 \times ... \times \mathcal{D}_n))$$

is equivalent to the composition

$$\phi : \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \simeq \mathrm{Fun}(\mathcal{C}_1, \mathcal{P}(\mathcal{D}_1)) \times ... \times \mathrm{Fun}(\mathcal{C}_n, \mathcal{P}(\mathcal{D}_n))$$

$$\xrightarrow{\theta^{\mathcal{P}(\mathcal{D}_1), ..., \mathcal{P}(\mathcal{D}_n)}_{\mathcal{C}_1, ..., \mathcal{C}_n}} \mathrm{Fun}(\mathcal{C}_1 \times ... \times \mathcal{C}_n, \mathcal{P}(\mathcal{D}_1) \times ... \times \mathcal{P}(\mathcal{D}_n)) \xrightarrow{\mathrm{Fun}(\mathcal{C}_1 \times ... \times \mathcal{C}_n, \beta_{\mathcal{D}_1, ..., \mathcal{D}_n})}$$

$$\mathrm{Fun}(\mathcal{C}_1 \times ... \times \mathcal{C}_n, \mathcal{P}(\mathcal{D}_1 \times ... \times \mathcal{D}_n)) \simeq \mathcal{P}((\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{D}_1 \times ... \times \mathcal{D}_n)).$$

The functors ϕ and ψ are equivalent if and only if their adjoint functors ϕ', ψ' :

$$((\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{D}_1 \times ... \times \mathcal{D}_n))^{\mathrm{op}} \times \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \to \mathcal{S}$$

are equivalent.
But both adjoint functors ϕ', ψ' are equivalent to the following composition:

$$((\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{D}_1 \times ... \times \mathcal{D}_n))^{\mathrm{op}} \times \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \simeq$$

$$((\mathcal{C}_1 \times ... \times \mathcal{C}_n) \times (\mathcal{D}_1 \times ... \times \mathcal{D}_n)^{\mathrm{op}}) \times \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \simeq$$
$$(\mathcal{C}_1 \times ... \times \mathcal{C}_n) \times ((\mathcal{D}_1)^{\mathrm{op}} \times ... \times (\mathcal{D}_n)^{\mathrm{op}}) \times \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \simeq$$
$$(\mathcal{C}_1 \times (\mathcal{D}_1)^{\mathrm{op}}) \times ... \times (\mathcal{C}_n \times (\mathcal{D}_n)^{\mathrm{op}}) \times \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1) \times ... \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n) \simeq$$
$$((\mathcal{C}_1 \times (\mathcal{D}_1)^{\mathrm{op}}) \times \mathcal{P}((\mathcal{C}_1)^{\mathrm{op}} \times \mathcal{D}_1)) \times ... \times ((\mathcal{C}_n \times (\mathcal{D}_n)^{\mathrm{op}}) \times \mathcal{P}((\mathcal{C}_n)^{\mathrm{op}} \times \mathcal{D}_n)) \to$$
$$\mathcal{S}^{\times n} \to \mathcal{S}.$$

□

The following lemma is an important ingredient in the proof of lemma 6.6:

Lemma 6.7. *Let* $n \in \mathbb{N}$ *be a natural and* $\mathcal{C}_1, \mathcal{C}_2, ..., \mathcal{C}_n$ *be small categories.*

The following three functors are equivalent:

1. *The functor* $\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n) \to \mathcal{P}(\mathcal{C}_1 \times ... \times \mathcal{C}_n)$ *induced by the lax symmetric monoidal functor* $\mathsf{Cat}_\infty^\times \to \widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}^\otimes$ *corresponding to the cocartesian fibration of symmetric monoidal categories* $q^\times : \mathcal{R}^\times \to \mathsf{Cat}_\infty^\times$.

2. *The induced functor*

$$\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n) \simeq \mathcal{R}_{\mathcal{C}_1} \times ... \times \mathcal{R}_{\mathcal{C}_n} \simeq \{(\mathcal{C}_1, ..., \mathcal{C}_n)\} \times_{\mathsf{Cat}_\infty^{\times n}} \mathcal{R}^{\times n} \to$$

$$\{\mathcal{C}_1 \times ... \times \mathcal{C}_n\} \times_{\mathsf{Cat}_\infty} \mathcal{R} \simeq \mathcal{P}(\mathcal{C}_1 \times ... \times \mathcal{C}_n)$$

on pullbacks, where the functors $\mathcal{R}^{\times n} \to \mathcal{R}$ *and* $\mathsf{Cat}_\infty^{\times n} \to \mathsf{Cat}_\infty$ *are induced by the cartesian structures.*

3. *The functor* $\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n) \to \mathcal{P}(\mathcal{C}_1 \times ... \times \mathcal{C}_n)$ *adjoint to the functor*

$(\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n)) \simeq (\mathcal{C}_1^{\mathrm{op}} \times ... \times \mathcal{C}_n^{\mathrm{op}}) \times (\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n)) \simeq (\mathcal{C}_1^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_1)) \times ... \times (\mathcal{C}_n^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_n)) \to \mathcal{S}^{\times n} \to \mathcal{S}$
induced by the evaluation functors $\mathcal{C}_i^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_i) \to \mathcal{S}$ *for* $i \in \{1, ..., n\}$ *and the functor* $\mathcal{S}^{\times n} \to \mathcal{S}$ *induced by the cartesian structure on* $\mathcal{S}$.

Proof. The equivalence of the functors in 1. and 2. follows from cor. 6.20.

By lemma 6.8 it is enough to see that the functors in 1. and 3. preserve small colimits in each variable and are compatible with the Yoneda-embeddings.
For the functor in 1. this follows from the fact that $q^\times : \mathcal{R}^\times \to \mathsf{Cat}_\infty^\times$ is compatible with small colimits.

Denote β the functor in 3.
Then it remains to check that β preserves small colimits in each variable and that the composition $\mathcal{C}_1 \times ... \times \mathcal{C}_n \to \mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n) \xrightarrow{\beta} \mathcal{P}(\mathcal{C}_1 \times ... \times \mathcal{C}_n)$ is the Yoneda-embedding of $\mathcal{C}_1 \times ... \times \mathcal{C}_n$.

As colimits in functor-categories are formed levelwise, we can assume that $\mathcal{C}_1, ..., \mathcal{C}_n$ are contractible when we proof that β preserves small colimits in each component. But in this case β is the functor $\mathcal{S}^{\times n} \to \mathcal{S}$ induced by the cartesian structure on $\mathcal{S}$ and $\mathcal{S}$ is cartesian closed.

We complete the proof by showing that the composition $\mathcal{C}_1 \times ... \times \mathcal{C}_n \to \mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n) \xrightarrow{\beta} \mathcal{P}(\mathcal{C}_1 \times ... \times \mathcal{C}_n)$ is the Yoneda-embedding of $\mathcal{C}_1 \times ... \times \mathcal{C}_n$.

By adjunction this is equivalent to the condition that the composition

$$\sigma : (\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{C}_1 \times ... \times \mathcal{C}_n) \to (\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n)) \simeq$$

$$(\mathcal{C}_1^{\mathrm{op}} \times ... \times \mathcal{C}_n^{\mathrm{op}}) \times (\mathcal{P}(\mathcal{C}_1) \times ... \times \mathcal{P}(\mathcal{C}_n)) \simeq$$
$$(\mathcal{C}_1^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_1)) \times ... \times (\mathcal{C}_n^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_n)) \to \mathcal{S}^{\times n} \to \mathcal{S}$$

is the mapping space functor $(\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{C}_1 \times ... \times \mathcal{C}_n) \to \mathcal{S}$ of $\mathcal{C}_1 \times ... \times \mathcal{C}_n$.

σ is equivalent to the composition

$$(\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{\mathrm{op}} \times (\mathcal{C}_1 \times ... \times \mathcal{C}_n) \simeq (\mathcal{C}_1^{\mathrm{op}} \times ... \times \mathcal{C}_n^{\mathrm{op}}) \times (\mathcal{C}_1 \times ... \times \mathcal{C}_n) \simeq$$

$$(\mathcal{C}_1^{\mathrm{op}} \times \mathcal{C}_1) \times ... \times (\mathcal{C}_n^{\mathrm{op}} \times \mathcal{C}_n) \to (\mathcal{C}_1^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_1)) \times ... \times (\mathcal{C}_n^{\mathrm{op}} \times \mathcal{P}(\mathcal{C}_n))$$

$$\to \mathcal{S}^{\times n} \to \mathcal{S}.$$

As $(\mathcal{C}_i^{op} \times \mathcal{C}_i) \to (\mathcal{C}_i^{op} \times \mathcal{P}(\mathcal{C}_i)) \to \mathcal{S}$ is the mapping space functor of $\mathcal{C}_i$ for $i \in \{1, ..., n\}$ and the mapping space functor of a small category $\mathcal{B}$ is classified by the twisted arrow category $\mathrm{Tw}(\mathcal{B}) \to \mathcal{B}^{op} \times \mathcal{B}$, we conclude by observing that we have a commutative square

$$\begin{array}{ccc} \mathrm{Tw}(\mathcal{C}_1 \times ... \times \mathcal{C}_n) & \xrightarrow{\simeq} & \mathrm{Tw}(\mathcal{C}_1) \times ... \times \mathrm{Tw}(\mathcal{C}_n) \\ \downarrow & & \downarrow \\ (\mathcal{C}_1 \times ... \times \mathcal{C}_n)^{op} \times (\mathcal{C}_1 \times ... \times \mathcal{C}_n) & \xrightarrow{\simeq} & (\mathcal{C}_1^{op} \times \mathcal{C}_1) \times ... \times (\mathcal{C}_n^{op} \times \mathcal{C}_n). \end{array}$$

□

Lemma 6.8. *Let $\mathcal{K} \subset \mathcal{K}' \subset \mathsf{Cat}_\infty$ be full subcategories.*
Let $n \in \mathbb{N}$ be a natural and $\mathcal{D}_1, \mathcal{D}_2, ..., \mathcal{D}_n \in \mathsf{Cat}_\infty^{coc}(\mathcal{K})$ be categories.

1. *Let $\alpha : \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n) \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ be a functor that preserves component-wise colimits indexed by categories that belong to $\mathcal{K}'$ such that that the composition*

   $$\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n) \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$$

   is the composition $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ of the canonical functor $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n$ and the Yoneda-embedding.
   Then α satisfies the following universal property:
   For every category $\mathcal{E} \in \widehat{\mathsf{Cat}_\infty^{coc}(\mathcal{K}')}$ the functor

   $$\mathrm{Fun}^{coc,\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n), \mathcal{E}) \to \mathrm{Fun}^{vcoc,\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n), \mathcal{E})$$

   given by composition with α is an equivalence.

   In other words α corresponds to a morphism $(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1), ..., \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n)) \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ of $\widehat{\mathsf{Cat}_\infty^{coc}(\mathcal{K}')}^{\otimes}$ that is cocartesian with respect to the cocartesian fibration $\widehat{\mathsf{Cat}_\infty^{coc}(\mathcal{K}')}^{\otimes} \to \mathcal{F}\mathrm{in}_*$.

2. *Let $\alpha' : \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n) \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ be another functor as in 1.*

   Then α and α' are equivalent in $\mathrm{Fun}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n), \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n))$.

Proof. By assumption the composition

$$\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n) \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$$

is the composition $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ of the canonical functor $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n$ and the Yoneda-embedding of $\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n$.

Hence the composition

$$\mathrm{Fun}^{coc,\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n), \mathcal{E}) \to \mathrm{Fun}^{vcoc,\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n), \mathcal{E})$$

$$\xrightarrow{\varepsilon} \mathrm{Fun}^{vcoc,\mathcal{K}}(\mathcal{D}_1 \times ... \times \mathcal{D}_n, \mathcal{E})$$

is equivalent to the composition

$$\mathrm{Fun}^{\mathrm{coc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n), \mathcal{E}) \to \mathrm{Fun}^{\mathrm{coc},\mathcal{K}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n, \mathcal{E})$$
$$\to \mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}}(\mathcal{D}_1 \times ... \times \mathcal{D}_n, \mathcal{E})$$

and thus is an equivalence.

Therefore for 1. it is enough to see that ε is an equivalence.

This follows by induction from the fact that ε admits a factorization

$$\mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n), \mathcal{E}) \simeq$$
$$\mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_{n-1}), \mathrm{Fun}^{\mathrm{coc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n), \mathcal{E})) \simeq$$
$$\mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_{n-1}), \mathrm{Fun}^{\mathrm{coc},\mathcal{K}}(\mathcal{D}_n, \mathcal{E})) \to$$
$$\mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}}(\mathcal{D}_1 \times ... \times \mathcal{D}_{n-1}, \mathrm{Fun}^{\mathrm{coc},\mathcal{K}}(\mathcal{D}_n, \mathcal{E})) \simeq \mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}}(\mathcal{D}_1 \times ... \times \mathcal{D}_n, \mathcal{E}),$$

where we use that $\mathrm{Fun}^{\mathrm{coc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n), \mathcal{E})$ belongs to $\widehat{\mathsf{Cat}_\infty^{\mathrm{coc}}}(\mathcal{K}')$.

As next we show 2.

It follows from 1. that α and α' satisfy the same universal property.

Consequently there is a unique autoequivalence μ of $\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ so that $\mu \circ \alpha$ is equivalent to α'.

Composing this equivalence from the right with the functor $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1) \times ... \times \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_n)$ we find that the composition

$$\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n) \xrightarrow{\mu} \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$$

is equivalent to the composition $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$.

Consequently μ and the identity of $\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ correspond under the equivalence

$$\mathrm{Fun}^{\mathrm{coc},\mathcal{K}'}(\widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n), \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)) \to$$
$$\mathrm{Fun}^{\mathrm{coc},\mathcal{K}}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n, \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)) \to$$
$$\mathrm{Fun}^{\mathrm{vcoc},\mathcal{K}}(\mathcal{D}_1 \times ... \times \mathcal{D}_n, \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n))$$

to the functor $\mathcal{D}_1 \times ... \times \mathcal{D}_n \to \mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n \to \widehat{\mathcal{P}}_{\mathcal{K}}^{\mathcal{K}'}(\mathcal{D}_1 \otimes ... \otimes \mathcal{D}_n)$ and so have to be equivalent.

□

Denote $\mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ the full subcategory spanned by the right fibrations and $\mathcal{U} \subset \mathcal{R}$ the full subcategory spanned by the representable right fibrations.

We will show the following:

The restriction $\mathcal{U} \subset \mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)$ is a cocartesian fibration and classifies the identity of Cat_∞ (prop. 6.9).

The full subcategory $\mathcal{U} \subset \mathcal{R}$ is closed under finite products so that the functor $\mathcal{U} \subset \mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)$ preserves finite products.

The induced symmetric monoidal functor $\mathcal{U}^\times \to \mathsf{Cat}_\infty^\times$ is a cocartesian fibration and corresponds to the identity of $\mathsf{Cat}_\infty^\times$ (corollary 6.12).

Proposition 6.9. *The restriction $\mathcal{U} \subset \mathcal{R} \to \mathsf{Cat}_\infty$ is a cocartesian fibration and classifies the identity of Cat_∞.*

Remark 6.10. *The cartesian fibration $\mathcal{R} \to \mathsf{Cat}_\infty$ is a bicartesian fibration as for every functor $\mathcal{C} \to \mathcal{D}$ the induced functor $\mathcal{R}_\mathcal{D} \to \mathcal{R}_\mathcal{C}$ admits a left adjoint.*

The left adjoint $\mathcal{R}_\mathcal{C} \to \mathcal{R}_\mathcal{D}$ preserves representable right fibrations.

Hence the cocartesian fibration $\mathcal{R} \to \mathsf{Cat}_\infty$ restricts to a cocartesian fibration $\mathcal{U} \to \mathsf{Cat}_\infty$ with the same cocartesian morphisms.

Especially the embedding $\mathcal{U} \subset \mathcal{R}$ is a map of cocartesian fibrations over Cat_∞.

Proof. Let $\mathcal{U}' \to \mathsf{Cat}_\infty$ be the cocartesian fibration classifying the identity of Cat_∞.

We will show that there is a canonical equivalence $\mathcal{U}' \simeq \mathcal{U}$ over Cat_∞.

By Yoneda it is enough to find for every functor $\mathrm{H} : \mathrm{S} \to \mathsf{Cat}_\infty$ a bijection between equivalence classes of objects of the categories $\mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}')$ and $\mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U})$ such that for every functor $\mathrm{T} \to \mathrm{S}$ over Cat_∞ the square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}') & \longrightarrow & \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{T}, \mathcal{U}') & \longrightarrow & \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{T}, \mathcal{U}) \end{array}$$

commutes on equivalence classes.

Denote $\mathcal{D} \to \mathrm{S}$ the cocartesian fibration classifying $\mathrm{H} : \mathrm{S} \to \mathsf{Cat}_\infty$ so that we have a canonical equivalence $\mathcal{D} \simeq \mathrm{S} \times_{\mathsf{Cat}_\infty} \mathcal{U}'$ over S.

We have a canonical equivalence $\mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}') \simeq \mathrm{Fun}_\mathrm{S}(\mathrm{S}, \mathcal{D})$ such that the square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}') & \longrightarrow & \mathrm{Fun}_\mathrm{S}(\mathrm{S}, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{T}, \mathcal{U}') & \longrightarrow & \mathrm{Fun}_\mathrm{T}(\mathrm{T}, \mathrm{T} \times_\mathrm{S} \mathcal{D}) \end{array}$$

commutes on equivalence classes.

We have a fully faithful functor

$$\mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}) \subset \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{R}) \subset \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty))$$

$$\simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty)_{/\mathrm{H}} \simeq (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathrm{S}})_{/\mathcal{D}},$$

whose essential image $\mathcal{W}(\mathrm{S}, \mathcal{D})$ consists of those maps $\mathcal{C} \to \mathcal{D}$ of cocartesian fibrations over S that induce on the fiber over every object of S a representable right fibration and the square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{S}, \mathcal{U}) & \longrightarrow & (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathrm{S}})_{/\mathcal{D}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathsf{Cat}_\infty}(\mathrm{T}, \mathcal{U}) & \longrightarrow & (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathrm{T}})_{/\mathrm{T} \times_\mathrm{S} \mathcal{D}} \end{array}$$

commutes on equivalence classes.

Consequently it is enough to find for every cocartesian fibration $\mathcal{D} \to S$ a bijection between equivalence classes of objects of the categories $\mathrm{Fun}_S(S, \mathcal{D})$ and $\mathcal{W}(S, \mathcal{D})$ such that the square

$$\begin{array}{ccc} \mathrm{Fun}_S(S, \mathcal{D}) & \longrightarrow & \mathcal{W}(S, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_T(T, T \times_S \mathcal{D}) & \longrightarrow & \mathcal{W}(T, T \times_S \mathcal{D}) \end{array} \tag{29}$$

commutes on equivalence classes.

Let X be a section of $\mathcal{D} \to S$.

By lemma 6.13 the map $\mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{1\}}$ of cocartesian fibrations over S is a cocartesian fibration, whose cocartesian morphisms are those that are sent by the map $\mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{0\}}$ of cocartesian fibrations over S to a cocartesian morphism of $\mathcal{D} \to S$.

So the pullback $\mathcal{D}^{/S}_{/X} := S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \to S$ along X is a cocartesian fibration and $\alpha : S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{0\}}$ is a map of cocartesian fibrations over S.

The map α of cocartesian fibrations over S induces on the fiber over every $\mathsf{s} \in S$ the representable right fibration $(\mathcal{D}_\mathsf{s})_{/X(\mathsf{s})} = \{X(\mathsf{s})\} \times_{\mathrm{Fun}(\{1\}, \mathcal{D}_\mathsf{s})} \mathrm{Fun}(\Delta^1, \mathcal{D}_\mathsf{s}) \to \mathrm{Fun}(\Delta^1, \mathcal{D}_\mathsf{s}) \to \mathrm{Fun}(\{0\}, \mathcal{D}_\mathsf{s})$. So α belongs to $\mathcal{W}(S, \mathcal{D})$.

Pulling back α along the functor $T \to S$ we get the map

$T \times_{(T \times_S \mathcal{D})^{\{1\}}} (T \times_S \mathcal{D})^{\Delta^1} \to (T \times_S \mathcal{D})^{\Delta^1} \to (T \times_S \mathcal{D})^{\{0\}}$ of cocartesian fibrations over T, where the pullback $T \times_{(T \times_S \mathcal{D})^{\{1\}}} (T \times_S \mathcal{D})^{\Delta^1}$ is taken over the functor $T \to T \times_S \mathcal{D}$ over T corresponding to the functor $T \to S \xrightarrow{X} \mathcal{D}$ over S. This shows the commutativity of square 29.

On the other hand let $\mathcal{C} \to \mathcal{D}$ be a map of cocartesian fibrations over S such that for every object s of S the induced functor $\mathcal{C}_\mathsf{s} \to \mathcal{D}_\mathsf{s}$ is a representable right fibration.

As for every object s of S the category $\mathcal{C}_\mathsf{s}$ admits a final object, by lemma 5.33 the category $\mathrm{Fun}_S(S, \mathcal{C})$ admits a final object Z such that for every object s of S the image $Z(\mathsf{s})$ is the final object of $\mathcal{C}_\mathsf{s}$.

The functor $\mathrm{Fun}_S(S, \mathcal{C}) \to \mathrm{Fun}_S(S, \mathcal{D})$ sends Z to the desired object Y of $\mathrm{Fun}_S(S, \mathcal{D})$.

We have a canonical equivalence $\mathrm{Fun}_S(S, S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1}) \simeq \mathrm{Fun}_S(S, \mathcal{D})_{/X}$ over $\mathrm{Fun}_S(S, \mathcal{D})$ so that the image of the final object of the category $\mathrm{Fun}_S(S, S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1})$ under the functor

$$\mathrm{Fun}_S(S, S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1}) \to \mathrm{Fun}_S(S, \mathcal{D}^{\Delta^1}) \to \mathrm{Fun}_S(S, \mathcal{D}^{\{0\}})$$

is X. So the functor $\mathrm{Fun}_S(S, \mathcal{D}) \to \mathcal{W}(S, \mathcal{D})$ induces a retract on equivalence classes.

Lemma 6.15 states that we have a canonical equivalence $\mathcal{C} \simeq \mathcal{D}^{/S}_{/Y}$ over $\mathcal{D}$.

□

The full subcategory $\mathcal{U} \subset \mathcal{R}$ spanned by the representable right fibrations is closed under finite products:

Given two representable right fibrations $\mathcal{C}_{/X} \to \mathcal{C}$ and $\mathcal{D}_{/Y} \to \mathcal{D}$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ the square

$$\begin{array}{ccc} \mathrm{Fun}(\Delta^1, \mathcal{C} \times \mathcal{D}) & \xrightarrow{\simeq} & \mathrm{Fun}(\Delta^1, \mathcal{C}) \times \mathrm{Fun}(\Delta^1, \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\partial\Delta^1, \mathcal{C} \times \mathcal{D}) & \xrightarrow{\simeq} & \mathrm{Fun}(\partial\Delta^1, \mathcal{C}) \times \mathrm{Fun}(\partial\Delta^1, \mathcal{D}) \end{array}$$

induces an equivalence $\mathcal{C} \times \mathcal{D}_{/(X,Y)} \to \mathcal{C}_{/X} \times \mathcal{D}_{/Y}$ over $\mathcal{C} \times \mathcal{D}$ after pulling back to $\mathcal{C} \times \mathcal{D} \times \{(X, Y)\}$.

So the functor $\mathcal{U} \subset \mathcal{R} \subset \mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\{1\}, \mathsf{Cat}_\infty)$ preserves finite products and so yields a symmetric monoidal functor $\mathcal{U}^\times \to \mathsf{Cat}_\infty^\times$.

The identity of $\mathsf{Cat}_\infty^\times$ corresponds to a $\mathsf{Cat}_\infty^\times$-monoid of Cat_∞ and so to a cocartesian fibration $\mathcal{U}^\otimes \to \mathsf{Cat}_\infty^\times$ of symmetric monoidal categories that lifts the cocartesian fibration $\mathcal{U} \to \mathsf{Cat}_\infty$ corresponding to the identity.

By the next lemma the symmetric monoidal structure $\mathcal{U}^\otimes$ is cartesian.

By the uniqueness of the cartesian structure we get the following corollary:

Corollary 6.11. *The symmetric monoidal functor $\mathcal{U}^\times \to \mathsf{Cat}_\infty^\times$ is a cocartesian fibration and corresponds to the identity of $\mathsf{Cat}_\infty^\times$.*

Lemma 6.12. *The symmetric monoidal category $\mathcal{U}^\otimes$ is cartesian.*

For every objects $(\mathcal{C}, X), (\mathcal{D}, Y)$ of $\mathcal{U}$ with $X \in \mathcal{U}_\mathcal{C} \simeq \mathcal{C}, Y \in \mathcal{U}_\mathcal{D} \simeq \mathcal{D}$ we have

$$(\mathcal{C}, X) \otimes (\mathcal{D}, Y) = (\mathcal{C} \times \mathcal{D}, (X, Y)).$$

Proof. We wish to see that the tensorunit $\mathbb{1}_\mathcal{U}$ of $\mathcal{U}$ is a final object of $\mathcal{U}$ and that for every two objects $(\mathcal{C}, X), (\mathcal{D}, Y) \in \mathcal{U}$ the induced morphisms

$$(\mathcal{C}, X) \otimes (\mathcal{D}, Y) \to (\mathcal{C}, X) \otimes \mathbb{1}_\mathcal{U} \simeq (\mathcal{C}, X), \ (\mathcal{C}, X) \otimes (\mathcal{D}, Y) \to \mathbb{1}_\mathcal{U} \otimes (\mathcal{D}, Y) \simeq (\mathcal{D}, Y)$$

exhibit $(\mathcal{C}, X) \otimes (\mathcal{D}, Y)$ as a product of $(\mathcal{C}, X)$ and $(\mathcal{D}, Y)$ in $\mathcal{U}$.

The symmetric monoidal functor $\mathcal{U}^\otimes \to \mathsf{Cat}_\infty^\times$ sends $\mathbb{1}_\mathcal{U}$ to the tensorunit $\mathbb{1}_{\mathsf{Cat}_\infty}$ of Cat_∞ being the final object of Cat_∞.
As $\mathcal{U} \to \mathsf{Cat}_\infty$ is a cocartesian fibration, $\mathbb{1}_\mathcal{U}$ is a final object of $\mathcal{U}$ if and only if $\mathbb{1}_\mathcal{U}$ is a final object of the contractible fiber $\mathcal{U}_{\mathbb{1}_{\mathsf{Cat}_\infty}} \simeq \mathbb{1}_{\mathsf{Cat}_\infty}$.

For every object $(\mathcal{E}, Z)$ of $\mathcal{U}$ we have a commutative square

$$\begin{array}{ccc} \mathcal{U}((\mathcal{E}, Z), (\mathcal{C}, X) \otimes (\mathcal{D}, Y)) & \longrightarrow & \mathcal{U}((\mathcal{E}, Z), (\mathcal{C}, X)) \times \mathcal{U}((\mathcal{E}, Z), (\mathcal{D}, Y)) \\ \downarrow & & \downarrow \\ \mathsf{Cat}_\infty(\mathcal{E}, \mathcal{C} \times \mathcal{D}) & \longrightarrow & \mathsf{Cat}_\infty(\mathcal{E}, \mathcal{C}) \times \mathsf{Cat}_\infty(\mathcal{E}, \mathcal{D}) \end{array} \tag{30}$$

that induces on the fiber over every functor $\phi : \mathcal{E} \to \mathcal{C} \times \mathcal{D}$ the map

$$\mathcal{U}_{\mathcal{C}\times\mathcal{D}}(\phi_*(\mathcal{E}, Z), (\mathcal{C}, X) \otimes (\mathcal{D}, Y)) \xrightarrow{\alpha}$$

$$\mathcal{U}_\mathcal{C}(\mathrm{pr}_{1*}(\phi_*(\mathcal{E}, Z)), \mathrm{pr}_{1*}((\mathcal{C}, X) \otimes (\mathcal{D}, Y))) \times$$
$$\mathcal{U}_\mathcal{D}(\mathrm{pr}_{2*}(\phi_*(\mathcal{E}, Z)), \mathrm{pr}_{2*}((\mathcal{C}, X) \otimes (\mathcal{D}, Y)))$$

$$\xrightarrow{\beta} \mathcal{U}_\mathcal{C}((\mathrm{pr}_1 \circ \phi)_*(\mathcal{E}, Z), (\mathcal{C}, X)) \times \mathcal{U}_\mathcal{D}((\mathrm{pr}_2 \circ \phi)_*(\mathcal{E}, Z), (\mathcal{D}, Y)),$$

where $\mathrm{pr}_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\mathrm{pr}_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ denote the projections.

Let $(\mathrm{A},\mathrm{B}) \in \mathcal{C} \times \mathcal{D}$ be the pair of objects corresponding to $(\mathcal{C},\mathrm{X}) \otimes (\mathcal{D},\mathrm{Y}) \in \mathcal{U}_{\mathcal{C}\times\mathcal{D}} \simeq \mathcal{C} \times \mathcal{D}$ so that we have $(\mathcal{C} \times \mathcal{D}, (\mathrm{A},\mathrm{B})) \simeq (\mathcal{C},\mathrm{X}) \otimes (\mathcal{D},\mathrm{Y})$.

The map α is equivalent to the canonical equivalence

$$(\mathcal{C} \times \mathcal{D})(\phi(\mathrm{Z}), (\mathrm{A},\mathrm{B})) \to \mathcal{C}(\mathrm{pr}_1(\phi(\mathrm{Z})), \mathrm{A}) \times \mathcal{D}(\mathrm{pr}_2(\phi(\mathrm{Z})), \mathrm{B}).$$

So it remains to show that β is an equivalence.

This follows from the fact that the induced morphisms

$$(\mathcal{C},\mathrm{X})\otimes(\mathcal{D},\mathrm{Y}) \to (\mathcal{C},\mathrm{X})\otimes\mathbb{1}_{\mathcal{U}} \simeq (\mathcal{C},\mathrm{X}) \quad (\mathcal{C},\mathrm{X})\otimes(\mathcal{D},\mathrm{Y}) \to \mathbb{1}_{\mathcal{U}}\otimes(\mathcal{D},\mathrm{Y}) \simeq (\mathcal{D},\mathrm{Y})$$

are cocartesian with respect to $\mathcal{U} \to \mathsf{Cat}_\infty$:

The morphisms $(\mathcal{C},\mathrm{X}) \to \mathbb{1}_{\mathcal{U}}$ and $(\mathcal{D},\mathrm{Y}) \to \mathbb{1}_{\mathcal{U}}$ are cocartesian with respect to $\mathcal{U} \to \mathsf{Cat}_\infty$ because the fiber $\mathcal{U}_{\mathbb{1}_{\mathsf{Cat}_\infty}} \simeq \mathbb{1}_{\mathsf{Cat}_\infty}$ is contractible.

As $\mathcal{U}^\otimes \to \mathsf{Cat}_\infty^\times$ is a cocartesian fibration, the collection of morphisms that are cocartesian with respect to $\mathcal{U} \to \mathsf{Cat}_\infty$ is closed under the tensor-product of $\mathcal{U}$.

□

For the proof of prop. 6.9 we used the following lemmata:

Lemma 6.13. *Let* $\rho : \mathrm{X} \to \mathrm{S}$ *be a functor. Denote* $\mathrm{ev}_0 : \mathrm{X}^{\Delta^1} \to \mathrm{X}^{\{0\}} \simeq \mathrm{X}$, $\mathrm{ev}_1 : \mathrm{X}^{\Delta^1} \to \mathrm{X}^{\{1\}} \simeq \mathrm{X}$ *the induced functors over* S.

If $\rho : \mathrm{X} \to \mathrm{S}$ *is a cocartesian fibration, then* $\mathrm{ev}_1 : \mathrm{X}^{\Delta^1} \to \mathrm{X}^{\{1\}} \simeq \mathrm{X}$ *is a cocartesian fibration, where a morphism of* X^{Δ^1} *is* ev_1*-cocartesian if and only if its image under* ev_0 *is* ρ*-cocartesian.*

This implies the following:

Given a functor $\mathrm{T} \to \mathrm{X}$ *over* S *such that the pullback* $\phi : \mathrm{T} \times_{\mathrm{S}} \mathrm{X} \to \mathrm{T}$ *is a cocartesian fibration. Then the pullback*

$$\mathrm{T} \times_{\mathrm{X}^{\{1\}}} \mathrm{X}^{\Delta^1} \simeq \mathrm{T} \times_{(\mathrm{T}\times_{\mathrm{S}}\mathrm{X})^{\{1\}}} (\mathrm{T} \times_{\mathrm{S}} \mathrm{X})^{\Delta^1} \to \mathrm{T}$$

is a cocartesian fibration, whose cocartesian morphism are those, whose image under the functor ev_0 *is* ϕ*-cocartesian.*

Especially for $\mathrm{T} = \Delta^1$ *we see that if* $\rho : \mathrm{X} \to \mathrm{S}$ *is a locally cocartesian fibration, then* $\mathrm{ev}_1 : \mathrm{X}^{\Delta^1} \to \mathrm{X}^{\{1\}} \simeq \mathrm{X}$ *is a locally cocartesian fibration, where a morphism of* X^{Δ^1} *is locally* ev_1*-cocartesian if and only if its image under* ev_0 *is locally* ρ*-cocartesian.*

Proof. If S is contractible, the statement of the lemma is well-known.

Assume that $\rho : \mathrm{X} \to \mathrm{S}$ is a cocartesian fibration.

Then $\mathrm{ev}_1 : \mathrm{X}^{\Delta^1} \to \mathrm{X}^{\{1\}}$ is a map of cocartesian fibrations over S. So for every morphism $\mathrm{Z} \to \mathrm{Z}'$ in S we have a commutative square

$$\begin{array}{ccc} \mathrm{X}_{\mathrm{Z}}^{\Delta^1} & \longrightarrow & \mathrm{X}_{\mathrm{Z}}^{\{0\}} \\ \downarrow{\scriptstyle \alpha} & & \downarrow \\ \mathrm{X}_{\mathrm{Z}'}^{\Delta^1} & \longrightarrow & \mathrm{X}_{\mathrm{Z}'}^{\{0\}}. \end{array}$$

As ev_1 is a map of cocartesian fibrations over S, it is enough to see that for every $Z \in S$ the induced functor $(ev_1)_Z : X_Z^{\Delta^1} \to X_Z^{\{1\}}$ is a cocartesian fibration and for every morphism $Z \to Z'$ in S the functor $\alpha : X_Z^{\Delta^1} \to X_{Z'}^{\Delta^1}$ sends $(ev_1)_Z$-cocartesian morphisms to $(ev_1)_{Z'}$-cocartesian morphisms.

This follows from the case that S is contractible.

It remains to characterize the ev_1-cocartesian morphisms:

Let $f : A \to B$ be a morphism of X^{Δ^1} lying over a morphism $g : s \to t$ in S.

Then we can factor f as a morphism $\alpha : A \to g_*(A)$ that is cocartesian with respect to the cocartesian fibration $X^{\Delta^1} \to S$ followed by a morphism $\beta : g_*(A) \to B$ in the fiber over t.

As $ev_1 : X^{\Delta^1} \to X^{\{1\}}$ is a map of cocartesian fibrations over S, the morphism $ev_1(\alpha) : ev_1(A) \to ev_1(g_*(A))$ is ρ-cocartesian and thus $\alpha : A \to g_*(A)$ is ev_1-cocartesian.

Therefore $f : A \to B$ is ev_1-cocartesian if and only if $\beta : g_*(A) \to B$ is ev_1-cocartesian which is equivalent to the condition that β is $(ev_1)_t$-cocartesian because ev_1 is a cocartesian fibration.

β is $(ev_1)_t$-cocartesian if and only if $ev_0(\beta) : ev_0(g_*(A)) \to ev_0(B)$ is an equivalence which is equivalent to the condition that $ev_0(f)$ is ρ-cocartesian.

□

Corollary 6.14. *Let $\rho : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of operads.*

1. *The induced $\mathcal{O}^\otimes$-monoidal functor $\xi : (\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{\{1\}}$ on cotensors in $\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ is a cocartesian fibration of $\mathcal{O}^\otimes$-monoidal categories, where a morphism of $(\mathcal{C}^\otimes)^{\Delta^1}$ is ξ-cocartesian if and only if its image under the $\mathcal{O}^\otimes$-monoidal functor $(\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{\{0\}}$ is ρ-cocartesian.*
2. *$\xi : (\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{\{1\}}$ is compatible with the same sort of colimits $\rho : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is compatible with.*

Proof. (1) follows immediately from lemma 6.13.

For (2) let $f \in \mathrm{Mul}_{\mathcal{C}}(Y_1, ..., Y_n, Z)$ be an operation for some $Y_1, ..., Y_n, Z \in \mathcal{C}$ corresponding to a morphism $f : Y \to Z$ in $\mathcal{C}^\otimes$ lying over the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}in_*$ with $Y \in \mathcal{C}^\otimes_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ corresponding to $(Y_1, ..., Y_n) \in \mathcal{C}^{\times n}$.

Let $g \in \mathrm{Mul}_{\mathcal{O}}(X_1, ..., X_n, T)$ be the image of f in $\mathcal{O}^\otimes$ corresponding to a morphism $g : X \to T$ in $\mathcal{O}^\otimes$ with $X \in \mathcal{O}^\otimes_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$ corresponding to $(X_1, ..., X_n) \in \mathcal{O}^{\times n}$.

Then we can factor $f : Y \to Z$ as a ρ-cocartesian morphism $Y \to \otimes_g(Y_1, ..., Y_n)$ followed by a morphism $\otimes_g(Y_1, ..., Y_n) \to Z$ in the fiber $\mathcal{C}_T$.

Thus the functor $\prod_{i=1}^n (\mathcal{C}_{X_i})_{/Y_i} \simeq \prod_{i=1}^n (\mathcal{C}^\otimes)^{\Delta^1}_{Y_i} \simeq (\mathcal{C}^\otimes)^{\Delta^1}_Y \to (\mathcal{C}^\otimes)^{\Delta^1}_Z \simeq (\mathcal{C}_T)_{/Z}$ induced by $f : Y \to Z$ factors as the functor

$$\prod_{i=1}^n (\mathcal{C}_{X_i})_{/Y_i} \simeq \prod_{i=1}^n (\mathcal{C}^\otimes)^{\Delta^1}_{Y_i} \simeq (\mathcal{C}^\otimes)^{\Delta^1}_Y \to (\mathcal{C}^\otimes)^{\Delta^1}_{\otimes_g(Y_1,...,Y_n)} \simeq (\mathcal{C}_T)_{/\otimes_g(Y_1,...,Y_n)}$$

induced by the ρ-cocartesian morphism $Y \to \otimes_g(Y_1, ..., Y_n)$ followed by the functor $(\mathcal{C}^\otimes)^{\Delta^1}_{\otimes_g(Y_1,...,Y_n)} \simeq (\mathcal{C}_T)_{/\otimes_g(Y_1,...,Y_n)} \to (\mathcal{C}^\otimes)^{\Delta^1}_Z \simeq (\mathcal{C}_T)_{/Z}$ induced by the morphism $\otimes_g(Y_1, ..., Y_n) \to Z$ in the fiber $\mathcal{C}_T$.

Moreover by corollary 6.20 the first induced functor in the composition

$$\prod_{i=1}^{n} (\mathcal{C}_{X_i})_{/Y_i} \simeq \prod_{i=1}^{n} (\mathcal{C}^{\otimes})^{\Delta^1}_{Y_i} \simeq (\mathcal{C}^{\otimes})^{\Delta^1}_{Y} \to (\mathcal{C}^{\otimes})^{\Delta^1}_{\otimes_g(Y_1,\dots,Y_n)} \simeq (\mathcal{C}_T)_{/\otimes_g(Y_1,\dots,Y_n)}$$

is equivalent to the induced functor $\prod_{i=1}^{n}(\{Y_i\} \times_{\mathcal{C}_{X_i}} \mathrm{Fun}(\Delta^1, \mathcal{C}_{X_i})) \simeq$

$\prod_{i=1}^{n}\{Y_i\} \times_{\prod_{i=1}^{n} \mathcal{C}_{X_i}} \prod_{i=1}^{n} \mathrm{Fun}(\Delta^1, \mathcal{C}_{X_i}) \to \{\otimes_g(Y_1,\dots,Y_n)\} \times_{\mathcal{C}_T} \mathrm{Fun}(\Delta^1, \mathcal{C}_T)$ on pullbacks.

Therefore we have a commutative diagram

$$\begin{array}{ccc}
\prod_{i=1}^{n} (\mathcal{C}_{X_i})_{/Y_i} & \longrightarrow & (\mathcal{C}_T)_{/\otimes_g(Y_1,\dots,Y_n)} \\
\downarrow & & \downarrow \\
\prod_{i=1}^{n} \mathrm{Fun}(\Delta^1, \mathcal{C}_{X_i}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}_T) \\
\downarrow & & \downarrow \\
\prod_{i=1}^{n} \mathrm{Fun}(\{0\}, \mathcal{C}_{X_i}) & \longrightarrow & \mathrm{Fun}(\{0\}, \mathcal{C}_T)
\end{array}$$

where the bottom square is induced by $g : X \to T$ and the $\mathcal{O}^{\otimes}$-monoidal functor $(\mathcal{C}^{\otimes})^{\Delta^1} \to (\mathcal{C}^{\otimes})^{\{0\}}$ and the vertical functors of the outer square are (induced by) the forgetful functors.

So (2) follows from the fact that for every category $\mathcal{B} \in \mathsf{Cat}_\infty$ and every object $C \in \mathcal{B}$ the category $\mathcal{B}_{/C}$ admits the same colimits like $\mathcal{B}$ which are preserved and reflected by the forgetful functor $\mathcal{B}_{/C} \to \mathcal{B}$.

□

Lemma 6.15. *Let* S *be a category and* $\phi : \mathcal{C} \to \mathcal{D}$ *a map of locally cocartesian fibrations over* S *that induces on the fiber over every object* s *of* S *a right fibration.*

Let X *be a section of* $\mathcal{C} \to S$ *such that for all* $\mathsf{s} \in S$ *the image* $X(\mathsf{s}) \in \mathcal{C}_{\mathsf{s}}$ *is a final object of* $\mathcal{C}_{\mathsf{s}}$.

The functor $\mathcal{C} \to \mathcal{D}$ *is canonically equivalent over* $\mathcal{D}$ *to the functor* $S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{0\}}$.

Proof. We have a commutative square

$$\begin{array}{ccc}
S \times_{\mathcal{C}^{\{1\}}} \mathcal{C}^{\Delta^1} & \longrightarrow & S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \\
\downarrow & & \downarrow \\
\mathcal{C}^{\{0\}} & \longrightarrow & \mathcal{D}^{\{0\}}
\end{array} \tag{31}$$

of categories over S that induces on the fiber over every object s of S the commutative square

$$\begin{array}{ccc}
(\mathcal{C}_{\mathsf{s}})_{/X_{\mathsf{s}}} & \longrightarrow & (\mathcal{D}_{\mathsf{s}})_{/\phi(X_{\mathsf{s}})} \\
\downarrow & & \downarrow \\
\mathcal{C}_{\mathsf{s}} & \longrightarrow & \mathcal{D}_{\mathsf{s}}.
\end{array} \tag{32}$$

As $\mathcal{C}_{\mathsf{s}} \to \mathcal{D}_{\mathsf{s}}$ is a right fibration, the top horizontal morphism of square 32 is an equivalence.

As $X(\mathsf{s})$ is a final object of $\mathcal{C}_{\mathsf{s}}$, the left vertical morphism of square 32 is an equivalence.

By lemma 6.13 square 31 is a square of locally cocartesian fibrations over S. Hence the left vertical and top horizontal map of locally cocartesian fibrations over S of square 31 are equivalences.

□

Let $\rho : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of operads and $\mathcal{B} \subset \mathcal{C}$ a full subcategory such that for every $X \in \mathcal{O}$ the full subcategory inclusion $\mathcal{B}_X \subset \mathcal{C}_X$ admits a left adjoint L_X.

Denote $\mathcal{B}^\otimes \subset \mathcal{C}^\otimes$ the full suboperad spanned by the objects of $\mathcal{B}$.

Assume that $\rho : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is compatible with the localization $\mathcal{B} \subset \mathcal{C}$ so that the restriction $\mathcal{B}^\otimes \subset \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ is a cocartesian fibration of operads and the full suboperad inclusion $j : \mathcal{B}^\otimes \subset \mathcal{C}^\otimes$ defines a lax $\mathcal{O}^\otimes$-monoidal functor that admits a left adojnt $L : \mathcal{C}^\otimes \to \mathcal{B}^\otimes$ relative to $\mathcal{O}^\otimes$ which is a $\mathcal{O}^\otimes$-monoidal functor.

Observation 6.16. *If $\mathcal{O}^\otimes$ is a symmetric monoidal category, $\mathcal{C}^\otimes$ is a cartesian symmetric monoidal category and $\mathcal{B}$ is closed under finite products in $\mathcal{C}$, then $\mathcal{B}^\otimes$ is a cartesian symmetric monoidal category.*

In this case the full suboperad inclusion $j : \mathcal{B}^\otimes \subset \mathcal{C}^\otimes$ is a symmetric monoidal functor and the relative left adjoint $\mathcal{C} \to \mathcal{B}$ over $\mathcal{O}$ of the embedding $\mathcal{B} \subset \mathcal{C}$ preserves finite products.

Proof. The tensorunit $\mathbb{1}_\mathcal{C}$ of the symmetric monoidal category $\mathcal{C}^\otimes$ is a final object of $\mathcal{C}$ and thus a final object of $\mathcal{B}$ so that its image $L(\mathbb{1}_\mathcal{C}) \simeq \mathbb{1}_\mathcal{C}$ is a final object of $\mathcal{B}$.

But $L(\mathbb{1}_\mathcal{C})$ is the tensorunit of the symmetric monoidal category $\mathcal{B}^\otimes$ because $L : \mathcal{C}^\otimes \to \mathcal{B}^\otimes$ is a $\mathcal{O}^\otimes$-monoidal functor and thus a symmetric monoidal functor as $\mathcal{O}^\otimes$ is a symmetric monoidal category.

Given two objects $X, Y \in \mathcal{B}$ we have the following chain of natural equivalences $X \otimes Y \simeq L(j(X)) \otimes L(j(Y)) \simeq L(j(X) \times j(Y)) \simeq L(j(X \times Y)) \simeq X \times Y$, so that the canonical morphisms $X \otimes Y \to X \otimes \mathbb{1}_\mathcal{B} \simeq X$ and $X \otimes Y \to \mathbb{1}_\mathcal{B} \otimes Y \simeq Y$ in $\mathcal{B}$ exhibit $X \otimes Y$ as a product of X and Y in $\mathcal{B}$.

□

Observation 6.17. *Let $q : \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration and $\mathcal{B} \subset \mathcal{C}$ a full subcategory such that for every $X \in \mathcal{D}$ the full subcategory inclusion $\mathcal{B}_X \subset \mathcal{C}_X$ admits a left adjoint.*

Then q is compatible with the localizations on the fibers $\mathcal{B}_X \subset \mathcal{C}_X$ for $X \in \mathcal{D}$ if and only if the restriction $\mathcal{B} \subset \mathcal{C} \xrightarrow{q} \mathcal{D}$ is a cartesian fibration and the full subcategory inclusion $\mathcal{B} \subset \mathcal{C}$ is a map of such.

Proof. For every morphism of $\mathcal{D}$ the functor on fibers induced by the cocartesian fibration q is a left adjoint of the functor on fibers induced by the cartesian fibration q.

A functor with a right adjoint between categories that admit localizations preserves local equivalences if and only if its right adjoint preserves local objects.

□

Let $\rho : \mathcal{C}^\otimes \to \mathcal{O}^\otimes$ be a cocartesian fibration of operads.

By lemma 6.13 the induced $\mathcal{O}^{\otimes}$-monoidal functor $\xi : (\mathcal{C}^{\otimes})^{\Delta^1} \to (\mathcal{C}^{\otimes})^{\{1\}}$ is a cocartesian fibration.

Let $\mathcal{L}, \mathcal{R} \subset \mathcal{C}^{\Delta^1}$ be full subcategories such that for every $X \in \mathcal{O}$ the full subcategories $\mathcal{L}_X, \mathcal{R}_X \subset (\mathcal{C}^{\Delta^1})_X \simeq \mathrm{Fun}(\Delta^1, \mathcal{C}_X)$ determine a factorization system on $\mathcal{C}_X$.

Then by [18] 5.2.8.19. for every $X \in \mathcal{O}$ the full subcategory $\mathcal{R}_X \subset (\mathcal{C}^{\Delta^1})_X \simeq \mathrm{Fun}(\Delta^1, \mathcal{C}_X)$ is a localization such that a morphism $F \to G$ of $\mathrm{Fun}(\Delta^1, \mathcal{C}_X)$ with $G \in \mathcal{R}_X$ is a local equivalence if and only if its image under evaluation at the source $\mathrm{Fun}(\Delta^1, \mathcal{C}_X) \to \mathrm{Fun}(\{0\}, \mathcal{C}_X)$ belongs to $\mathcal{L}_X$ and its image under evaluation at the target $\mathrm{Fun}(\Delta^1, \mathcal{C}_X) \to \mathrm{Fun}(\{1\}, \mathcal{C}_X)$ is an equivalence.

So the cocartesian fibration $\mathrm{Fun}(\Delta^1, \mathcal{C}_X) \to \mathrm{Fun}(\{1\}, \mathcal{C}_X)$ is compatible with the localization $\mathcal{R}_X \subset \mathrm{Fun}(\Delta^1, \mathcal{C}_X)$.

Denote $\mathcal{R}^{\otimes} \subset (\mathcal{C}^{\otimes})^{\Delta^1}$ the full suboperad spanned by the objects of $\mathcal{R} \subset \mathcal{C}^{\Delta^1}$.

Lemma 6.18. *If for every natural* $n \in \mathbb{N}$, *every objects* $X_1, ..., X_n, T \in \mathcal{O}$ *and every operation* $g \in \mathrm{Mul}_{\mathcal{O}}(X_1, ..., X_n; T)$ *the induced functor* $\prod_{i=1}^n \mathcal{C}_{X_i} \simeq \mathcal{C}^{\otimes}_X \to \mathcal{C}_T$ *sends objects of* $\prod_{i=1}^n \mathcal{L}_{X_i}$ *to objects of* $\mathcal{L}_T$ *and objects of* $\prod_{i=1}^n \mathcal{R}_{X_i}$ *to objects of* $\mathcal{R}_T$, *the cocartesian fibration* $(\mathcal{C}^{\otimes})^{\Delta^1} \to (\mathcal{C}^{\otimes})^{\{1\}}$ *of* $\mathcal{O}^{\otimes}$*-monoidal categories is compatible with the localization* $\mathcal{R}^{\otimes} \subset (\mathcal{C}^{\otimes})^{\Delta^1}$.

Thus the restriction $\mathcal{R}^{\otimes} \subset (\mathcal{C}^{\otimes})^{\Delta^1} \to (\mathcal{C}^{\otimes})^{\{1\}}$ *is a cocartesian fibration of* $\mathcal{O}^{\otimes}$*-monoidal categories and the full suboperad embedding* $\mathcal{R}^{\otimes} \subset (\mathcal{C}^{\otimes})^{\Delta^1}$ *is a lax* $\mathcal{O}^{\otimes}$*-monoidal functor over* $\mathcal{C}^{\otimes}$ *that admits a left adjoint relative to* $\mathcal{C}^{\otimes}$ *which is a map of cocartesian fibrations of* $\mathcal{O}^{\otimes}$*-monoidal categories over* $\mathcal{C}^{\otimes}$.

Proof. Let $\mathsf{f} \in \mathrm{Mul}_{\mathcal{C}}(Y_1, ..., Y_n, Z)$ be an operation for some $Y_1, ..., Y_n, Z \in \mathcal{C}$ corresponding to a morphism $\mathsf{f} : Y \to Z$ in $\mathcal{C}^{\otimes}$ lying over the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$ with $Y \in \mathcal{C}^{\otimes}_{\langle n \rangle} \simeq \mathcal{C}^{\times n}$ corresponding to $(Y_1, ..., Y_n) \in \mathcal{C}^{\times n}$.

Let $g \in \mathrm{Mul}_{\mathcal{O}}(X_1, ..., X_n; T)$ be the image of f in $\mathcal{O}^{\otimes}$ corresponding to a morphism $g : X \to T$ in $\mathcal{O}^{\otimes}$ with $X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$ corresponding to $(X_1, ..., X_n) \in \mathcal{O}^{\times n}$.

Then we can factor $\mathsf{f} : Y \to Z$ as a ρ-cocartesian morphism $Y \to \otimes_g(Y_1, ..., Y_n)$ followed by a morphism $\otimes_g(Y_1, ..., Y_n) \to Z$ in the fiber $\mathcal{C}_T$.

Thus the functor

$$\prod_{i=1}^n (\mathcal{C}_{X_i})_{/Y_i} \simeq \prod_{i=1}^n (\mathcal{C}^{\otimes})^{\Delta^1}_{Y_i} \simeq (\mathcal{C}^{\otimes})^{\Delta^1}_{Y} \to (\mathcal{C}^{\otimes})^{\Delta^1}_{Z} \simeq (\mathcal{C}_T)_{/Z}$$

induced by $\mathsf{f} : Y \to Z$ factors as the functor

$$\alpha : \prod_{i=1}^n (\mathcal{C}_{X_i})_{/Y_i} \simeq \prod_{i=1}^n (\mathcal{C}^{\otimes})^{\Delta^1}_{Y_i} \simeq (\mathcal{C}^{\otimes})^{\Delta^1}_{Y} \to (\mathcal{C}^{\otimes})^{\Delta^1}_{\otimes_g(Y_1,...,Y_n)} \simeq (\mathcal{C}_T)_{/\otimes_g(Y_1,...,Y_n)}$$

induced by the ρ-cocartesian morphism $Y \to \otimes_g(Y_1, ..., Y_n)$ followed by the functor

$$(\mathcal{C}^{\otimes})^{\Delta^1}_{\otimes_g(Y_1,...,Y_n)} \simeq (\mathcal{C}_T)_{/\otimes_g(Y_1,...,Y_n)} \to (\mathcal{C}^{\otimes})^{\Delta^1}_{Z} \simeq (\mathcal{C}_T)_{/Z}$$

induced by the morphism $\otimes_g(Y_1, ..., Y_n) \to Z$ in the fiber $\mathcal{C}_T$.

The second functor in the composition preserves local equivalences because $\mathrm{Fun}(\Delta^1, \mathcal{C}_T) \to \mathrm{Fun}(\{1\}, \mathcal{C}_T)$ is compatible with the localization $\mathcal{R}_T \subset \mathrm{Fun}(\Delta^1, \mathcal{C}_T)$.

By corollary 6.20 the functor α is equivalent to the induced functor

$$\prod_{i=1}^{n}(\{Y_i\} \times_{\mathcal{C}_{X_i}} \mathrm{Fun}(\Delta^1, \mathcal{C}_{X_i})) \simeq \prod_{i=1}^{n}\{Y_i\} \times_{\prod_{i=1}^{n} \mathcal{C}_{X_i}} \prod_{i=1}^{n}\mathrm{Fun}(\Delta^1, \mathcal{C}_{X_i})$$

$$\to \{\otimes_g(Y_1, ..., Y_n)\} \times_{\mathcal{C}_T} \mathrm{Fun}(\Delta^1, \mathcal{C}_T)$$

on pullbacks so that we have a commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^{n} (\mathcal{C}_{X_i})_{/Y_i} & \longrightarrow & (\mathcal{C}_T)_{/\otimes_g(Y_1,...,Y_n)} \\ \downarrow & & \downarrow \\ \prod_{i=1}^{n} \mathrm{Fun}(\Delta^1, \mathcal{C}_{X_i}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{C}_T) \\ \downarrow & & \downarrow \\ \prod_{i=1}^{n} \mathrm{Fun}(\{0\}, \mathcal{C}_{X_i}) & \longrightarrow & \mathrm{Fun}(\{0\}, \mathcal{C}_T). \end{array} \tag{33}$$

By the assumption that the middle horizontal functor in diagram 33 sends objects of $\prod_{i=1}^{n} \mathcal{R}_{X_i}$ to objects of $\mathcal{R}_T$ the top horizontal functor in diagram 33 sends objects of $\prod_{i=1}^{n}(\mathcal{R}_{X_i})_{Y_i}$ to objects of $(\mathcal{R}_T)_{\otimes_g(Y_1,...,Y_n)}$.

The assumption that the bottom horizontal functor in diagram 33 sends objects of $\prod_{i=1}^{n} \mathcal{L}_{X_i}$ to objects of $\mathcal{L}_T$ implies that the top horizontal functor in diagram 33 preserves local equivalences.

□

For the proof of lemma 6.18 we needed the following lemma:

Lemma 6.19. *Let* S *be a category and* $\xi : X \to Y$ *a morphism of cocartesian fibrations over* S.

Let $\alpha : T \to S$ *be a functor and* $\beta : T \to T \times_S Y$ *a cocartesian section of the cocartesian fibration* $T \times_S Y \to T$.

Then the pullback $T \times_Y X \to T$ *of* $X \to Y$ *along the functor* $T \xrightarrow{\beta} T \times_S Y \to Y$ *is equivalent over* T *to the the pullback of the maps* $T \times_S X \to T \times_S Y$ *and* $\beta : T \to T \times_S Y$ *of cocartesian fibrations over* T *and is therefore in particular a cocartesian fibration.*

Consequently if $\xi : X \to Y$ *is itself a cocartesian fibration classifying a functor* $H : Y \to \mathsf{Cat}_\infty$ *and classifying a natural transformation* $\tau : F \to G$ *of functors* $S \to \mathsf{Cat}_\infty$ *then the composition* $T \xrightarrow{\beta} T \times_S Y \to Y \xrightarrow{H} \mathsf{Cat}_\infty$ *is equivalent to the pullback* $* \times_{G \circ \alpha} F \circ \alpha$ *in* $\mathrm{Fun}(T, \mathsf{Cat}_\infty)$ *formed by the natural transformations* $\tau \circ \alpha : F \circ \alpha \to G \circ \alpha$ *and* $* \to G \circ \alpha$ *classified by* $\beta : T \to T \times_S Y$.

In particular if $\beta : \Delta^1 \to \Delta^1 \times_S Y$ *corresponds to a cocartesian morphism* $Z \to Z'$ *in* Y *lying over a morphism* $A \to B$ *in* S *corresponding to* $\alpha : \Delta^1 \to S$ *the induced functor* $X_Z \to X_{Z'}$ *on the fiber is equivalent to the induced functor* $\{Z\} \times_{Y_A} X_A \to \{Z'\} \times_{Y_B} X_B$ *between the fibers.*

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
T \times_Y X & \longrightarrow & T \times_S X & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
T & \xrightarrow{\beta} & T \times_S Y & \longrightarrow & Y \\
 & & \downarrow & & \downarrow \\
 & & T & \longrightarrow & S
\end{array}$$

The lower right square and the outer right square are pullback squares and thus also the upper right square.

As the outer upper square is a pullback square, the upper left square is, too.

□

Corollary 6.20. *Let* $\mathcal{O}^{\otimes}$ *be an operad and* $\xi : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *a cocartesian fibration of* $\mathcal{O}^{\otimes}$*-monoidal categories.*

Let $n \in \mathbb{N}$ *and* $X \in \mathcal{D}^{\otimes}_{\langle n \rangle} \simeq \mathcal{D}^{\times n}$ *be an object corresponding to the family* $(X_1, ..., X_n) \in \mathcal{D}^{\times n}$ *that lies over an object* $Y \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$ *corresponding to the family* $(Y_1, ..., Y_n) \in \mathcal{O}^{\times n}$ *and let* $\mathsf{f} \in \mathrm{Mul}_{\mathcal{O}}(Y_1, ..., Y_n, Z)$ *be an operation.*

Then the functor

$$\prod_{i=1}^{n} \mathcal{C}_{X_i} \to \mathcal{C}_{\otimes_{\mathsf{f}}(X_1,...,X_n)}$$

induced by a cocartesian lift $h : X \to \otimes_{\mathsf{f}}(X_1, ..., X_n)$ *of* $\mathsf{f} : Y \to Z$ *is equivalent to the induced functor*

$$\prod_{i=1}^{n}(\{X_i\} \times_{\mathcal{D}_{Y_i}} \mathcal{C}_{Y_i}) \simeq \prod_{i=1}^{n}\{X_i\} \times_{\prod_{i=1}^{n} \mathcal{D}_{Y_i}} \prod_{i=1}^{n} \mathcal{C}_{Y_i} \to \{\otimes_{\mathsf{f}}(X_1, ..., X_n)\} \times_{\mathcal{D}_Z} \mathcal{C}_Z$$

on pullbacks.

6.1.2 Presheaves with Day convolution are the free monoidal category compatible with small colimits

Let $\mathcal{O}^\otimes$ be an operad and $\mathcal{C}^\otimes$ a small $\mathcal{O}^\otimes$-monoidal category.

In this chapter we characterize the Day convolution on $\mathcal{P}(\mathcal{C})$ as the free $\mathcal{O}^\otimes$-monoidal category compatible with small colimits (proposition 6.21):

For every small $\mathcal{O}^\otimes$-monoidal category $\mathcal{E}^\otimes$ compatible with small colimits composition with the $\mathcal{O}^\otimes$-monoidal Yoneda-embedding $\mathcal{C}^\otimes \to \mathcal{P}(\mathcal{C})^\otimes$ yields an equivalence

$$\mathrm{Fun}^{\otimes,\mathrm{coc}}_{\mathcal{O}}(\mathcal{P}(\mathcal{C}), \mathcal{E}) \to \mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathcal{C}, \mathcal{E})$$

between the category of $\mathcal{O}^\otimes$-monoidal functors $\mathcal{P}(\mathcal{C})^\otimes \to \mathcal{E}^\otimes$ that preserve small colimits and the category of $\mathcal{O}^\otimes$-monoidal functors $\mathcal{C}^\otimes \to \mathcal{E}^\otimes$.

To do so, we show that for $\mathcal{K} = \varnothing, \mathcal{K}' = \mathsf{Cat}_\infty$ the symmetric monoidal functor $\widehat{\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}} : \widehat{\mathsf{Cat}_\infty}^\times \to (\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty})^\otimes$ corresponding to the cocartesian fibration $\widehat{\mathcal{R}^{\mathcal{K}'}_{\mathcal{K}}}^\otimes \to \widehat{\mathsf{Cat}_\infty}^\times$ of symmetric monoidal categories is left adjoint relative to $\mathcal{F}\mathrm{in}_*$ to the suboperad inclusion $\iota : \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty}^\otimes \subset \widehat{\mathsf{Cat}_\infty}^\times$.

If this is shown, we obtain an induced adjunction $\mathrm{Alg}_{\mathcal{O}}(\widehat{\mathsf{Cat}_\infty}) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty})$, where the left adjoint restricts to the functor

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{P}) : \mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \to \mathrm{Alg}_{\mathcal{O}}(\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty}).$$

Together with remark 6.22 this implies that the Day convolution $\mathcal{P}(\mathcal{C})^\otimes$ is the free $\mathcal{O}^\otimes$-monoidal category compatible with small colimits.

More generally we show the following proposition:

Proposition 6.21. *Let $\mathcal{K} \subset \mathcal{K}' \subset \mathsf{Cat}_\infty$ be full subcategories.*

The symmetric monoidal functor $\widehat{\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}} : \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \to \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}')}^\otimes$ corresponding to the cocartesian fibration

$$\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \times_{\widehat{\mathsf{Cat}_\infty}^\times} \widehat{\mathcal{R}^{\mathcal{K}'}_{\mathcal{K}}}^\otimes \to \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes$$

of symmetric monoidal categories is left adjoint relative to $\mathcal{F}\mathrm{in}_*$ to the suboperad inclusion $\iota : \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}')}^\otimes \subset \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes$.

Proof. Due to 6.1.1 the full subcategory $\widehat{\mathcal{U}} \subset \widehat{\mathcal{R}}$ spanned by the representable right fibrations is closed under finite products.

Thus the full subcategory inclusion $\widehat{\mathcal{U}} \subset \widehat{\mathcal{R}}$ induces a symmetric monoidal embedding $\widehat{\mathcal{U}}^\times \subset \widehat{\mathcal{R}}^\times$ which is a map of cocartesian fibrations of symmetric monoidal categories over $\widehat{\mathsf{Cat}_\infty}^\times$.

Pulling back this map of cocartesian fibrations of symmetric monoidal categories over $\widehat{\mathsf{Cat}_\infty}^\times$ along the suboperad inclusion $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \subset \mathsf{Cat}^\times_\infty$ we obtain a map

$$\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \times_{\widehat{\mathsf{Cat}_\infty}^\times} \widehat{\mathcal{U}}^\times \subset \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \times_{\widehat{\mathsf{Cat}_\infty}^\times} \widehat{\mathcal{R}}^\times$$

of cocartesian fibrations of symmetric monoidal categories over $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes$ that induces a map

$$\zeta : \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \times_{\widehat{\mathsf{Cat}_\infty}^\times} \widehat{\mathcal{U}}^\times \subset \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^\otimes \times_{\widehat{\mathsf{Cat}_\infty}^\times} \widehat{\mathcal{R}^{\mathcal{K}'}_{\mathcal{K}}}^\otimes$$

of symmetric monoidal categories over $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ as $\widehat{\mathcal{R}^{\mathcal{K}'}_{\mathcal{K}}} \subset \widehat{\mathcal{R}}$ contains all representable presheaves.

By lemma 6.25 it is enough to construct a symmetric monoidal natural transformation λ from the identity of $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ to the lax symmetric monoidal functor $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes} \xrightarrow{\widehat{\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}}} \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}')}^{\otimes} \subset \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ such that the underlying natural transformation $\mathrm{id}_{\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}} \to \iota\widehat{\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}}$ exhibits $\widehat{\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}}$ as left adjoint to the subcategory inclusion $\iota : \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}')} \subset \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}$.

According to corollary 6.12 such a symmetric monoidal natural transformation λ corresponds to a map of cocartesian fibrations

$$\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes} \times_{\widehat{\mathsf{Cat}_\infty}^{\times}} \widehat{\mathcal{U}}^{\times} \subset \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes} \times_{\widehat{\mathsf{Cat}_\infty}^{\times}} \widehat{\mathcal{R}^{\mathcal{K}'}_{\mathcal{K}}}^{\otimes}$$

of symmetric monoidal categories over $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ compatible with colimits indexed by categories that belong to $\mathcal{K}'$ such that that for every category $\mathcal{C} \in \widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ the induced functor on the fiber $\mathcal{C} \simeq \widehat{\mathcal{U}}_{\mathcal{C}} \to \widehat{\mathcal{R}^{\mathcal{K}'}_{\mathcal{K}}}^{\otimes}_{\mathcal{C}} \simeq \widehat{\mathcal{P}}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})$ factors as an autoequivalence of $\mathcal{C}$ followed by the Yoneda-embedding.

Consequently λ corresponds to a map of cocartesian fibrations of symmetric monoidal categories over $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ compatible with colimits indexed by categories that belong to $\mathcal{K}'$ that factors as an autoequivalence of $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes} \times_{\widehat{\mathsf{Cat}_\infty}^{\times}} \widehat{\mathcal{U}}^{\times}$ in the category of cocartesian fibrations of symmetric monoidal categories over $\widehat{\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})}^{\otimes}$ followed by ζ.

So we choose λ to be the symmetric monoidal natural transformation corresponding to ζ.

□

Remark 6.22. *Let $\mathcal{K} \subset \mathcal{K}' \subset \mathsf{Cat}_\infty$ be full subcategories and $\mathcal{O}^{\otimes}$ an operad. The adjunction*

$$\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}} : \mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}) \rightleftarrows \mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}') : \iota$$

of proposition 6.21 yields an adjunction

$$\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})) \rightleftarrows \mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}')).$$

Given a small $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{C}^{\otimes}$ compatible with colimits indexed by categories that belong to $\mathcal{K}$ corresponding to a $\mathcal{O}^{\otimes}$-algebra $\mathcal{O}^{\otimes} \to \mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})^{\otimes}$ of $\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K})^{\otimes}$ the unit $\eta_{\mathcal{C}^{\otimes}} : \mathcal{C}^{\otimes} \to \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ is equivalent to the induced $\mathcal{O}^{\otimes}$-monoidal functor

$$\mathcal{C}^{\otimes} \simeq \mathcal{O}^{\otimes} \times_{\mathsf{Cat}^{\times}_\infty} \mathcal{U}^{\times} \to \mathcal{O}^{\otimes} \times_{\mathsf{Cat}^{\times}_\infty} \mathcal{R}^{\mathcal{K}'}_{\mathcal{K}} \simeq \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})^{\otimes}.$$

We call $\eta_{\mathcal{C}^{\otimes}} : \mathcal{C}^{\otimes} \to \mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C})^{\otimes}$ the $\mathcal{O}^{\otimes}$-monoidal Yoneda-embedding of $\mathcal{C}^{\otimes}$.

For every small $\mathcal{O}^{\otimes}$-monoidal category $\mathcal{D}^{\otimes}$ compatible with colimits indexed by categories that belong to $\mathcal{K}'$ the functor

$$\mathrm{Fun}^{\otimes \mathrm{coc}, \mathcal{K}'}_{\mathcal{O}}(\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\otimes \mathrm{coc}, \mathcal{K}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$$

given by composition with $\eta_{\mathcal{C}^{\otimes}}$ is an equivalence:

For every small category $\mathcal{W}$ the cotensor $\mathcal{D}^{\otimes^{\mathcal{W}}}$ in $\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ is compatible with colimits indexed by categories that belong to $\mathcal{K}'$ and for every small $\mathcal{O}^\otimes$-monoidal category $\mathcal{B}^\otimes$ compatible with colimits indexed by categories that belong to $\mathcal{K}'$ the canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}_\infty)(\mathcal{B}, \mathcal{D}^{\mathcal{W}}) \simeq \mathsf{Cat}_\infty(\mathcal{W}, \mathrm{Fun}^{\otimes}_{\mathcal{O}}(\mathcal{B}, \mathcal{D}))$$

restricts to equivalences

$$\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}))(\mathcal{B}, \mathcal{D}^{\mathcal{W}}) \simeq \mathsf{Cat}_\infty(\mathcal{W}, \mathrm{Fun}^{\otimes\,\mathrm{coc},\mathcal{K}}_{\mathcal{O}}(\mathcal{B}, \mathcal{D}))$$

and

$$\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}'))(\mathcal{B}, \mathcal{D}^{\mathcal{W}}) \simeq \mathsf{Cat}_\infty(\mathcal{W}, \mathrm{Fun}^{\otimes\,\mathrm{coc},\mathcal{K}'}_{\mathcal{O}}(\mathcal{B}, \mathcal{D})).$$

So by Yoneda the commutativity of the square

$$\begin{array}{ccc} \mathsf{Cat}_\infty(\mathcal{W}, \mathrm{Fun}^{\otimes\,\mathrm{coc},\mathcal{K}'}_{\mathcal{O}}(\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C}), \mathcal{D})) & \longrightarrow & \mathsf{Cat}_\infty(\mathcal{W}, \mathrm{Fun}^{\otimes\,\mathrm{coc},\mathcal{K}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}'))(\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C}), \mathcal{D}^{\mathcal{W}}) & \longrightarrow & \mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}^{\mathrm{coc}}_\infty(\mathcal{K}))(\mathcal{C}, \mathcal{D}^{\mathcal{W}}) \end{array}$$

implies that the functor

$$\mathrm{Fun}^{\otimes\,\mathrm{coc},\mathcal{K}'}_{\mathcal{O}}(\mathcal{P}^{\mathcal{K}'}_{\mathcal{K}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}^{\otimes\,\mathrm{coc},\mathcal{K}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D})$$

given by composition with $\eta_{\mathcal{C}^\otimes}$ is an equivalence.

Let $\mathcal{O}^\otimes$ be an operad, $\mathcal{C}^\otimes$ a small $\mathcal{O}^\otimes$-monoidal category and $\mathcal{D}^\otimes$ a $\mathcal{O}^\otimes$-monoidal category compatible with small colimits.

We complete this section by showing that there is a canonical $\mathcal{O}^\otimes$-monoidal equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\otimes \simeq \mathcal{P}(\mathcal{C}^{\mathrm{rev}})^\otimes \otimes \mathcal{D}^\otimes$$

(prop. 6.23).

In the proof of prop. 6.23 we use the following fact:

Let $\mathcal{C}$ be a small category and $\mathcal{D}$ a presentable category.
Denote $\alpha : \mathcal{P}(\mathcal{C}) \times \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ the functor adjoint to the functor

$$\mathcal{C}^{\mathrm{op}} \times (\mathcal{P}(\mathcal{C}) \times \mathcal{D}) \simeq (\mathcal{C}^{\mathrm{op}} \times \mathcal{P}(\mathcal{C})) \times \mathcal{D} \to \mathcal{S} \times \mathcal{D} \to \mathcal{D},$$

where the functor $\mathcal{S} \times \mathcal{D} \to \mathcal{D}$ is the left action map of the canonical left $\mathcal{S}$-module structure on $\mathcal{D}$.

With $\mathcal{S} \times \mathcal{D} \to \mathcal{D}$ also α preserves small colimits in both variables and so induces a small colimits preserving functor $\alpha' : \mathcal{P}(\mathcal{C}) \otimes \mathcal{D} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D})$ that is an equivalence by [18] 4.8.1.16.

Proposition 6.23. *Let $\mathcal{O}^\otimes$ be an operad, $\mathcal{C}^\otimes$ a small $\mathcal{O}^\otimes$-monoidal category and $\mathcal{D}^\otimes$ a $\mathcal{O}^\otimes$-monoidal category compatible with small colimits.*

The canonical $\mathcal{O}^\otimes$-monoidal equivalence $\psi : \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times \mathcal{D})^\otimes \simeq \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}})^\otimes \widehat{\otimes}\, \widehat{\mathcal{P}}(\mathcal{D})^\otimes$ restricts to a $\mathcal{O}^\otimes$-monoidal equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\otimes \simeq \mathcal{P}(\mathcal{C}^{\mathrm{rev}})^\otimes \otimes \mathcal{D}^\otimes.$$

Proof. Assume that $\mathcal{D}^\otimes$ is compatible with τ-small colimits for some uncountable regular cardinal τ.

By prop. 6.5 we have $\mathcal{O}^\otimes$-monoidal localizations

$$\widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}} \times \mathcal{D})^\otimes \rightleftarrows \mathrm{Fun}(\mathcal{C}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}))^\otimes, \ \widehat{\mathcal{P}}(\mathcal{D})^\otimes \rightleftarrows \widehat{\mathrm{Ind}}_\tau(\mathcal{D})^\otimes.$$

The $\mathcal{O}^\otimes$-monoidal functor $\widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}})^\otimes \widehat{\otimes}\ \widehat{\mathcal{P}}(\mathcal{D})^\otimes \rightleftarrows \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}})^\otimes \widehat{\otimes}\ \widehat{\mathrm{Ind}}_\tau(\mathcal{D})^\otimes$ induces on the fiber over every $\mathrm{X} \in \mathcal{O}$ the localization

$$\mathrm{Fun}(\mathcal{C}_\mathrm{X}, \widehat{\mathcal{P}}(\mathcal{D}_\mathrm{X})) \rightleftarrows \mathrm{Fun}(\mathcal{C}_\mathrm{X}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}_\mathrm{X}))$$

and is thus a $\mathcal{O}^\otimes$-monoidal localization.

Hence ψ restricts to a $\mathcal{O}^\otimes$-monoidal equivalence

$$\xi : \mathrm{Fun}(\mathcal{C}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}))^\otimes \simeq \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}})^\otimes \widehat{\otimes}\ \widehat{\mathrm{Ind}}_\tau(\mathcal{D})^\otimes.$$

We have a canonical $\mathcal{O}^\otimes$-monoidal equivalence

$$\mathrm{Fun}(\mathcal{C}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}))^\otimes \simeq \widehat{\mathcal{P}}(\mathcal{C}^{\mathrm{rev}})^\otimes \widehat{\otimes}\ \widehat{\mathrm{Ind}}_\tau(\mathcal{D})^\otimes \simeq \widehat{\mathrm{Ind}}_\tau(\mathcal{P}(\mathcal{C}^{\mathrm{rev}}))^\otimes \widehat{\otimes}\ \widehat{\mathrm{Ind}}_\tau(\mathcal{D})^\otimes$$
$$\simeq \widehat{\mathrm{Ind}}_\tau(\mathcal{P}(\mathcal{C}^{\mathrm{rev}}) \otimes \mathcal{D})^\otimes$$

that induces on the fiber over every $\mathrm{X} \in \mathcal{O}$ the canonical equivalence

$$\mathrm{Fun}(\mathcal{C}_\mathrm{X}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}_\mathrm{X})) \simeq \widehat{\mathcal{P}}(\mathcal{C}_\mathrm{X}^{\mathrm{op}}) \widehat{\otimes}\ \widehat{\mathrm{Ind}}_\tau(\mathcal{D}_\mathrm{X}) \simeq \widehat{\mathrm{Ind}}_\tau(\mathcal{P}(\mathcal{C}_\mathrm{X}^{\mathrm{op}})) \widehat{\otimes}\ \widehat{\mathrm{Ind}}_\tau(\mathcal{D}_\mathrm{X})$$
$$\simeq \widehat{\mathrm{Ind}}_\tau(\mathcal{P}(\mathcal{C}_\mathrm{X}^{\mathrm{op}}) \otimes \mathcal{D}_\mathrm{X})$$

that restricts to an equivalence

$$\mathrm{Fun}(\mathcal{C}_\mathrm{X}, \mathcal{D}_\mathrm{X}) = \mathrm{Fun}(\mathcal{C}_\mathrm{X}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}_\mathrm{X})^\tau) = \mathrm{Fun}(\mathcal{C}_\mathrm{X}, \widehat{\mathrm{Ind}}_\tau(\mathcal{D}_\mathrm{X}))^\tau \simeq$$
$$\widehat{\mathrm{Ind}}_\tau(\mathcal{P}(\mathcal{C}_\mathrm{X}^{\mathrm{op}}) \otimes \mathcal{D}_\mathrm{X})^\tau = \mathcal{P}(\mathcal{C}_\mathrm{X}^{\mathrm{op}}) \otimes \mathcal{D}_\mathrm{X}.$$

Consequently ξ restricts to a $\mathcal{O}^\otimes$-monoidal equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{D})^\otimes \simeq \mathcal{P}(\mathcal{C}^{\mathrm{rev}}) \otimes \mathcal{D}^\otimes.$$

□

Lemma 6.24. *Let $\mathcal{B}$ be a category and $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ a map of cocartesian fibrations over $\mathcal{B}$.*
Let $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ be a functor over $\mathcal{B}$ and $\lambda : \mathrm{id}_\mathcal{C} \to \mathrm{G} \circ \mathrm{F}$ a natural transformation relative to $\mathcal{B}$, i.e. a morphism of the category $\mathrm{Fun}_\mathcal{B}(\mathcal{C}, \mathcal{C})$.

Assume that for every $\mathrm{X} \in \mathcal{B}$ the induced natural transformation $\lambda_\mathrm{X} : \mathrm{id}_{\mathcal{C}_\mathrm{X}} \to \mathrm{G}_\mathrm{X} \circ \mathrm{F}_\mathrm{X}$ exhibits F_X as a left adjoint of G_X.

Then $\lambda : \mathrm{id}_\mathcal{C} \to \mathrm{G} \circ \mathrm{F}$ exhibits F as a left adjoint of G relative to $\mathcal{B}$.

Proof. We want to see that for arbitrary objects $\mathrm{C} \in \mathcal{C}$ and $\mathrm{D} \in \mathcal{D}$ the canonical map

$$\theta : \mathcal{D}(\mathrm{F}(\mathrm{C}), \mathrm{D}) \to \mathcal{C}(\mathrm{G}(\mathrm{F}(\mathrm{C})), \mathrm{G}(\mathrm{D})) \to \mathcal{C}(\mathrm{C}, \mathrm{G}(\mathrm{D}))$$

is an equivalence.

Let Y be the image of C in $\mathcal{B}$ and Z the image of D in $\mathcal{B}$.

Then we have a commutative square of spaces

$$\begin{array}{ccccc} \mathcal{D}(F(C),D) & \longrightarrow & \mathcal{C}(G(F(C)),G(D)) & \longrightarrow & \mathcal{C}(C,G(D)) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{B}(Y,Z). & & \end{array}$$

Therefore θ will be an equivalence if and only if for every morphism $\phi : Y \to Z$ in $\mathcal{B}$ the map induced by θ on the fiber over ϕ is an equivalence.

As $\mathcal{C} \to \mathcal{B}$ and $\mathcal{D} \to \mathcal{B}$ are cocartesian fibrations, this map is equivalent to the map

$$\rho : \mathcal{D}_Z(\phi_*(F_Y(C)),D) \to \mathcal{C}_Z(G_Z(\phi_*(F_Y(C))),G_Z(D)) \to$$

$$\mathcal{C}_Z(\phi_*(G_Y(F_Y(C))),G_Z(D)) \to \mathcal{C}_Z(\phi_*(C),G_Z(D)).$$

By our assumption that $F : \mathcal{C} \to \mathcal{D}$ preserves cocartesian morphisms, the canonical morphism $\phi_*(F_Y(C)) \to F_Z(\phi_*(C))$ in $\mathcal{D}_Z$ is an equivalence.

We will complete the proof by showing that the composition

$$\xi : \mathcal{D}_Z(F_Z(\phi_*(C)),D) \simeq \mathcal{D}_Z(\phi_*(F_Y(C)),D) \xrightarrow{\rho} \mathcal{C}_Z(\phi_*(C),G_Z(D))$$

is equivalent to the map

$$\gamma : \mathcal{D}_Z(F_Z(\phi_*(C)),D) \to \mathcal{C}_Z(G_Z(F_Z(\phi_*(C))),G_Z(D)) \to \mathcal{C}_Z(\phi_*(C),G_Z(D))$$

that is an equivalence because we assumed that $\lambda_Z : \mathrm{id}_{\mathcal{C}_Z} \to G_Z \circ F_Z$ exhibits F_Z as a left adjoint of G_Z.

ξ is equivalent to the map

$$\mathcal{D}_Z(F_Z(\phi_*(C)),D) \to \mathcal{C}_Z(G_Z(F_Z(\phi_*(C))),G_Z(D)) \to$$

$$\mathcal{C}_Z(G_Z(\phi_*(F_Y(C))),G_Z(D)) \to \mathcal{C}_Z(\phi_*(G_Y(F_Y(C))),G_Z(D)) \to$$
$$\mathcal{C}_Z(\phi_*(C),G_Z(D))$$

and is therefore equivalent to γ if $\lambda_Z(\phi_*(C))$ factors as

$$\phi_*(C) \to \phi_*(G_Y(F_Y(C))) \to G_Z(\phi_*(F_Y(C))) \to G_Z(F_Z(\phi_*(C))).$$

The composition

$$\phi_*(G_Y(F_Y(C))) \to G_Z(\phi_*(F_Y(C))) \to G_Z(F_Z(\phi_*(C)))$$

is the canonical morphism $\phi_*((G \circ F)_Y(C)) \to (G \circ F)_Z(\phi_*(C))$.

So it remains to show that

$$\phi_*(C) \to \phi_*((G \circ F)_Y(C)) \to (G \circ F)_Z(\phi_*(C))$$

and $\phi_*(C) \to (G \circ F)_Z(\phi_*(C))$ are equivalent in $\mathcal{C}_Z(\phi_*(C),(G \circ F)_Z(\phi_*(C)))$.

By the equivalence

$$\mathcal{C}_Z(\phi_*(C),(G \circ F)_Z(\phi_*(C))) \simeq \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C,(G \circ F)_Z(\phi_*(C)))$$

this is equivalent to the condition that

$$\alpha : C \to \phi_*(C) \to \phi_*((G \circ F)_Y(C)) \to (G \circ F)_Z(\phi_*(C))$$

and $\beta : C \to \phi_*(C) \to (G \circ F)_Z(\phi_*(C))$ are equivalent in $\{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C,(G \circ F)_Z(\phi_*(C)))$.

By definition $\phi_*(C) \to \phi_*((G \circ F)_Y(C))$ corresponds under the equivalence

$$\mathcal{C}_Z(\phi_*(C), \phi_*((G \circ F)_Y(C))) \simeq \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, \phi_*((G \circ F)_Y(C)))$$

to $C \to (G \circ F)_Y(C) \to \phi_*((G \circ F)_Y(C))$.

Thus by the commutativity of

$$\begin{array}{ccc} \mathcal{C}_Z(\phi_*(C), \phi_*((G \circ F)_Y(C))) & \longrightarrow & \mathcal{C}_Z(\phi_*(C), (G \circ F)_Z(\phi_*(C))) \\ \downarrow \simeq & & \downarrow \simeq \\ \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, \phi_*((G \circ F)_Y(C))) & \longrightarrow & \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, (G \circ F)_Z(\phi_*(C))) \end{array}$$

the morphisms α and $\alpha' : C \to (G \circ F)_Y(C) \to \phi_*((G \circ F)_Y(C)) \to (G \circ F)_Z(\phi_*(C))$ are equivalent in $\{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, (G \circ F)_Z(\phi_*(C)))$.

Similarly $\phi_*((G \circ F)_Y(C)) \to (G \circ F)_Z(\phi_*(C))$ corresponds under the equivalence

$$\mathcal{C}_Z(\phi_*((G\circ F)_Y(C)), (G\circ F)_Z(\phi_*(C))) \simeq \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}((G\circ F)_Y(C), (G\circ F)_Z(\phi_*(C)))$$

to $(G \circ F)_Y(C) \to (G \circ F)_Z(\phi_*(C))$.

Therefore $\alpha'' : C \to (G \circ F)_Y(C) \to (G \circ F)_Z(\phi_*(C))$ is the image of $\phi_*((G \circ F)_Y(C)) \to (G \circ F)_Z(\phi_*(C))$ under the map

$$\mathcal{C}_Z(\phi_*((G\circ F)_Y(C)), (G\circ F)_Z(\phi_*(C))) \simeq \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}((G\circ F)_Y(C), (G\circ F)_Z(\phi_*(C)))$$
$$\to \{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, (G \circ F)_Z(\phi_*(C))).$$

Consequently α', α'' are equivalent in $\{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, (G\circ F)_Z(\phi_*(C)))$.

By the naturality of λ relative to $\mathcal{B}$ we have a commutative square

$$\begin{array}{ccc} C & \longrightarrow & (G \circ F)_Y(C) \\ \downarrow & & \downarrow \\ \phi_*(C) & \longrightarrow & (G \circ F)_Z(\phi_*(C)) \end{array}$$

in $\mathcal{C}$ which lies over the identity of ϕ in $\mathcal{B}(Y, Z)$ and therefore yields an equivalence in $\{\phi\} \times_{\mathcal{B}(Y,Z)} \mathcal{C}(C, (G \circ F)_Z(\phi_*(C)))$ from

$$\alpha'' : C \to (G \circ F)_Y(C) \to (G \circ F)_Z(\phi_*(C))$$

to $\beta : C \to \phi_*(C) \to (G \circ F)_Z(\phi_*(C))$.

□

Corollary 6.25. *Let $\mathcal{O}^\otimes$ be an operad, $F : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ a $\mathcal{O}^\otimes$-monoidal functor and $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ a lax $\mathcal{O}^\otimes$-monoidal functor.*

Let $\lambda : \mathrm{id}_{\mathcal{C}^\otimes} \to G \circ F$ be a $\mathcal{O}^\otimes$-monoidal natural transformation, i.e. a morphism of the category $\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{O}^\otimes}(\mathcal{C}^\otimes, \mathcal{C}^\otimes)$.

Assume that for every $X \in \mathcal{O}$ the induced natural transformation $\lambda_X : \mathrm{id}_{\mathcal{C}_X} \to G_X \circ F_X$ on the fiber over X exhibits F_X as a left adjoint of G_X.

Then $\lambda : \mathrm{id}_{\mathcal{C}} \to G \circ F$ exhibits F as a left adjoint of G relative to $\mathcal{O}^\otimes$.

Proof. By lemma 6.24 we have to check that $\lambda_X : \mathrm{id}_{\mathcal{C}_X} \to G_X \circ F_X$ exhibits F_X as a left adjoint of G_X for every $X \in \mathcal{O}^{\otimes}$.

Let $X \in \mathcal{O}^{\otimes}_{\langle n \rangle} \simeq \mathcal{O}^{\times n}$ for some $n \in \mathbb{N}$ corresponding to the family $(X_1, ..., X_n)$ and let $C \in \mathcal{C}^{\otimes}_X \simeq \prod_{i=1}^n \mathcal{C}_{X_i}$ and $D \in \mathcal{D}^{\otimes}_X \simeq \prod_{i=1}^n \mathcal{D}_{X_i}$ be objects corresponding to the families $(C_1, ..., C_n)$ and $(D_1, ..., D_n)$.

Then we have a commutative square of spaces, where all vertical maps are equivalences:

$$\begin{array}{ccccc}
\mathcal{D}^{\otimes}_X(F_X(C), D) & \longrightarrow & \mathcal{C}^{\otimes}_X(G_X(F_X(C)), G_X(D)) & \longrightarrow & \mathcal{C}^{\otimes}_X(C, G_X(D)) \\
\downarrow & & \downarrow & & \downarrow \\
\prod_{i=1}^n \mathcal{D}_{X_i}(F_{X_i}(C_i), D_i) & \longrightarrow & \prod_{i=1}^n \mathcal{C}_{X_i}(G_{X_i}(F_{X_i}(C_i)), G_{X_i}(D_i)) & \longrightarrow & \prod_{i=1}^n \mathcal{C}_{X_i}(C_i, G_{X_i}(D_i))
\end{array}$$

□

6.1.3 Algebras in the Day convolution are lax monoidal functors

Let $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of operads, $\mathcal{C}^{\otimes}$ a small $\mathcal{O}^{\otimes}$-monoidal category and $\mathcal{D}^{\otimes}$ a $\mathcal{O}^{\otimes}$-monoidal category compatible with small colimits.

In this chapter we show that there is a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{D}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$$

between maps of operads $\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $\mathcal{O}^{\otimes}$ and $\mathcal{O}'^{\otimes}$-algebras relative to $\mathcal{O}^{\otimes}$ in the Day convolution $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ (prop. 6.28).

Especially if we choose the map $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ to be the identity, we get a canonical equivalence

$$\mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D}) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D}))$$

between lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and $\mathcal{O}^{\otimes}$-algebras relative to $\mathcal{O}^{\otimes}$ in the Day convolution $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$.

Here we don't need to assume that $\mathcal{C}^{\otimes}$ is a small $\mathcal{O}^{\otimes}$-monoidal category and $\mathcal{D}^{\otimes}$ is compatible with small colimits as we don't need the Day convolution $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes}$ to be a $\mathcal{O}^{\otimes}$-monoidal category but only an operad over $\mathcal{O}^{\otimes}$.

The strategy to construct this equivalence is as follows:

We show in proposition 6.26 that there is a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathcal{S}).$$

Applying this equivalence twice we obtain an equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} (\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\mathrm{rev}}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C} \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathcal{S})$$
$$\simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{P}(\mathcal{D}))$$

that restricts to the desired equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{D}).$$

For later applications we work with cocartesian S-families of operads for some category S.

We start with some notation:

Given S-families of operads $\mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \times \mathrm{S}$, $\mathcal{O}''^{\otimes} \to \mathcal{O}^{\otimes} \times \mathrm{S}$ denote $\mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'}(\mathcal{O}'') \subset \mathrm{Fun}_{\mathcal{F}\mathrm{in}_* \times \mathrm{S}}(\mathcal{O}'^{\otimes}, \mathcal{O}''^{\otimes})$ the full subcategory spanned by the functors over $\mathcal{F}\mathrm{in}_* \times \mathrm{S}$ that induce on the fiber over every $\mathsf{s} \in \mathrm{S}$ a map of operads $\mathcal{O}'^{\otimes}_{\mathsf{s}} \to \mathcal{O}''^{\otimes}_{\mathsf{s}}$.

Given maps of S-families of operads $\alpha : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}, \mathcal{O}''^{\otimes} \to \mathcal{O}^{\otimes}$ we write $\mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{O}'')$ for the pullback $\{\alpha\} \times_{\mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'}(\mathcal{O})} \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'}(\mathcal{O}'')$.

Given an operad $\mathcal{B}^{\otimes}$ denote $\mathrm{Alg}_{\mathcal{O}}(\mathcal{B}) \subset \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{O}^{\otimes}, \mathcal{B}^{\otimes})$ the full subcategory spanned by the functors over $\mathcal{F}\mathrm{in}_*$ such that for every $\mathsf{s} \in \mathrm{S}$ the composition $\mathcal{O}^{\otimes}_{\mathsf{s}} \to \mathcal{O}^{\otimes} \to \mathcal{B}^{\otimes}$ preserves inert morphisms so that we have a canonical equivalence $\mathrm{Alg}_{\mathcal{O}}(\mathcal{B}) \simeq \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}}(\mathcal{B} \times \mathrm{S})$.

Given a category with finite products $\mathcal{C}$ denote $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{O}^{\otimes}, \mathcal{C})$ the full subcategory spanned by the functors such that for every $\mathsf{s} \in \mathrm{S}$ the composition $\mathcal{O}^{\otimes}_{\mathsf{s}} \to \mathcal{O}^{\otimes} \to \mathcal{C}$ is a $\mathcal{O}^{\otimes}_{\mathsf{s}}$-monoid of $\mathcal{C}$.

For $\mathcal{C} = \mathsf{Cat}_\infty$ the objects of $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_\infty)$ are exactly classified by cocartesian fibrations $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ of cocartesian S-families of operads.

The universal $\mathcal{C}^\times$-monoid $\mathcal{C}^\times \to \mathcal{C}$ of $\mathcal{C}$ yields an equivalence

$$\beta : \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}}(\mathcal{C} \times \mathrm{S}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}).$$

Proof. The map

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}}(\mathcal{C}\times\mathrm{S}) \subset \mathrm{Fun}^{/\mathrm{S}}_{\mathcal{F}\mathrm{in}_*\times\mathrm{S}}(\mathcal{O}^\otimes, \mathcal{C}^\times\times\mathrm{S}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{O}^\otimes, \mathcal{C}^\times\times\mathrm{S}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{O}^\otimes, \mathcal{C}\times\mathrm{S})$$

of cartesian fibrations over S induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ the fully faithful functor

$$\mathrm{Alg}_{\mathcal{O}_{\mathsf{s}}}(\mathcal{C}) \subset \mathrm{Fun}_{\mathcal{F}\mathrm{in}_*}(\mathcal{O}^\otimes_{\mathsf{s}}, \mathcal{C}^\times) \to \mathrm{Fun}(\mathcal{O}^\otimes_{\mathsf{s}}, \mathcal{C}^\times) \to \mathrm{Fun}(\mathcal{O}^\otimes_{\mathsf{s}}, \mathcal{C})$$

with essential image $\mathrm{Mon}_{\mathcal{O}_{\mathsf{s}}}(\mathcal{C})$ and so induces on sections a fully faithful functor given by $\beta : \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}}(\mathcal{C} \times \mathrm{S}) \to \mathrm{Mon}_{\mathcal{O}}(\mathcal{C}) \subset \mathrm{Fun}(\mathcal{O}^\otimes, \mathcal{C}) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathcal{O}^\otimes, \mathcal{C} \times \mathrm{S})$ with essential image $\mathrm{Mon}_{\mathcal{O}}(\mathcal{C})$.

□

Especially for $\mathcal{C} = \mathsf{Cat}_\infty$ every cocartesian fibration $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ of cocartesian S-families of operads classifies an object of $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_\infty) \simeq$
$\mathrm{Alg}_{\mathcal{O}}(\mathsf{Cat}_\infty) \simeq \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}}(\mathsf{Cat}_\infty \times \mathrm{S})$, i.e. a map of S-families of operads $\mathcal{O}^\otimes \to \mathsf{Cat}^\times_\infty \times \mathrm{S}$.

Given a cocartesian fibration $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ of cocartesian S-families of operads corresponding to a map $\mathcal{O}^\otimes \to \mathsf{Cat}^\times_\infty \times \mathrm{S}$ of S-families of operads denote $\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})^\otimes \to \mathcal{O}^\otimes$ the pullback of the symmetric monoidal functor $\mathrm{Cart}^\times \to \mathsf{Cat}^\times_\infty$ along $\mathcal{O}^\otimes \to \mathsf{Cat}^\times_\infty \times \mathrm{S} \to \mathsf{Cat}^\times_\infty$ and $\mathcal{P}^{/\mathrm{S}}(\mathcal{D})^\otimes \to \mathcal{O}^\otimes$ the pullback of the symmetric monoidal functor $\mathcal{R}^\times \to \mathsf{Cat}^\times_\infty$ along $\mathcal{O}^\otimes \to \mathsf{Cat}^\times_\infty \times \mathrm{S} \to \mathsf{Cat}^\times_\infty$.

If S is contractible, we write $\mathcal{P}_{\mathsf{Cat}_\infty}(\mathcal{D})^\otimes \to \mathcal{O}^\otimes$ and $\mathcal{P}(\mathcal{D})^\otimes \to \mathcal{O}^\otimes$.

Proposition 6.26.

Let S *be a category,* $\varphi : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ *a map of cocartesian* S*-families of operads and* $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ *a cocartesian fibration of cocartesian* S*-families of operads.*

There is a canonical equivalence

$$\beta : \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})$$

over S.

For every $\mathrm{X} \in \mathcal{O}$ *and* $\mathsf{s} \in \mathrm{S}$ *the following square commutes:*

$$\begin{array}{ccc} \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D}))_{\mathsf{s}} & \xrightarrow{\simeq} & \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})_{\mathsf{s}} \\ \downarrow & & \downarrow \\ \mathrm{Fun}((\mathcal{O}'_{\mathsf{s}})_{\mathrm{X}}, \mathrm{Fun}((\mathcal{D}_{\mathsf{s}})^{\mathrm{op}}_{\mathrm{X}}, \mathsf{Cat}_\infty)) & \xrightarrow{\simeq} & \mathrm{Fun}((\mathcal{O}'_{\mathsf{s}})_{\mathrm{X}} \times (\mathcal{D}_{\mathsf{s}})^{\mathrm{op}}_{\mathrm{X}}, \mathsf{Cat}_\infty). \end{array} \tag{34}$$

Hence β *restricts to an equivalence*

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}(\mathcal{D})) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathcal{S} \times \mathrm{S})$$

over S.

Proof. The cocartesian fibration $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ of cocartesian S-families of operads classifies a functor $\phi' : \mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty}$ that corresponds to a map of S-families of operads $\mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty}^{\times} \times \mathrm{S}$.

Denote ϕ the composition $\mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty}^{\times} \times \mathrm{S} \to \mathsf{Cat}_{\infty}^{\times}$.

We have a canonical embedding

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})) \simeq$$

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})} \mathrm{Fun}(\Delta^1, \mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})) \subset$$

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Cat}_{\infty})} \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathcal{O}'^{\otimes}, \mathsf{Cat}_{\infty})) \simeq$$

$$\{\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}\} \times_{\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes}}} \mathrm{Fun}(\Delta^1, \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes}}) \simeq$$

$$(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes}})_{/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}}.$$

The subcategory inclusion

$$\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}} \subset \mathsf{Cat}_{\infty/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}} \simeq$$

$$(\mathsf{Cat}_{\infty/\mathcal{O}'^{\otimes}})_{/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}}$$

restricts to a subcategory inclusion

$$\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}} \subset (\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes}})_{/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}}$$

and the embedding

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})) \subset$$

$$(\mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes}})_{/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}}$$

restricts to an embedding

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Cocart}) \subset \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}} :$$

A map $\mathcal{B} \to \mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ of cocartesian fibrations over $\mathcal{O}'^{\otimes}$ is a cocartesian fibration if and only if it classifes a functor $\mathcal{O}'^{\otimes} \to \mathrm{Fun}(\Delta^1, \mathsf{Cat}_{\infty})$ that factors through Cocart.

Given cocartesian fibrations $\mathcal{B} \to \mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}, \mathcal{C} \to \mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ a functor $\mathcal{B} \to \mathcal{C}$ over $\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ that is a map of cocartesian fibrations over $\mathcal{O}'^{\otimes}$ is a map of cocartesian fibrations over $\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ if and only if for every $\mathrm{X} \in \mathcal{O}'^{\otimes}$ the induced functor $\mathcal{B}_{\mathrm{X}} \to \mathcal{C}_{\mathrm{X}}$ over $\mathcal{D}^{\mathrm{op}}_{\varphi(\mathrm{X})}$ is a map of cocartesian fibrations over $\mathcal{D}^{\mathrm{op}}_{\varphi(\mathrm{X})}$.

Moreover the embedding

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Cocart}) \subset \mathsf{Cat}^{\mathrm{cocart}}_{\infty/\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}}$$

$$\simeq \mathrm{Fun}(\mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}, \mathsf{Cat}_{\infty})$$

restricts to an equivalence

$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_{\infty})} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Cocart}) \simeq \mathrm{Mon}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_{\infty}) :$$

Given a cocartesian fibration $\mathcal{B} \to \mathcal{O}'^{\otimes} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ and an object $\mathsf{s} \in \mathrm{S}$ the induced cocartesian fibration $\mathcal{B}_{\mathsf{s}} \to \mathcal{O}'^{\otimes}_{\mathsf{s}} \times_{\mathcal{O}^{\otimes}_{\mathsf{s}}} (\mathcal{D}^{\otimes}_{\mathsf{s}})^{\mathrm{rev}}$ exhibits $\mathcal{B}_{\mathsf{s}}$ as a $\mathcal{O}'^{\otimes}_{\mathsf{s}} \times_{\mathcal{O}^{\otimes}_{\mathsf{s}}} (\mathcal{D}^{\otimes}_{\mathsf{s}})^{\mathrm{rev}}$-monoidal category if and only if the composition $\mathcal{B}_{\mathsf{s}} \to \mathcal{O}'^{\otimes}_{\mathsf{s}} \times_{\mathcal{O}^{\otimes}_{\mathsf{s}}} (\mathcal{D}^{\otimes}_{\mathsf{s}})^{\mathrm{rev}} \to \mathcal{O}'^{\otimes}_{\mathsf{s}}$ exhibits $\mathcal{B}_{\mathsf{s}}$ as a $\mathcal{O}'^{\otimes}_{\mathsf{s}}$-monoidal category.

The duality involution on Cat_∞ induces an involution on $\mathrm{Fun}(\Delta^1, \mathsf{Cat}_\infty)$ that restricts to an equivalence $\mathrm{Cart} \simeq \mathrm{Cocart}$.

So we obtain a canonical equivalence

$$\beta : \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})) \simeq \{\phi \circ \varphi\} \times_{\mathrm{Alg}_{\mathcal{O}'}(\mathsf{Cat}_\infty)} \mathrm{Alg}_{\mathcal{O}'}(\mathrm{Cart}) \simeq$$
$$\{\phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_\infty)} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Cart}) \simeq$$
$$\{(-)^{\mathrm{op}} \circ \phi' \circ \varphi\} \times_{\mathrm{Mon}_{\mathcal{O}'}(\mathsf{Cat}_\infty)} \mathrm{Mon}_{\mathcal{O}'}(\mathrm{Cocart}) \simeq$$
$$\mathrm{Mon}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty).$$

For every map $\mathcal{O}''^{\otimes} \to \mathcal{O}'^{\otimes}$ of cocartesian S-families of operads over $\mathcal{O}^{\otimes}$ we have a commutative square

$$\begin{array}{ccc} \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty) \\ \downarrow & & \downarrow \\ \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}''/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathcal{O}'' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty) \end{array} \tag{35}$$

and for every functor $\mathrm{T} \to \mathrm{S}$ we have a commutative square

$$\begin{array}{ccc} \mathrm{Alg}^{\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty) \\ \downarrow & & \downarrow \\ \mathrm{Alg}^{\mathrm{T}}_{\mathrm{T} \times_{\mathrm{S}} \mathcal{O}'/\mathrm{T} \times_{\mathrm{S}} \mathcal{O}}(\mathcal{P}^{/\mathrm{T}}_{\mathsf{Cat}_\infty}(\mathrm{T} \times_{\mathrm{S}} \mathcal{D})) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathrm{T} \times_{\mathrm{S}} \mathcal{O}' \times_{\mathrm{T} \times_{\mathrm{S}} \mathcal{O}} \mathrm{T} \times_{\mathrm{S}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty). \end{array} \tag{36}$$

Given a functor $\mathrm{K} \to \mathrm{S}$ we have canonical equivalences

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D}))) \simeq$$
$$\mathrm{Fun}_{\mathrm{K}}(\mathrm{K}, \mathrm{K} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D}))) \simeq$$
$$\mathrm{Fun}_{\mathrm{K}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{K}}_{\mathrm{K} \times_{\mathrm{S}} \mathcal{O}'/\mathrm{K} \times_{\mathrm{S}} \mathcal{O}}(\mathcal{P}^{/\mathrm{K}}_{\mathsf{Cat}_\infty}(\mathrm{K} \times_{\mathrm{S}} \mathcal{D}))) \simeq$$
$$\mathrm{Alg}^{\mathrm{K}}_{\mathrm{K} \times_{\mathrm{S}} \mathcal{O}'/\mathrm{K} \times_{\mathrm{S}} \mathcal{O}}(\mathcal{P}^{/\mathrm{K}}_{\mathsf{Cat}_\infty}(\mathrm{K} \times_{\mathrm{S}} \mathcal{D}))$$

and

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})) \simeq$$
$$\mathrm{Fun}_{\mathrm{K}}(\mathrm{K}, \mathrm{K} \times_{\mathrm{S}} \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})) \simeq$$
$$\mathrm{Fun}_{\mathrm{K}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{K}}_{\mathrm{K} \times_{\mathrm{S}} \mathcal{O}' \times_{\mathrm{K} \times_{\mathrm{S}} \mathcal{O}} \mathrm{K} \times_{\mathrm{S}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{K})) \simeq$$
$$\mathrm{Alg}^{\mathrm{K}}_{\mathrm{K} \times_{\mathrm{S}} \mathcal{O}' \times_{\mathrm{K} \times_{\mathrm{S}} \mathcal{O}} \mathrm{K} \times_{\mathrm{S}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{K}).$$

So we obtain a canonical equivalence

$$\psi : \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D}))) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})).$$

Moreover the commutativity of square 36 implies that for every functor $\mathrm{K} \to \mathrm{K}'$ over S we have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}', \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D}))) & \xrightarrow{\simeq} & \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}', \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D}))) & \xrightarrow{\simeq} & \mathrm{Fun}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})). \end{array}$$

Thus ψ represents an equivalence

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}_{\mathsf{Cat}_\infty}(\mathcal{D})) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})$$

over S that fits into square 34 and so restricts to an equivalence

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}(\mathcal{D})) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathcal{S} \times \mathrm{S})$$

over S.

□

Corollary 6.27. *Let* $\mathcal{O}^\otimes$ *be an operad and* $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ *a* $\mathcal{O}^\otimes$*-monoidal category.*

There is a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}_{\mathsf{Cat}_\infty}(\mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty)$$

natural in every map of operads $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ *that restricts to an equivalence*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathcal{S}).$$

Proof. For $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes \times \mathrm{S}$ the map of cocartesian S-families of operads that classifies the identity of $\mathrm{S} = \mathrm{Op}_{\infty/\mathcal{O}^\otimes}$ we get an equivalence

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}\times\mathrm{S}}(\mathcal{P}_{\mathsf{Cat}_\infty}(\mathcal{D}) \times \mathrm{S}) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty \times \mathrm{S})$$

of cartesian fibrations over S that classifies an equivalence

$$\mathrm{Alg}_{(-)/\mathcal{O}}(\mathcal{P}_{\mathsf{Cat}_\infty}(\mathcal{D})) \simeq \mathrm{Alg}_{(-\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}})}(\mathsf{Cat}_\infty)$$

of functors $(\mathrm{Op}_{\infty/\mathcal{O}^\otimes})^{\mathrm{op}} \to \mathsf{Cat}_\infty$ (theorem 5.23) that sends a map of operads $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ to the canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}_{\mathsf{Cat}_\infty}(\mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}}}(\mathsf{Cat}_\infty)$$

of prop. 6.26 (for S contractible).

□

Proposition 6.28. *Let* S *be a category,* $\varphi : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ *a map of cocartesian* S*-families of operads and* $\mathcal{C}^\otimes \to \mathcal{O}^\otimes, \mathcal{D}^\otimes \to \mathcal{O}^\otimes$ *cocartesian fibrations of cocartesian* S*-families of operads.*

There is a canonical equivalence

$$\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})) \simeq \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{C}/\mathcal{O}}(\mathcal{P}^{/\mathrm{S}}(\mathcal{D}))$$

over S.

For S *contractible we get the following:*

Let $\varphi : \mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ *be a map of operads and* $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ *small* $\mathcal{O}^\otimes$*-monoidal categories.*

There is a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}'\times_{\mathcal{O}}\mathcal{C}/\mathcal{O}}(\mathcal{P}(\mathcal{D}))$$

natural in $\varphi : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.

For every $X \in \mathcal{O}$ *the following square commutes:*

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})) & \xrightarrow{\simeq} & \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{P}(\mathcal{D})) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{O}'_{X}, \mathcal{P}((\mathcal{C}_{X})^{\mathrm{op}} \times \mathcal{D}_{X})) & \xrightarrow{\simeq} & \mathrm{Fun}(\mathcal{O}'_{X} \times \mathcal{C}_{X}, \mathcal{P}(\mathcal{D}_{X})) \end{array} \tag{37}$$

So this equivalence restricts to an equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{D})$$

natural in $\varphi : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.

Proof. By proposition 6.26 applied twice there is a canonical equivalence

$$\mathrm{Alg}^{/S}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}^{/S}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})) \simeq \mathrm{Alg}^{/S}_{\mathcal{O}' \times_{\mathcal{O}} (\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\mathrm{rev}}}(\mathcal{S} \times S) \simeq$$

$$\mathrm{Alg}^{/S}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C} \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathcal{S} \times S) \simeq \mathrm{Alg}^{/S}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{P}^{/S}(\mathcal{D}))$$

over S.

For S contratible we get an equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} (\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\mathrm{rev}}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C} \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}}(\mathcal{S})$$

$$\simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{P}(\mathcal{D}))$$

natural in $\varphi : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ and making square 37 commutative.

So this equivalence restricts to an equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{D})$$

natural in $\varphi : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$.

□

Corollary 6.29.

1. *Let* $\mathcal{O}^{\otimes}$ *be an operad and* $\mathcal{C}^{\otimes}$ *a small* $\mathcal{O}^{\otimes}$*-monoidal category corresponding to a* $\mathcal{O}^{\otimes}$*-monoid* ϕ *of* Cat_{∞}.
There are canonical $\mathcal{O}^{\otimes}$*-monoidal equivalences*

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{rev}}, \mathsf{Cat}_{\infty} \times \mathcal{O})^{\otimes} \simeq \mathcal{P}_{\mathsf{Cat}_{\infty}}(\mathcal{C})^{\otimes}$$

and

$$\mathrm{Fun}(\mathcal{C}^{\mathrm{rev}}, \mathcal{S} \times \mathcal{O})^{\otimes} \simeq \mathcal{P}(\mathcal{C})^{\otimes}$$

represented by the canonical equivalences

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{rev}}, \mathsf{Cat}_{\infty} \times \mathcal{O})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}^{\mathrm{rev}}/\mathcal{O}}(\mathsf{Cat}_{\infty} \times \mathcal{O}) \simeq$$

$$\mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}^{\mathrm{rev}}}(\mathsf{Cat}_{\infty}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}_{\mathsf{Cat}_{\infty}}(\mathcal{C}))$$

respectively

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{rev}}, \mathcal{S} \times \mathcal{O})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}^{\mathrm{rev}}/\mathcal{O}}(\mathcal{S} \times \mathcal{O}) \simeq$$

$$\mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}^{\mathrm{rev}}}(\mathcal{S}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{C}))$$

natural in the operad $\mathcal{O}'^{\otimes}$ *over* $\mathcal{O}^{\otimes}$ *provided by the second part of proposition 6.28 and proposition 6.26.*

2. *Let $\mathcal{D}^{\otimes}$ be a small $\mathcal{O}^{\otimes}$-monoidal category.*

   There is a canonical $\mathcal{O}^{\otimes}$-monoidal equivalence

$$\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))^{\otimes} \simeq \mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\otimes}$$

   such that we have a commutative square

$$\begin{array}{ccc} \mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))^{\otimes} & \longrightarrow & \mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D})^{\otimes} \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\mathcal{C}_{\mathrm{X}}, \mathcal{P}(\mathcal{D}_{\mathrm{X}})) & \xrightarrow{\simeq} & \mathcal{P}((\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \times \mathcal{D}_{\mathrm{X}}) \end{array}$$

   represented by the canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{P}(\mathcal{D}))) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{P}(\mathcal{D}))$$
$$\simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{P}(\mathcal{C}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{D}))$$

   natural in the operad $\mathcal{O}'^{\otimes}$ over $\mathcal{O}^{\otimes}$ provided by the first and second part of proposition 6.28.

Corollary 6.30.
Let $\varphi : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of small operads and $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ a small $\mathcal{O}^{\otimes}$-monoidal category.

The functor $- \times_{\mathcal{O}^{\otimes}} \mathcal{C}^{\otimes} : \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}} \to \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}$ admits a right adjoint.

Proof. Given a map of small operads $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}$ denote $\mathrm{Env}_{\mathcal{O}}(\mathcal{D})^{\otimes} \to \mathcal{O}^{\otimes}$ its enveloping $\mathcal{O}^{\otimes}$-monoidal category that comes equipped with an embedding $\mathcal{D}^{\otimes} \to \mathrm{Env}_{\mathcal{O}}(\mathcal{D})^{\otimes}$ of operads over $\mathcal{O}^{\otimes}$.

Denote $\mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\otimes} \subset \mathrm{Fun}(\mathcal{C}, \mathrm{Env}_{\mathcal{O}}(\mathcal{D}))^{\otimes}$ the full suboperad spanned by the objects of $\mathrm{Fun}(\mathcal{C}_{\mathrm{X}}, \mathrm{Env}_{\mathcal{O}}(\mathcal{D})_{\mathrm{X}})$ for some $\mathrm{X} \in \mathcal{O}$ that belong to $\mathrm{Fun}(\mathcal{C}_{\mathrm{X}}, \mathcal{D}_{\mathrm{X}})$.

By proposition 6.28 there is a canonical equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathrm{Env}_{\mathcal{O}}(\mathcal{D}))) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{D}))$$

natural in $\varphi : \mathcal{O}'^{\otimes} \to \mathcal{O}^{\otimes} \in \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}$ that restricts to an equivalence

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathrm{Fun}(\mathcal{C}, \mathcal{D})) \simeq \mathrm{Alg}_{\mathcal{O}' \times_{\mathcal{O}} \mathcal{C}/\mathcal{O}}(\mathcal{D}).$$

□

6.1.4 $\mathcal{O}^{\otimes}$-monoidal adjointness

Denote $\mathsf{Cat}_\infty^{\mathrm{L}}, \mathsf{Cat}_\infty^{\mathrm{R}} \subset \mathsf{Cat}_\infty$ the subcategories with the same objects and with morphisms the left respectively right adjoint functors.

Recall that there is a canonical equivalence $\mathsf{Cat}_\infty^{\mathrm{L}} \simeq (\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}}$ that sends a category to itself and a left adjoint functor to its right adjoint.

This equivalence is represented by the equivalence

$$\widehat{\mathsf{Cat}_\infty}(-, \mathsf{Cat}_\infty^{\mathrm{L}}) \simeq (\mathsf{Cat}_{\infty/(-)}^{\mathrm{bicart}})^{\simeq} \simeq \widehat{\mathsf{Cat}_\infty}((-)^{\mathrm{op}}, \mathsf{Cat}_\infty^{\mathrm{R}}) \simeq \widehat{\mathsf{Cat}_\infty}(-, (\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}}).$$

Given two categories $\mathcal{C}, \mathcal{D} \in \mathsf{Cat}_\infty$ there is a canonical equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}}$$

that sends a right adjoint functor to its left adjoint.
Taking the opposite category defines a functor $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$.

The canonical equivalence $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}})$ is induced by the canonical equivalence

$$\mathrm{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathrm{op}}, \mathcal{S}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S}) \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{P}(\mathcal{D}^{\mathrm{op}}))$$

as $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C}))$ corresponds to the full subcategory of $\mathrm{Fun}(\mathcal{D} \times \mathcal{C}^{\mathrm{op}}, \mathcal{S})$ spanned by the functors $\mathcal{D} \times \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ that are representable in both variables and so $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{P}(\mathcal{D}^{\mathrm{op}}))$ corresponds to the full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{D}, \mathcal{S})$ spanned by the functors $\mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{S}$ that are representable in both variables.

Moreover we have a canonical equivalence $(\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}} \simeq \mathsf{Cat}_\infty^{\mathrm{L}}$, under which a right adjoint functor corresponds to its left adjoint.

The canonical involution $(-)^{\mathrm{op}}$ on Cat_∞ restricts to an equivalence $\mathsf{Cat}_\infty^{\mathrm{L}} \simeq \mathsf{Cat}_\infty^{\mathrm{R}}$ so that we obtain a canonical equivalence $(\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}} \simeq \mathsf{Cat}_\infty^{\mathrm{R}}$.

In this section we generalize the notion of adjunction to the notion of $\mathcal{O}^{\otimes}$-monoidal adjunction for every operad $\mathcal{O}^{\otimes}$ and construct similar equivalences.

We show in prop. 6.35 that for every operad $\mathcal{O}^{\otimes}$ and arbitrary $\mathcal{O}^{\otimes}$-monoidal categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ there is a canonical equivalence

$$\mathrm{Fun}^{\otimes, \mathrm{R}, \mathrm{lax}}(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\otimes, \mathrm{L}, \mathrm{oplax}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}}$$

between the category of lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ that admit fiberwise a left adjoint and the opposite category of the category of oplax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ that admit fiberwise a right adjoint.

We show in proposition 6.40 that there is a canonical equivalence

$$(\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_\infty)^{\mathrm{R}}$$

that sends a right adjoint lax $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ to the right adjoint lax $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ representing the oplax $\mathcal{O}^{\otimes}$-monoidal left adjoint $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ of G.

Definition 6.31.
Let $\mathcal{O}^{\otimes}$ be an operad and $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ be $\mathcal{O}^{\otimes}$-monoidal categories.

1. *Let* $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *be an oplax* $\mathcal{O}^{\otimes}$*-monoidal functor corresponding to a lax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ *and* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *a lax* $\mathcal{O}^{\otimes}$*-monoidal functor.*

   We say that $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *is* $\mathcal{O}^{\otimes}$*-monoidally left adjoint to* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *or* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *is* $\mathcal{O}^{\otimes}$*-monoidally right adjoint to* $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *if the lax* $\mathcal{O}^{\otimes}$*-monoidal functors*

   $$\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}} \subset \mathcal{P}(\mathcal{D}^{\mathrm{rev}})^{\otimes}, \qquad \mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}$$

   correspond to equivalent lax $\mathcal{O}^{\otimes}$*-monoidal functors*

   $$(\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{S}^{\times}.$$

2. *Let* $\mathcal{E}^{\otimes}$ *be a* $\mathcal{O}^{\otimes}$*-monoidal category,* $\mathrm{F} : \mathcal{C}^{\otimes} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes} \to \mathcal{E}^{\otimes}$ *an oplax* $\mathcal{O}^{\otimes}$*-monoidal functor corresponding to a lax* $\mathcal{O}^{\otimes}$*-monoidal functor*

   $$\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}} \to (\mathcal{E}^{\otimes})^{\mathrm{rev}}$$

   and $\mathrm{G} : (\mathcal{D}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes} \to \mathcal{C}^{\otimes}, \mathrm{H} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes} \to \mathcal{D}^{\otimes}$ *lax* $\mathcal{O}^{\otimes}$*-monoidal functors.*

   We call the triple $(\mathrm{F}, \mathrm{G}, \mathrm{H})$ *a* $\mathcal{O}^{\otimes}$*-monoidal adjunction of two variables if the lax* $\mathcal{O}^{\otimes}$*-monoidal functors*

   $$\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}} \to (\mathcal{E}^{\otimes})^{\mathrm{rev}} \subset \mathcal{P}(\mathcal{E}^{\mathrm{rev}})^{\otimes},$$

   $$\mathrm{G} : (\mathcal{D}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes} \to \mathcal{C}^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}, \qquad \mathrm{H} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes} \to \mathcal{D}^{\otimes} \subset \mathcal{P}(\mathcal{D})^{\otimes}$$

   correspond to equivalent lax $\mathcal{O}^{\otimes}$*-monoidal functors*

   $$(\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} (\mathcal{D}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{S}^{\times}.$$

Remark 6.32. *Let* $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *be an oplax* $\mathcal{O}^{\otimes}$*-monoidal functor corresponding to a lax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ *and* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *a lax* $\mathcal{O}^{\otimes}$*-monoidal functor corresponding to an oplax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{G}^{\mathrm{rev}} : (\mathcal{D}^{\otimes})^{\mathrm{rev}} \to (\mathcal{C}^{\otimes})^{\mathrm{rev}}$.

The oplax $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *is* $\mathcal{O}^{\otimes}$*-monoidally left adjoint to* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *if and only if* $\mathrm{G}^{\mathrm{rev}} : (\mathcal{D}^{\otimes})^{\mathrm{rev}} \to (\mathcal{C}^{\otimes})^{\mathrm{rev}}$ *is* $\mathcal{O}^{\otimes}$*-monoidally left adjoint to* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$.

Remark 6.33. *An oplax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *corresponding to a lax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}}$ *is* $\mathcal{O}^{\otimes}$*-monoidally left adjoint to* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *if and only if the lax* $\mathcal{O}^{\otimes}$*-monoidal functors*

$$\mathcal{D}^{\otimes} \subset \mathcal{P}(\mathcal{D})^{\otimes} \xrightarrow{(\mathrm{F}^{\mathrm{rev}})^*} \mathcal{P}(\mathcal{C})^{\otimes}, \qquad \mathcal{D}^{\otimes} \xrightarrow{\mathrm{G}} \mathcal{C}^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}$$

are equivalent.

Dually $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *is* $\mathcal{O}^{\otimes}$*-monoidally left adjoint to* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ *if and only if the lax* $\mathcal{O}^{\otimes}$*-monoidal functors*

$$(\mathcal{C}^{\otimes})^{\mathrm{rev}} \subset \mathcal{P}(\mathcal{C}^{\mathrm{rev}})^{\otimes} \xrightarrow{\mathrm{G}^*} \mathcal{P}(\mathcal{D}^{\mathrm{rev}})^{\otimes}, \; (\mathcal{C}^{\otimes})^{\mathrm{rev}} \xrightarrow{\mathrm{F}^{\mathrm{rev}}} (\mathcal{D}^{\otimes})^{\mathrm{rev}} \subset \mathcal{P}(\mathcal{D}^{\mathrm{rev}})^{\otimes}$$

are equivalent.

Proof. Let α be an equivalence of lax $\mathcal{O}^{\otimes}$-monoidal functors

$$(\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{S}^{\times}$$

between the lax $\mathcal{O}^{\otimes}$-monoidal functor $(\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{S}^{\times}$ adjoint to $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^{\otimes})^{\mathrm{rev}} \to (\mathcal{D}^{\otimes})^{\mathrm{rev}} \subset \mathcal{P}(\mathcal{D}^{\mathrm{rev}})^{\otimes}$ and the lax $\mathcal{O}^{\otimes}$-monoidal functor $(\mathcal{C}^{\otimes})^{\mathrm{rev}} \times_{\mathcal{O}^{\otimes}} \mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes} \times_{\mathcal{F}\mathrm{in}_*} \mathcal{S}^{\times}$ adjoint to $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}$.

α is adjoint to an equivalence of lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{D}^{\otimes} \to \mathcal{P}(\mathcal{C})^{\otimes}$ between $\mathcal{D}^{\otimes} \subset \mathcal{P}(\mathcal{D})^{\otimes} \xrightarrow{(\mathrm{F}^{\mathrm{rev}})^*} \mathcal{P}(\mathcal{C})^{\otimes}$ and $\mathcal{D}^{\otimes} \xrightarrow{\mathrm{G}} \mathcal{C}^{\otimes} \subset \mathcal{P}(\mathcal{C})^{\otimes}$.

□

As the $\mathcal{O}^{\otimes}$-monoidal Yoneda-embeddings are fully faithful, $\mathcal{O}^{\otimes}$-monoidal left respectively right adjoints are unique if they exist.

Hence an oplax $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ admits a lax $\mathcal{O}^{\otimes}$-monoidal right adjoint $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ in the sense of definition 6.31 if and only if for all $\mathrm{X} \in \mathcal{O}$ there is a functor $\mathrm{G}_{\mathrm{X}} : \mathcal{D}_{\mathrm{X}} \to \mathcal{C}_{\mathrm{X}}$ such that the functors

$$\mathcal{D}_{\mathrm{X}} \subset \mathcal{P}(\mathcal{D}_{\mathrm{X}}) \xrightarrow{(\mathrm{F}_{\mathrm{X}}^{\mathrm{op}})^*} \mathcal{P}(\mathcal{C}_{\mathrm{X}}), \qquad \mathcal{D}_{\mathrm{X}} \xrightarrow{\mathrm{G}_{\mathrm{X}}} \mathcal{C}_{\mathrm{X}} \subset \mathcal{P}(\mathcal{C}_{\mathrm{X}})$$

are equivalent or equivalently if for all $\mathrm{X} \in \mathcal{O}$ the induced functor $\mathrm{F}_{\mathrm{X}} : \mathcal{C}_{\mathrm{X}} \to \mathcal{D}_{\mathrm{X}}$ admits a right adjoint $\mathrm{G}_{\mathrm{X}} : \mathcal{D}_{\mathrm{X}} \to \mathcal{C}_{\mathrm{X}}$.

Dually a lax $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ admits an oplax $\mathcal{O}^{\otimes}$-monoidal left adjoint $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ if and only if for all $\mathrm{X} \in \mathcal{O}$ there is a functor $\mathrm{F}_{\mathrm{X}} : \mathcal{C}_{\mathrm{X}} \to \mathcal{D}_{\mathrm{X}}$ such that the functors

$$(\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \subset \mathcal{P}((\mathcal{C}_{\mathrm{X}})^{\mathrm{op}}) \xrightarrow{(\mathrm{G}_{\mathrm{X}})^*} \mathcal{P}((\mathcal{D}_{\mathrm{X}})^{\mathrm{op}}), \ (\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \xrightarrow{(\mathrm{F}_{\mathrm{X}})^{\mathrm{op}}} (\mathcal{D}_{\mathrm{X}})^{\mathrm{op}} \subset \mathcal{P}((\mathcal{D}_{\mathrm{X}})^{\mathrm{op}})$$

are equivalent or equivalently if for all $\mathrm{X} \in \mathcal{O}$ the induced functor $\mathrm{G}_{\mathrm{X}} : \mathcal{D}_{\mathrm{X}} \to \mathcal{C}_{\mathrm{X}}$ admits a left adjoint $\mathrm{F}_{\mathrm{X}} : \mathcal{C}_{\mathrm{X}} \to \mathcal{D}_{\mathrm{X}}$.

So an oplax $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ admits a lax $\mathcal{O}^{\otimes}$-monoidal right adjoint $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ if and only if for all $\mathrm{X} \in \mathcal{O}$ the induced functor

$$(\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \times \mathcal{D}_{\mathrm{X}} \xrightarrow{(\mathrm{F}_{\mathrm{X}})^{\mathrm{op}} \times \mathcal{D}_{\mathrm{X}}} (\mathcal{D}_{\mathrm{X}})^{\mathrm{op}} \times \mathcal{D}_{\mathrm{X}} \to \mathcal{S}$$

is representable in both variables.

Dually a lax $\mathcal{O}^{\otimes}$-monoidal functor $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ admits an oplax $\mathcal{O}^{\otimes}$-monoidal left adjoint $\mathrm{F} : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ if and only if for all $\mathrm{X} \in \mathcal{O}$ the induced functor

$$(\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \times \mathcal{D}_{\mathrm{X}} \xrightarrow{(\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \times \mathrm{G}_{\mathrm{X}}} (\mathcal{C}_{\mathrm{X}})^{\mathrm{op}} \times \mathcal{C}_{\mathrm{X}} \to \mathcal{S}$$

is representable in both variables.

Given categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$ denote

- $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \subset \mathrm{Fun}(\mathcal{D}, \mathcal{P}(\mathcal{C})) \simeq \mathcal{P}(\mathcal{D}^{\mathrm{op}} \times \mathcal{C})$ the full subcategory spanned by the functors $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ that are representable in both variables, i.e. that for every $\mathrm{X} \in \mathcal{C}$ the functor $\mathcal{D} \xrightarrow{\mathrm{G}} \mathcal{C} \xrightarrow{\mathcal{C}(\mathrm{X},-)} \mathcal{S}$ is corepresentable.

- $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}\times\mathcal{E},\mathcal{C}) \subset \mathrm{Fun}(\mathcal{D}\times\mathcal{E},\mathcal{C}) \subset \mathrm{Fun}(\mathcal{D}\times\mathcal{E},\mathcal{P}(\mathcal{C})) \simeq \mathcal{P}(\mathcal{D}^{\mathrm{op}}\times\mathcal{E}^{\mathrm{op}}\times\mathcal{C})$ (by abuse of notation) the full subcategory spanned by the functors $\gamma : \mathcal{D}\times\mathcal{E} \to \mathcal{C}$ that are representable in all three variables, i.e. that for every $X \in \mathcal{C}, Y \in \mathcal{D}, Z \in \mathcal{E}$ the functors $\mathcal{D} \xrightarrow{\gamma(-,Z)} \mathcal{C} \xrightarrow{\mathcal{C}(X,-)} \mathcal{S}$ and $\mathcal{E} \xrightarrow{\gamma(Y,-)} \mathcal{C} \xrightarrow{\mathcal{C}(X,-)} \mathcal{S}$ are representable.

If a functor $G : \mathcal{D} \to \mathcal{C}$ belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})$, the functor

$$\mathcal{C} \subset \mathcal{P}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{(G^*)^{\mathrm{op}}} \mathcal{P}(\mathcal{D}^{\mathrm{op}})^{\mathrm{op}}$$

induces a functor $F : \mathcal{C} \to \mathcal{D}$ so that we have an equivalence

$$\mathcal{D}(F(X),Y) \simeq \mathcal{C}(X,G(Y))$$

natural in $X \in \mathcal{C}, Y \in \mathcal{D}$.

Remark 6.34. *If $\phi : \mathcal{M} \to \Delta^1$ denotes the cartesian fibration classifying $G : \mathcal{D} \to \mathcal{C}$, for every $X \in \mathcal{C}$ there is a canonical equivalence*

$$\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/} \simeq \mathcal{D} \times_{\mathcal{M}} \mathcal{M}_{X/} \simeq \{1\} \times_{\Delta^1} \mathcal{M}_{X/},$$

where the final objects of the category $\mathcal{D}\times_{\mathcal{C}}\mathcal{C}_{X/}$ are the corepresentations of the functor $\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{\mathcal{C}(X,-)} \mathcal{S}$ and the final objects of the category $\{1\} \times_{\Delta^1} \mathcal{M}_{X/}$ are the ϕ-cocartesian lifts of the canonical morphism $0 \to 1$ in Δ^1 starting at X.

So a functor $\mathcal{D} \to \mathcal{C}$ belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})$ if and only if $\phi : \mathcal{M} \to \Delta^1$ is a bicartesian fibration, i.e. G admits a left adjoint.

If a functor $G : \mathcal{D} \times \mathcal{E} \to \mathcal{C}$ belongs to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D} \times \mathcal{E},\mathcal{C})$, the functors

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{P}(\mathcal{C}^{\mathrm{op}}) \times \mathrm{Fun}(\mathcal{D},\mathcal{C}) \to \mathcal{P}(\mathcal{D}^{\mathrm{op}})$$

and

$$\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{P}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \times \mathrm{Fun}(\mathcal{E},\mathcal{C})^{\mathrm{op}} \to \mathcal{P}(\mathcal{E}^{\mathrm{op}})^{\mathrm{op}}$$

induce functors $\beta : \mathcal{C}^{\mathrm{op}} \times \mathcal{E} \to \mathcal{D}^{\mathrm{op}}$ respectively $\alpha : \mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{E}$ so that we have equivalences $\mathcal{E}(\alpha(X,Y),Z) \simeq \mathcal{D}^{\mathrm{op}}(Y,\beta(X,Z)) \simeq \mathcal{C}(X,\gamma(Y,Z))$ natural in $X \in \mathcal{C}, Y \in \mathcal{D}, Z \in \mathcal{E}$.

Let $\mathcal{O}^{\otimes}$ be an operad and $\mathcal{C}^{\otimes},\mathcal{D}^{\otimes},\mathcal{E}^{\otimes}$ be $\mathcal{O}^{\otimes}$-monoidal categories.
We introduce the following abbreviations:

Denote

- $\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})^{\otimes} \subset \mathrm{Fun}(\mathcal{D},\mathcal{C})^{\otimes} \subset \mathcal{P}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{C})^{\otimes}$ the full suboperad spanned by the objects that belong to

  $$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}_X,\mathcal{C}_X) \subset \mathrm{Fun}(\mathcal{D}_X,\mathcal{C}_X) \subset \mathrm{Fun}(\mathcal{D}_X,\mathcal{P}(\mathcal{C}_X)) \simeq \mathcal{P}((\mathcal{D}_X)^{\mathrm{op}}\times\mathcal{C}_X) \simeq \mathcal{P}(\mathcal{D}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{C})^{\otimes}_X$$

  for some $X \in \mathcal{O}$
- $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})^{\otimes} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}},\mathcal{D}^{\mathrm{rev}})^{\otimes}$
- $\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}_{\mathcal{O}}(\mathcal{D},\mathcal{C}) := \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})) \subset \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}(\mathcal{D},\mathcal{C})) \simeq \mathrm{Fun}^{\otimes,\mathrm{lax}}_{\mathcal{O}}(\mathcal{D},\mathcal{C})$

- $\mathrm{Fun}^{\otimes,\mathrm{oplax},\mathrm{L}}_{\mathcal{O}}(\mathcal{D},\mathcal{C}) := \mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}_{\mathcal{O}}(\mathcal{D}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} = \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}}))^{\mathrm{op}} = \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{L}}(\mathcal{D},\mathcal{C}))^{\mathrm{op}} \subset \mathrm{Fun}^{\otimes,\mathrm{lax}}_{\mathcal{O}}(\mathcal{D}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} = \mathrm{Fun}^{\otimes,\mathrm{oplax}}_{\mathcal{O}}(\mathcal{D},\mathcal{C})$.

- $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})^{\otimes} \subset \mathrm{Fun}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})^{\otimes}$ the full suboperad spanned by the objects that belong to $\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}_{\mathrm{X}}\times\mathcal{D}_{\mathrm{X}},\mathcal{C}_{\mathrm{X}}) \subset \mathrm{Fun}(\mathcal{E}_{\mathrm{X}}\times\mathcal{D}_{\mathrm{X}},\mathcal{C}_{\mathrm{X}}) \simeq \mathrm{Fun}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})^{\otimes}_{\mathrm{X}}$ for some $\mathrm{X}\in\mathcal{O}$
- $\mathrm{Fun}^{\mathrm{L}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})^{\otimes} := \mathrm{Fun}^{\mathrm{R}}(\mathcal{E}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}})^{\otimes}$
- $\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}_{\mathcal{O}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C}) := \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})) \subset \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})) \simeq \mathrm{Fun}^{\otimes,\mathrm{lax}}_{\mathcal{O}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})$
- $\mathrm{Fun}^{\otimes,\mathrm{oplax},\mathrm{L}}_{\mathcal{O}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C}) := \mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}_{\mathcal{O}}(\mathcal{E}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} = \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{L}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C}))^{\mathrm{op}} \subset \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}(\mathcal{E}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}}))^{\mathrm{op}} \simeq \mathrm{Fun}^{\otimes,\mathrm{oplax}}_{\mathcal{O}}(\mathcal{E}\times_{\mathcal{O}}\mathcal{D},\mathcal{C})$

Proposition 6.35.

Let $\mathcal{O}^{\otimes}$ be an operad and $\mathcal{C}^{\otimes},\mathcal{D}^{\otimes},\mathcal{E}^{\otimes}$ be $\mathcal{O}^{\otimes}$-monoidal categories.

There are canonical equivalences

$$\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}_{\mathcal{O}}(\mathcal{D},\mathcal{C}) \simeq \mathrm{Fun}^{\otimes,\mathrm{oplax},\mathrm{L}}_{\mathcal{O}}(\mathcal{C},\mathcal{D})^{\mathrm{op}}$$

and

$$\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{D}) \simeq \mathrm{Fun}^{\otimes,\mathrm{oplax},\mathrm{L}}(\mathcal{C}\times_{\mathcal{O}}\mathcal{D},\mathcal{E})^{\mathrm{op}} \simeq \mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{C}).$$

Proof. The canonical equivalence

$$(\mathcal{D}^{\otimes})^{\mathrm{rev}}\times_{\mathcal{O}^{\otimes}}\mathcal{C}^{\otimes} \simeq \mathcal{C}^{\otimes}\times_{\mathcal{O}^{\otimes}}(\mathcal{D}^{\otimes})^{\mathrm{rev}}$$

of $\mathcal{O}^{\otimes}$-monoidal categories induces an equivalence

$$\mathcal{P}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{C})^{\otimes} \simeq \mathcal{P}(\mathcal{C}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}})^{\otimes}$$

of $\mathcal{O}^{\otimes}$-monoidal categories.

For every $\mathrm{X}\in\mathcal{O}$ the categories

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}_{\mathrm{X}},\mathcal{C}_{\mathrm{X}}) \subset \mathrm{Fun}(\mathcal{D}_{\mathrm{X}},\mathcal{P}(\mathcal{C}_{\mathrm{X}})) \simeq \mathcal{P}((\mathcal{D}_{\mathrm{X}})^{\mathrm{op}}\times\mathcal{C}_{\mathrm{X}})$$

and

$$\mathrm{Fun}^{\mathrm{R}}((\mathcal{C}_{\mathrm{X}})^{\mathrm{op}},(\mathcal{D}_{\mathrm{X}})^{\mathrm{op}}) \subset \mathrm{Fun}((\mathcal{C}_{\mathrm{X}})^{\mathrm{op}},\mathcal{P}((\mathcal{D}_{\mathrm{X}})^{\mathrm{op}})) \simeq \mathcal{P}(\mathcal{C}_{\mathrm{X}}\times(\mathcal{D}_{\mathrm{X}})^{\mathrm{op}})$$

correspond to the full subcategory of

$$\mathcal{P}((\mathcal{D}_{\mathrm{X}})^{\mathrm{op}}\times\mathcal{C}_{\mathrm{X}}) \simeq \mathcal{P}(\mathcal{C}_{\mathrm{X}}\times(\mathcal{D}_{\mathrm{X}})^{\mathrm{op}})$$

spanned by the presheaves that are representable in both variables.

So the equivalence $\mathcal{P}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{C})^{\otimes} \simeq \mathcal{P}(\mathcal{C}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}})^{\otimes}$ restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})^{\otimes} \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}},\mathcal{D}^{\mathrm{rev}})^{\otimes} = \mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})^{\otimes}$$

of operads over $\mathcal{O}^{\otimes}$ that yields an equivalence

$$\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}_{\mathcal{O}}(\mathcal{D},\mathcal{C}) = \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{D},\mathcal{C})) \simeq \mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{L}}(\mathcal{C},\mathcal{D})) =$$

$$\mathrm{Fun}_{\mathcal{O}}^{\otimes,\mathrm{oplax},\mathrm{L}}(\mathcal{C},\mathcal{D})^{\mathrm{op}}.$$

The canonical equivalence

$$\mathcal{C}\times_{\mathcal{O}}(\mathcal{E}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D})\simeq\mathcal{C}\times_{\mathcal{O}}(\mathcal{D}\times_{\mathcal{O}}\mathcal{E}^{\mathrm{rev}})\simeq(\mathcal{D}\times_{\mathcal{O}}\mathcal{E}^{\mathrm{rev}})\times_{\mathcal{O}}\mathcal{C}$$

of $\mathcal{O}^{\otimes}$-monoidal categories yields an equivalence

$$\mathcal{P}(\mathcal{C}\times_{\mathcal{O}}(\mathcal{E}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}))\simeq\mathcal{P}(\mathcal{C}\times_{\mathcal{O}}(\mathcal{D}\times_{\mathcal{O}}\mathcal{E}^{\mathrm{rev}}))\simeq\mathcal{P}((\mathcal{D}\times_{\mathcal{O}}\mathcal{E}^{\mathrm{rev}})\times_{\mathcal{O}}\mathcal{C})$$

of $\mathcal{O}^{\otimes}$-monoidal categories that induces on the fiber over every $X\in\mathcal{O}$ the canonical equivalence

$$\mathcal{P}(\mathcal{C}_X\times((\mathcal{E}_X)^{\mathrm{op}}\times\mathcal{D}_X))\simeq\mathcal{P}(\mathcal{C}_X\times(\mathcal{D}_X\times(\mathcal{E}_X)^{\mathrm{op}}))\simeq$$
$$\mathcal{P}((\mathcal{D}_X\times(\mathcal{E}_X)^{\mathrm{op}})\times\mathcal{C}_X).$$

The full subcategories

$$\mathrm{Fun}^{\mathrm{R}}((\mathcal{C}_X)^{\mathrm{op}}\times\mathcal{E}_X,\mathcal{D}_X)\simeq\mathrm{Fun}^{\mathrm{R}}((\mathcal{C}_X)^{\mathrm{op}}\times(\mathcal{D}_X)^{\mathrm{op}},(\mathcal{E}_X)^{\mathrm{op}})\simeq$$
$$\mathrm{Fun}^{\mathrm{R}}((\mathcal{D}_X)^{\mathrm{op}}\times\mathcal{E}_X,\mathcal{C}_X)$$

correspond to the full subcategory of

$$\mathcal{P}(\mathcal{C}_X\times((\mathcal{E}_X)^{\mathrm{op}}\times\mathcal{D}_X))\simeq\mathcal{P}(\mathcal{C}_X\times(\mathcal{D}_X\times(\mathcal{E}_X)^{\mathrm{op}}))\simeq$$
$$\mathcal{P}((\mathcal{D}_X\times(\mathcal{E}_X)^{\mathrm{op}})\times\mathcal{C}_X)$$

spanned by the presheaves on

$$\mathcal{C}_X\times((\mathcal{E}_X)^{\mathrm{op}}\times\mathcal{D}_X)\simeq\mathcal{C}_X\times(\mathcal{D}_X\times(\mathcal{E}_X)^{\mathrm{op}})\simeq$$
$$(\mathcal{D}_X\times(\mathcal{E}_X)^{\mathrm{op}})\times\mathcal{C}_X$$

that are representable in all three variables.

Thus the canonical equivalence

$$\mathcal{P}(\mathcal{C}\times_{\mathcal{O}}(\mathcal{E}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}))\simeq\mathcal{P}(\mathcal{C}\times_{\mathcal{O}}(\mathcal{D}\times_{\mathcal{O}}\mathcal{E}^{\mathrm{rev}}))\simeq\mathcal{P}((\mathcal{D}\times_{\mathcal{O}}\mathcal{E}^{\mathrm{rev}})\times_{\mathcal{O}}\mathcal{C})$$

restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{D})^{\otimes}\simeq\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}\times_{\mathcal{O}}\mathcal{D},\mathcal{E})^{\otimes}=\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}},\mathcal{E}^{\mathrm{rev}})^{\otimes}\simeq$$
$$\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{C})^{\otimes}$$

of operads over $\mathcal{O}^{\otimes}$ that induces a canonical equivalence

$$\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{D})=\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{D}))\simeq$$
$$\mathrm{Fun}^{\otimes,\mathrm{oplax},\mathrm{L}}(\mathcal{C}\times_{\mathcal{O}}\mathcal{D},\mathcal{E})^{\mathrm{op}}\simeq\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{D}^{\mathrm{rev}},\mathcal{E}^{\mathrm{rev}}))\simeq$$
$$\mathrm{Fun}^{\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{C})=\mathrm{Alg}_{/\mathcal{O}}(\mathrm{Fun}^{\mathrm{R}}(\mathcal{D}^{\mathrm{rev}}\times_{\mathcal{O}}\mathcal{E},\mathcal{C})).$$

□

Given cocartesian fibrations $\mathcal{C}\to S,\mathcal{D}\to S$ denote

$$\mathrm{Fun}_S^{\mathrm{R}}(\mathcal{D},\mathcal{C})\subset\mathrm{Fun}_S(\mathcal{D},\mathcal{C})$$

the full subcategory spanned by the functors over S that induce on the fiber over every object of S a right adjoint functor.

Set $\mathrm{Fun}_S^{\mathrm{L}}(\mathcal{C},\mathcal{D}):=\mathrm{Fun}_S^{\mathrm{R}}(\mathcal{C}^{\mathrm{rev}},\mathcal{D}^{\mathrm{rev}})^{\mathrm{op}}$.

Corollary 6.36.

Let S *be a category and* $\mathcal{C} \to \mathrm{S}, \mathcal{D} \to \mathrm{S}$ *be cocartesian fibrations.*

There is a canonical equivalence

$$\mathrm{Fun}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\mathrm{L}}_{\mathrm{S}}(\mathcal{C}, \mathcal{D})^{\mathrm{op}} = \mathrm{Fun}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}}).$$

Proof. Let $\mathcal{O}^{\otimes}$ be the trivial operad associated to S.

Then we have a canonical equivalence

$$\mathrm{Fun}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{D}, \mathcal{C}) \simeq \mathrm{Fun}^{\otimes, \mathrm{lax}, \mathrm{R}}_{\mathcal{O}}(\mathcal{D}, \mathcal{C})$$

and so a canonical equivalence

$$\mathrm{Fun}^{\mathrm{L}}_{\mathrm{S}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})^{\mathrm{op}} \simeq \mathrm{Fun}^{\otimes, \mathrm{lax}, \mathrm{R}}_{\mathcal{O}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{D}^{\mathrm{rev}})^{\mathrm{op}} = \mathrm{Fun}^{\otimes, \mathrm{oplax}, \mathrm{L}}_{\mathcal{O}}(\mathcal{C}, \mathcal{D}).$$

So the assertion follows from proposition 6.35.

□

Definition 6.31 generalizes the concept of an adjunction as a pair of functors $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ and $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ equipped with an equivalence $\mathcal{D}(\mathrm{F}(\mathrm{X}), \mathrm{Y}) \simeq \mathcal{C}(\mathrm{X}, \mathrm{G}(\mathrm{Y}))$ natural in $\mathrm{X} \in \mathcal{C}$ and $\mathrm{Y} \in \mathcal{D}$.

Equivalently an adjunction can be defined as a pair of functors $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ and $\mathrm{G} : \mathcal{D} \to \mathcal{C}$ such that there is a bicartesian fibration $\mathcal{M} \to \Delta^1$ that classifies F as a cocartesian fibration over Δ^1 and classifies G as a cartesian fibration over Δ^1.

In the following we will also generalize this concept to the setting of $\mathcal{O}^{\otimes}$-monoidal categories and show that it gives an alternative description of $\mathcal{O}^{\otimes}$-monoidal adjointness.

Definition 6.37. *Let* $\mathcal{O}^{\otimes}$ *be an operad and* $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{O}^{\otimes}$ *a functor.*

By cor. 6.43 the following conditions are equivalent:

1. γ *is a map of cartesian fibrations over* Δ^1 *classifying a functor* $(\Delta^1)^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ *that corresponds to a lax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.
2. γ *is a map of cocartesian fibrations over* $\mathcal{O}^{\otimes}$ *that classifies a* $\mathcal{O}^{\otimes}$*-monoid* ϕ *of* $\mathsf{Cat}_{\infty/\Delta^1}$ *that sends every* $\mathrm{X} \in \mathcal{O}$ *to a cartesian fibration over* Δ^1.

If there is a functor $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{O}^{\otimes}$ *satisfying condition 1. or 2., we say that* ϕ *classifies the lax* $\mathcal{O}^{\otimes}$*-monoidal functor* $\mathrm{G} : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.

The duality involution $(-)^{\mathrm{op}}$ on Cat_{∞} gives rise to an involution

$$\mathsf{Cat}_{\infty/\Delta^1} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/(\Delta^1)^{\mathrm{op}}} \simeq \mathsf{Cat}_{\infty/\Delta^1}$$

on $\mathsf{Cat}_{\infty/\Delta^1}$, where we use the unique equivalence $\Delta^1 \simeq (\Delta^1)^{\mathrm{op}}$.

Remark 6.38.

Let $\phi : \mathcal{O}^\otimes \to \mathsf{Cat}_{\infty/\Delta^1}$ *be a* $\mathcal{O}^\otimes$*-monoid that sends every* $\mathrm{X} \in \mathcal{O}$ *to a bicartesian fibration corresponding to an adjunction* $\mathrm{H_X} : \mathcal{C}_\mathrm{X} \rightleftarrows \mathcal{D}_\mathrm{X} : \mathrm{G_X}$.

ϕ *induces a* $\mathcal{O}^\otimes$*-monoid*

$$\phi' : \mathcal{O}^\otimes \xrightarrow{\phi} \mathsf{Cat}_{\infty/\Delta^1} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/(\Delta^1)^{\mathrm{op}}} \simeq \mathsf{Cat}_{\infty/\Delta^1}$$

that sends every $\mathrm{X} \in \mathcal{O}$ *to a bicartesian fibration corresponding to the adjunction* $(\mathrm{G_X})^{\mathrm{op}} : (\mathcal{D}_\mathrm{X})^{\mathrm{op}} \rightleftarrows (\mathcal{C}_\mathrm{X})^{\mathrm{op}} : (\mathrm{H_X})^{\mathrm{op}}$.

ϕ *classifies a lax* $\mathcal{O}^\otimes$*-monoidal functor* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ *and* ϕ' *classifies a lax* $\mathcal{O}^\otimes$*-monoidal functor* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$ *corresponding to an oplax* $\mathcal{O}^\otimes$*-monoidal functor* $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *such that for every* $\mathrm{X} \in \mathcal{O}$ *the functor* $\mathrm{F_X} = (\mathrm{F_X^{rev}})^{\mathrm{op}} : \mathcal{C}_\mathrm{X} \to \mathcal{D}_\mathrm{X}$ *is left adjoint to* $\mathrm{G_X}$.

Proof. The functors ϕ and ϕ' are classified by maps $\gamma, \gamma' : \mathcal{M} \to \Delta^1 \times \mathcal{O}^\otimes$ of cocartesian fibrations over $\mathcal{O}^\otimes$.

So γ and γ' are maps of cartesian fibrations over Δ^1 classifying functors $(\Delta^1)^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$ respectively $\Delta^1 \simeq (\Delta^1)^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{O}^\otimes}$ that correspond to lax $\mathcal{O}^\otimes$-monoidal functors $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ and $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$ with the desired properties.

□

Now we are able to state the following characterization of $\mathcal{O}^\otimes$-monoidal adjointness:

Proposition 6.39. *Let* $\mathcal{O}^\otimes$ *be an operad,* $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ *be* $\mathcal{O}^\otimes$*-monoidal categories,* $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *an oplax* $\mathcal{O}^\otimes$*-monoidal functor corresponding to a lax* $\mathcal{O}^\otimes$*-monoidal functor* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$ *and* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ *a lax* $\mathcal{O}^\otimes$*-monoidal functor.*

The following conditions are equivalent:

1. $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *is* $\mathcal{O}^\otimes$*-monoidally left adjoint to* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$.
2. *There is a* $\mathcal{O}^\otimes$*-monoid* $\phi : \mathcal{O}^\otimes \to \mathsf{Cat}_{\infty/\Delta^1}$ *in* $\mathsf{Cat}_{\infty/\Delta^1}$ *that sends every* $\mathrm{X} \in \mathcal{O}$ *to a bicartesian fibration over* Δ^1 *such that* $\phi : \mathcal{O}^\otimes \to \mathsf{Cat}_{\infty/\Delta^1}$ *classifies the lax* $\mathcal{O}^\otimes$*-monoidal functor* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ *and*

$$\phi' : \mathcal{O}^\otimes \xrightarrow{\phi} \mathsf{Cat}_{\infty/\Delta^1} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/(\Delta^1)^{\mathrm{op}}}$$

classifies the lax $\mathcal{O}^\otimes$*-monoidal functor* $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$.

Proof. The lax $\mathcal{O}^\otimes$-monoidal functor $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$ gives rise to a $\mathcal{O}^\otimes$-monoid $\psi : \mathcal{O}^\otimes \to \mathsf{Cat}_{\infty/\Delta^1}$ that sends every $\mathrm{X} \in \mathcal{O}$ to a bicartesian fibration over Δ^1 such that $\psi : \mathcal{O}^\otimes \to \mathsf{Cat}_{\infty/\Delta^1}$ classifies the lax $\mathcal{O}^\otimes$-monoidal functor $(\mathrm{F}^{\mathrm{rev}})^* : \mathcal{P}(\mathcal{D})^\otimes \to \mathcal{P}(\mathcal{C})^\otimes$ and

$$\mathcal{O}^\otimes \xrightarrow{\psi} \mathsf{Cat}_{\infty/\Delta^1} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/(\Delta^1)^{\mathrm{op}}}$$

classifies the lax $\mathcal{O}^\otimes$-monoidal functor $\mathcal{P}(\mathrm{F})^{\mathrm{rev}} : (\mathcal{P}(\mathcal{C})^\otimes)^{\mathrm{rev}} \to (\mathcal{P}(\mathcal{D})^\otimes)^{\mathrm{rev}}$.

Moreover the lax $\mathcal{O}^\otimes$-monoidal functor $\mathcal{P}(\mathrm{F})^{\mathrm{rev}} : (\mathcal{P}(\mathcal{C})^\otimes)^{\mathrm{rev}} \to (\mathcal{P}(\mathcal{D})^\otimes)^{\mathrm{rev}}$ restricts to the lax $\mathcal{O}^\otimes$-monoidal functor $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$.

If 1. holds, the lax $\mathcal{O}^\otimes$-monoidal functor $(\mathrm{F}^{\mathrm{rev}})^* : \mathcal{P}(\mathcal{D})^\otimes \to \mathcal{P}(\mathcal{C})^\otimes$ restricts to the lax $\mathcal{O}^\otimes$-monoidal functor $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$.

Consequently $\psi : \mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/\Delta^1}$ induces the desired $\mathcal{O}^{\otimes}$-monoid $\phi : \mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/\Delta^1}$. Hence 1. implies 2.

Conversely assume that 2. holds.

Then F admits a $\mathcal{O}^{\otimes}$-monoidal right adjoint $G' : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ and by the first part of the proof F and G' are classified by a $\mathcal{O}^{\otimes}$-monoid $\varphi : \mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/\Delta^1}$ that sends every $X \in \mathcal{O}$ to a bicartesian fibration over Δ^1.

Hence ϕ and φ both classify F and thus are equivalent so that $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ and $G' : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ are equivalent lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$. This shows 1.

□

We have a canonical equivalence $(\mathsf{Cat}_{\infty}^{\mathrm{R}})^{\mathrm{op}} \simeq \mathsf{Cat}_{\infty}^{\mathrm{L}}$, under which a right adjoint functor corresponds to its left adjoint.
This equivalence is represented by the canonical equivalence

$$\mathrm{Fun}(S, (\mathsf{Cat}_{\infty}^{\mathrm{R}})^{\mathrm{op}})^{\simeq} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathsf{Cat}_{\infty}^{\mathrm{R}})^{\simeq} \simeq (\mathsf{Cat}_{\infty/S}^{\mathrm{bicart}})^{\simeq} \simeq \mathrm{Fun}(S, \mathsf{Cat}_{\infty}^{\mathrm{L}})^{\simeq}.$$

The canonical involution $(-)^{\mathrm{op}}$ on Cat_{∞} restricts to an equivalence $\mathsf{Cat}_{\infty}^{\mathrm{L}} \simeq \mathsf{Cat}_{\infty}^{\mathrm{R}}$ so that we obtain a canonical equivalence $(\mathsf{Cat}_{\infty}^{\mathrm{R}})^{\mathrm{op}} \simeq \mathsf{Cat}_{\infty}^{\mathrm{R}}$.

In the following we will construct a $\mathcal{O}^{\otimes}$-monoidal version of this equivalence for every operad $\mathcal{O}^{\otimes}$.

To do so, we need some notation:

Let $\mathcal{O}^{\otimes}$ be an operad and S a category.

Denote

- $\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})^{\mathrm{R}} \subset \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})$ the wide subcategory with morphisms the lax $\mathcal{O}^{\otimes}$-monoidal functors that induce on the fiber over every object of $\mathcal{O}$ a right adjoint functor
- $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})^{\mathrm{cart}}$, $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})^{\mathrm{bicart}} \subset \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})$ the subcategories spanned by the $\mathcal{O}^{\otimes}$-monoids of $\mathsf{Cat}_{\infty/S}$ that send every $X \in \mathcal{O}$ to a cartesian respectively bicartesian fibration over S and with morphisms the natural transformations, whose components are maps of cartesian respectively bicartesian fibrations over S
- $\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty}) \subset \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}$ the full subcategory spanned by the $\mathcal{O}^{\otimes}$-monoidal categories
- $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty}))^{\mathrm{cocart}} \subset \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty}))$ the wide subcategory with morphisms the natural transformations $\alpha : F \to G$ such that for all $\mathsf{s} \in S$ the component $\alpha(\mathsf{s}) : F(\mathsf{s}) \to G(\mathsf{s})$ is a $\mathcal{O}^{\otimes}$-monoidal functor
- $\mathrm{Fun}^{\mathrm{lad}}(S^{\mathrm{op}}, \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})) \subset \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty}))^{\mathrm{cocart}}$ the subcategory with objects
  - the functors $S^{\mathrm{op}} \to \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})$ such that for every $X \in \mathcal{O}$ the composition $S^{\mathrm{op}} \to \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty}) \to \mathsf{Cat}_{\infty}$ sends every morphism of S to a right adjoint functor
  - and with morphisms the natural transformations such that for every $X \in \mathcal{O}$ the induced natural transformation of functors $S^{\mathrm{op}} \to \mathsf{Cat}_{\infty}$ sends every morphism of S to a left adjointable square (classified by a map of bicartesian fibrations over Δ^1).

Proposition 6.40. *Let $\mathcal{O}^\otimes$ be an operad and S a category.*

1. *There is a canonical equivalence*
$$\mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)) \simeq \mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)).$$

2. *This equivalence induces an equivalence*
$$\alpha : \mathrm{Fun}(\mathrm{S}, (\mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\mathrm{op}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\simeq} \simeq$$
$$\mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq} \simeq \mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq} \simeq$$
$$\mathrm{Fun}(\mathrm{S}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\simeq}$$
that represents an equivalence
$$(\mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}}$$
that sends a lax $\mathcal{O}^\otimes$-monoidal functor $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ to the lax $\mathcal{O}^\otimes$-monoidal functor $\mathrm{F}^{\mathrm{rev}} : (\mathcal{C}^\otimes)^{\mathrm{rev}} \to (\mathcal{D}^\otimes)^{\mathrm{rev}}$ representing the oplax $\mathcal{O}^\otimes$-monoidal left adjoint $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ of G.

3. *For $\mathcal{O}^\otimes$ the trivial operad the equivalence of 2. is the canonical equivalence*
$$(\mathsf{Cat}^{\mathrm{R}}_\infty)^{\mathrm{op}} \simeq \mathsf{Cat}^{\mathrm{R}}_\infty.$$

Proof. 1: By lemma 6.43 there is a canonical equivalence
$$\mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\mathrm{cocart}} \simeq \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{cart}}$$
that restricts to an equivalence
$$\mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{bicart}}.$$

The duality involution on Cat_∞ yields an equivalence $\mathsf{Cat}_{\infty/\mathrm{S}} \simeq \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ and so an equivalence
$$\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}}) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}})$$
that restricts to an equivalence $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{bicart}} \simeq \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}})^{\mathrm{bicart}}$.
So we get a canonical equivalence
$$\mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)) \simeq \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{bicart}} \simeq \mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}})^{\mathrm{bicart}}$$
$$\simeq \mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)).$$

2: The full subspaces
$$\mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\simeq} \subset \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq},$$
$$\mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq} \subset \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq}$$
coincide.
So we get an equivalence
$$\alpha : \mathrm{Fun}(\mathrm{S}, (\mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\mathrm{op}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\simeq} \simeq$$
$$\mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq} \simeq \mathrm{Fun}^{\mathrm{lad}}(\mathrm{S}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty))^{\simeq} \simeq$$
$$\mathrm{Fun}(\mathrm{S}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\simeq}$$

that represents an equivalence

$$(\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_\infty)^{\mathrm{R}})^{\mathrm{op}} \simeq \mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_\infty)^{\mathrm{R}}.$$

3: The canonical equivalence $(\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}} \simeq \mathsf{Cat}_\infty^{\mathrm{L}} \simeq \mathsf{Cat}_\infty^{\mathrm{R}}$ is represented by the canonical equivalence

$$\mathrm{Fun}(\mathrm{S}, (\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_\infty^{\mathrm{R}})^{\simeq} \simeq (\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{bicart}})^{\simeq} \simeq$$

$$\mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty^{\mathrm{L}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty^{\mathrm{R}})^{\simeq}$$

that is equivalent to the equivalence

$$\mathrm{Fun}(\mathrm{S}, (\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_\infty^{\mathrm{R}})^{\simeq} \simeq (\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{bicart}})^{\simeq} \xrightarrow{(-)^{\mathrm{op}}}$$

$$(\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}^{\mathrm{bicart}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty^{\mathrm{R}})^{\simeq}$$

as the equivalence

$$\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty) \xrightarrow{(-)^{\mathrm{op}}} \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty)$$

factors as

$$\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}^{\mathrm{cart}} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty).$$

For $\mathcal{O}^{\otimes}$ the trivial operad α is the canonical equivalence

$$\mathrm{Fun}(\mathrm{S}, (\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_\infty^{\mathrm{R}})^{\simeq} \simeq (\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{bicart}})^{\simeq} \xrightarrow{(-)^{\mathrm{op}}}$$

$$(\mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}^{\mathrm{bicart}})^{\simeq} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_\infty^{\mathrm{R}})^{\simeq}$$

and so represents the canonical equivalence $(\mathsf{Cat}_\infty^{\mathrm{R}})^{\mathrm{op}} \simeq \mathsf{Cat}_\infty^{\mathrm{R}}$.

□

For the proof of proposition 6.40 we needed the following lemmata and their corollaries:

Lemma 6.41. *Let* S, T *be categories and* $\mathcal{C} \to \mathrm{S} \times \mathrm{T}$ *a map of locally cocartesian fibrations over* T.

Then for all objects $\mathrm{t} \in \mathrm{T}$ *the functor* $\mathcal{C}_\mathrm{t} \to \mathcal{C}$ *over* S *preserves cartesian morphisms.*

Dually, let $\mathcal{C} \to \mathrm{S} \times \mathrm{T}$ *be a map of locally cartesian fibrations over* T.

Then for all objects $\mathrm{t} \in \mathrm{T}$ *the functor* $\mathcal{C}_\mathrm{t} \to \mathcal{C}$ *over* S *preserves cocartesian morphisms.*

Proof. Let $\mathsf{f} : \mathrm{X} \to \mathrm{Y}$ be a cartesian morphism with respect to the functor $\mathcal{C}_\mathrm{t} \to \mathrm{S}$ lying over a morphism $\mathsf{s} \to \mathsf{s}'$ of S.

Let Z be an object of $\mathcal{C}$ lying over an object t' of T and s'' of S.

For 1) we have to show that the commutative square

$$\begin{array}{ccc} \mathcal{C}(\mathrm{Z}, \mathrm{X}) & \longrightarrow & \mathcal{C}(\mathrm{Z}, \mathrm{Y}) \\ \downarrow & & \downarrow \\ \mathrm{S}(\mathsf{s}'', \mathsf{s}) \times \mathrm{T}(\mathrm{t}', \mathrm{t}) & \longrightarrow & \mathrm{S}(\mathsf{s}'', \mathsf{s}') \times \mathrm{T}(\mathrm{t}', \mathrm{t}) \end{array} \tag{38}$$

of spaces is a pullback square.

Considering square 38 as a square of spaces over $T(t', t)$ it is enough to see that square 38 induces on the fiber over every object $\varphi \in T(t', t)$ a pullback square.

Using that the functor $\mathcal{C} \to T$ is a locally cocartesian fibration, whose cocartesian morphisms get equivalences in S, the fiber of square 38 over an object $\varphi \in T(t', t)$ is the following commutative square of spaces:

$$\begin{array}{ccc} \mathcal{C}_t(\varphi_*(Z), X) & \longrightarrow & \mathcal{C}_t(\varphi_*(Z), Y) \\ \downarrow & & \downarrow \\ S(s'', s) & \longrightarrow & S(s'', s') \end{array}$$

But this square is a pullback square because $f : X \to Y$ is a cartesian morphism with respect to the functor $\mathcal{C}_t \to S$.

□

Corollary 6.42.

1. *Let* $\mathcal{C} \to S \times T$ *be a functor corresponding to a functor* $\mathcal{C} \to S \times T$ *over* T *and a functor* $\mathcal{C} \to S \times T$ *over* S *and* $\mathcal{E} \subset \mathrm{Fun}(\Delta^1, S)$ *a full subcategory.*

   If $\mathcal{C} \to S \times T$ *is a map of (locally) cocartesian fibrations over* T *which induces on the fiber over every* $t \in T$ *a cartesian fibration* $\mathcal{C}_t \to S$ *relative to* $\mathcal{E}$*, then* $\mathcal{C} \to S \times T$ *is a map of cartesian fibrations relative to* $\mathcal{E}$ *which induces on the fiber over every* $s \in S$ *a (locally) cocartesian fibration* $\mathcal{C}_s \to T$.

   Dually, if $\mathcal{C} \to S \times T$ *is a map of (locally) cartesian fibrations over* T *which induces on the fiber over every* $t \in T$ *a cocartesian fibration* $\mathcal{C}_t \to S$ *relative to* $\mathcal{E}$*, then* $\mathcal{C} \to S \times T$ *is a map of cocartesian fibrations relative to* $\mathcal{E}$ *which induces on the fiber over every* $s \in S$ *a (locally) cartesian fibration* $\mathcal{C}_s \to T$.

2. *Let* $\mathcal{C} \to S \times T$ *be a functor corresponding to a functor* $\mathcal{C} \to S \times T$ *over* T *and a functor* $\mathcal{C} \to S \times T$ *over* S.
   Then the following two conditions are equivalent:
   (a) $\mathcal{C} \to S \times T$ *is a map of cocartesian fibrations over* T *which induces on the fiber over every* $t \in T$ *a cartesian fibration* $\mathcal{C}_t \to S$.
   (b) $\mathcal{C} \to S \times T$ *is a map of cartesian fibrations over* S *which induces on the fiber over every* $s \in S$ *a cocartesian fibration* $\mathcal{C}_s \to T$.

3. *Let* $\mathcal{C} \to S \times T, \mathcal{D} \to S \times T$ *be functors satisfying the equivalent conditions of 2. and let* $\mathcal{C} \to \mathcal{D}$ *be a functor over* $S \times T$.
   Then the following two conditions are equivalent:
   (a) $\mathcal{C} \to \mathcal{D}$ *is a map of cocartesian fibrations over* T *which induces on the fiber over every* $t \in T$ *a map of cartesian fibrations over* S.
   (b) $\mathcal{C} \to \mathcal{D}$ *is a map of cartesian fibrations over* S *which induces on the fiber over every* $s \in S$ *a map of cocartesian fibrations over* T.

4. *Consequently the following two subcategories of* $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ *coincide:*

   The subcategory with objects the functors $\mathcal{C} \to \mathrm{S} \times \mathrm{T}$ *satisfying the condition of 2. (a) and with morphisms the functors* $\mathcal{C} \to \mathcal{D}$ *over* $\mathrm{S} \times \mathrm{T}$ *satisfying the condition of 3. (a).*

   The subcategory with objects the functors $\mathcal{C} \to \mathrm{S} \times \mathrm{T}$ *satisfying the condition of 2. (b) and with morphisms the functors* $\mathcal{C} \to \mathcal{D}$ *over* $\mathrm{S} \times \mathrm{T}$ *satisfying the condition of 3. (b).*

5. *Given categories* $\mathcal{C}, \mathcal{D}$ *denote* $\mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_{\infty/\mathcal{D}})^{\mathrm{cocart}}$ *the subcategory of* $\mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_{\infty/\mathcal{D}})$ *with objects the functors* $\mathcal{C} \to \mathsf{Cat}_{\infty/\mathcal{D}}$ *that send every object of* $\mathcal{C}$ *to a cocartesian fibration over* $\mathcal{D}$ *and with morphisms the natural transformations of functors* $\mathcal{C} \to \mathsf{Cat}_{\infty/\mathcal{D}}$ *whose components are maps of cocartesian fibrations over* $\mathcal{D}$.
   Similarly we define $\mathrm{Fun}(\mathcal{C}, \mathsf{Cat}_{\infty/\mathcal{D}})^{\mathrm{cart}}$.

   The category $\mathrm{Fun}(\mathrm{T}, \mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{cart}}$ *is equivalent to the first subcategory of* $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ *of 4., the category* $\mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{cocart}}$ *is equivalent to the second subcategory of* $\mathsf{Cat}_{\infty/\mathrm{S}\times\mathrm{T}}$ *of 4.*
   Thus we obtain a canonical equivalence

$$\mathrm{Fun}(\mathrm{T}, \mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{cart}} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{cocart}}.$$

   By composing the last equivalence with the equivalence $\mathsf{Cat}_{\infty/\mathrm{S}} \simeq \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}}$ *induced by the duality involution on* Cat_{∞} *(and replacing* S *by* S^{op} *) we get canonical equivalences*

$$\mathrm{Fun}(\mathrm{T}, \mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{cocart}} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_{\infty/\mathrm{T}})^{\mathrm{cocart}}$$

   and

$$\mathrm{Fun}(\mathrm{T}, \mathsf{Cat}_{\infty/\mathrm{S}^{\mathrm{op}}})^{\mathrm{cart}} \simeq \mathrm{Fun}(\mathrm{S}, \mathsf{Cat}_{\infty/\mathrm{T}^{\mathrm{op}}})^{\mathrm{cart}}.$$

Proof. Let $\psi : \mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ be an object of $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{cart}}$ and $\mathrm{H} : \mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ be an object of $\mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}})^{\mathrm{cocart}}$ that correspond under the canonical equivalence

$$\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat}_{\infty/\mathrm{S}})^{\mathrm{cart}} \simeq \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}})^{\mathrm{cocart}}$$

of corollary 6.42.

Then there is a functor $\gamma : \mathcal{C} \to \mathcal{O}^{\otimes} \times \mathrm{S}$ that is a map of cocartesian fibrations over $\mathcal{O}^{\otimes}$ classifying $\mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ and is a map of cartesian fibrations over S classifying $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$.

We have to see that $\mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/\mathrm{S}}$ is a $\mathcal{O}^{\otimes}$-monoid object of $\mathsf{Cat}_{\infty/\mathrm{S}}$ if and only if $\mathrm{S}^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$ factors through the subcategory $\mathrm{Mon}_{\mathcal{O}}^{\mathrm{lax}}(\mathsf{Cat}_{\infty})$ of $\mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}}$.

Let $\mathrm{n} \in \mathbb{N}$ and let for every $\mathrm{i} \in \{1, ..., \mathrm{n}\}$ an inert morphism $\mathrm{X} \to \mathrm{X}_{\mathrm{i}}$ of $\mathcal{O}^{\otimes}$ be given lying over the unique inert morphism $\langle \mathrm{n} \rangle \to \langle 1 \rangle$ of $\mathcal{F}\mathrm{in}_{*}$ that sends i to 1.

Then the following two conditions are equivalent:

1. The induced functors $\psi(\mathrm{X}) \to \psi(\mathrm{X}_{\mathrm{i}})$ over S for $\mathrm{i} \in \{1, ..., \mathrm{n}\}$ form a product diagram in $\mathsf{Cat}_{\infty/\mathrm{S}}$.
2. Each of the functors $\psi(\mathrm{X}) \to \psi(\mathrm{X}_{\mathrm{i}})$ is a map of cartesian fibrations over S and for every $\mathsf{s} \in \mathrm{S}$ the induced functors $\psi(\mathrm{X})_{\mathsf{s}} \to \psi(\mathrm{X}_{\mathrm{i}})_{\mathsf{s}}$ on the fiber over s form a product diagram.

By the naturality of the canonical equivalence $\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat}_{\infty/S})^{\mathrm{cart}} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}})^{\mathrm{cocart}}$ the induced functor $\psi(X)_{\mathsf{s}} \to \psi(X_i)_{\mathsf{s}}$ on the fiber over s is classified by $H(\mathsf{s})_X \to H(\mathsf{s})_{X_i}$.

Consequently it is enough to show that for every morphism $h : t \to \mathsf{s}$ of S and every inert morphism $\mathsf{f} : X \to Y$ of $\mathcal{O}^{\otimes}$ the following two conditions are equivalent:

1. The induced functor $\psi(\mathsf{f}) : \psi(X) \to \psi(Y)$ over S preserves cartesian morphisms lying over the morphism $h : t \to \mathsf{s}$.
2. The induced functor $H(h) : H(\mathsf{s}) \to H(t)$ over $\mathcal{O}^{\otimes}$ preserves cocartesian morphisms lying over the morphism $\mathsf{f} : X \to Y$.

This follows from lemma 6.44.

□

Let S be a category and $\mathcal{O}^{\otimes}$ an operad.

Denote $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})^{\mathrm{cart}}$ the subcategory of $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})$ with objects the $\mathcal{O}^{\otimes}$-monoids of $\mathsf{Cat}_{\infty/S}$ that send every object X of $\mathcal{O}$ to a cartesian fibration over S and with morphisms the natural transformations of functors $\mathcal{O}^{\otimes} \to \mathsf{Cat}_{\infty/S}$, whose components on objects of $\mathcal{O}$ are maps of cartesian fibrations over S. Let $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})^{\mathrm{cocart}}$ be defined similarly.

Denote $\mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_{\infty}))^{\mathrm{cocart}} \subset \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_{\infty}))$ the subcategory with the same objects and with morphisms the natural transformations of functors $S^{\mathrm{op}} \to \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_{\infty})$, whose components are $\mathcal{O}^{\otimes}$-monoidal functors.

Corollary 6.43. *Let* S *be a category and* $\mathcal{O}^{\otimes}$ *an operad.*

The canonical equivalence

$$\mathrm{Fun}(\mathcal{O}^{\otimes}, \mathsf{Cat}_{\infty/S})^{\mathrm{cart}} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathsf{Cat}_{\infty/\mathcal{O}^{\otimes}})^{\mathrm{cocart}}$$

of corollary 6.42 restricts to an equivalence

$$\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})^{\mathrm{cart}} \simeq \mathrm{Fun}(S^{\mathrm{op}}, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_{\infty}))^{\mathrm{cocart}}.$$

By composing this equivalence with the equivalence $\mathsf{Cat}_{\infty/S} \simeq \mathsf{Cat}_{\infty/S^{\mathrm{op}}}$ *induced by the duality involution on* Cat_{∞} (*and replacing* S *by* S^{op}) *we get an equivalence*

$$\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_{\infty/S})^{\mathrm{cocart}} \simeq \mathrm{Fun}(S, \mathrm{Mon}^{\mathrm{lax}}_{\mathcal{O}}(\mathsf{Cat}_{\infty}))^{\mathrm{cocart}}.$$

Lemma 6.44. *Let* $S, T, \mathcal{C}$ *be categories and* $p : \mathcal{C} \to T,\ q : \mathcal{C} \to S$ *functors.*
Assume that the functor $\rho = (p, q) : \mathcal{C} \to T \times S$ *is a map of cocartesian fibrations over* T, *which is fiberwise a cartesian fibration, classifying a functor* $\psi : T \to \mathsf{Cat}_{\infty/S}$.
By corollary 6.42 $\varrho = (q, p) : \mathcal{C} \to S \times T$ *is a map of cartesian fibrations over* S, *which is fiberwise a cocartesian fibration, classifying a functor* $H : S^{\mathrm{op}} \to \mathsf{Cat}_{\infty/T}$.

Let $H(h)(B) \to B$ *and* $H(h)(\psi(\mathsf{f})(B)) \to \psi(\mathsf{f})(B)$ *be* q*-cartesian lifts of the morphism* $h : \mathsf{s} \to \mathsf{s}'$ *of* S *and let* $H(h)(B) \to \psi(\mathsf{f})(H(h)(B))$ *and* $B \to \psi(\mathsf{f})(B)$ *be* p*-cocartesian lifts of the morphism* $\mathsf{f} : t' \to t$ *of* T.

1. *The morphisms*

$$\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})) \to \psi(\mathsf{f})(\mathrm{B}),$$
$$\mathrm{H}(\mathrm{h})(\mathrm{B}) \to \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))$$

   induce the same morphism $\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})) \to \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))$ *in the fiber* $\psi(\mathrm{t})_{\mathsf{s}} \simeq \mathrm{H}(\mathsf{s})_{\mathrm{t}}$.

   Consequently the morphism $\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})) \to \psi(\mathsf{f})(\mathrm{B})$ *is* q*-cartesian if and only if* $\mathrm{H}(\mathrm{h})(\mathrm{B}) \to \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))$ *is* p*-cocartesian.*

2. *This implies the following:*

   The functor $\psi(\mathsf{f}) : \psi(\mathrm{t}') \to \psi(\mathrm{t})$ *sends* $\rho_{\mathrm{t}'}$*-cartesian lifts of* $\mathrm{h} : \mathsf{s} \to \mathsf{s}'$ *to* ρ_{t}*-cartesian morphisms if and only if* $\mathrm{H}(\mathrm{h}) : \mathrm{H}(\mathsf{s}) \to \mathrm{H}(\mathsf{s}')$ *sends* ϱ_{s}*-cocartesian lifts of* $\mathsf{f} : \mathrm{t}' \to \mathrm{t}$ *to* $\varrho_{\mathsf{s}'}$*-cocartesian morphisms.*

Proof. Denote β the composition $\mathrm{H}(\mathrm{h})(\mathrm{B}) \to \mathrm{B} \to \psi(\mathsf{f})(\mathrm{B})$ of morphisms of $\mathcal{C}$ so that β lies over f and h.

By definition the morphism $\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})) \to \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))$ in the fiber $\psi(\mathrm{t})_{\mathsf{s}} \simeq \mathrm{H}(\mathsf{s})_{\mathrm{t}} \simeq \mathcal{C}_{\mathrm{t},\mathsf{s}}$ induced by $\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})) \to \psi(\mathsf{f})(\mathrm{B})$ corresponds to β under the top horizontal functor of the following diagram of pullback squares:

$$\begin{array}{ccccc}
\mathcal{C}(\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})), \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))) & \longrightarrow & \mathcal{C}(\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})), \psi(\mathsf{f})(\mathrm{B})) & \longrightarrow & \mathcal{C}(\mathrm{H}(\mathrm{h})(\mathrm{B}), \psi(\mathsf{f})(\mathrm{B})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{T}(\mathrm{t},\mathrm{t}) \times \mathrm{S}(\mathsf{s},\mathsf{s}) & \longrightarrow & \mathrm{T}(\mathrm{t},\mathrm{t}) \times \mathrm{S}(\mathsf{s},\mathsf{s}') & \longrightarrow & \mathrm{T}(\mathrm{t}',\mathrm{t}) \times \mathrm{S}(\mathsf{s},\mathsf{s}')
\end{array}$$

By definition the morphism $\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})) \to \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))$ in the fiber $\psi(\mathrm{t})_{\mathsf{s}} \simeq \mathrm{H}(\mathsf{s})_{\mathrm{t}} \simeq \mathcal{C}_{\mathrm{t},\mathsf{s}}$ induced by $\mathrm{H}(\mathrm{h})(\mathrm{B}) \to \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))$ corresponds to β under the top horizontal functor of the following diagram of pullback squares:

$$\begin{array}{ccccc}
\mathcal{C}(\psi(\mathsf{f})(\mathrm{H}(\mathrm{h})(\mathrm{B})), \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))) & \longrightarrow & \mathcal{C}(\mathrm{H}(\mathrm{h})(\mathrm{B}), \mathrm{H}(\mathrm{h})(\psi(\mathsf{f})(\mathrm{B}))) & \longrightarrow & \mathcal{C}(\mathrm{H}(\mathrm{h})(\mathrm{B}), \psi(\mathsf{f})(\mathrm{B})) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{T}(\mathrm{t},\mathrm{t}) \times \mathrm{S}(\mathsf{s},\mathsf{s}) & \longrightarrow & \mathrm{T}(\mathrm{t}',\mathrm{t}) \times \mathrm{S}(\mathsf{s},\mathsf{s}) & \longrightarrow & \mathrm{T}(\mathrm{t}',\mathrm{t}) \times \mathrm{S}(\mathsf{s},\mathsf{s}')
\end{array}$$

So both induced morphisms coincide as both outer squares of the two diagrams coincide.

□

Finally we compare definition 6.31 to the following definition equivalent to that of [18] 7.3.2.2.:

Definition 6.45.

Let $\mathcal{B}$ *be a category,* $\mathcal{C}, \mathcal{D}$ *categories over* $\mathcal{B}$ *and* $\mathrm{F} : \mathcal{C} \to \mathcal{D}, \mathrm{G} : \mathcal{D} \to \mathcal{C}$ *functors over* $\mathcal{B}$.

We call F *a left adjoint of* G *relative to* $\mathcal{B}$ *or* G *a right adjoint of* F *relative to* $\mathcal{B}$ *if there is a map* $\mathcal{M} \to \Delta^1 \times \mathcal{B}$ *of bicartesian fibrations over* Δ^1 *classifying functors* $\Delta^1 \to \mathsf{Cat}_{\infty/\mathcal{B}}$ *and* $(\Delta^1)^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{B}}$ *corresponding to* $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *respectively* $\mathrm{G} : \mathcal{D} \to \mathcal{C}$.

Especially we apply def. 6.45 to the situation that $\mathcal{B}$ is an operad $\mathcal{O}^{\otimes}$, the categories $\mathcal{C}, \mathcal{D}$ are $\mathcal{O}^{\otimes}$-monoidal categories $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}$ and F, G are lax $\mathcal{O}^{\otimes}$-monoidal functors.

In this case by remark 6.47 F is a $\mathcal{O}^{\otimes}$-monoidal functor.

Remark 6.46.

1. *As bicartesian fibrations are self-dual,* F *is a left adjoint of* G *relative to* $\mathcal{B}$ *if and only if* F^{op} *is a right adjoint of* G^{op} *relative to* $\mathcal{B}^{\mathrm{op}}$.

2. *Let* $\phi : \mathcal{B}' \to \mathcal{B}$ *be a functor.*
   As bicartesian fibrations are stable under pullback, one has:

   If F *is a left adjoint of* G *relative to* $\mathcal{B}$, *then the pullback*

   $$\mathcal{B}' \times_{\mathcal{B}} \mathrm{F} : \mathcal{B}' \times_{\mathcal{B}} \mathcal{C} \to \mathcal{B}' \times_{\mathcal{B}} \mathcal{D}$$

   is a left adjoint of $\mathcal{B}' \times_{\mathcal{B}} \mathrm{G} : \mathcal{B}' \times_{\mathcal{B}} \mathcal{D} \to \mathcal{B}' \times_{\mathcal{B}} \mathcal{C}$ *relative to* $\mathcal{B}'$.

Remark 6.47. *Let* $\mathcal{B}$ *be a category,* $\mathcal{C}, \mathcal{D}$ *cocartesian fibrations over* $\mathcal{B}$ *and* $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *a functor over* $\mathcal{B}$.

The following conditions are equivalent:

- F *admits a right adjoint relative to* $\mathcal{B}$.
- F *is a map of cocartesian fibrations over* $\mathcal{B}$ *and induces on the fiber over every object of* $\mathcal{B}$ *a left adjoint functor.*

This is equivalent to the following observation:

Let $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ *be the map of cocartesian fibrations over* Δ^1 *classifying* $\mathrm{F} : \mathcal{C} \to \mathcal{D}$.

The following conditions are equivalent:

- $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ *is a map of bicartesian fibrations over* Δ^1.
- $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ *is a map of cocartesian fibrations over* $\mathcal{B}$ *and for every object* X *of* $\mathcal{B}$ *the induced functor* $\mathcal{M}_{\mathrm{X}} \to \Delta^1$ *is a bicartesian fibration.*

Proof. By 6.42 2. $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ is a map of bicartesian fibrations over Δ^1 if and only if $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ is a map of cocartesian fibrations over $\mathcal{B}$ and for every object X of $\mathcal{B}$ the induced functor $\mathcal{M}_{\mathrm{X}} \to \Delta^1$ is a bicartesian fibration.

The functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ is a map of cocartesian fibrations over $\mathcal{B}$ if and only if γ is a cocartesian fibration.

As $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ is a map of cocartesian fibrations over Δ^1, the functor $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ is a cocartesian fibration if and only if $\gamma : \mathcal{M} \to \Delta^1 \times \mathcal{B}$ is a map of cocartesian fibrations over $\mathcal{B}$:

Every morphism of $\Delta^1 \times \mathcal{B}$ factors as a cocartesian morphism of $\Delta^1 \times \mathcal{B} \to \mathcal{B}$ followed by a cocartesian morphism of $\Delta^1 \times \mathcal{B} \to \Delta^1$.

□

The next proposition tells us that both definitions of monoidal adjointness coincide:

Proposition 6.48. *Let $\mathcal{O}^\otimes$ be an operad, $\mathcal{C}^\otimes, \mathcal{D}^\otimes$ be $\mathcal{O}^\otimes$-monoidal categories,* $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *a $\mathcal{O}^\otimes$-monoidal functor and* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ *a lax $\mathcal{O}^\otimes$-monoidal functor.*

The following conditions are equivalent:

1. $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *is $\mathcal{O}^\otimes$-monoidally left adjoint to* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$.
2. $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ *is left adjoint to* $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ *relative to* $\mathcal{O}^\otimes$.
3. *There is a $\mathcal{O}^\otimes$-monoidal natural transformation* $\eta : \mathrm{id}_{\mathcal{C}^\otimes} \to \mathrm{G} \circ \mathrm{F}$ *such that for all* $\mathrm{X} \in \mathcal{O}$ *the induced natural transformation* $\eta_\mathrm{X} : \mathrm{id}_{\mathcal{C}_\mathrm{X}} \to \mathrm{G}_\mathrm{X} \circ \mathrm{F}_\mathrm{X}$ *exhibits* $\mathrm{F}_\mathrm{X} : \mathcal{C}_\mathrm{X} \to \mathcal{D}_\mathrm{X}$ *as a left adjoint of* $\mathrm{G}_\mathrm{X} : \mathcal{D}_\mathrm{X} \to \mathcal{C}_\mathrm{X}$.
4. *There is a $\mathcal{O}^\otimes$-monoidal natural transformation* $\varepsilon : \mathrm{F} \circ \mathrm{G} \to \mathrm{id}_{\mathcal{D}^\otimes}$ *such that for all* $\mathrm{X} \in \mathcal{O}$ *the induced natural transformation* $\varepsilon_\mathrm{X} : \mathrm{F}_\mathrm{X} \circ \mathrm{G}_\mathrm{X} \to \mathrm{id}_{\mathcal{D}_\mathrm{X}}$ *exhibits* $\mathrm{F}_\mathrm{X} : \mathcal{C}_\mathrm{X} \to \mathcal{D}_\mathrm{X}$ *as a left adjoint of* $\mathrm{G}_\mathrm{X} : \mathcal{D}_\mathrm{X} \to \mathcal{C}_\mathrm{X}$.

Proof. The $\mathcal{O}^\otimes$-monoidal functor $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ gives rise to an adjunction

$$\mathcal{P}(\mathrm{F}) : \mathcal{P}(\mathcal{C})^\otimes \rightleftarrows \mathcal{P}(\mathcal{D})^\otimes : (\mathrm{F}^{\mathrm{rev}})^*$$

relative to $\mathcal{O}^\otimes$.

We start by showing that 1. implies 2:

If $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ is $\mathcal{O}^\otimes$-monoidally left adjoint to $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$, then by remark 6.33 the lax $\mathcal{O}^\otimes$-monoidal functor $(\mathrm{F}^{\mathrm{rev}})^* : \mathcal{P}(\mathcal{D})^\otimes \to \mathcal{P}(\mathcal{C})^\otimes$ restricts to the lax $\mathcal{O}^\otimes$-monoidal functor $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$.

So the adjunction $\mathcal{P}(\mathrm{F}) : \mathcal{P}(\mathcal{C})^\otimes \rightleftarrows \mathcal{P}(\mathcal{D})^\otimes : (\mathrm{F}^{\mathrm{rev}})^*$ relative to $\mathcal{O}^\otimes$ restricts to the adjunction $\mathrm{F} : \mathcal{C}^\otimes \rightleftarrows \mathcal{D}^\otimes : \mathrm{G}$ relative to $\mathcal{O}^\otimes$. This implies 2.

By remark 6.46 2. condition 2. implies 3. and dually 4.

As next we show that 4. implies 1.
The $\mathcal{O}^\otimes$-monoidal natural transformation $\varepsilon : \mathrm{F} \circ \mathrm{G} \to \mathrm{id}_{\mathcal{D}^\otimes}$ gives rise to a $\mathcal{O}^\otimes$-monoidal natural transformation

$$\alpha : \mathsf{y}_\mathcal{C} \circ \mathrm{G} \to (\mathrm{F}^{\mathrm{rev}})^* \circ \mathcal{P}(\mathrm{F}) \circ \mathsf{y}_\mathcal{C} \circ \mathrm{G} \simeq (\mathrm{F}^{\mathrm{rev}})^* \circ \mathsf{y}_\mathcal{D} \circ \mathrm{F} \circ \mathrm{G} \xrightarrow{(\mathrm{F}^{\mathrm{rev}})^* \circ \mathsf{y}_\mathcal{D} \circ \varepsilon} (\mathrm{F}^{\mathrm{rev}})^* \circ \mathsf{y}_\mathcal{D}$$

of lax $\mathcal{O}^\otimes$-monoidal functors $\mathcal{D}^\otimes \to \mathcal{P}(\mathcal{C})^\otimes$.

By remark 6.33 it is enough to see that α is an equivalence.

For all $\mathrm{X} \in \mathcal{O}$ the induced natural transformation α_X of functors $\mathcal{D}_\mathrm{X} \to \mathcal{P}(\mathcal{C}_\mathrm{X})$ on the fiber over X is homotopic to the natural transformation

$$\mathsf{y}_{\mathcal{C}_\mathrm{X}} \circ \mathrm{G}_\mathrm{X} \to (\mathrm{F}_\mathrm{X})^* \circ \mathcal{P}(\mathrm{F}_\mathrm{X}) \circ \mathsf{y}_{\mathcal{C}_\mathrm{X}} \circ \mathrm{G}_\mathrm{X} \simeq (\mathrm{F}_\mathrm{X})^* \circ \mathsf{y}_{\mathcal{D}_\mathrm{X}} \circ \mathrm{F}_\mathrm{X} \circ \mathrm{G}_\mathrm{X} \xrightarrow{(\mathrm{F}_\mathrm{X})^* \circ \mathsf{y}_{\mathcal{D}_\mathrm{X}} \circ \varepsilon_\mathrm{X}} (\mathrm{F}_\mathrm{X})^* \circ \mathsf{y}_{\mathcal{D}_\mathrm{X}}$$

of functors $\mathcal{D}_\mathrm{X} \to \mathcal{P}(\mathcal{C}_\mathrm{X})$.
So for all $\mathrm{Y} \in \mathcal{C}_\mathrm{X}$ and $\mathrm{Z} \in \mathcal{D}_\mathrm{X}$ the map

$$\alpha_\mathrm{X}(\mathrm{Z})(\mathrm{Y}) : \mathcal{C}_\mathrm{X}(\mathrm{Y}, \mathrm{G}_\mathrm{X}(\mathrm{Z})) \to \mathcal{D}_\mathrm{X}(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{Z})$$

is homotopic to the map

$$\mathcal{C}_\mathrm{X}(\mathrm{Y}, \mathrm{G}_\mathrm{X}(\mathrm{Z})) \to \mathcal{D}_\mathrm{X}(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{F}_\mathrm{X}(\mathrm{G}_\mathrm{X}(\mathrm{Z}))) \to \mathcal{D}_\mathrm{X}(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{Z})$$

given by composition with $\varepsilon_\mathrm{X} : \mathrm{F}_\mathrm{X} \circ \mathrm{G}_\mathrm{X} \to \mathrm{id}_{\mathcal{D}_\mathrm{X}}$.
The implication 3. to 1. is dual.

□

6.2 Appendix B: Lurie-enriched category theory

6.2.1 Enveloping enriched categories

In this subsection we show that every operad $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ over $\mathrm{LM}^\otimes$ embeds into a $\mathrm{LM}^\otimes$-monoidal category $\mathcal{M}'^\otimes \to \mathrm{LM}^\otimes$ that exhibits a category $\mathcal{M}'$ as enriched over a monoidal category $\mathcal{C}'^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}'^\otimes$.

Moreover the operad $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ over $\mathrm{LM}^\otimes$ exhibits a category $\mathcal{M}$ as enriched over a locally cocartesian fibration of operads $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ if and only if $\mathcal{M}^\otimes$ is closed in $\mathcal{M}'^\otimes$ under morphism objects, i.e. the morphism object in $\mathcal{C}'$ of every objects X, Y of $\mathcal{M} \subset \mathcal{M}'$ belongs to $\mathcal{C}$ (prop. 6.49).

More generally we will show that every cocartesian S-family $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ of operads over $\mathrm{LM}^\otimes$ for a small category S embeds into a cocartesian S-family $\mathcal{M}'^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ of $\mathrm{LM}^\otimes$-monoidal categories that is a cocartesian S-family of categories enriched over $\mathcal{C}'^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}'^\otimes$.

We start by showing that the enveloping $\mathrm{LM}^\otimes$-monoidal category $\mathrm{Env}_{\mathrm{LM}}(\mathcal{M})^\otimes \to \mathrm{LM}^\otimes$ exhibits the category $\mathrm{Env}_{\mathrm{LM}}(\mathcal{M}) := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})^\otimes$ as a left module over the monoidal category $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^\otimes \to \mathrm{Ass}^\otimes$ (lemma 6.52) and define $\mathcal{M}'^\otimes := \mathcal{P}(\mathrm{Env}_{\mathrm{LM}}(\mathcal{M}))^\otimes$ to be the $\mathrm{LM}^\otimes$-monoidal Day-convolution on $\mathrm{Env}_{\mathrm{LM}}(\mathcal{M})^\otimes$.

We start with recalling some basic facts about the enveloping $\mathcal{O}^\otimes$-monoidal category for some operad $\mathcal{O}^\otimes$ and then prove prop. 6.49:

Given a map of operad $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ we have an enveloping $\mathcal{O}^\otimes$-monoidal category $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^\otimes$ equipped with a $\mathcal{O}^\otimes$-monoidal functor $\mathcal{C}^\otimes \to \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^\otimes$ such that for every $\mathcal{O}^\otimes$-monoidal category $\mathcal{D}^\otimes$ the induced functor

$$\mathrm{Fun}^\otimes_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{C}), \mathcal{D}) \to \mathrm{Alg}_{\mathcal{C}/\mathcal{O}}(\mathcal{D})$$

is an equivalence.

Moreover we have an equivalence $\mathrm{Env}_{\mathcal{O}}(\mathcal{C})^\otimes \simeq \mathrm{Act}(\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{C}^\otimes$ over $\mathrm{Fun}(\{1\}, \mathcal{O}^\otimes)$ and the unit $\mathcal{C}^\otimes \to \mathrm{Env}_{\mathcal{O}}(\mathcal{C})^\otimes \simeq \mathrm{Act}(\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{C}^\otimes$ is the pullback of the diagonal embedding $\mathcal{O}^\otimes \to \mathrm{Act}(\mathcal{O}^\otimes) \subset \mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$ along the functor $\mathrm{Act}(\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{C}^\otimes \to \mathrm{Act}(\mathcal{O}^\otimes)$ and is thus fully faithful.

We adapt this definition to cocartesian families of operads:

Let S be a category and $\mathcal{C}^\otimes \to \mathrm{S} \times \mathcal{O}^\otimes$ a cocartesian S-family of operads over $\mathcal{O}^\otimes$.

We set $\mathrm{Env}^{/\mathrm{S}}_{\mathcal{O}}(\mathcal{C})^\otimes := \mathrm{Act}(\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{C}^\otimes$.

So $\mathrm{Env}^{/\mathrm{S}}_{\mathcal{O}}(\mathcal{C})^\otimes \to \mathrm{Fun}(\{1\}, \mathcal{O}^\otimes) \times \mathrm{S}$ is a map of cocartesian fibrations over S that induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ the cocartesian fibration $\mathrm{Env}_{\mathcal{O}}(\mathcal{C}_\mathsf{s})^\otimes \to \mathcal{O}^\otimes$ of operads.

Every morphism $\mathsf{s} \to \mathrm{t}$ in S yields a $\mathcal{O}^\otimes$-monoidal functor $\mathrm{Env}_{\mathcal{O}}(\mathcal{C}_\mathsf{s})^\otimes \to \mathrm{Env}_{\mathcal{O}}(\mathcal{C}_\mathrm{t})^\otimes$ so that the functor $\mathrm{Env}^{/\mathrm{S}}_{\mathcal{O}}(\mathcal{C})^\otimes \to \mathrm{Fun}(\{1\}, \mathcal{O}^\otimes) \times \mathrm{S}$ is a cocartesian S-family of operads over $\mathcal{O}^\otimes$.

Pulling back the diagonal embedding $\mathcal{O}^\otimes \to \mathrm{Act}(\mathcal{O}^\otimes) \subset \mathrm{Fun}(\Delta^1, \mathcal{O}^\otimes)$ along the functor $\mathrm{Act}(\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{C}^\otimes \to \mathrm{Act}(\mathcal{O}^\otimes)$ we get an embedding

$\mathcal{C}^{\otimes} \to \mathrm{Env}_{\mathcal{O}}^{/\mathrm{S}}(\mathcal{C})^{\otimes}$ of cocartesian S-families of operads over $\mathcal{O}^{\otimes}$ that is the unit of the adjunction

$$\mathrm{Fun}(\mathrm{S}, \mathrm{Env}_{\mathcal{O}}) : \mathrm{Fun}(\mathrm{S}, \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}) \rightleftarrows \mathrm{Fun}(\mathrm{S}, \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}^{\mathrm{cocart}})$$

induced by the adjunction $\mathrm{Env}_{\mathcal{O}} : \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}} \rightleftarrows \mathrm{Op}_{\infty/\mathcal{O}^{\otimes}}^{\mathrm{cocart}}$.

We call $\mathrm{Env}_{\mathcal{O}}^{/\mathrm{S}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes} \times \mathrm{S}$ the enveloping cocartesian S-family of operads over $\mathcal{O}^{\otimes}$ of $\mathcal{C}^{\otimes} \to \mathrm{S} \times \mathcal{O}^{\otimes}$.

For $\mathcal{O}^{\otimes} = \mathrm{LM}^{\otimes}$ we see that given a cocartesian S-family $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ of operads over $\mathrm{LM}^{\otimes}$ with $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$ there is an enveloping cocartesian S-family $\mathrm{Env}_{\mathrm{LM}}^{/\mathrm{S}}(\mathcal{C})^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ of $\mathrm{LM}^{\otimes}$-monoidal categories that exhibits the functor $\{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathrm{Env}_{\mathrm{LM}}^{/\mathrm{S}}(\mathcal{C})^{\otimes} \to \mathrm{S}$ as a left module in $\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}$ over the cocartesian S-family $\mathrm{Env}_{\mathrm{Ass}}^{/\mathrm{S}}(\mathcal{C})^{\otimes} \to \mathrm{Ass}^{\otimes} \times \mathrm{S}$ of monoidal categories (lemma 6.52).

The lax monoidal functor $\mathcal{P} : \mathsf{Cat}_{\infty} \to \widehat{\mathsf{Cat}_{\infty}}$ that takes preshaves yields a functor $\mathcal{P}^{/\mathrm{S}} : \mathrm{Fun}(\mathrm{S}, \mathrm{Alg}_{\mathrm{LM}}(\mathsf{Cat}_{\infty})) \to \mathrm{Fun}(\mathrm{S}, \mathrm{Alg}_{\mathrm{LM}}(\widehat{\mathsf{Cat}_{\infty}}))$.

We call $\mathcal{P}^{/\mathrm{S}}(\mathrm{Env}_{\mathrm{LM}}^{/\mathrm{S}}(\mathcal{M}))^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ the enveloping cocartesian S-family of $\mathcal{P}^{/\mathrm{S}}(\mathrm{Env}_{\mathrm{Ass}}^{/\mathrm{S}}(\mathcal{C}))^{\otimes}$-enriched categories associated to $\mathcal{M}^{\otimes}$.

Now we are ready to prove the following proposition:

Proposition 6.49. *Let $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes}$ be a map of operads such that the map $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes} \to \mathrm{Ass}^{\otimes}$ is a locally cocartesian fibration.*
Set $\mathcal{M} := \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$.

Let X, Y be objects of $\mathcal{M}$ and $\beta \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{B}, \mathrm{X}; \mathrm{Y})$ an operation that exhibits B as the morphism object of X and Y in $\mathcal{C}$.

Denote $\sigma \in \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ the unique object and for every $\alpha \in \mathrm{Ass}_{\mathrm{n}}$ for some $\mathrm{n} \in \mathbb{N}$ denote α' the image of α, the identity of $\mathfrak{m}$ and σ under the operadic composition

$$\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m}) \times (\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}; \mathfrak{a}) \times \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{m}; \mathfrak{m})) \to \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m}).$$

The following conditions are equivalent:

1. *For every objects $\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}} \in \mathcal{C}$ for some $\mathrm{n} \in \mathbb{N}$ and every $\alpha \in \mathrm{Ass}_{\mathrm{n}}$ the canonical map*

$$\mathrm{Mul}_{\mathcal{M}}(\otimes_{\alpha}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}), \mathrm{X}; \mathrm{Y}) \simeq$$
$$\{\sigma\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\otimes_{\alpha}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}), \mathrm{X}; \mathrm{Y}) \to$$
$$\{\alpha'\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}, \mathrm{X}; \mathrm{Y})$$

   is an equivalence.

2. *The embedding $\mathcal{M}^{\otimes} \subset \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ preserves the morphism object of X and Y, i.e. $\beta \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{B}, \mathrm{X}; \mathrm{Y}) \simeq \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})(\mathrm{B} \otimes \mathrm{X}, \mathrm{Y})$ exhibits B as the morphism object of X and Y in $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$.*

3. *The morphism object of X and Y in $\mathcal{P}(\mathrm{Env}_{\mathrm{LM}}(\mathcal{M}))^{\otimes}$ belongs to $\mathcal{C}^{\otimes} \subset \mathcal{P}(\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C}))^{\otimes}$.*

Proof. Write $[X, Y]$ for B and let A be an object of $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ corresponding to objects $A_1, ..., A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and an operation $\alpha \in \mathrm{Ass}_n$.

The canonical map

$$\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})(A, [X, Y]) \to \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})(A \otimes X, [X, Y] \otimes X)$$
$$\to \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})(A \otimes X, Y)$$

induced by $\beta \in \mathrm{Mul}_{\mathcal{M}}(B, X; Y) \simeq \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})(B \otimes X, Y)$ factors as the composition of canonical maps

$$\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})(A, [X, Y]) \simeq \{\alpha\} \times_{\mathrm{Ass}(n)} \mathrm{Mul}_{\mathcal{C}}((A_1, ..., A_n), [X, Y]) \simeq$$
$$\mathcal{C}(\otimes_\alpha(A_1, ..., A_n), [X, Y]) \simeq \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(A_1, ..., A_n), X; Y) \simeq$$
$$\{\sigma\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(A_1, ..., A_n), X; Y) \to$$
$$\{\alpha'\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(A_1, ..., A_n, X; Y) \simeq \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})(A \otimes X, Y)$$

as for $A = [X, Y]$ both maps send the identity to β.

The equivalence between 2. and 3. follows from the fact that the embedding $\mathcal{M}^\otimes \subset \mathcal{P}(\mathrm{Env}_{\mathrm{LM}}(\mathcal{M}))^\otimes$ of operads over $\mathrm{LM}^\otimes$ preserves morphism objects according to lemma 6.51.

□

Corollary 6.50. *Let* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ *be a map of operads such that the map* $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes \to \mathrm{Ass}^\otimes$ *is a locally cocartesian fibration. Set* $\mathcal{M} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$.

Let X *be an object of* $\mathcal{M}$ *that admits an endomorphism object corresponding to a final object of the category* $\mathcal{C}[X]$.

Set $\bar{\mathcal{M}}^\otimes := \mathcal{P}(\mathrm{Env}_{\mathrm{LM}}(\mathcal{M}))^\otimes, \bar{\mathcal{C}}^\otimes := \mathcal{P}(\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C}))^\otimes$ *and* $\bar{\mathcal{M}} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \bar{\mathcal{M}}^\otimes$.

Denote $\sigma \in \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ *the unique object and for every* $\alpha \in \mathrm{Ass}_n$ *for some* $n \in \mathbb{N}$ *denote* α' *the image of* α, *the identity of* $\mathfrak{m}$ *and* σ *under the operadic composition*

$$\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m}) \times (\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}; \mathfrak{a}) \times \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{m}; \mathfrak{m})) \to \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m}).$$

The following conditions are equivalent:

1. *For every objects* $A_1, ..., A_n \in \mathcal{C}$ *for some* $n \in \mathbb{N}$ *and every* $\alpha \in \mathrm{Ass}_n$ *the canonical map*

   $$\mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(A_1, ..., A_n), X; X) \simeq$$
   $$\{\sigma\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(A_1, ..., A_n), X; X) \to$$
   $$\{\alpha'\} \times_{\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_{\mathcal{M}}(A_1, ..., A_n, X; X)$$

   is an equivalence.

2. *The full inclusion* $\mathcal{M}^\otimes \subset \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})^\otimes$ *of operads over* $\mathrm{LM}^\otimes$ *preserves the endomorphism object, in other words the full subcategory inclusion* $\mathcal{C}[X] \subset \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})[X]$ *preserves the final object.*

3. *The final object of* $\mathcal{C}[X]$ *lifts to a final object of* $\{X\} \times_{\mathcal{M}} \mathrm{LMod}(\mathcal{M})$, *which is preserved by the embedding* $\{X\} \times_{\mathcal{M}} \mathrm{LMod}(\mathcal{M}) \subset \{X\} \times_{\bar{\mathcal{M}}} \mathrm{LMod}(\bar{\mathcal{M}})$.

Proof. Lemma 6.49 implies that 1. and 2. are equivalent.

Let $\mathcal{N}^\otimes \to \mathrm{LM}^\otimes$ be a $\mathrm{LM}^\otimes$-monoidal category and $\mathrm{Z} \in \mathcal{N} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$. Set $\mathcal{B}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$.

By corollary [18] 4.7.2.40. we know that if $\mathcal{B}[\mathrm{X}]$ admits a final object, this final object lifts to a final object of $\{\mathrm{X}\} \times_{\mathcal{N}} \mathrm{LMod}(\mathcal{N})$.

As the forgetful functor $\{\mathrm{X}\} \times_{\mathcal{N}} \mathrm{LMod}(\mathcal{N}) \to \mathcal{B}[\mathrm{X}]$ is conservative, in this case an object of $\{\mathrm{X}\} \times_{\mathcal{N}} \mathrm{LMod}(\mathcal{N})$ is final if and only if its image in $\mathcal{B}[\mathrm{X}]$ is.

As $\bar{\mathcal{M}}^\otimes$ exhibits $\bar{\mathcal{M}}$ as closed left module over $\bar{\mathcal{C}}^\otimes$, the category $\bar{\mathcal{C}}[\mathrm{X}]$ admits a final object that lifts to a final object of $\{\mathrm{X}\} \times_{\bar{\mathcal{M}}} \mathrm{LMod}(\bar{\mathcal{M}})$.

We have a pullback square

$$\begin{array}{ccc} \{\mathrm{X}\} \times_{\mathcal{M}} \mathrm{LMod}(\mathcal{M}) & \longrightarrow & \{\mathrm{X}\} \times_{\bar{\mathcal{M}}} \mathrm{LMod}(\bar{\mathcal{M}}) \\ \downarrow & & \downarrow \\ \mathcal{C}[\mathrm{X}] & \longrightarrow & \bar{\mathcal{C}}[\mathrm{X}]. \end{array}$$

The functor $\mathcal{C}[\mathrm{X}] \subset \bar{\mathcal{C}}[\mathrm{X}]$ factors as $\mathcal{C}[\mathrm{X}] \subset \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})[\mathrm{X}] \subset \bar{\mathcal{C}}[\mathrm{X}]$.

By lemma 6.51 the functor $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})[\mathrm{X}] \subset \bar{\mathcal{C}}[\mathrm{X}]$ preserves the final object.

So 2. is equivalent to the condition that the functor $\mathcal{C}[\mathrm{X}] \to \bar{\mathcal{C}}[\mathrm{X}]$ preserves the final object.

Hence 2. and 3. are equivalent.

□

The following two lemmata were used in the proof of prop. 6.49:

Lemma 6.51. *Let $\iota : \mathcal{M}^\otimes \subset \mathcal{M}'^\otimes$ be an embedding of operads over $\mathrm{LM}^\otimes$.*

Set $\mathcal{M} = \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes, \mathcal{M}' = \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}'^\otimes$ and $\mathcal{C}^\otimes = \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes, \mathcal{C}'^\otimes = \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}'^\otimes$.

Let X, Y be objects of $\mathcal{M}$ and $\beta \in \mathrm{Mul}_{\mathcal{M}}(\mathrm{B}, \mathrm{X}; \mathrm{Y})$ an operation that exhibits B as the morphism object of X and Y in $\mathcal{C}$.

Assume that $\mathcal{C}'$ is the only full subcategory of $\mathcal{C}'$ containing $\mathcal{C}$ and closed under small colimits and that the functor $\mathrm{Mul}_{\mathcal{M}'}(-, \iota(\mathrm{X}); \iota(\mathrm{Y})) : \mathcal{C}'^{\mathrm{op}} \to \mathcal{S}$ preserves small limits.

Then $\iota(\beta) \in \mathrm{Mul}_{\mathcal{M}'}(\iota(\mathrm{B}), \iota(\mathrm{X}); \iota(\mathrm{Y}))$ exhibits $\iota(\mathrm{B})$ as the morphism object of $\iota(\mathrm{X})$ and $\iota(\mathrm{Y})$ in $\mathcal{C}'$:

Proof. For every object A of $\mathcal{C}'$ denote ξ_{A} the canonical map

$$\mathcal{C}'(\mathrm{A}, \iota([\mathrm{X}, \mathrm{Y}])) \to$$

$$\mathrm{Mul}_{\mathcal{M}'}(\iota([\mathrm{X}, \mathrm{Y}]), \iota(\mathrm{X}); \iota(\mathrm{Y})) \times (\mathcal{C}'(\mathrm{A}, \iota([\mathrm{X}, \mathrm{Y}])) \times \mathcal{M}'(\iota(\mathrm{X}), \iota(\mathrm{X}))$$
$$\to \mathrm{Mul}_{\mathcal{M}'}(\mathrm{A}, \iota(\mathrm{X}); \iota(\mathrm{Y}))$$

induced by $\iota(\beta)$.

If A belongs to $\mathcal{C}$, the map ξ_{A} is canonically equivalent to the canonical map

$$\mathcal{C}(\mathrm{A}, [\mathrm{X}, \mathrm{Y}]) \to \mathrm{Mul}_{\mathcal{M}}([\mathrm{X}, \mathrm{Y}], \mathrm{X}; \mathrm{Y}) \times (\mathcal{C}(\mathrm{A}, [\mathrm{X}, \mathrm{Y}]) \times \mathcal{M}(\mathrm{X}, \mathrm{X}))$$
$$\to \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}, \mathrm{X}; \mathrm{Y})$$

induced by β and is thus an equivalence.

Thus $\mathcal{C}$ is contained in the full subcategory $\mathcal{W}$ of $\mathcal{C}'$ spanned by the objects A such that ξ_A is an equivalence.

But $\mathcal{W}$ is closed under small colimits as the functor $\mathrm{Mul}_{\mathcal{M}'}(-, \iota(X); \iota(Y)) : \mathcal{C}'^{op} \to \mathcal{S}$ preserves small limits. So by assumption $\mathcal{W} = \mathcal{C}'$.

□

Lemma 6.52. *Let* S *be a small category and* $\mathcal{M}^\otimes \to LM^\otimes \times S$ *a cocartesian* S*-family of operads over* $LM^\otimes$.

Set $\mathcal{M} := \{\mathfrak{m}\} \times_{LM^\otimes} \mathcal{M}^\otimes$ *and* $\mathcal{C}^\otimes := Ass^\otimes \times_{LM^\otimes} \mathcal{M}^\otimes$.

Denote $\mathrm{Env}_{LM}(\mathcal{M})^\otimes \to LM^\otimes \times S$ *the enveloping cocartesian* S*-family of* $LM^\otimes$*-monoidal categories and* $\mathrm{Env}_{Ass}(\mathcal{C})^\otimes \to Ass^\otimes \times S$ *the enveloping cocartesian* S*-family of monoidal categories.*

Denote ζ *the canonical map*

$$\mathrm{Env}_{Ass}(\mathcal{C})^\otimes \to Ass^\otimes \times_{LM^\otimes} \mathrm{Env}_{LM}(\mathcal{M})^\otimes$$

of cocartesian S*-families of monoidal categories adjoint to the map* $\mathcal{C}^\otimes = Ass^\otimes \times_{LM^\otimes} \mathcal{M}^\otimes \subset Ass^\otimes \times_{LM^\otimes} \mathrm{Env}_{LM}(\mathcal{M})^\otimes$ *of cocartesian* S*-families of operads over* $Ass^\otimes$.

Then ζ *is an equivalence.*

Proof. As ζ is a map of cocartesian S-families of operads over $Ass^\otimes$, it is an equivalence if it induces on the fiber over every $s \in S$ an equivalence.

ζ induces on the fiber over every $s \in S$ the monoidal functor

$$\mathrm{Env}_{Ass}(\mathcal{C}_s)^\otimes \to Ass^\otimes \times_{LM^\otimes} \mathrm{Env}_{LM}(\mathcal{M}_s)^\otimes$$

adjoint to the map $\mathcal{C}_s^\otimes = Ass^\otimes \times_{LM^\otimes} \mathcal{M}_s^\otimes \subset Ass^\otimes \times_{LM^\otimes} \mathrm{Env}_{LM}(\mathcal{M}_s)^\otimes$ of operads over $Ass^\otimes$. So we can reduce to the case that S is contractible.

We have canonical equivalences

$$\mathrm{Env}_{LM}(\mathcal{M})^\otimes \simeq \mathrm{Act}(LM^\otimes) \times_{\mathrm{Fun}(\{0\}, LM^\otimes)} \mathcal{M}^\otimes$$

over $\mathrm{Fun}(\{1\}, LM^\otimes)$ and

$$\mathrm{Env}_{Ass}(\mathcal{C})^\otimes \simeq \mathrm{Act}(Ass^\otimes) \times_{\mathrm{Fun}(\{0\}, Ass^\otimes)} \mathcal{C}^\otimes$$

over $\mathrm{Fun}(\{1\}, Ass^\otimes)$.

We have a canonical equivalence

$$\mathrm{Act}(Ass^\otimes) \times_{\mathrm{Fun}(\{0\}, Ass^\otimes)} \mathcal{C}^\otimes \simeq \mathrm{Act}(Ass^\otimes) \times_{\mathrm{Fun}(\{0\}, Ass^\otimes)} Ass^\otimes \times_{LM^\otimes} \mathcal{M}^\otimes \simeq$$

$$\mathrm{Act}(Ass^\otimes) \times_{\mathrm{Act}(LM^\otimes)} \mathrm{Act}(LM^\otimes) \times_{\mathrm{Fun}(\{0\}, LM^\otimes)} \mathcal{M}^\otimes$$

over $\mathrm{Act}(Ass^\otimes)$ and thus get a pullback square

$$\begin{array}{ccc} \mathrm{Env}_{Ass}(\mathcal{C})^\otimes & \longrightarrow & \mathrm{Env}_{LM}(\mathcal{M})^\otimes \\ \downarrow & & \downarrow \\ \mathrm{Act}(Ass^\otimes) & \longrightarrow & \mathrm{Act}(LM^\otimes). \end{array}$$

Consequently it is enough to see that the commutative square

$$\begin{array}{ccc} \mathrm{Act}(Ass^\otimes) & \longrightarrow & \mathrm{Act}(LM^\otimes) \\ \downarrow & & \downarrow \\ \mathrm{Fun}(\{1\}, Ass^\otimes) & \longrightarrow & \mathrm{Fun}(\{1\}, LM^\otimes) \end{array}$$

is a pullback square.

To do so we have to show that for every active morphism $h : Y \to X$ of $LM^{\otimes}$ with X also Y belongs to $Ass^{\otimes}$.

But if $h : Y \to X$ lies over the active morphism $f : \langle m \rangle \to \langle n \rangle$ we have a canonical equivalence

$$\{f\} \times_{\mathcal{F}in_*(\langle m \rangle, \langle n \rangle)} LM^{\otimes}(Y, X) \simeq \prod_{i=1}^{n} Mul_{LM}((Y_j)_{j \in f^{-1}\{i\}}, X_i).$$

Containing h the space $\{f\} \times_{\mathcal{F}in_*(\langle m \rangle, \langle n \rangle)} LM^{\otimes}(Y, X)$ is not empty so that for all $i \in \{1, ..., n\}$ the space $Mul_{LM}((Y_j)_{j \in f^{-1}\{i\}}, X_i)$ is not empty.

So for every $j \in \{1, ..., m\}$ the object Y_j is the unique color of $Ass^{\otimes}$.

□

Lemma 6.53. *Let $\varphi : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ be a map of operads.*

Denote $Env_{\mathcal{O}}(\mathcal{C})^{\otimes} \to \mathcal{O}^{\otimes}$ the enveloping $\mathcal{O}^{\otimes}$-monoidal category of $\mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$.

$\varphi : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a locally cocartesian fibration if and only if for all objects Y of $\mathcal{O}$ the full subcategory inclusion $\mathcal{C}_Y \subset Env_{\mathcal{O}}(\mathcal{C})_Y$ admits a left adjoint.

Proof. We have a canonical equivalence

$$Env_{\mathcal{O}}(\mathcal{C})^{\otimes} \simeq Act(\mathcal{O}^{\otimes}) \times_{Fun(\{0\}, \mathcal{O}^{\otimes})} \mathcal{C}^{\otimes}$$

over $Fun(\{1\}, \mathcal{O}^{\otimes})$. So for every object Y of $\mathcal{O}$ we get a canonical equivalence

$$Env_{\mathcal{O}}(\mathcal{C})_Y \simeq (\mathcal{O}^{\otimes})^{act}_{/Y} \times_{Fun(\{0\}, \mathcal{O}^{\otimes})} \mathcal{C}^{\otimes}$$

and given an object $B \in \mathcal{C}_Y$ and an object A of $Env_{\mathcal{O}}(\mathcal{C})_Y$ corresponding to objects $A_1, ..., A_n$ of $\mathcal{C}$ for $n \in \mathbb{N}$ and an object $\alpha \in Mul_{\mathcal{O}}(\varphi(A_1), ..., \varphi(A_n), Y)$ we get a canonical equivalence

$$Env_{\mathcal{O}}(\mathcal{C})_Y(A, B) \simeq \{\alpha\} \times_{Mul_{\mathcal{O}}(\varphi(A_1), ..., \varphi(A_n), Y)} Mul_{\mathcal{C}}(A_1, ..., A_n, B).$$

To show that the full subcategory inclusion $\mathcal{C}_Y \subset Env_{\mathcal{O}}(\mathcal{C})_Y$ admits a left adjoint we have to find a morphism $\theta : A \to B$ of $Env_{\mathcal{O}}(\mathcal{C})_Y$ with $B \in \mathcal{C}_Y$ such that for every object V of $\mathcal{C}_Y$ composition with θ induces an equivalence

$$\mathcal{C}_Y(B, V) \simeq Env_{\mathcal{O}}(\mathcal{C})_Y(B, V) \to Env_{\mathcal{O}}(\mathcal{C})_Y(A, V) \simeq$$

$$\{\alpha\} \times_{Mul_{\mathcal{O}}(\varphi(A_1), ..., \varphi(A_n), Y)} Mul_{\mathcal{C}}(A_1, ..., A_n, V).$$

If $\varphi : \mathcal{C}^{\otimes} \to \mathcal{O}^{\otimes}$ is a locally cocartesian fibration, we have a locally φ-cocartesian lift $h : (A_1, ..., A_n) \to \otimes_{\alpha}(A_1, ..., A_n)$ in $\mathcal{C}^{\otimes}$ of the active morphism α of $\mathcal{O}^{\otimes}$.

Define $\theta : A \to \otimes_{\alpha}(A_1, ..., A_n)$ to correspond to the morphism h under the equivalence $Env_{\mathcal{O}}(\mathcal{C})_Y(A, \otimes_{\alpha}(A_1, ..., A_n)) \simeq$
$\{\alpha\} \times_{Mul_{\mathcal{O}}(\varphi(A_1), ..., \varphi(A_n), Y)} Mul_{\mathcal{C}}(A_1, ..., A_n, \otimes_{\alpha}(A_1, ..., A_n))$.

For every object V of $\mathcal{C}_Y$ composition with $\theta : A \to \otimes_{\alpha}(A_1, ..., A_n)$

$$Env_{\mathcal{O}}(\mathcal{C})_Y(\otimes_{\alpha}(A_1, ..., A_n), V) \to Env_{\mathcal{O}}(\mathcal{C})_Y(A, V)$$

is equivalent to composition with $h : (A_1, ..., A_n) \to \otimes_\alpha(A_1, ..., A_n)$

$\varsigma : \mathcal{C}_Y(\otimes_\alpha(A_1, ..., A_n), V) \to \{\alpha\} \times_{Mul_{\mathcal{O}}(\varphi(A_1),...,\varphi(A_n),Y)} Mul_{\mathcal{C}}(A_1, ..., A_n, V)$

as for $V = \otimes_\alpha(A_1, ..., A_n)$ both maps send the identity to equivalent objects.

As h is locally φ-cocartesian, ς is an equivalence.

The if-direction follows from cor. 6.54.

□

Lemma 6.54. *Let* $p : \mathcal{C} \to \mathcal{D}$ *be a locally cocartesian fibration and* $\mathcal{B} \subset \mathcal{C}$ *a full subcategory such that for every* $X \in \mathcal{D}$ *the full subcategory inclusion* $\mathcal{B}_X \subset \mathcal{C}_X$ *admits a left adjoint* L_X.

Then the restriction $q : \mathcal{B} \subset \mathcal{C} \xrightarrow{p} \mathcal{D}$ *is a locally cocartesian fibration with the following class of locally cocartesian morphisms:*

A morphism in $\mathcal{B}$ *is locally* q*-cocartesian if and only if it factors as a locally* p*-cocartesian morphism of* $\mathcal{C}$ *followed by a* L_X*-equivalence for some* $X \in \mathcal{D}$.

Proof. Given an object $X \in \mathcal{B}$ and a morphism $g : Y := q(X) \to Z$ one can find a locally p-cocartesian morphism $\mathsf{f} : X \to T$ in $\mathcal{C}$ lying over g.

The composition $\mathsf{f}' : X \xrightarrow{\mathsf{f}} T \xrightarrow{\eta} L_Z(T)$ of f with the unit $\eta : T \to U := L_Z(T)$ is a lift of g with the desired properties.

So it remains to show that f' is locally q-cocartesian, i.e. that for every $W \in \mathcal{B}_Z$ the induced map

$$\psi : \{id_Z\} \times_{\mathcal{D}(Z,Z)} \mathcal{B}(U, W) \to \{g\} \times_{\mathcal{D}(Y,Z)} \mathcal{B}(X, W)$$

is an equivalence. But ψ factors as

$$\{id_Z\} \times_{\mathcal{D}(Z,Z)} \mathcal{C}(U, W) \xrightarrow{\phi} \{id_Z\} \times_{\mathcal{D}(Z,Z)} \mathcal{C}(T, W) \xrightarrow{\varphi} \{g\} \times_{\mathcal{D}(Y,Z)} \mathcal{C}(X, W)$$

and ϕ is equivalent to the map

$$\mathcal{C}_Z(U, W) \to \mathcal{C}_Z(T, W)$$

given by composition with η.

ϕ is an equivalence because $W \in \mathcal{B}_Z$ and η is a L_Z-equivalence and φ is an equivalence because f is locally p-cocartesian.

□

6.2.2 Functoriality of morphism objects

Let $\mathcal{M}^\otimes \to LM^\otimes$ be an operad over $LM^\otimes$ that exhibits $\mathcal{M} := \{\mathfrak{m}\} \times_{LM^\otimes} \mathcal{M}^\otimes$ as category enriched over $\mathcal{C}^\otimes := Ass^\otimes \times_{LM^\otimes} \mathcal{M}^\otimes$ and let X, Y be objects of $\mathcal{M}$.

We will construct a canonical left module structure on $[Y, X]$ over $[X, X]$.

As $\mathcal{M}^\otimes$ is closed in $\mathcal{M}'^\otimes$ under morphism objects, we can reduce to the case that $\mathcal{M}^\otimes$ is an enriched left module over a monoidal category $\mathcal{C}^\otimes := Ass^\otimes \times_{LM^\otimes} \mathcal{M}^\otimes$.

In this case the functor $- \otimes Y : \mathcal{C} \to \mathcal{M}$ is $\mathcal{C}$-linear and admits a right adjoint $[Y, -] : \mathcal{M} \to \mathcal{C}$ that is lax $\mathcal{C}$-linear in a canonical way.

Being lax $\mathcal{C}$-linear the functor $[Y, -] : \mathcal{M} \to \mathcal{C}$ sends the endomorphism left module structure on X over $[X, X]$ to a left module structure on $[Y, X]$ over $[X, X]$.

More coherently we will show the following:

Let $\mathcal{M}^{\otimes} \to LM^{\otimes}$ be a cocartesian S-family of categories enriched over $\mathcal{C}^{\otimes} := Ass^{\otimes} \times_{LM^{\otimes}} \mathcal{M}^{\otimes}$.

Due to 5.22 we have a multi-mapping space functor $Mul_{\mathcal{M}}(-, -; -) : \mathcal{C}^{rev} \times_S \mathcal{M}^{rev} \times_S \mathcal{M} \to \mathcal{S}$ relative to S with $Mul_{\mathcal{M}}(\mathbb{1}_{\mathcal{C}}^{rev}, -; -) : \mathcal{M}^{rev} \times_S \mathcal{M} \to \mathcal{S}$ the mapping space functor of $\mathcal{M}$, where $\mathbb{1}_{\mathcal{C}} : S \to \mathcal{C}$ denotes the unit of $\mathcal{C}^{\otimes} \to S \times Ass^{\otimes}$.

The functor $Mul_{\mathcal{M}}(-, -; -)$ is adjoint to a functor $\beta : \mathcal{M}^{rev} \times_S \mathcal{M} \to Map_S(\mathcal{C}^{rev}, \mathcal{S} \times S)$ over S.

As $\mathcal{M}^{\otimes} \to LM^{\otimes}$ is a cocartesian S-family of categories enriched over $\mathcal{C}^{\otimes}$, this functor β over S induces a functor $\mathcal{M}^{rev} \times_S \mathcal{M} \to \mathcal{C}$ over S adjoint to a functor $\theta : \mathcal{M} \to Map_S(\mathcal{M}^{rev}, \mathcal{C})$ that sends an object X of $\mathcal{M}$ lying over some $s \in S$ to the functor $[-, X] : \mathcal{M}_s^{op} \to \mathcal{C}_s$.

θ lifts the Yoneda-embedding relative to S along the functor $\mathcal{C} \xrightarrow{\mathbb{1}_{\mathcal{C}}^{rev} \times_S \mathcal{C}} \mathcal{C}^{rev} \times_S \mathcal{C} \to \mathcal{S} \times S$, where the last functor is the mapping space functor of $\mathcal{C}$ relative to S.

We will show in the following that θ lifts to a map

$$\gamma : \mathcal{M}^{\otimes} \to Map_S(\mathcal{M}^{rev}, \mathcal{C})^{\otimes}$$

of S-families of operads over $LM^{\otimes}$, whose pullback to $Ass^{\otimes}$ is the diagonal map $\delta : \mathcal{C}^{\otimes} \to Map_S(\mathcal{M}^{rev}, \mathcal{C})^{\otimes}$ of S-families of operads over $Ass^{\otimes}$.

For S contractible this especially guarantees the following:

Let X be an object of $\mathcal{M}$ and $\beta \in Mul_{\mathcal{M}}(B, X; X)$ an operation that exhibits $B = [X, X]$ as the endomorphism object of X in $\mathcal{C}$.

As γ is a map of operads over $LM^{\otimes}$, it sends the endomorphism $[X, X]$-left module structure on X to a $\delta([X, X])$-left module structure on $[-, X] : \mathcal{M}^{op} \to \mathcal{C}$ corresponding to a lift $\mathcal{M}^{op} \to LMod_{[X,X]}(\mathcal{C})$ of $[-, X] : \mathcal{M}^{op} \to \mathcal{C}$.

So for every object Y of $\mathcal{M}$ the morphism object $[Y, X]$ is a left-module over the endomorphism object $[X, X]$ in $\mathcal{C}$ and for every morphism $Y \to Z$ in $\mathcal{M}$ the induced morphism $[Z, X] \to [Y, X]$ is $[X, X]$-linear.

Proposition 6.55. *Let* S *be a category and* $\mathcal{M}^{\otimes} \to LM^{\otimes} \times S$ *a cocartesian* S*-family of small categories enriched in* $\mathcal{C}^{\otimes} := Ass^{\otimes} \times_{LM^{\otimes}} \mathcal{M}^{\otimes}$.

Set $\mathcal{M} := \{\mathfrak{m}\} \times_{LM^{\otimes}} \mathcal{M}^{\otimes}$.

There is a map $\gamma : \mathcal{M}^{\otimes} \to Map_S(\mathcal{M}^{rev}, \mathcal{C})^{\otimes}$ *of* S*-families of operads over* $LM^{\otimes}$*, whose underlying functor is the functor* $\theta : \mathcal{M} \to Map_S(\mathcal{M}^{rev}, \mathcal{C})$ *over* S *and whose pullback to* $Ass^{\otimes}$ *is the diagonal map* $\delta : \mathcal{C}^{\otimes} \to Map_S(\mathcal{M}^{rev}, \mathcal{C})^{\otimes}$ *of* S*-families of operads over* $Ass^{\otimes}$.

γ *corresponds to a* $\mathcal{C}$*-linear map* $\mathcal{M}^{\otimes} \to \delta^*(Map_S(\mathcal{M}^{rev}, \mathcal{C})^{\otimes})$ *of* S*-families, i.e. a map of* S*-families of operads over* $LM^{\otimes}$*, whose pullback to* $Ass^{\otimes}$ *is the identity of* $\mathcal{C}^{\otimes}$.

For $S = \Delta^1$ we obtain the following corollary:

Corollary 6.56. *Let* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes, \mathcal{N}^\otimes \to \mathrm{LM}^\otimes$ *be operads over* $\mathrm{LM}^\otimes$ *that exhibit categories* $\mathcal{M}, \mathcal{N}$ *as enriched over locally cocartesian fibrations of operads* $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes, \mathcal{D}^\otimes \to \mathrm{Ass}^\otimes$ *and let* $\mathrm{F} : \mathcal{M}^\otimes \to \mathcal{N}^\otimes$ *be a map of operads over* $\mathrm{LM}^\otimes$.

The natural transformation $\mathrm{Fun}(\mathcal{M}^{\mathrm{op}}, \mathrm{F}) \circ \theta \to \mathrm{Fun}(\mathrm{F}^{\mathrm{op}}, \mathcal{D}) \circ \theta \circ \mathrm{F}$ *of functors* $\mathcal{M} \to \mathrm{Fun}(\mathcal{M}^{\mathrm{op}}, \mathcal{D})$ *adjoint to the canonical natural transformation*

$$\mathrm{F} \circ [-,-] \to [-,-] \circ (\mathrm{F}^{\mathrm{op}} \times \mathrm{F})$$

of functors $\mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{D}$ *lifts to a natural transformation*

$$\mathrm{F}^{\mathcal{M}^{\mathrm{op}}} \circ \gamma \to (\mathcal{D}^\otimes)^{\mathrm{F}^{\mathrm{op}}} \circ \gamma \circ \mathrm{F}$$

over $\mathrm{LM}^\otimes$ *of maps of operads* $\mathcal{M}^\otimes \to (\mathcal{D}^\otimes)^{\mathcal{M}^{\mathrm{op}}}$ *over* $\mathrm{LM}^\otimes$.

Proof. We first show that we can reduce to the case that $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ is a cocartesian S-family of $\mathrm{LM}^\otimes$-monoidal categories.

Let $\mathcal{M}'^\otimes := \mathcal{P}^{/\mathrm{S}}(\mathrm{Env}^{/\mathrm{S}}_{\mathrm{LM}^\otimes}(\mathcal{M}))^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ be the enveloping cocartesian S-family of $\mathcal{C}'^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}'^\otimes$-enriched categories associated to $\mathcal{M}^\otimes$.

The embedding $\mathcal{M}^\otimes \subset \mathcal{M}'^\otimes$ of cocartesian S-families of operads over $\mathrm{LM}^\otimes$ yields an equivalence from the multi-mapping space functor $\mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \to \mathcal{S}$ to the restricted multi-mapping space functor $\mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \subset \mathcal{C}'^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}'^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}' \to \mathcal{S}$ adjoint to an equivalence ψ from the functor $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S})$ over S to the composition $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \subset \mathcal{M}'^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}' \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}'^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S})$.

The map $\mathrm{Map}_{\mathrm{S}}(\mathcal{C}'^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S})$ of cartesian fibrations over S induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ the right adjoint restriction functor $\mathrm{Fun}(\mathcal{C}'^{\mathrm{op}}, \mathcal{S}) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$ and thus admits a left adjoint relative to S.

ψ is adjoint to a natural transformation over S from the functor $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}'^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S})$ to the functor $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \subset \mathcal{M}'^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}' \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}'^{\mathrm{rev}}, \mathrm{S} \times \mathcal{S})$ that restricts to an equivalence over S between the functor $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \xrightarrow{\theta} \mathcal{C} \subset \mathcal{C}'$ and the functor $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \subset \mathcal{M}'^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}' \xrightarrow{\theta'} \mathcal{C}'$ by proposition 6.49.

Assume that there is a map $\mathcal{M}'^\otimes \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}')^\otimes$ of S-families of operads over $\mathrm{LM}^\otimes$, whose underlying functor is the functor $\theta' : \mathcal{M}' \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}')$ over S and whose pullback to $\mathrm{Ass}^\otimes$ is the diagonal map $\delta' : \mathcal{C}'^\otimes \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}')^\otimes$ of S-families of operads over $\mathrm{Ass}^\otimes$.

Then the underlying functor over S of the map

$$\mathcal{M}^\otimes \subset \mathcal{M}'^\otimes \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}')^\otimes \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}')^\otimes$$

of S-families of operads over $\mathrm{LM}^\otimes$ is equivalent to

$$\mathcal{M} \subset \mathcal{M}' \xrightarrow{\theta'} \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}') \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}')$$

being equivalent to $\mathcal{M} \xrightarrow{\theta} \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}) \subset \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}')$ and whose pullback to $\mathrm{Ass}^\otimes$ is the map

$$\mathcal{C}^\otimes \subset \mathcal{C}'^\otimes \xrightarrow{\delta'} \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}')^\otimes \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}')^\otimes$$

of S-families of operads over $\mathrm{Ass}^{\otimes}$ being equivalent to

$$\mathcal{C}^{\otimes} \xrightarrow{\delta} \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})^{\otimes} \subset \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}')^{\otimes}.$$

Hence the map $\mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}'^{\mathrm{rev}}, \mathcal{C}')^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}')^{\otimes}$ of S-families of operads over $\mathrm{LM}^{\otimes}$ induces a map $\mathcal{M}^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})^{\otimes}$ of S-families of operads over $\mathrm{LM}^{\otimes}$, whose underlying functor over S is the functor $\theta : \mathcal{M} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})$ and whose pullback to $\mathrm{Ass}^{\otimes}$ is the diagonal functor $\delta : \mathcal{C}^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})^{\otimes}$.

So we can assume that $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ is a cocartesian S-family of $\mathrm{LM}^{\otimes}$-monoidal categories.

Given an operad $\mathcal{O}^{\otimes}$ and cocartesian S-families of $\mathcal{O}^{\otimes}$-monoidal categories $\mathcal{D}^{\otimes} \to \mathcal{O}^{\otimes}, \mathcal{E}^{\otimes} \to \mathcal{O}^{\otimes}$ denote

- $\mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{D}, \mathcal{E}) := \mathrm{Alg}_{\mathcal{D}/\mathrm{S}\times\mathcal{O}}^{/\mathrm{S}}(\mathcal{E})$,
- $\mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes}(\mathcal{D}, \mathcal{E}) \subset \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{D}, \mathcal{E})$ the full subcategory spanned by the lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{D}_{\mathsf{s}} \to \mathcal{E}_{\mathsf{s}}$ for some $\mathrm{S} \in \mathrm{S}$ that are $\mathcal{O}^{\otimes}$-monoidal.
- $\mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{D}, \mathcal{E}) \subset \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{D}, \mathcal{E})$ the full subcategory spanned by the lax $\mathcal{O}^{\otimes}$-monoidal functors $\mathcal{D}_{\mathsf{s}} \to \mathcal{E}_{\mathsf{s}}$ for some $\mathrm{S} \in \mathrm{S}$ that induce on the fiber over every $\mathrm{X} \in \mathcal{O}$ a right adjoint functor.

By prop. 6.28 we have a canonical equivalence

$$\mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{E}, \mathcal{P}(\mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{E} \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}, \mathcal{S} \times \mathrm{S})$$

over S.

Especially for $\mathcal{O}^{\otimes} = \mathrm{Triv}^{\otimes}$ we get a canonical equivalence $\mathcal{P}(\mathcal{D}) \simeq \mathrm{Map}_{\mathrm{S}}(\mathcal{D}^{\mathrm{rev}}, \mathcal{S} \times \mathrm{S})$ over S.

So we get a canonical equivalence

$$\mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{E}, \mathcal{P}(\mathcal{D})) \simeq \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{E} \times_{\mathcal{O}} \mathcal{D}^{\mathrm{rev}}, \mathcal{S} \times \mathrm{S})$$

$$\simeq \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{D}^{\mathrm{rev}} \times_{\mathcal{O}} \mathcal{E}, \mathcal{S} \times \mathrm{S}) \simeq \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{D}^{\mathrm{rev}}, \mathcal{P}(\mathcal{E}^{\mathrm{rev}}))$$

over S that restricts to an equivalence

$$\mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{E}, \mathcal{D}) \simeq \mathrm{Fun}_{\mathcal{O}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{D}^{\mathrm{rev}}, \mathcal{E}^{\mathrm{rev}})$$

over S.

Specializing to our situation we make the following definitions:

Given cocartesian S-families of $\mathrm{LM}^{\otimes}$-monoidal categories $\mathcal{N}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}, \mathcal{N}'^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ we write

- $\mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{N}, \mathcal{N}') := \{\mathrm{id}\} \times_{\mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{C},\mathcal{C})} \mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes,\mathrm{lax}}(\mathcal{N}, \mathcal{N}')$,
- $\mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S}}(\mathcal{N}, \mathcal{N}') := \{\mathrm{id}\} \times_{\mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes}(\mathcal{C},\mathcal{C})} \mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes}(\mathcal{N}, \mathcal{N}')$,
- $\mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}(\mathcal{N}, \mathcal{N}') := \{\mathrm{id}\} \times_{\mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C},\mathcal{C})} \mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{N}, \mathcal{N}')$.

So we get canonical equivalences

$$\mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{N}^{\mathrm{rev}}, \mathcal{N}'^{\mathrm{rev}}) \simeq \mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{N}', \mathcal{N})$$

and

$$\mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{C}^{\mathrm{rev}}) \simeq \mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C}, \mathcal{C})$$

over S and so a canonical equivalence

$$\mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}(\mathcal{N}^{\mathrm{rev}}, \mathcal{N'}^{\mathrm{rev}}) =$$

$$\{\mathrm{id}\} \times_{\mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C}^{\mathrm{rev}},\mathcal{C}^{\mathrm{rev}})} \mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{N}^{\mathrm{rev}}, \mathcal{N'}^{\mathrm{rev}}) \simeq$$

$$\{\mathrm{id}\} \times_{\mathrm{Fun}_{\mathrm{Ass}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{C},\mathcal{C})} \mathrm{Fun}_{\mathrm{LM}}^{/\mathrm{S},\otimes,\mathrm{lax},\mathrm{R}}(\mathcal{N}', \mathcal{N}) = \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}(\mathcal{N}', \mathcal{N})$$

over S.

Especially we get a canonical equivalence

$$\mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \simeq \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}(\mathcal{M}, \mathcal{C})$$

over S.

By lemma 6.57 we have a canonical equivalence

$$\mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \mathcal{C})) \simeq$$

$$\mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \delta^*(\mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})))$$

over S that induces on sections a canonical equivalence

$$\mathrm{Fun}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \mathcal{C})) \simeq$$

$$\mathrm{LinFun}_{\mathcal{C}}^{\mathrm{lax}}(\mathcal{M}, \delta^*(\mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})))$$

over $\mathrm{Fun}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{C})) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathcal{M}, \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C}))$.

Consequently it is enough to find a canonical functor

$$\mathcal{M}^{\mathrm{rev}} \to \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \mathcal{C})$$

over S such that the composition

$$\mathcal{M}^{\mathrm{rev}} \to \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \mathcal{C}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{C})$$

corresponds to $\theta : \mathcal{M} \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}^{\mathrm{rev}}, \mathcal{C})$.

By lemma 6.58 we have a canonical equivalence

$$\alpha : \mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \simeq \mathcal{M}^{\mathrm{rev}}$$

over S.

The composition

$$\mathcal{M}^{\mathrm{rev}} \simeq \mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}})$$

is adjoint to the left action functor $\mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}^{\mathrm{rev}} \to \mathcal{M}^{\mathrm{rev}}$ over S of the $\mathcal{C}^{\mathrm{rev}}$-left module $\mathcal{M}^{\mathrm{rev}}$ in $\mathsf{Cat}_{\infty/\mathrm{S}}$.

α induces on the fiber over every $\mathsf{s} \in \mathrm{S}$ the canonical equivalence

$$\mathrm{LinFun}_{\mathcal{C}_{\mathsf{s}}^{\mathrm{op}}}(\mathcal{C}_{\mathsf{s}}^{\mathrm{op}}, \mathcal{M}_{\mathsf{s}}^{\mathrm{op}}) \simeq \mathcal{M}_{\mathsf{s}}^{\mathrm{op}}.$$

So every $\mathcal{C}_{\mathsf{s}}$-linear functor $\mathcal{C}_{\mathsf{s}} \to \mathcal{M}_{\mathsf{s}}$ is of the form $-\otimes \mathrm{X}$ for some $\mathrm{X} \in \mathcal{M}_{\mathsf{s}}$ and so admits a right adjoint as $\mathcal{M}_{\mathsf{s}}$ is enriched in $\mathcal{C}_{\mathsf{s}}$. So every $\mathcal{C}_{\mathsf{s}}^{\mathrm{op}}$-linear functor $\mathcal{C}_{\mathsf{s}}^{\mathrm{op}} \to \mathcal{M}_{\mathsf{s}}^{\mathrm{op}}$ admits a left adjoint.

Thus the full subcategory inclusion

$$\mathcal{M}^{\mathrm{rev}} \simeq \mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \subset \mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}})$$

induces a full subcategory inclusion

$$\mathcal{M}^{\mathrm{rev}} \simeq \mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \subset \mathrm{LinFun}_{\mathcal{C}^{\mathrm{rev}}}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}).$$

So we get a full subcategory inclusion

$$\varphi : \mathcal{M}^{\mathrm{rev}} \simeq \mathrm{LinFun}^{/\mathrm{S}}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \subset \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}_{\mathcal{C}^{\mathrm{rev}}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}})$$

$$\simeq \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax},\mathrm{R}}_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \subset \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{C})$$

over S.

The functor

$$\mathcal{M}^{\mathrm{rev}} \xrightarrow{\varphi} \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{C})$$

over S is equivalent to the composition

$$\beta : \mathcal{M}^{\mathrm{rev}} \to \mathrm{Map}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}}) \simeq \mathrm{Map}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{M}, \mathcal{C}) \subset \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{C})$$

of functors over S, where the functor $\mathcal{M}^{\mathrm{rev}} \to \mathrm{Map}^{\mathrm{R}}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}}, \mathcal{M}^{\mathrm{rev}})$ is adjoint to the left action functor $\mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}^{\mathrm{rev}} \to \mathcal{M}^{\mathrm{rev}}$ over S of the $\mathcal{C}^{\mathrm{rev}}$-left module $\mathcal{M}^{\mathrm{rev}}$ in $\mathsf{Cat}_{\infty/\mathrm{S}}$.

So $\mathcal{M}^{\mathrm{rev}} \xrightarrow{\beta} \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{C}) \subset \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{P}(\mathcal{C})) \simeq \mathrm{Map}_{\mathrm{S}}(\mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}, \mathcal{S} \times \mathrm{S})$ is adjoint to the functor $\mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \simeq \mathcal{C}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \to \mathcal{M}^{\mathrm{rev}} \times_{\mathrm{S}} \mathcal{M} \to \mathcal{S}$.

Thus the functor

$$\mathcal{M}^{\mathrm{rev}} \xrightarrow{\varphi} \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \to \mathrm{Map}_{\mathrm{S}}(\mathcal{M}, \mathcal{C})$$

over S is adjoint to θ.

□

The following two lemmata were used for the proof of proposition 6.55, where we use the following notation:

Given S-families of operads $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}, \mathcal{N}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ for some category S with $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes} \simeq \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{N}^{\otimes}$ we set

$$\mathrm{LinFun}^{/\mathrm{S},\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) := \mathrm{S} \times_{\mathrm{Alg}^{/\mathrm{S}}_{\mathcal{C}/\mathrm{Ass}}(\mathcal{C})} \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{M}/\mathrm{LM}}(\mathcal{N})$$

and write $\mathrm{LinFun}^{\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ if S is contractible.

Denote $\mathrm{LinFun}^{/\mathrm{S}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \subset \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ the full subcategory spanned by the $\mathcal{C}_{\mathsf{s}}$-linear functors $\mathcal{M}_{\mathsf{s}} \to \mathcal{N}_{\mathsf{s}}$ for some $\mathsf{s} \in \mathrm{S}$.

We have a canonical equivalence

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{LinFun}^{/\mathrm{S},\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}))^{\simeq} \simeq \{\mathrm{id}\} \times_{\mathrm{Op}^{\mathrm{S}}_{\infty/\mathrm{Ass}^{\otimes}}(\mathcal{C}^{\otimes}, \mathcal{C}^{\otimes})} \mathrm{Op}^{\mathrm{S}}_{\infty/\mathrm{LM}^{\otimes}}(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}) \simeq$$

$$(\{\mathcal{C}^{\otimes}\} \times_{\mathrm{Op}^{\mathrm{S}}_{\infty/\mathrm{Ass}^{\otimes}}} \mathrm{Op}^{\mathrm{S}}_{\infty/\mathrm{LM}^{\otimes}})(\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}),$$

where for every operad $\mathcal{O}^{\otimes}$ the category $\mathrm{Op}^{\mathrm{S}}_{\infty/\mathcal{O}^{\otimes}}$ denotes the category of S-families of operads over $\mathcal{O}^{\otimes}$.

If the functors $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}, \mathcal{N}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ are maps of cocartesian fibrations over $\mathrm{LM}^{\otimes}$ so that $\mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}$ classify $\mathrm{LM}^{\otimes}$-monoids of $\mathsf{Cat}_{\infty/\mathrm{S}}$, the last equivalence restricts to an equivalence

$$\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{LinFun}^{/\mathrm{S}}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}))^{\simeq} \simeq (\{\mathcal{C}^{\otimes}\} \times_{\mathrm{Alg}_{\mathrm{Ass}}(\mathsf{Cat}_{\infty/\mathrm{S}})} \mathrm{Alg}_{\mathrm{LM}}(\mathsf{Cat}_{\infty/\mathrm{S}}))(\mathcal{M}, \mathcal{N})$$

$$\simeq \mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/\mathrm{S}})(\mathcal{M}, \mathcal{N}).$$

Lemma 6.57. *Let* S *be a category and* $\mathcal{N}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ *a* $\mathrm{LM}^{\otimes}$*-monoid of* $\mathsf{Cat}_{\infty/\mathrm{S}}$ *that exhibits a cartesian fibration* $\mathcal{N} \to \mathrm{S}$ *as a left-module over some cartesian fibration* $\mathcal{C} \to \mathrm{S}$*. Set* $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{N}^{\otimes}$*.*

Let $\mathcal{M}^{\otimes} \to \mathrm{LM}^{\otimes} \times \mathrm{S}$ *be a cocartesian* S*-family of operads over* $\mathrm{LM}^{\otimes}$ *such that we have an equivalence* $\mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes} \simeq \mathcal{C}^{\otimes}$ *over* S*.*

Let $\psi : \mathrm{K} \to \mathrm{S}$ *be a cocartesian fibration.*

Denote $\delta : \mathcal{C}^{\otimes} \simeq \mathrm{Map}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{C})^{\otimes}$ *the monoidal functor over* S *induced by* ψ*.*

Denote $\delta^*(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})^{\otimes}) \to \mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})^{\otimes}$ *a cartesian lift of* δ *with respect to the cartesian fibration* $\mathrm{LMod}(\mathsf{Cat}_{\infty/\mathrm{S}}) \to \mathrm{Alg}(\mathsf{Cat}_{\infty/\mathrm{S}})$*.*

So $\delta^*(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})^{\otimes})$ *is a* $\mathrm{LM}^{\otimes}$*-monoid of* $\mathsf{Cat}_{\infty/\mathrm{S}}$ *that exhibits the cartesian fibration* $\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N}) \to \mathrm{S}$ *as a left module over the cartesian fibration* $\mathcal{C} \to \mathrm{S}$*.*

There is a canonical equivalence

$$\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \mathcal{N})) \simeq \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \delta^*(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})))$$

over S *that induces on the fiber over* $\mathsf{s} \in \mathrm{S}$ *the canonical equivalence*

$$\mathrm{Fun}(\mathrm{K}_{\mathsf{s}}, \mathrm{LinFun}_{\mathcal{C}}^{\mathrm{lax}}(\mathcal{M}_{\mathsf{s}}, \mathcal{N}_{\mathsf{s}})) \simeq \mathrm{LinFun}_{\mathcal{C}}^{\mathrm{lax}}(\mathcal{M}_{\mathsf{s}}, \delta_{\mathsf{s}}^*(\mathcal{N}_{\mathsf{s}}^{\mathrm{K}_{\mathsf{s}}})).$$

Proof. By remark 5.5 1. $\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ is a $\mathrm{LM}^{\otimes}$-monoid of $\mathsf{Cat}_{\infty/\mathrm{S}}$ that exhibits the cartesian fibration $\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N}) \to \mathrm{S}$ as a left module over the cartesian fibration $\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{C}) \to \mathrm{S}$.

$\delta : \mathcal{C}^{\otimes} \simeq \mathrm{Map}_{\mathrm{S}}(\mathrm{S}, \mathcal{C})^{\otimes} \to \mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{C})^{\otimes}$ is a map of associative monoids in $\mathsf{Cat}_{\infty/\mathrm{S}}$, whose underlying functor $\mathcal{C} \simeq \mathrm{Map}_{\mathrm{S}}(\mathrm{S}, \mathcal{C}) \to \mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{C})$ is a map of cartesian fibrations over S induced by the unique map $\mathrm{K} \to \mathrm{S}$ of cocartesian fibrations over S.

Hence we can apply lemma 6.62 to deduce that the commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}/\mathrm{LM}}^{/\mathrm{S}}(\delta^*(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N}))) & \longrightarrow & \mathrm{Alg}_{\mathcal{M}/\mathrm{LM}}^{/\mathrm{S}}(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})) \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{\mathcal{C}/\mathrm{Ass}}^{/\mathrm{S}}(\mathcal{C}) & \longrightarrow & \mathrm{Alg}_{\mathcal{C}/\mathrm{Ass}}^{/\mathrm{S}}(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{C})) \end{array}$$

over S is a pullback square.

Pulling back this square along the section of $\mathrm{Alg}_{\mathcal{C}/\mathrm{Ass}}^{/\mathrm{S}}(\mathcal{C}) \to \mathrm{S}$ corresponding to the identity of $\mathcal{C}^{\otimes}$ we get a canonical equivalence

$$\mathrm{S} \times_{\mathrm{Alg}_{\mathcal{C}/\mathrm{Ass}}^{/\mathrm{S}}(\mathcal{C})} \mathrm{Alg}_{\mathcal{M}/\mathrm{LM}}^{/\mathrm{S}}(\delta^*(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N}))) \simeq$$

$$\mathrm{S} \times_{\mathrm{Alg}_{\mathcal{C}/\mathrm{Ass}}^{/\mathrm{S}}(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{C}))} \mathrm{Alg}_{\mathcal{M}/\mathrm{LM}}^{/\mathrm{S}}(\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathcal{N})).$$

The desired equivalence over S is the composition of canonical equivalences

$$\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathrm{LinFun}_{\mathcal{C}}^{/\mathrm{S},\mathrm{lax}}(\mathcal{M}, \mathcal{N})) \simeq$$

$$\mathrm{S} \times_{\mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}_{\mathcal{C}/\mathrm{Ass}}^{/\mathrm{S}}(\mathcal{C}))} \mathrm{Map}_{\mathrm{S}}(\mathrm{K}, \mathrm{Alg}_{\mathcal{M}/\mathrm{LM}}^{/\mathrm{S}}(\mathcal{N})) \simeq$$

$$S \times_{\mathrm{Alg}^{/S}_{\mathcal{C}/\mathrm{Ass}}(\mathrm{Map}_S(K,\mathcal{C}))} \mathrm{Alg}^{/S}_{\mathcal{M}/\mathrm{LM}}(\mathrm{Map}_S(K,\mathcal{N})) \simeq$$

$$S \times_{\mathrm{Alg}^{/S}_{\mathcal{C}/\mathrm{Ass}}(\mathcal{C})} \mathrm{Alg}^{/S}_{\mathcal{M}/\mathrm{LM}}(\delta^*(\mathrm{Map}_S(K,\mathcal{N}))) =$$

$$\mathrm{LinFun}^{/S,\mathrm{lax}}_{\mathcal{C}}(\mathcal{M}, \delta^*(\mathrm{Map}_S(K,\mathcal{N})))$$

over S, where the first equivalence exists as the functor $\mathrm{Map}_S(K,-) : \mathsf{Cat}_{\infty/S} \to \mathsf{Cat}_{\infty/S}$ preserves pullbacks being the right adjoint of the functor $K \times_S - : \mathsf{Cat}_{\infty/S} \to \mathsf{Cat}_{\infty/S}$, the second equivalence is due to remark 5.11 and the third equivalence is those from above.

□

Lemma 6.58. *Let* S *be a category and* $\mathcal{N} \to S$ *a left module in* $\mathsf{Cat}_{\infty/S}$ *over some functor* $\mathcal{C} \to S$.

Denote

$$\psi_{\mathcal{N}} : \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{C},\mathcal{N}) \to \mathrm{Map}^{/S}(\mathcal{C},\mathcal{N}) \to \mathrm{Map}^{/S}(S,\mathcal{N}) \simeq \mathcal{N}$$

the composition of the forgetful functor over S *and the functor over* S *induced by the unit* $S \to \mathcal{C}$ *of the associative monoid* $\mathcal{C}$ *of* $\mathsf{Cat}_{\infty/S}$.

$\psi_{\mathcal{N}}$ *is an equivalence.*

Proof. By Yoneda it is enough to show that for every category K over S the induced map

$$\mathsf{Cat}_{\infty/S}(K,\psi_{\mathcal{N}}) : \mathsf{Cat}_{\infty/S}(K, \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{C},\mathcal{N})) \to \mathsf{Cat}_{\infty/S}(K,\mathcal{N})$$

is an equivalence.

The map $\mathsf{Cat}_{\infty/S}(K,\psi_{\mathcal{N}})$ is equivalent to the map

$$\mathsf{Cat}_{\infty/K}(K, K\times_S\psi_{\mathcal{N}}) : \mathsf{Cat}_{\infty/K}(K, K\times_S\mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{C},\mathcal{N})) \simeq \mathsf{Cat}_{\infty/K}(K, K\times_S\mathcal{N}).$$

The functor $K \times_S \psi_{\mathcal{N}} : K \times_S \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{C},\mathcal{N}) \to K \times_S \mathcal{N}$ over K factors as

$$K \times_S \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{C},\mathcal{N}) \simeq \mathrm{LinFun}^{/K}_{K\times_S\mathcal{C}}(K \times_S \mathcal{C}, K \times_S \mathcal{N}) \xrightarrow{\psi_{K\times_S\mathcal{N}}} K \times_S \mathcal{N}$$

over K so that we can reduce to the case K = S.

Let $\mathcal{M},\mathcal{M}' \in \mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/S})$. We have a canonical equivalence

$$\mathsf{Cat}_{\infty/S}(S, \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{M},\mathcal{M}')) \simeq \mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/S})(\mathcal{M},\mathcal{M}')$$

and the forgetful functor $\mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{M},\mathcal{M}') \to \mathrm{Map}^{/S}(\mathcal{M},\mathcal{M}')$ over S induces the forgetful map

$$\mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/S})(\mathcal{M},\mathcal{M}') \simeq \mathsf{Cat}_{\infty/S}(S, \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{M},\mathcal{M}'))$$

$$\to \mathsf{Cat}_{\infty/S}(S, \mathrm{Map}^{/S}(\mathcal{M},\mathcal{M}')) \simeq \mathsf{Cat}_{\infty/S}(\mathcal{M},\mathcal{M}').$$

So $\mathsf{Cat}_{\infty/S}(S,\psi_{\mathcal{N}})$ factors as

$$\mathsf{Cat}_{\infty/S}(S, \mathrm{LinFun}^{/S}_{\mathcal{C}}(\mathcal{C},\mathcal{N})) \simeq \mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/S})(\mathcal{C},\mathcal{N}) \to \mathsf{Cat}_{\infty/S}(\mathcal{C},\mathcal{N})$$

$$\to \mathsf{Cat}_{\infty/S}(S,\mathcal{N}),$$

where the last map is induced by the unit of $\mathcal{C}$.
But the map $\mathrm{LMod}_{\mathcal{C}}(\mathsf{Cat}_{\infty/S})(\mathcal{C},\mathcal{N}) \to \mathsf{Cat}_{\infty/S}(\mathcal{C},\mathcal{N}) \to \mathsf{Cat}_{\infty/S}(S,\mathcal{N})$ is an equivalence as the unit $S \to \mathcal{C}$ of $\mathcal{C}$ exhibits $\mathcal{C}$ as the free left $\mathcal{C}$-module on the tensorunit S of $\mathsf{Cat}_{\infty/S}$. □

6.2.3 Pulling back enriched categories

Let $\mathcal{M}^\otimes$ be an operad over $\mathrm{LM}^\otimes$ that exhibits a category $\mathcal{M}$ as enriched over a locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.

Let $\mathcal{B}^\otimes \to \mathrm{Ass}^\otimes$ be a locally cocartesian fibration of operads and $\mathrm{F} : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ a map of operads over $\mathrm{Ass}^\otimes$ that admits a right adjoint $\mathrm{G} : \mathcal{C}^\otimes \to \mathcal{B}^\otimes$ relative to $\mathrm{Ass}^\otimes$.

We will show in the following that one can pullback $\mathcal{M}^\otimes$ along $\mathrm{F} : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ to obtain an operad $\mathrm{F}^*(\mathcal{M})^\otimes$ over $\mathrm{LM}^\otimes$ that exhibits $\mathcal{M}$ as enriched over the locally cocartesian fibration of operads $\mathcal{B}^\otimes \to \mathrm{Ass}^\otimes$.

We start with the following construction:

Construction 6.59. *Let* $\mathcal{M}^\otimes$ *be an operad over* $\mathrm{LM}^\otimes$. *Set* $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$.

Let $\mathrm{F} : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ *be a map of operads over* $\mathrm{Ass}^\otimes$.

Pulling back the $\mathrm{LM}^\otimes$*-monoidal category* $\bar{\mathcal{M}}^\otimes := \mathrm{Env}_{\mathrm{LM}^\otimes}(\mathcal{M})^\otimes \to \mathrm{LM}^\otimes$ *along the monoidal functor*

$$\bar{\mathrm{F}} := \mathrm{Env}_{\mathrm{Ass}^\otimes}(\mathrm{F}) : \bar{\mathcal{B}}^\otimes := \mathrm{Env}_{\mathrm{Ass}^\otimes}(\mathcal{B})^\otimes \to \bar{\mathcal{C}}^\otimes := \mathrm{Env}_{\mathrm{Ass}^\otimes}(\mathcal{C})^\otimes$$

we get a $\mathrm{LM}^\otimes$*-monoidal category* $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \mathrm{LM}^\otimes$ *that exhibits* $\bar{\mathcal{D}} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \bar{\mathcal{M}}^\otimes$ *as left module over the monoidal category* $\bar{\mathcal{B}}$.

Denote $\mathrm{F}^*(\mathcal{M})^\otimes \subset \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes$ *the full suboperad spanned by the objects that belong to* $\mathcal{B}$ *or* $\mathcal{D}$.

Then we have a canonical equivalence $\mathcal{B}^\otimes \simeq \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathrm{F}^*(\mathcal{M})^\otimes$ *of operads over* $\mathrm{Ass}^\otimes$ *and a canonical equivalence* $\mathcal{D} \simeq \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathrm{F}^*(\mathcal{M})^\otimes$.

The map $\mathrm{F}^*(\mathcal{M})^\otimes \subset \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \bar{\mathcal{M}}^\otimes$ *of operads over* $\mathrm{LM}^\otimes$ *induces a map* $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$ *of operads over* $\mathrm{LM}^\otimes$, *whose fiber over* $\mathfrak{m} \in \mathrm{LM}$ *is the identity of* $\mathcal{D}$ *and whose pullback to* $\mathrm{Ass}^\otimes$ *is* $\mathrm{F} : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$.

Remark 6.60. *If* $\mathcal{M}^\otimes$ *is a* $\mathrm{LM}^\otimes$*-monoidal category and* $\mathrm{F} : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ *is a monoidal functor, the definition of* $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$ *given in construction 6.59 extends the usual one.*

Proof. By lemma 6.53 the full subcategory inclusion $\mathcal{B}^\otimes \subset \bar{\mathcal{B}}^\otimes$ admits a left adjoint $\bar{\mathcal{B}}^\otimes \to \mathcal{B}^\otimes$ relative to $\mathrm{Ass}^\otimes$ and the full subcategory inclusion $\mathcal{M}^\otimes \subset \bar{\mathcal{M}}^\otimes$ admits a left adjoint $\mathrm{L} : \bar{\mathcal{M}}^\otimes \to \mathcal{M}^\otimes$ relative to $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is a left adjoint $\bar{\mathcal{C}}^\otimes \to \mathcal{C}^\otimes$ of the full subcategory inclusion $\mathcal{C}^\otimes \subset \bar{\mathcal{C}}^\otimes$ relative to $\mathrm{Ass}^\otimes$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is a left adjoint $\bar{\mathcal{D}} \to \mathcal{D}$ of the full subcategory inclusion $\mathcal{D} \subset \bar{\mathcal{D}}$.

The monoidal functor F extends to a monoidal functor $\bar{\mathrm{F}} : \bar{\mathcal{B}}^\otimes \to \bar{\mathcal{C}}^\otimes$ that commutes with the left adjoints $\bar{\mathcal{B}}^\otimes \to \mathcal{B}^\otimes$ and $\bar{\mathcal{C}}^\otimes \to \mathcal{C}^\otimes$ as monoidal functors.

Thus we get a $\mathrm{LM}^\otimes$-monoidal functor $\psi : \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \mathrm{F}^*(\mathcal{M})^\otimes$ commuting with the $\mathrm{LM}^\otimes$-monoidal functors $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \bar{\mathcal{M}}^\otimes$ and $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the monoidal functor $\bar{\mathcal{B}}^\otimes \to \mathcal{B}^\otimes$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is the functor $\bar{\mathcal{D}} \to \mathcal{D}$.

Thus the $\mathrm{LM}^\otimes$-monoidal functor $\psi : \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \mathrm{F}^*(\mathcal{M})^\otimes$ induces on the fiber over every object of LM a localization and so admits a fully faithful right adjoint $\mathrm{F}^*(\mathcal{M})^\otimes \to \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes$ relative to $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^{\otimes}$ is the embedding $\mathcal{B}^{\otimes} \subset \bar{\mathcal{B}}^{\otimes}$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is the embedding $\mathcal{D} \subset \bar{\mathcal{D}}$.

Moreover as $\psi : \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^{\otimes} \to \mathrm{F}^*(\mathcal{M})^{\otimes}$ commutes with the $\mathrm{LM}^{\otimes}$-monoidal functors $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^{\otimes} \to \bar{\mathcal{M}}^{\otimes}$ and $\mathrm{F}^*(\mathcal{M})^{\otimes} \to \mathcal{M}^{\otimes}$ and $\bar{\mathrm{F}} : \bar{\mathcal{B}}^{\otimes} \to \bar{\mathcal{C}}^{\otimes}$ restricts to F, the lax $\mathrm{LM}^{\otimes}$-monoidal embedding $\mathrm{F}^*(\mathcal{M})^{\otimes} \to \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^{\otimes}$ also commutes with the $\mathrm{LM}^{\otimes}$-monoidal functors $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^{\otimes} \to \bar{\mathcal{M}}^{\otimes}$ and $\mathrm{F}^*(\mathcal{M})^{\otimes} \to \mathcal{M}^{\otimes}$.

□

As next we show that the canonical map $\mathrm{F}^*(\mathcal{M})^{\otimes} \to \mathcal{M}^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ is cartesian with respect to the forgetful functor $\mathrm{Op}_{\infty/\mathrm{LM}^{\otimes}} \to \mathrm{Op}_{\infty/\mathrm{Ass}^{\otimes}}$ so that the forgetful functor $\mathrm{Op}_{\infty/\mathrm{LM}^{\otimes}} \to \mathrm{Op}_{\infty/\mathrm{Ass}^{\otimes}}$ is a cartesian fibration.

Finally lemma 6.64 states that if $\mathcal{M}^{\otimes}$ exhibits a category $\mathcal{M}$ as enriched over a locally cocartesian fibration of operads $\mathcal{C}^{\otimes} \to \mathrm{Ass}^{\otimes}$ and $\mathrm{F} : \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes}$ admits a right adjoint relative to $\mathrm{Ass}^{\otimes}$, then $\mathrm{F}^*(\mathcal{M})^{\otimes}$ exhibits $\mathcal{M}$ as enriched over $\mathcal{B}^{\otimes} \to \mathrm{Ass}^{\otimes}$.

Proposition 6.61. *Let $\mathcal{M}^{\otimes}$ be an operad over $\mathrm{LM}^{\otimes}$. Set $\mathcal{C}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$.*

Let $\mathrm{F} : \mathcal{B}^{\otimes} \to \mathcal{C}^{\otimes}$ be a map of operads over $\mathrm{Ass}^{\otimes}$.

For every operad $\mathcal{Q}^{\otimes}$ over $\mathrm{LM}^{\otimes}$, where we set $\mathcal{A}^{\otimes} := \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{Q}^{\otimes}$, the canonical map $\mathrm{F}^*(\mathcal{M})^{\otimes} \to \mathcal{M}^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ induces a pullback square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\mathrm{F}^*(\mathcal{M})) & \longrightarrow & \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\mathcal{B}) & \longrightarrow & \mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\mathcal{C}). \end{array} \tag{39}$$

Especially the canonical map $\mathrm{F}^*(\mathcal{M})^{\otimes} \to \mathcal{M}^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ is cartesian with respect to the forgetful functor $\mathrm{Op}_{\infty/\mathrm{LM}^{\otimes}} \to \mathrm{Op}_{\infty/\mathrm{Ass}^{\otimes}}$.

Thus the forgetful functor $\theta : \mathrm{Op}_{\infty/\mathrm{LM}^{\otimes}} \to \mathrm{Op}_{\infty/\mathrm{Ass}^{\otimes}}$ is a cartesian fibration.

Moreover by remark 6.60 the subcategory inclusion $\mathrm{LMod}(\mathsf{Cat}_{\infty}) \simeq \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}^{\otimes}} \subset \mathrm{Op}_{\infty/\mathrm{LM}^{\otimes}}$ sends γ-cartesian morphisms to θ-cartesian morphisms, where γ denotes the cartesian fibration $\mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}^{\otimes}} \to \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}^{\otimes}}$ equivalent to the cartesian fibration $\mathrm{LMod}(\mathsf{Cat}_{\infty}) \to \mathrm{Alg}(\mathsf{Cat}_{\infty})$.

Proof. Square 39 embeds into the commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}^{\otimes}}(\bar{\mathrm{F}}^*(\bar{\mathcal{M}})) & \longrightarrow & \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\bar{\mathcal{M}}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\bar{\mathcal{B}}) & \longrightarrow & \mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}/}(\bar{\mathcal{C}}). \end{array} \tag{40}$$

Assume that we have already shown that square 39 is a pullback square.

Then the full subcategory inclusion $\mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\mathrm{F}^*(\mathcal{M})) \subset \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\bar{\mathrm{F}}^*(\bar{\mathcal{M}}))$ factors as

$$\mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\mathrm{F}^*(\mathcal{M})) \xrightarrow{\chi} \mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\mathcal{B}) \times_{\mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\mathcal{C})} \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\mathcal{M})$$

$$\subset \mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\bar{\mathcal{B}}) \times_{\mathrm{Alg}_{\mathcal{A}/\mathrm{Ass}}(\bar{\mathcal{C}})} \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\bar{\mathcal{M}}) \simeq \mathrm{Alg}_{\mathcal{Q}/\mathrm{LM}}(\bar{\mathrm{F}}^*(\bar{\mathcal{M}})).$$

Hence the canonical functor χ is fully faithful.

Let $\psi : \mathcal{Q}^\otimes \to \bar{\mathrm{F}}^*(\bar{\mathcal{M}}^\otimes)$ be a map of operads over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ induces a map $\mathcal{A}^\otimes \to \mathcal{B}^\otimes$ of operads over $\mathrm{Ass}^\otimes$ and such that the composition $\psi' : \mathcal{Q}^\otimes \xrightarrow{\psi} \bar{\mathrm{F}}^*(\bar{\mathcal{M}}^\otimes) \to \bar{\mathcal{M}}^\otimes$ factors through $\mathcal{M}^\otimes$.

ψ and ψ' induce on the fiber over $\mathfrak{m} \in \mathrm{LM}$ the same functor $\mathcal{Q} \to \bar{\mathcal{M}}$ that factors through $\mathcal{M}$. Hence $\psi : \mathcal{Q}^\otimes \to \bar{\mathrm{F}}^*(\bar{\mathcal{M}}^\otimes)$ factors through $\mathrm{F}^*(\mathcal{M})^\otimes$.

Thus χ is essentially surjective and so an equivalence.

So it remains to show that square 40 is a pullback square.
Set $\bar{\mathcal{Q}}^\otimes := \mathrm{Env}_{\mathrm{LM}}(\mathcal{Q})^\otimes$ and $\bar{\mathcal{A}}^\otimes := \mathrm{Env}_{\mathrm{LM}}(\mathcal{A})^\otimes$.

Using lemma 6.52 square 40 is equivalent to the commutative square

$$\begin{array}{ccc}
\mathrm{Fun}^\otimes_{\mathrm{LM}}(\bar{\mathcal{Q}}, \bar{\mathrm{F}}^*(\bar{\mathcal{M}})) & \longrightarrow & \mathrm{Fun}^\otimes_{\mathrm{LM}}(\bar{\mathcal{Q}}, \bar{\mathcal{M}}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^\otimes_{\mathrm{Ass}}(\bar{\mathcal{A}}, \bar{\mathcal{B}}) & \longrightarrow & \mathrm{Fun}^\otimes_{\mathrm{Ass}}(\bar{\mathcal{A}}, \bar{\mathcal{C}}).
\end{array}$$

So it is enough to check the following:

Let $\mathcal{M}^\otimes$ and $\mathcal{Q}^\otimes$ be $\mathrm{LM}^\otimes$-monoidal categories and $\mathrm{F} : \mathcal{C}^\otimes \to \mathcal{D}^\otimes$ a monoidal functor. Let $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$ be a γ-cartesian lift of F.
Then the commutative square

$$\begin{array}{ccc}
\mathrm{Fun}^\otimes_{\mathrm{LM}}(\mathcal{Q}, \mathrm{F}^*(\mathcal{M})) & \longrightarrow & \mathrm{Fun}^\otimes_{\mathrm{LM}}(\mathcal{Q}, \mathcal{M}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}^\otimes_{\mathrm{Ass}}(\mathcal{A}, \mathcal{B}) & \longrightarrow & \mathrm{Fun}^\otimes_{\mathrm{Ass}}(\mathcal{A}, \mathcal{C})
\end{array}$$

is a pullback square.
This square is a pullback square if and only if for every category T the induced square

$$\begin{array}{ccc}
\mathsf{Cat}_\infty(\mathrm{T}, \mathrm{Fun}^\otimes_{\mathrm{LM}}(\mathcal{Q}, \mathrm{F}^*(\mathcal{M}))) & \longrightarrow & \mathsf{Cat}_\infty(\mathrm{T}, \mathrm{Fun}^\otimes_{\mathrm{LM}}(\mathcal{Q}, \mathcal{M})) \\
\downarrow & & \downarrow \\
\mathsf{Cat}_\infty(\mathrm{T}, \mathrm{Fun}^\otimes_{\mathrm{Ass}}(\mathcal{A}, \mathcal{B})) & \longrightarrow & \mathsf{Cat}_\infty(\mathrm{T}, \mathrm{Fun}^\otimes_{\mathrm{Ass}}(\mathcal{A}, \mathcal{C}))
\end{array}$$

is a pullback square.
This square is equivalent to the commutative square

$$\begin{array}{ccc}
\mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}}(\mathcal{Q}, \mathrm{F}^*(\mathcal{M})^{\mathrm{T}}) & \longrightarrow & \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}}(\mathcal{Q}, \mathcal{M}^{\mathrm{T}}) \\
\downarrow & & \downarrow \\
\mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}}(\mathcal{A}, \mathcal{B}^{\mathrm{T}}) & \longrightarrow & \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}}(\mathcal{A}, \mathcal{C}^{\mathrm{T}}).
\end{array} \quad (41)$$

Being right adjoint to the functor $\mathrm{T} \times - : \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ the functor $\mathrm{Fun}(\mathrm{T}, -) : \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ preserves finite products and so lifts to a symmetric monoidal functor that induces functors

$$\beta : \mathrm{LMod}(\mathsf{Cat}_\infty) \to \mathrm{LMod}(\mathsf{Cat}_\infty), \ \mathrm{Alg}(\mathsf{Cat}_\infty) \to \mathrm{Alg}(\mathsf{Cat}_\infty)$$

that are equivalent to the functors $(-)^{\mathrm{T}} : \mathrm{LMod}(\mathsf{Cat}_\infty) \to \mathrm{LMod}(\mathsf{Cat}_\infty)$ respectively $(-)^{\mathrm{T}} : \mathrm{Alg}(\mathsf{Cat}_\infty) \to \mathrm{Alg}(\mathsf{Cat}_\infty)$.

The γ-cartesian morphisms are those that get equivalences in Cat_∞.

Thus β sends γ-cartesian morphisms to γ-cartesian morphisms so that $(\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes)^{\mathrm{T}} \to (\bar{\mathcal{M}}^\otimes)^{\mathrm{T}}$ factors as

$$(\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes)^{\mathrm{T}} \simeq (\bar{\mathrm{F}}^{\mathrm{T}})^*((\bar{\mathcal{M}}^\otimes)^{\mathrm{T}}) \to (\bar{\mathcal{M}}^\otimes)^{\mathrm{T}}$$

in $\mathrm{LMod}(\mathsf{Cat}_\infty)$.

Thus square 41 is equivalent to the commutative square

$$\begin{array}{ccc} \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}}(\mathcal{Q}, (\mathrm{F}^{\mathrm{T}})^*(\mathcal{M}^{\mathrm{T}})) & \longrightarrow & \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}}(\mathcal{Q}, \mathcal{M}^{\mathrm{T}}) \\ \downarrow & & \downarrow \\ \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}}(\mathcal{A}, \mathcal{B}^{\mathrm{T}}) & \longrightarrow & \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}}(\mathcal{A}, \mathcal{C}^{\mathrm{T}}). \end{array}$$

Consequently it is enough to see that the commutative square

$$\begin{array}{ccc} \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}}(\mathcal{Q}, \mathrm{F}^*(\mathcal{M})) & \longrightarrow & \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{LM}}(\mathcal{Q}, \mathcal{M}) \\ \downarrow & & \downarrow \\ \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}}(\mathcal{A}, \mathcal{B}) & \longrightarrow & \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}}(\mathcal{A}, \mathcal{C}) \end{array}$$

is a pullback square, which follows from the fact that $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$ is γ-cartesian.

□

Corollary 6.62. *Let* S *be a category and* $\mathcal{N}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ *a cartesian* S*-family of operads over* $\mathrm{LM}^\otimes$. *Set* $\mathcal{C}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{N}^\otimes$.

Let $\phi : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ *be a map of cartesian* S*-families of operads over* $\mathrm{Ass}^\otimes$.

Let $\chi : \phi^*(\mathcal{N}^\otimes) \to \mathcal{N}^\otimes$ *be a map of cartesian* S*-families of operads over* $\mathrm{LM}^\otimes$ *that is a cartesian lift of* $\phi : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ *with respect to the cartesian fibration* $\mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Op}_{\infty/\mathrm{LM}^\otimes}) \to \mathrm{Fun}(\mathrm{S}^{\mathrm{op}}, \mathrm{Op}_{\infty/\mathrm{Ass}^\otimes})$ *induced by taking pullback along the map of operads* $\mathrm{Ass}^\otimes \to \mathrm{LM}^\otimes$.

For every cocartesian S*-family* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ *of operads over* $\mathrm{LM}^\otimes$, *where we set* $\mathcal{D}^\otimes := \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$, *the commutative square*

$$\begin{array}{ccc} \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{M}/\mathrm{LM}}(\phi^*(\mathcal{N})) & \longrightarrow & \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{M}/\mathrm{LM}}(\mathcal{N}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{D}/\mathrm{Ass}}(\mathcal{B}) & \longrightarrow & \mathrm{Alg}^{/\mathrm{S}}_{\mathcal{D}/\mathrm{Ass}}(\mathcal{C}) \end{array} \tag{42}$$

of cartesian fibrations over S *is a pullback square.*

Proof. By remark 5.9 1. square 42 is a square of cartesian fibrations over S.

Consequently it is enough to see that square 42 induces a pullback square on the fiber over every object s of S.

Square 42 induces on the fiber over every object s of S the commutative square

$$\begin{array}{ccc} \mathrm{Alg}_{\mathcal{M}_\mathsf{s}/\mathrm{LM}}(\phi_\mathsf{s}^*(\mathcal{N}_\mathsf{s})) & \longrightarrow & \mathrm{Alg}_{\mathcal{M}_\mathsf{s}/\mathrm{LM}}(\mathcal{N}_\mathsf{s}) \\ \downarrow & & \downarrow \\ \mathrm{Alg}_{\mathcal{D}_\mathsf{s}/\mathrm{Ass}}(\mathcal{B}_\mathsf{s}) & \longrightarrow & \mathrm{Alg}_{\mathcal{D}_\mathsf{s}/\mathrm{Ass}}(\mathcal{C}_\mathsf{s}) \end{array}$$

of categories.

Consequently we can reduce to the case that S is contractible.

But then the statement follows from proposition 6.61.

□

Remark 6.63. *Let a commutative square*

$$\begin{array}{ccc} \mathcal{B}^\otimes & \longrightarrow & \mathcal{B}'^\otimes \\ \downarrow{\scriptstyle \mathrm{F}} & & \downarrow{\scriptstyle \mathrm{F}'} \\ \mathcal{C}^\otimes & \longrightarrow & \mathcal{C}'^\otimes \end{array}$$

of operads over $\mathrm{Ass}^\otimes$ *be given and a map* $\mathcal{M}^\otimes \to \mathcal{M}'^\otimes$ *of operads over* $\mathrm{LM}^\otimes$*, whose pullback to* $\mathrm{Ass}^\otimes$ *is the map* $\mathcal{C}^\otimes \to \mathcal{C}'^\otimes$ *of operads over* $\mathrm{Ass}^\otimes$ *and whose fiber over* $\mathfrak{m} \in \mathrm{LM}$ *is a functor* $\mathcal{D} \to \mathcal{D}'$.

If the maps $\mathcal{M}^\otimes \to \mathcal{M}'^\otimes$ *of operads over* $\mathrm{LM}^\otimes$ *and* $\mathcal{B}^\otimes \to \mathcal{B}'^\otimes$ *of operads over* $\mathrm{Ass}^\otimes$ *are fully faithful, the map* $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathrm{F}'^*(\mathcal{M}')^\otimes$ *of operads over* $\mathrm{LM}^\otimes$ *is fully faithful.*

Proof. The map $\mathcal{M}^\otimes \to \mathcal{M}'^\otimes$ of operads over $\mathrm{LM}^\otimes$ extends to a $\mathrm{LM}^\otimes$-monoidal functor $\bar{\mathcal{M}}^\otimes \to \bar{\mathcal{M}}'^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the monoidal functor $\bar{\mathcal{C}}^\otimes \to \bar{\mathcal{C}}'^\otimes$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is a functor $\bar{\mathcal{D}} \to \bar{\mathcal{D}}'$.

The induced square

$$\begin{array}{ccc} \bar{\mathcal{B}}^\otimes & \longrightarrow & \bar{\mathcal{B}}'^\otimes \\ \downarrow{\scriptstyle \bar{\mathrm{F}}} & & \downarrow{\scriptstyle \bar{\mathrm{F}}'} \\ \bar{\mathcal{C}}^\otimes & \longrightarrow & \bar{\mathcal{C}}'^\otimes \end{array}$$

of monoidal categories yields a $\mathrm{LM}^\otimes$-monoidal functor $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \bar{\mathrm{F}}'^*(\bar{\mathcal{M}}')^\otimes$ that commutes with the $\mathrm{LM}^\otimes$-monoidal functors $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \bar{\mathcal{M}}^\otimes$ and

$\bar{\mathrm{F}}'^*(\bar{\mathcal{M}}')^\otimes \to \bar{\mathcal{M}}'^\otimes$ and whose pullback to $\mathrm{Ass}^\otimes$ is the monoidal functor $\bar{\mathcal{B}}^\otimes \to \bar{\mathcal{B}}'^\otimes$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is the functor $\bar{\mathcal{D}} \to \bar{\mathcal{D}}'$.

Thus the $\mathrm{LM}^\otimes$-monoidal functor $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \bar{\mathrm{F}}'^*(\bar{\mathcal{M}}')^\otimes$ restricts to a map $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathrm{F}'^*(\mathcal{M}')^\otimes$ of operads over $\mathrm{LM}^\otimes$ that commutes with the maps $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathcal{M}^\otimes$ and $\mathrm{F}'^*(\mathcal{M}')^\otimes \to \mathcal{M}'^\otimes$ of operads over $\mathrm{LM}^\otimes$ and whose pullback to $\mathrm{Ass}^\otimes$ is the map $\mathcal{B}^\otimes \to \mathcal{B}'^\otimes$ of operads over $\mathrm{Ass}^\otimes$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is the functor $\mathcal{D} \to \mathcal{D}'$.

If the maps $\mathcal{M}^\otimes \to \mathcal{M}'^\otimes$ of operads over $\mathrm{LM}^\otimes$ and $\mathcal{B}^\otimes \to \mathcal{B}'^\otimes$ of operads over $\mathrm{Ass}^\otimes$ are fully faithful (so that the $\mathrm{LM}^\otimes$-monoidal functor $\bar{\mathcal{M}}^\otimes \to \bar{\mathcal{M}}'^\otimes$ and monoidal functor $\bar{\mathcal{B}}^\otimes \to \bar{\mathcal{B}}'^\otimes$ are fully faithful), the $\mathrm{LM}^\otimes$-monoidal functor $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \bar{\mathrm{F}}'^*(\bar{\mathcal{M}}')^\otimes$ and so its restriction $\mathrm{F}^*(\mathcal{M})^\otimes \to \mathrm{F}'^*(\mathcal{M}')^\otimes$ are fully faithful.

□

Lemma 6.64. *Let $\mathcal{M}^\otimes$ be an operad over $\mathrm{LM}^\otimes$ that exhibits a category $\mathcal{D}$ as enriched over a locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.*

Let $\mathcal{B}^\otimes \to \mathrm{Ass}^\otimes$ be a locally cocartesian fibration of operads and $\mathrm{F} : \mathcal{B}^\otimes \to \mathcal{C}^\otimes$ a map of operads over $\mathrm{Ass}^\otimes$ that admits a right adjoint $\mathrm{G} : \mathcal{C}^\otimes \to \mathcal{B}^\otimes$ relative to $\mathrm{Ass}^\otimes$.

The operad $\mathrm{F}^*(\mathcal{M})^\otimes$ over $\mathrm{LM}^\otimes$ exhibits $\mathcal{D}$ as enriched over $\mathcal{B}^\otimes \to \mathrm{Ass}^\otimes$.

The morphism object of $\mathrm{F}^*(\mathcal{M})^\otimes$ of two objects X, Y of $\mathcal{D}$ is given by $\mathrm{G}([\mathrm{X},\mathrm{Y}]) \in \mathcal{B}$, where $[\mathrm{X},\mathrm{Y}] \in \mathcal{C}$ denotes the morphism object of X and Y of $\mathcal{M}^\otimes$.

Proof. Pulling back the $\mathrm{LM}^\otimes$-monoidal category $\bar{\mathcal{M}}^\otimes := \mathrm{Env}_{\mathrm{LM}}(\mathcal{M})^\otimes \to \mathrm{LM}^\otimes$ along the monoidal functor $\bar{\mathrm{F}} := \mathrm{Env}_{\mathrm{Ass}}(\mathrm{F}) : \bar{\mathcal{B}}^\otimes := \mathrm{Env}_{\mathrm{Ass}}(\mathcal{B})^\otimes \to \bar{\mathcal{C}}^\otimes := \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^\otimes$ we get a $\mathrm{LM}^\otimes$-monoidal category $\bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes \to \mathrm{LM}^\otimes$ that exhibits $\bar{\mathcal{D}} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \bar{\mathcal{M}}^\otimes$ as a left module over the monoidal category $\bar{\mathcal{B}}^\otimes$ and $\mathrm{F}^*(\mathcal{M})^\otimes \subset \bar{\mathrm{F}}^*(\bar{\mathcal{M}})^\otimes$ is defined to be the full suboperad spanned by the objects that belong to $\mathcal{D}$ or $\mathcal{B}$.

Being a 2-functor $\mathrm{Env}_{\mathrm{Ass}} : \mathrm{Op}_{\infty/\mathrm{Ass}^\otimes} \to \mathrm{Op}^{\mathrm{cocart}}_{\infty/\mathrm{Ass}^\otimes}$ sends the adjunction $\mathrm{F} : \mathcal{B}^\otimes \rightleftarrows \mathcal{C}^\otimes : \mathrm{G}$ relative to $\mathrm{Ass}^\otimes$ to an adjunction $\bar{\mathrm{F}} : \bar{\mathcal{B}}^\otimes \rightleftarrows \bar{\mathcal{C}}^\otimes : \bar{\mathrm{G}}$ relative to $\mathrm{Ass}^\otimes$.

Given two objects X, Y of $\mathcal{D}$ by lemma 6.49 the morphism object $[\mathrm{X},\mathrm{Y}] \in \mathcal{C}$ of $\mathcal{M}^\otimes$ is a morphism object of $\bar{\mathcal{M}}^\otimes$.
So given an object $\mathrm{A} \in \bar{\mathcal{B}}$ we have a canonical equivalence

$$\bar{\mathcal{B}}(\mathrm{A}, \mathrm{G}([\mathrm{X},\mathrm{Y}])) \simeq \bar{\mathcal{C}}(\bar{\mathrm{F}}(\mathrm{A}), [\mathrm{X},\mathrm{Y}]) \simeq \bar{\mathcal{D}}(\bar{\mathrm{F}}(\mathrm{A}) \otimes \mathrm{X}, \mathrm{Y}) \simeq \bar{\mathrm{F}}^*(\bar{\mathcal{D}})(\mathrm{A} \otimes \mathrm{X}, \mathrm{Y}).$$

Thus the statement follows from lemma 6.65.

□

Lemma 6.65. *Let $\varphi : \mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$ be a locally cocartesian fibration of operads and $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes$ an operad over $\mathrm{LM}^\otimes$ that exhibits a category $\mathcal{M}$ as pseudo-enriched over the monoidal category $\varphi' : \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^\otimes \to \mathrm{Ass}^\otimes$.*

Let $\mathcal{N} \subset \mathcal{M}$ be a full subcategory. Denote $\mathcal{N}^\otimes \subset \mathcal{M}^\otimes$ the full suboperad spanned by the objects that belong to $\mathcal{N}$ or $\mathcal{C}$.

If every objects $\mathrm{X}, \mathrm{Y} \in \mathcal{N}$ admit a morphism object $[\mathrm{X},\mathrm{Y}]$ in $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ that belongs to $\mathcal{C}$, the map of operads $\mathcal{N}^\otimes \to \mathrm{LM}^\otimes$ exhibits the category $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\varphi : \mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.

Proof. Let $\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n} \in \mathcal{C}$ be objects of $\mathcal{C}$ for some $\mathrm{n} \in \mathbb{N}$ and $\alpha \in \mathrm{Ass}_\mathrm{n}$ an operation.

We have a canonical equivalence $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^\otimes \simeq \mathrm{Act}(\mathrm{Ass}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathrm{Ass}^\otimes)} \mathcal{C}^\otimes$ over $\mathrm{Fun}(\{1\},\mathrm{Ass}^\otimes)$ and the full suboperad inclusion $\mathcal{C}^\otimes \subset \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^\otimes \simeq \mathrm{Act}(\mathrm{Ass}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathrm{Ass}^\otimes)} \mathcal{C}^\otimes$ over $\mathrm{Ass}^\otimes$ is the pullback of the diagonal embedding $\mathrm{Ass}^\otimes \subset \mathrm{Act}(\mathrm{Ass}^\otimes)$ along the functor $\mathrm{Act}(\mathrm{Ass}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathrm{Ass}^\otimes)} \mathcal{C}^\otimes \to \mathrm{Act}(\mathrm{Ass}^\otimes)$.

Thus $(\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n}, \alpha)$ corresponds to an object A of $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$, which can be obtained as $\mathrm{A} \simeq \otimes_\alpha(\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n})$, where we consider $\mathrm{A}_1, ..., \mathrm{A}_\mathrm{n}$ as objects of $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ via the natural embedding $\mathcal{C} \subset \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ and form the tensorproduct of the monoidal category $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$.

Denote $\beta : (A_1, ..., A_n) \to \otimes_\alpha(A_1, ..., A_n) \simeq A$ a φ'-cocartesian lift of α.

Denote $\sigma \in \mathrm{Mul}_{LM}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ the unique object and α' the image of α, the identity of $\mathfrak{m}$ and σ under the operadic composition

$$\mathrm{Mul}_{LM}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m}) \times (\mathrm{Mul}_{LM}(\mathfrak{a}, ..., \mathfrak{a}; \mathfrak{a}) \times \mathrm{Mul}_{LM}(\mathfrak{m}; \mathfrak{m})) \to \mathrm{Mul}_{LM}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m}).$$

Let X, Y be objects of $\mathcal{N}$.

As $\mathcal{M}^\otimes \to LM^\otimes$ exhibits $\mathcal{M}$ as pseudo-enriched over the monoidal category $\varphi' : \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^\otimes \to \mathrm{Ass}^\otimes$ composition with β

$$\mathrm{Mul}_\mathcal{M}(A, X; Y) \to \{\alpha'\} \times_{\mathrm{Mul}_{LM}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_\mathcal{M}(A_1, ..., A_n, X; Y)$$

is an equivalence.

Denote $\gamma : (A_1, ..., A_n) \to \otimes_\alpha(A_1, ..., A_n)$ a locally φ-cocartesian lift of α.

We have to see that composition with γ

$$\mathrm{Mul}_\mathcal{M}(\otimes_\alpha(A_1, ..., A_n), X; Y) \to$$

$$\{\alpha'\} \times_{\mathrm{Mul}_{LM}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_\mathcal{M}(A_1, ..., A_n, X; Y)$$

is an equivalence.

If this is shown, $\mathcal{N}^\otimes \to LM^\otimes$ exhibits the category $\mathcal{N}$ as pseudo-enriched over the locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.

As every objects $X, Y \in \mathcal{N}$ admit a morphism object $[X, Y]$ in $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ that belongs to $\mathcal{C}$, then $\mathcal{N}^\otimes \to LM^\otimes$ exhibits the category $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.

By 6.53 1. the full suboperad inclusion $\mathcal{C} \subset \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ admits a left adjoint L, where the unit $\eta : A \to L(A)$ corresponds to $\gamma : (A_1, ..., A_n) \to \otimes_\alpha(A_1, ..., A_n)$ together with the commutative square

$$\begin{array}{ccc} \langle n \rangle & \xrightarrow{\alpha} & \langle 1 \rangle \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \mathrm{id}} \\ \langle 1 \rangle & \xrightarrow{\mathrm{id}} & \langle 1 \rangle \end{array}$$

in $\mathrm{Ass}^\otimes$.

β corresponds to the identity of $(A_1, ..., A_n)$ in $\mathcal{C}^\otimes$ together with the commutative square

$$\begin{array}{ccc} \langle n \rangle & \xrightarrow{\mathrm{id}} & \langle n \rangle \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \alpha} \\ \langle n \rangle & \xrightarrow{\alpha} & \langle 1 \rangle \end{array}$$

in $\mathrm{Ass}^\otimes$.

So the image of γ in $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ factors as $(A_1, ..., A_n) \xrightarrow{\beta} A \to L(A)$ so that composition with γ factors as

$$\mathrm{Mul}_\mathcal{M}(L(A), X; Y) \to \mathrm{Mul}_\mathcal{M}(A, X; Y) \simeq$$

$$\{\alpha'\} \times_{\mathrm{Mul}_{LM}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})} \mathrm{Mul}_\mathcal{M}(A_1, ..., A_n, X; Y).$$

Consequently it is enough to see that composition with $\eta : A \to L(A)$

$$\mathrm{Mul}_\mathcal{M}(L(A), X; Y) \to \mathrm{Mul}_\mathcal{M}(A, X; Y)$$

is an equivalence.

As we assumed that X, Y admit a morphism object [X, Y] in $\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ this map factors as

$$\mathrm{Mul}_{\mathcal{M}}(\mathrm{L}(\mathrm{A}), \mathrm{X}; \mathrm{Y}) \simeq \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})(\mathrm{L}(\mathrm{A}), [\mathrm{X}, \mathrm{Y}]) \to \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})(\mathrm{A}, [\mathrm{X}, \mathrm{Y}])$$

$$\simeq \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}, \mathrm{X}; \mathrm{Y}).$$

As we assumed that [X, Y] belongs to $\mathcal{C}$, composition with $\eta : \mathrm{A} \to \mathrm{L}(\mathrm{A})$

$$\mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})(\mathrm{L}(\mathrm{A}), [\mathrm{X}, \mathrm{Y}]) \to \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})(\mathrm{A}, [\mathrm{X}, \mathrm{Y}])$$

is an equivalence.

□

6.2.4 Enriched over- and undercategories

Let $\mathcal{C}$ be a monoidal category with final tensorunit, $\mathcal{D}$ a $\mathcal{C}$-enriched category and $\mathrm{X} \in \mathcal{D}$.

In this section we show that the categories $\mathcal{D}_{/\mathrm{X}}, \mathcal{D}_{\mathrm{X}/}$ are $\mathcal{C}$-enriched categories and the forgetful functors $\mathcal{D}_{/\mathrm{X}} \to \mathcal{D}, \mathcal{D}_{\mathrm{X}/} \to \mathcal{D}$ are $\mathcal{C}$-enriched functors (prop. 6.69).

Moreover we show that every morphism $\mathrm{X} \to \mathrm{Y}$ in $\mathcal{D}$ yields $\mathcal{C}$-enriched functors $\mathcal{D}_{/\mathrm{X}} \to \mathcal{D}_{/\mathrm{Y}}$ and $\mathcal{D}_{\mathrm{Y}/} \to \mathcal{D}_{\mathrm{X}/}$.

Let $\mathcal{C}$ be a monoidal category. The unique monoidal functor $\mathcal{C} \to *$ makes the final category $*$ to a left module over $\mathcal{C}$ encoded by a $\mathrm{LM}^\otimes$-monoidal category $\mathcal{B}^\otimes \to \mathrm{LM}^\otimes$.

If $\mathcal{C}$ admits a final object $*_\mathcal{C}$, this left module structure on $*$ over $\mathcal{C}$ is closed with endomorphism object $*_\mathcal{C}$ as we have $*(\mathrm{A} \otimes *, *) \simeq * \simeq \mathcal{C}(\mathrm{A}, *_\mathcal{C})$ for every $\mathrm{A} \in \mathcal{C}$.

Lemma 6.66. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ a cocartesian* S*-family of operads over* $\mathrm{LM}^\otimes$ *for some category* S.

Set $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ and assume that we have an equivalence $\mathrm{S} \times \mathcal{C}^\otimes \simeq \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ over $\mathrm{S} \times \mathrm{Ass}^\otimes$.

If the tensorunit of $\mathcal{C}$ is a final object, every section X *of $\mathcal{D} \to \mathrm{S}$ lifts to a map $\mathrm{S} \times \mathcal{B}^\otimes \to \mathcal{M}^\otimes$ of* S*-families of operads over* $\mathrm{LM}^\otimes$*, whose pullback to* $\mathrm{Ass}^\otimes$ *is the identity of* $\mathrm{S} \times \mathcal{C}^\otimes$.

To prove lemma 6.66 we use the following lemma:

Lemma 6.67. *Let $\mathcal{C}$ be a monoidal category and $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ a cocartesian* S*-family of operads over* $\mathrm{LM}^\otimes$ *for some category* S.

Set $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ and assume that we have an equivalence $\mathrm{S} \times \mathcal{C}^\otimes \simeq \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ over $\mathrm{S} \times \mathrm{Ass}^\otimes$.

Assume that the tensorunit of $\mathcal{C}$ is a final object.

Denote $\mathcal{C}^\otimes_{/\mathbb{1}}$ the pullback of the cocartesian fibration $(\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{\{1\}}$ of monoidal categories along the unique monoidal functor $\mathbb{1} : \mathrm{Ass}^\otimes \to \mathcal{C}^\otimes$ so that the monoidal forgetful functor $\mathcal{C}^\otimes_{/\mathbb{1}} \to (\mathcal{C}^\otimes)^{\Delta^1} \to (\mathcal{C}^\otimes)^{\{0\}}$ is an equivalence.

Denote $\alpha : \mathrm{S} \times \mathrm{LM}^{\otimes} \to \mathcal{M}^{\otimes}$ *the map of* S*-families of operads over* $\mathrm{LM}^{\otimes}$ *adjoint to the functor*

$$\beta : \mathrm{S} \xrightarrow{\mathrm{X}} \mathcal{D} \simeq \mathrm{LMod}_{\mathbb{1}}^{/\mathrm{S}}(\mathcal{D}) \to \mathrm{LMod}^{/\mathrm{S}}(\mathcal{D}) = \mathrm{Alg}_{\mathrm{LM}/\mathrm{LM}}^{/\mathrm{S}}(\mathcal{M}) \subset$$

$$\mathrm{Fun}_{\mathrm{S} \times \mathrm{LM}^{\otimes}}^{/\mathrm{S}}(\mathrm{S} \times \mathrm{LM}^{\otimes}, \mathcal{M}^{\otimes}).$$

Denote $\mathcal{X}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ *the pullback of the map* $(\mathcal{M}^{\otimes})^{\Delta^1} \to (\mathcal{M}^{\otimes})^{\{1\}}$ *of cocartesian* S*-families of operads over* $\mathrm{LM}^{\otimes}$ *along* α.

We have canonical equivalences

$$\mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{X}^{\otimes} \simeq \mathrm{S} \times \mathcal{C}^{\otimes}_{/\mathbb{1}} \simeq \mathrm{S} \times \mathcal{C}^{\otimes}$$

over $\mathrm{S} \times \mathrm{Ass}^{\otimes}$ *and* $\{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{X}^{\otimes} \simeq \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} = \mathrm{S} \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1}$ *over* $\mathcal{D}$.

The functor $\mathcal{X}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ *is a cocartesian* S*-family of operads over* $\mathrm{LM}^{\otimes}$ *and the forgetful functor* $\mathcal{X}^{\otimes} \to (\mathcal{M}^{\otimes})^{\Delta^1} \to (\mathcal{M}^{\otimes})^{\{0\}}$ *is a map of such.*

Proof. The cocartesian S-family $\mathcal{M}^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$ embeds into its enveloping cocartesian S-family $\mathcal{M}'^{\otimes} := \mathrm{Env}_{\mathrm{LM}}^{/\mathrm{S}}(\mathcal{M})^{\otimes}$ of $\mathrm{LM}^{\otimes}$-monoidal categories classifying a $\mathrm{LM}^{\otimes}$-monoid in $\mathsf{Cat}_{\infty/\mathrm{S}}^{\mathrm{cocart}}$ that exhibits a cocartesian fibration $\mathcal{D}' \to \mathrm{S}$ as a left module over $\mathcal{C}'^{\otimes} \times \mathrm{S} \to \mathrm{Ass}^{\otimes} \times \mathrm{S}$ with $\mathcal{C}'^{\otimes} := \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})^{\otimes}$.

Denote $\mathcal{C}'^{\otimes}_{/\mathbb{1}}$ the pullback of the cocartesian fibration $(\mathcal{C}'^{\otimes})^{\Delta^1} \to (\mathcal{C}'^{\otimes})^{\{1\}}$ of monoidal categories along the lax monoidal functor $\mathbb{1} : \mathrm{Ass}^{\otimes} \to \mathcal{C}^{\otimes} \subset \mathcal{C}'^{\otimes}$.

Denote $\mathcal{X}'^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ the pullback of the map $(\mathcal{M}'^{\otimes})^{\Delta^1} \to (\mathcal{M}'^{\otimes})^{\{1\}}$ of cocartesian S-families of $\mathrm{LM}^{\otimes}$-monoidal categories along $\mathrm{S} \times \mathrm{LM}^{\otimes} \xrightarrow{\alpha} \mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes}$.

The embedding $\mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes}$ of cocartesian S-families of operads over $\mathrm{LM}^{\otimes}$ yields an embedding $\mathcal{X}^{\otimes} \subset \mathcal{X}'^{\otimes}$ of S-families of operads over $\mathrm{LM}^{\otimes}$.

The embedding $\mathcal{X}^{\otimes} \subset \mathcal{X}'^{\otimes}$ induces on the fiber over $\mathfrak{m} \in \mathrm{LM}$ the map

$$\mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} = \mathrm{S} \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \subset \mathcal{D}'^{/\mathrm{S}}_{/\mathrm{X}} = \mathrm{S} \times_{\mathcal{D}'^{\{1\}}} \mathcal{D}'^{\Delta^1}$$

of cocartesian fibrations over S according to ... and on the fiber over $\mathfrak{a} \in \mathrm{LM}$ the map $\mathrm{S} \times \mathcal{C}^{\otimes}_{/\mathbb{1}} \subset \mathrm{S} \times \mathcal{C}'^{\otimes}_{/\mathbb{1}}$ of bicartesian fibrations over S.

Consequently it is enough to see that that $\mathcal{X}'^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ is a cocartesian S-family of $\mathrm{LM}^{\otimes}$-monoidal categories and the forgetful functor $\mathcal{X}'^{\otimes} \to (\mathcal{M}'^{\otimes})^{\Delta^1} \to (\mathcal{M}'^{\otimes})^{\{0\}}$ is a map of such.

When we have shown this, the embedding $\mathcal{X}^{\otimes} \subset \mathcal{X}'^{\otimes}$ is a map of cocartesian S-families of operads over $\mathrm{LM}^{\otimes}$ so that the forgetful functor $\mathcal{X}^{\otimes} \to (\mathcal{M}^{\otimes})^{\Delta^1} \to (\mathcal{M}^{\otimes})^{\{0\}}$ is a map of such.

By lemma 6.13 the map $(\mathcal{M}'^{\otimes})^{\Delta^1} \to (\mathcal{M}'^{\otimes})^{\{1\}}$ of cocartesian S-families of $\mathrm{LM}^{\otimes}$-monoidal categories is a cocartesian fibration, whose cocartesian morphisms are sent to cocartesian morphisms of $\mathcal{M}'^{\otimes} \to \mathrm{S}$ by the functor $(\mathcal{M}'^{\otimes})^{\Delta^1} \to (\mathcal{M}'^{\otimes})^{\{0\}}$.

Thus also the pullback $\mathcal{X}'^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ is a cocartesian fibration, whose cocartesian morphisms are sent to cocartesian morphisms of $\mathcal{M}'^{\otimes} \to \mathrm{S}$ by the functor $\mathcal{X}'^{\otimes} \to (\mathcal{M}'^{\otimes})^{\Delta^1} \to (\mathcal{M}'^{\otimes})^{\{0\}}$.

□

By lemma 6.67 the functor $\mathcal{X}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ is a cocartesian S-family of operads over $\mathrm{LM}^{\otimes}$ and the forgetful functor $\mathcal{X}^{\otimes} \to (\mathcal{M}^{\otimes})^{\Delta^1} \to (\mathcal{M}^{\otimes})^{\{0\}}$ is a map of such.

By prop. 6.61 the unique map $\mathcal{X}^{\otimes} \to \mathrm{S} \times \mathrm{LM}^{\otimes}$ of cocartesian S-families of operads over $\mathrm{LM}^{\otimes}$ corresponds to a map $\varphi : \mathcal{X}^{\otimes} \to \mathrm{S} \times \mathcal{B}^{\otimes}$ of cocartesian S-families of operads over $\mathrm{LM}^{\otimes}$, whose pullback to $\mathrm{Ass}^{\otimes}$ is the identity of $\mathcal{C}^{\otimes} \times \mathrm{S}$.

Remark 6.68. *The map* $\varphi : \mathcal{X}^{\otimes} \to \mathrm{S} \times \mathcal{B}^{\otimes}$ *of cocartesian* S*-families of operads over* $\mathrm{LM}^{\otimes}$ *admits a right adjoint* $\gamma : \mathrm{S} \times \mathcal{B}^{\otimes} \to \mathcal{X}^{\otimes}$ *relative to* $\mathrm{S} \times \mathrm{LM}^{\otimes}$*, whose pullback to* $\mathrm{Ass}^{\otimes}$ *is the identity adjunction of* $\mathcal{C}^{\otimes} \times \mathrm{S}$ *and whose pullback to* $\mathfrak{m} \in \mathrm{LM}$ *is a right adjoint* $\psi : \mathrm{S} \to \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}}$ *of the functor* $\mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} \to \mathrm{S}$ *relative to* S.

ψ *sends every* $\mathsf{s} \in \mathrm{S}$ *to the final object of the category* $(\mathcal{D}_{\mathsf{s}})_{/\mathrm{X}(\mathsf{s})}$ *and is thus the final object of the category* $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}})$ *by 5.33.*

Proof. For this we can reduce to the case that S is contractible.

The map $\varphi : \mathcal{X}^{\otimes} \to \mathcal{B}^{\otimes}$ induces on the fiber over $\{\mathfrak{m}\} \in \mathrm{LM}$ the adjunction $\mathcal{D}_{/\mathrm{X}} \rightleftarrows * : \mathrm{id}_{\mathrm{X}}$.

By cor. 6.74 it is enough to see that for every $\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}} \in \mathcal{C}$ for some $\mathrm{n} \in \mathbb{N}$ and $\mathrm{Y} \in \mathcal{D}_{/\mathrm{X}}$ the space $\mathrm{Mul}_{\mathcal{X}}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}, \mathrm{Y}; \mathrm{X})$ is contractible.

This follows from the fact that for every $\mathrm{Z} \in \mathcal{D}_{/\mathrm{X}}$ we have a canonical equivalence

$$\mathrm{Mul}_{\mathcal{X}}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}, \mathrm{Y}; \mathrm{Z}) \simeq \{\mathrm{id}_{\mathrm{X}}\} \times_{\mathrm{Mul}_{\mathcal{M}}(\mathbb{1}, ..., \mathbb{1}, \mathrm{X}; \mathrm{X})}$$

$$\mathrm{Mul}_{\mathcal{M}}(\mathbb{1}, ..., \mathbb{1}, \mathrm{X}; \mathrm{X}) \times_{\mathrm{Mul}_{\mathcal{M}}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}, \mathrm{Y}; \mathrm{X})} \mathrm{Mul}_{\mathcal{M}}(\mathrm{A}_1, ..., \mathrm{A}_{\mathrm{n}}, \mathrm{Y}; \mathrm{Z}).$$

□

Proof. 6.66

By remark 6.68 the map $\varphi : \mathcal{X}^{\otimes} \to \mathrm{S} \times \mathcal{B}^{\otimes}$ of cocartesian S-families of operads over $\mathrm{LM}^{\otimes}$ admits a right adjoint $\gamma : \mathrm{S} \times \mathcal{B}^{\otimes} \to \mathcal{X}^{\otimes}$ relative to $\mathrm{S} \times \mathrm{LM}^{\otimes}$, whose pullback to $\mathrm{Ass}^{\otimes}$ is the identity adjunction of $\mathcal{C}^{\otimes} \times \mathrm{S}$ and whose pullback to $\mathfrak{m} \in \mathrm{LM}$ is a right adjoint $\psi : \mathrm{S} \to \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}}$ of the functor $\mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} \to \mathrm{S}$ relative to S that is the final object of the category $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}})$.

We have a canonical equivalence $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}}) \simeq \mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D})_{/\mathrm{X}}$ over $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D}^{\{0\}})$, under which ψ corresponds to the identity of X as ψ is the final object of the category $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}})$.

So the section X factors as $\mathrm{S} \xrightarrow{\psi} \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} \to \mathcal{D}^{\{0\}}$.

The composition $\mathrm{S} \times \mathcal{B}^{\otimes} \xrightarrow{\gamma} \mathcal{X}^{\otimes} \to (\mathcal{M}^{\otimes})^{\{0\}}$ is the desired map of S-families of operads over $\mathrm{LM}^{\otimes}$, whose pullback to $\mathrm{Ass}^{\otimes}$ is the identity of $\mathrm{S} \times \mathcal{C}^{\otimes}$ and whose pullback to $\mathfrak{m} \in \mathrm{LM}$ is the section $\mathrm{X} : \mathrm{S} \xrightarrow{\psi} \mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} \to \mathcal{D}^{\{0\}}$ of $\mathcal{D} \to \mathrm{S}$.

□

Proposition 6.69. *Let $\mathcal{C}$ be a monoidal category, whose tensorunit is a final object and $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ a cocartesian* S*-family of operads over* $\mathrm{LM}^\otimes$.

Set $\mathcal{D} := \{\mathfrak{m}\} \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ and assume that we have an equivalence $\mathrm{S} \times \mathcal{C}^\otimes \simeq \mathrm{Ass}^\otimes \times_{\mathrm{LM}^\otimes} \mathcal{M}^\otimes$ over $\mathrm{S} \times \mathrm{Ass}^\otimes$.

Denote $\mathcal{N}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ the pullback of $(\mathcal{M}^\otimes)^{\Delta^1} \to \mathrm{S} \times \mathrm{LM}^\otimes$ along the diagonal map $\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ of monoidal categories.

The maps $(\mathcal{M}^\otimes)^{\Delta^1} \to (\mathcal{M}^\otimes)^{\{1\}}$, $(\mathcal{M}^\otimes)^{\Delta^1} \to (\mathcal{M}^\otimes)^{\{0\}}$ of cocartesian S*-families of operads over $\mathrm{LM}^\otimes$ correspond to maps of cocartesian* S*-families of operads over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the identity of $(\mathcal{C}^\otimes)^{\Delta^1} \times \mathrm{S}$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ are the maps $\mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{1\}}$ and $\mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{0\}}$ of cocartesian fibrations over* S.

Pulling back along the diagonal map $\mathcal{C} \to \mathrm{Fun}(\Delta^1, \mathcal{C})$ of monoidal categories we get maps $\alpha, \beta : \mathcal{N}^\otimes \to \mathcal{M}^\otimes$ of cocartesian S*-families of operads over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the identity of $\mathcal{C}^\otimes \times \mathrm{S}$.*

The monoidal functor $\mathcal{C} \to *$ makes the final category $*$ to a left module over $\mathcal{C}$ encoded by a $\mathrm{LM}^\otimes$-monoidal category $\mathcal{B}^\otimes \to \mathrm{LM}^\otimes$.

By lemma 6.66 every section X *of $\mathcal{D} \to \mathrm{S}$ lifts to a map $\mathrm{S} \times \mathcal{B}^\otimes \to \mathcal{M}^\otimes$ of* S*-families of operads over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the identity of $\mathrm{S} \times \mathcal{C}^\otimes$.*

Denote $\mathcal{W}_1^\otimes, \mathcal{W}_2^\otimes$ the pullbacks $(\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ along α respectively β.
We have forgetful maps

$$\mathcal{W}_1^\otimes \to \mathcal{N}^\otimes \xrightarrow{\beta} \mathcal{M}^\otimes, \ \mathcal{W}_2^\otimes \to \mathcal{N}^\otimes \xrightarrow{\alpha} \mathcal{M}^\otimes$$

of S*-families of operads over $\mathrm{LM}^\otimes$, whose pullback to $\mathrm{Ass}^\otimes$ is the identity of $\mathcal{C}^\otimes \times \mathrm{S}$ and whose pullback to $\mathfrak{m} \in \mathrm{LM}$ are the forgetful functors*

$$\mathcal{D}^{/\mathrm{S}}_{/\mathrm{X}} = \mathrm{S} \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{0\}}, \ \mathcal{D}^{/\mathrm{S}}_{\mathrm{X}/} = \mathrm{S} \times_{\mathcal{D}^{\{0\}}} \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\Delta^1} \to \mathcal{D}^{\{1\}}.$$

1. *The pullback $\mathcal{W}_1^\otimes := (\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ along α is a cocartesian* S*-family of operads over $\mathrm{LM}^\otimes$ and the forgetful functor $(\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathcal{N}^\otimes \xrightarrow{\beta} \mathcal{M}^\otimes$ is a map of such.*

2. *If $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ is a cocartesian* S*-family of $\mathcal{C}$-enriched categories and $\mathcal{C}$ admits pullbacks, the pullback $(\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ along α is a cocartesian* S*-family of $\mathcal{C}$-enriched categories.*

   Let $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ be a bicartesian S*-family of operads over $\mathrm{LM}^\otimes$.*

3. *The pullback $\mathcal{W}_2^\otimes := (\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ along β is a cartesian* S*-family of operads over $\mathrm{LM}^\otimes$ and the forgetful functor $(\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathcal{N}^\otimes \xrightarrow{\alpha} \mathcal{M}^\otimes$ is a map of such.*

4. *If $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes \times \mathrm{S}$ is a bicartesian* S*-family of $\mathcal{C}$-enriched categories and $\mathcal{C}$ admits pullbacks, the pullback $(\mathrm{S} \times \mathcal{B}^\otimes) \times_{\mathcal{M}^\otimes} \mathcal{N}^\otimes \to \mathrm{S} \times \mathrm{LM}^\otimes$ along β is a cartesian* S*-family of $\mathcal{C}$-enriched categories.*

For S *contractible and* $A, B \in \mathcal{D}_{/X}$ *and* $A', B' \in \mathcal{D}_{X/}$ *the morphism objects are*

$$[A,B]_{/X} := \mathbb{1} \times_{[A,X]} [A,B] \simeq \mathbb{1} \times_{[X,X]} [X,X] \times_{[A,X]} [A,B],$$

$$[A',B']_{X/} := \mathbb{1} \times_{[X,B']} [A',B'] \simeq \mathbb{1} \times_{[X,X]} [X,X] \times_{[X,B']} [A',B'].$$

Proof. We start by showing 1. and 3.

Denote $\mathcal{M}'^{\otimes} := \mathcal{P}^{/S}(\mathrm{Env}^{/S}_{LM}(\mathcal{M}))^{\otimes} \to LM^{\otimes} \times S$ the enveloping cocartesian S-family of $LM^{\otimes}$-monoidal categories so that we have a canonical equivalence $S \times \mathcal{C}'^{\otimes} \simeq Ass^{\otimes} \times_{LM^{\otimes}} \mathcal{M}'^{\otimes}$ over $S \times Ass^{\otimes}$ with $\mathcal{C}'^{\otimes} := \mathcal{P}(\mathrm{Env}_{Ass}(\mathcal{C}))^{\otimes}$.

Set $\mathcal{D}' := \{\mathfrak{m}\} \times_{LM^{\otimes}} \mathcal{M}'^{\otimes}$.

If $\mathcal{M}^{\otimes} \to LM^{\otimes} \times S$ is a bicartesian S-family of operads over $LM^{\otimes}$, the embedding $\mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes}$ is a map of bicartesian S-families of operads over $LM^{\otimes}$.

We define $\mathcal{N}'^{\otimes}$ similarly using $\mathcal{M}'^{\otimes}$ so that we have maps $\alpha', \beta' : \mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$ of cocartesian S-families of $LM^{\otimes}$-monoidal categories, whose pullback to $Ass^{\otimes}$ is the identity of $\mathcal{C}'^{\otimes} \times S$ and whose fiber over $\mathfrak{m} \in LM$ are the maps $\mathcal{D}'^{\Delta^1} \to \mathcal{D}'^{\{1\}}$ respectively $\mathcal{D}'^{\Delta^1} \to \mathcal{D}'^{\{0\}}$ of cocartesian fibrations over S.

The embedding $\mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes}$ of cocartesian S-families of operads over $LM^{\otimes}$ yields an embedding $\mathcal{N}^{\otimes} \subset \mathcal{N}'^{\otimes}$ of cocartesian S-families of operads over $LM^{\otimes}$, whose pullback to $Ass^{\otimes}$ is the embedding $S \times \mathcal{C}^{\otimes} \subset S \times \mathcal{C}'^{\otimes}$ and whose fiber over $\mathfrak{m} \in LM$ is the embedding $\mathcal{D}^{\Delta^1} \subset \mathcal{D}'^{\Delta^1}$.

We write $\mathcal{W}_1'^{\otimes}, \mathcal{W}_2'^{\otimes}$ for the pullbacks $(S \times \mathcal{B}^{\otimes}) \times_{\mathcal{M}'^{\otimes}} \mathcal{N}'^{\otimes}$ of $S \times \mathcal{B}^{\otimes} \to \mathcal{M}^{\otimes} \subset \mathcal{M}'^{\otimes}$ along $\alpha', \beta' : \mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$.

So we have embeddings $\mathcal{W}_1^{\otimes} \subset \mathcal{W}_1'^{\otimes}, \mathcal{W}_2^{\otimes} \subset \mathcal{W}_2'^{\otimes}$ of S-families of operads over $LM^{\otimes}$, whose pullback to $Ass^{\otimes}$ is the identity of $S \times \mathcal{C}^{\otimes}$ and whose fiber over $\mathfrak{m} \in LM$ are the embeddings

$$\mathcal{D}^{/S}_{/X} = S \times_{\mathcal{D}^{\{1\}}} \mathcal{D}^{\Delta^1} \subset \mathcal{D}'^{/S}_{/X} = S \times_{\mathcal{D}'^{\{1\}}} \mathcal{D}'^{\Delta^1}$$

respectively $\mathcal{D}^{/S}_{X/} = S \times_{\mathcal{D}^{\{0\}}} \mathcal{D}^{\Delta^1} \subset \mathcal{D}'^{/S}_{X/} = S \times_{\mathcal{D}'^{\{0\}}} \mathcal{D}'^{\Delta^1}$.

By 6.13 the functor $\mathcal{D}^{/S}_{/X} \to S$ is a cocartesian fibration, whose cocartesian morphisms are those that are sent by the functor $\mathcal{D}^{/S}_{/X} \to \mathcal{D}^{\{0\}}$ to cocartesian morphisms of the cocartesian fibration $\mathcal{D} \to S$.

Dually the functor $\mathcal{D}^{/S}_{X/} \to S$ is a cartesian fibration, whose cartesian morphisms are those that are sent by the functor $\mathcal{D}^{/S}_{X/} \to \mathcal{D}^{\{1\}}$ to cartesian morphisms of the cartesian fibration $\mathcal{D} \to S$.

Thus the embedding $\mathcal{D}^{/S}_{/X} \subset \mathcal{D}'^{/S}_{/X}$ is a map of cocartesian fibrations over S and dually the embedding $\mathcal{D}^{/S}_{X/} \subset \mathcal{D}'^{/S}_{X/}$ is a map of cartesian fibrations over S if $\mathcal{M}^{\otimes} \to LM^{\otimes} \times S$ is a cartesian S-family of operads over $LM^{\otimes}$.

Consequently to verify 1. and 3., it is enough to show that $\mathcal{W}_1'^{\otimes} \to S \times LM^{\otimes}$ is a cocartesian S-family of operads over $LM^{\otimes}$ and $\mathcal{W}_2'^{\otimes} \to S \times LM^{\otimes}$ is a cartesian S-family of operads over $LM^{\otimes}$ if $\mathcal{M}^{\otimes} \to LM^{\otimes} \times S$ is a bicartesian S-family of operads over $LM^{\otimes}$.

If this is shown, the embeddings $\mathcal{W}_1^{\otimes} \subset \mathcal{W}_1'^{\otimes}, \mathcal{W}_2^{\otimes} \subset \mathcal{W}_2'^{\otimes}$ are maps of cocartesian respectively cartesian S-families of operads over $LM^{\otimes}$.

Then the second part of 1. and 3. follows from the fact that the functor $\mathcal{D}^{/S}_{/X} \to \mathcal{D}^{\{0\}}$ is a map of cocartesian fibrations over S and dually the functor $\mathcal{D}^{/S}_{X/} \to \mathcal{D}^{\{1\}}$ is a map of cartesian fibrations over S.

In the following we will show that $\mathcal{W}'^{\otimes}_1 = (S \times \mathcal{B}^{\otimes}) \times_{\mathcal{M}'^{\otimes}} \mathcal{N}'^{\otimes} \to S \times \mathcal{B}^{\otimes}$ is a cocartesian fibration and $\mathcal{W}'^{\otimes}_2 \to S \times \mathcal{B}^{\otimes}$ is a map of cartesian fibrations over S if $\mathcal{M}^{\otimes} \to LM^{\otimes} \times S$ and so $\mathcal{M}'^{\otimes} \to LM^{\otimes} \times S$ is a bicartesian S-family of operads over $LM^{\otimes}$.

We start with verifying that $\mathcal{W}'^{\otimes}_1 = (S \times \mathcal{B}^{\otimes}) \times_{\mathcal{M}'^{\otimes}} \mathcal{N}'^{\otimes} \to S \times \mathcal{B}^{\otimes}$ is a cocartesian fibration.

For this it is enough to check that for every morphism $\Delta^1 \to S$ and every active morphism $\Delta^1 \to \mathcal{B}^{\otimes}$, whose target belongs to $\mathcal{C}$ or $*$, the pullback $\Delta^1 \times_{\mathcal{M}'^{\otimes}} \mathcal{N}'^{\otimes} \to \Delta^1$ is a cocartesian fibration, whose cocartesian morphisms are cocartesian with respect to $\mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$.

The morphism corresponding to the composition $\Delta^1 \to S \times \mathcal{B}^{\otimes} \to \mathcal{M}'^{\otimes}$ factors as a cocartesian active morphism followed by a morphism in $S \times \mathcal{C}'$ or $\mathcal{D}'$.

The functor $\alpha' : \mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$ is a map of cocartesian fibrations over $LM^{\otimes}$ and the functor $\mathcal{D}'^{\Delta^1} \to \mathcal{D}'^{\{1\}}$ is a cocartesian fibration.

Hence we are reduced to show that every morphism that is cocartesian with respect to $\mathcal{D}'^{\Delta^1} \to \mathcal{D}'^{\{1\}}$ or the identity of $\mathcal{C}' \times S$ is cocartesian with respect to $\mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$.

This follows from the fact that every morphism $W \to Z$ in the set LM induces a map $\mathcal{N}'^{\otimes}_W \to \mathcal{N}'^{\otimes}_Z$ of cocartesian fibrations.

As next we prove that $\mathcal{W}'^{\otimes}_2 \to S \times \mathcal{B}^{\otimes}$ is a map of cartesian fibrations over S if $\mathcal{M}^{\otimes} \to LM^{\otimes} \times S$ and so $\mathcal{M}'^{\otimes} \to LM^{\otimes} \times S$ is a bicartesian S-family of operads over $LM^{\otimes}$.

To do so, it is enough to check that for every morphism $\Delta^1 \to S$ and every object of $\mathcal{B}^{\otimes}$ lying over some object Z of $LM^{\otimes}$ the pullback $\Delta^1 \times_{\mathcal{M}'^{\otimes}_Z} \mathcal{N}'^{\otimes}_Z \to \Delta^1$ is a cartesian fibration, whose cartesian morphisms are cartesian with respect to $\mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$.

The map $\mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$ of cocartesian fibrations over $LM^{\otimes}$ induces on the fiber over $\mathfrak{a} \in LM$ the identity of $\mathcal{C}' \times S$ and on the fiber over $\mathfrak{m} \in LM$ the cartesian fibration $\mathcal{D}'^{\Delta^1} \to \mathcal{D}'^{\{0\}}$.

By lemma 6.41 every morphism that is cartesian with respect to the cartesian fibration $\mathcal{N}'^{\otimes}_Z \to \mathcal{M}'^{\otimes}_Z$ for some $Z \in LM^{\otimes}$ is cartesian with respect to $\mathcal{N}'^{\otimes} \to \mathcal{M}'^{\otimes}$.

As next we prove 2. and 4.: For this we can reduce to the case that S is contractible.

We want to see that the pullbacks $\mathcal{B}^{\otimes} \times_{\mathcal{M}^{\otimes}} \mathcal{N}^{\otimes} \to LM^{\otimes}$ along α respectively β exhibit $\mathcal{D}_{/X}$ respectively $\mathcal{D}_{X/}$ as a $\mathcal{C}$-enriched categories.

For this it is enough to show that the operads $\mathcal{B}^{\otimes}, \mathcal{M}^{\otimes}, \mathcal{N}^{\otimes}$ over $LM^{\otimes}$ exhibit $*, \mathcal{D}, \mathrm{Fun}(\Delta^1, \mathcal{D})$ as $\mathcal{C}$-enriched categories.

By assumption the tensorunit $\mathbb{1}$ of $\mathcal{C}$ is a final object.
So $\mathcal{B}^{\otimes}$ is a closed $LM^{\otimes}$-monoidal category with endomorphism object the final object $\mathbb{1}$ of $\mathcal{C}$.

So it remains to show that $\mathcal{N}^{\otimes}$ exhibits $\mathrm{Fun}(\Delta^1, \mathcal{D})$ as $\mathcal{C}$-enriched category.

We have an embedding $\mathcal{N}^{\otimes} \subset \mathcal{N}'^{\otimes}$ of operads over $\mathrm{LM}^{\otimes}$, whose pullback to $\mathrm{Ass}^{\otimes}$ is the embedding $\mathcal{C}^{\otimes} \subset \mathcal{C}'^{\otimes}$ and whose fiber over $\mathfrak{m} \in \mathrm{LM}$ is the embedding $\mathrm{Fun}(\Delta^1, \mathcal{D}) \subset \mathrm{Fun}(\Delta^1, \mathcal{D}')$.

The $\mathrm{LM}^{\otimes}$-monoidal category $\mathcal{N}'^{\otimes}$ is the pullback of the $\mathrm{LM}^{\otimes}$-monoidal category $(\mathcal{M}'^{\otimes})^{\Delta^1} \to \mathrm{LM}^{\otimes}$ along the monoidal diagonal functor $\mathcal{C}' \to \mathrm{Fun}(\Delta^1, \mathcal{C}')$.

Given morphisms $\mathsf{f}: \mathrm{A} \to \mathrm{B}, \mathsf{g}: \mathrm{Y} \to \mathrm{Z}$ in $\mathcal{D}'$ we set $[\mathsf{f}, \mathsf{g}] \coloneqq [\mathrm{B}, \mathrm{Z}] \times_{[\mathrm{A},\mathrm{Z}]} [\mathrm{A}, \mathrm{Y}] \in \mathcal{C}'$, where $[-.-]$ denotes the morphism object of the closed $\mathrm{LM}^{\otimes}$-monoidal category $\mathcal{M}'^{\otimes}$.

For every $\mathrm{K} \in \mathcal{C}'$ we have a canonical equivalence

$$\mathcal{D}'(\mathrm{K} \otimes \mathsf{f}, \mathsf{g}) \simeq \mathcal{D}'(\mathrm{K} \otimes \mathrm{B}, \mathrm{Z}) \times_{\mathcal{D}'(\mathrm{K}\otimes\mathrm{A};\mathrm{Z})} \mathcal{D}'(\mathrm{K} \otimes \mathrm{A}; \mathrm{Y})$$

$$\simeq \mathcal{C}'(\mathrm{K}, [\mathrm{B}, \mathrm{Z}]) \times_{\mathcal{C}'(\mathrm{K},[\mathrm{A},\mathrm{Z}])} \mathcal{C}'(\mathrm{K}, [\mathrm{A}, \mathrm{Y}]) \simeq \mathcal{C}'(\mathrm{K}, [\mathsf{f}, \mathsf{g}]).$$

Thus $\mathcal{N}'^{\otimes}$ is a closed $\mathrm{LM}^{\otimes}$-monoidal category.

By lemma 6.65 it is enough to see that $\mathcal{N}^{\otimes}$ is closed under morphism objects in $\mathcal{N}'^{\otimes}$, i.e. that for every morphisms $\mathsf{f}: \mathrm{A} \to \mathrm{B}, \mathsf{g}: \mathrm{Y} \to \mathrm{Z}$ in $\mathcal{D} \subset \mathcal{D}'$ the morphism object $[\mathsf{f}, \mathsf{g}] = [\mathrm{B}, \mathrm{Z}] \times_{[\mathrm{A},\mathrm{Z}]} [\mathrm{A}, \mathrm{Y}] \in \mathcal{C}'$ belongs to $\mathcal{C}$.

This follows from the following two facts:

- By prop. 6.49 $\mathcal{M}^{\otimes}$ is closed under morphism objects in $\mathcal{M}'^{\otimes}$ as $\mathcal{M}^{\otimes}$ exhibits $\mathcal{D}$ as $\mathcal{C}$-enriched category, i.e. for every $\mathrm{A}, \mathrm{B} \in \mathcal{D}$ the morphism object $[\mathrm{A}, \mathrm{B}] \in \mathcal{C}'$ of $\mathcal{M}'^{\otimes}$ belongs to $\mathcal{C}$.
- By 6.13 the full subcategory inclusion $\mathcal{C} \subset \mathrm{Env}_{\mathrm{Ass}}(\mathcal{C})$ admits a left adjoint so that $\mathcal{C}$ is closed under pullbacks in $\mathcal{C}'$.

More explicitely given $\mathrm{A}, \mathrm{B} \in \mathcal{D}_{/\mathrm{X}}$ and $\mathrm{A}', \mathrm{B}' \in \mathcal{D}_{\mathrm{X}/}$ the morphism objects are

$$[\mathrm{A}, \mathrm{B}]_{/\mathrm{X}} \coloneqq \mathbb{1} \times_{[\mathrm{A},\mathrm{X}]} [\mathrm{A}, \mathrm{B}] \simeq \mathbb{1} \times_{[\mathrm{X},\mathrm{X}]} [\mathrm{X}, \mathrm{X}] \times_{[\mathrm{A},\mathrm{X}]} [\mathrm{A}, \mathrm{B}],$$

$$[\mathrm{A}', \mathrm{B}']_{\mathrm{X}/} \coloneqq \mathbb{1} \times_{[\mathrm{X},\mathrm{B}']} [\mathrm{A}', \mathrm{B}'] \simeq \mathbb{1} \times_{[\mathrm{X},\mathrm{X}]} [\mathrm{X}, \mathrm{X}] \times_{[\mathrm{X},\mathrm{B}']} [\mathrm{A}', \mathrm{B}'].$$

□

6.2.5 Endomorphism objects

Let $\mathcal{M}^{\otimes}$ be an operad over $\mathrm{LM}^{\otimes}$. Set $\mathcal{M} \coloneqq \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$ and $\mathcal{C}^{\otimes} \coloneqq \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$.

We show that the right fibration $\mathcal{C}[\mathrm{X}] \to \mathcal{C}$ classifies the functor $\mathrm{Mul}_{\mathrm{LM}}(-, \mathrm{X}; \mathrm{X}): \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ (lemma 6.70).

Lemma 6.70. *Let* $\mathcal{M}^{\otimes}$ *be an operad over* $\mathrm{LM}^{\otimes}$*. Set* $\mathcal{M} \coloneqq \{\mathfrak{m}\} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$ *and* $\mathcal{C}^{\otimes} \coloneqq \mathrm{Ass}^{\otimes} \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$*.*

Denote $\alpha: \Delta^1 \to \mathrm{LM}^{\otimes}$ *the morphism of* $\mathrm{LM}^{\otimes}$ *corresponding to the unique object of* $\mathrm{Mul}_{\mathrm{LM}^{\otimes}}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$*.*

α gives rise to a category $\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes})$*.*

We have canonical functors $\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \to \mathcal{M}^{\otimes}_{(\mathfrak{a},\mathfrak{m})} \simeq \mathcal{C} \times \mathcal{M}$ *and*

$\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \to \mathcal{M}^{\otimes}_{\mathfrak{m}} \simeq \mathcal{M}$ *evaluating at 0 respectively 1.*

There is a canonical equivalence

$$\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \simeq (\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathcal{M}^{\otimes} \times \mathcal{M}^{\otimes})} \mathrm{Act}(\mathcal{M}^{\otimes})$$

over $\mathcal{C} \times \mathcal{M} \times \mathcal{M}$.

In particular we have a canonical equivalence of right fibrations

$$\{\mathrm{X}\} \times_{\mathcal{M}} \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \simeq (\mathcal{C} \times \mathcal{M}) \times_{\mathcal{M}^{\otimes}} (\mathcal{M}^{\otimes})^{\mathrm{act}}_{/\mathrm{X}}$$

over $\mathcal{C} \times \mathcal{M}$, *where* $(\mathcal{M}^{\otimes})^{\mathrm{act}}_{/\mathrm{X}} \subset \mathcal{M}^{\otimes}_{/\mathrm{X}}$ *denotes the full subcategory spanned by the active morphisms with target* X, *and so a canonical equivalence of right fibrations*

$$\mathcal{C}[\mathrm{X}] \coloneqq \{(\mathrm{X}, \mathrm{X})\} \times_{\mathcal{M} \times \mathcal{M}} \mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \simeq (\mathcal{C} \times \{\mathrm{X}\}) \times_{\mathcal{M}^{\otimes}} (\mathcal{M}^{\otimes})^{\mathrm{act}}_{/\mathrm{X}}$$

over $\mathcal{C}$.

So the right fibration $\mathcal{C}[\mathrm{X}] \to \mathcal{C}$ *classifies the functor* $\mathrm{Mul}_{\mathrm{LM}}(-, \mathrm{X}; \mathrm{X}) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$.

Proof. Set $\mathcal{X} \coloneqq \Delta^1 \times_{\mathrm{LM}^{\otimes}} \mathcal{M}^{\otimes}$.
By lemma 6.71 we have a canonical equivalence

$$\mathrm{Fun}_{\mathrm{LM}^{\otimes}}(\Delta^1, \mathcal{M}^{\otimes}) \simeq \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{X}) \simeq (\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathcal{X} \times \mathcal{X})} \mathrm{Fun}(\Delta^1, \mathcal{X})$$

over $\mathcal{C} \times \mathcal{M} \times \mathcal{M}$.

Moreover we have a canonical equivalence

$$(\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathcal{X} \times \mathcal{X})} \mathrm{Fun}(\Delta^1, \mathcal{X}) \simeq$$
$$((\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\Delta^1 \times \Delta^1)} \mathrm{Fun}(\Delta^1, \Delta^1)) \times_{((\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathrm{LM}^{\otimes} \times \mathrm{LM}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathrm{LM}^{\otimes}))}$$
$$((\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathcal{M}^{\otimes} \times \mathcal{M}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{M}^{\otimes})) \simeq$$
$$\{\alpha\} \times_{\mathrm{LM}^{\otimes}((\mathfrak{a}, \mathfrak{m}), \mathfrak{m})} ((\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathcal{M}^{\otimes} \times \mathcal{M}^{\otimes})} \mathrm{Fun}(\Delta^1, \mathcal{M}^{\otimes})) \simeq$$
$$(\mathcal{C} \times \mathcal{M} \times \mathcal{M}) \times_{(\mathcal{M}^{\otimes} \times \mathcal{M}^{\otimes})} \mathrm{Act}(\mathcal{M}^{\otimes})$$

over $\mathcal{C} \times \mathcal{M} \times \mathcal{M}$, where we use that α is the unique active morphism $(\mathfrak{a}, \mathfrak{m}) \to \mathfrak{m}$ of $\mathrm{LM}^{\otimes}$.

□

In the proof of lemma 6.70 we needed the following lemma:

Lemma 6.71. *Let* $\mathcal{M}$ *be a category and* $\gamma : \mathcal{M} \to \Delta^1$ *a functor with* $\mathcal{M}_0 = \mathcal{C}$ *and* $\mathcal{M}_1 = \mathcal{D}$.

The commutative square

$$\begin{array}{ccc}
\mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M}) & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathcal{M}) \\
\downarrow & & \downarrow \\
\mathrm{Fun}_{\Delta^1}(\{0\}, \mathcal{M}) \times \mathrm{Fun}_{\Delta^1}(\{1\}, \mathcal{M}) & \longrightarrow & \mathrm{Fun}(\{0\}, \mathcal{M}) \times \mathrm{Fun}(\{1\}, \mathcal{M}) \\
\downarrow \simeq & & \downarrow \simeq \\
\mathcal{C} \times \mathcal{D} & \longrightarrow & \mathcal{M} \times \mathcal{M}
\end{array}$$

is a pullback square.

Proof. We will show that the induced functor

$$\rho : \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M}) \to (\mathcal{C} \times \mathcal{D}) \times_{(\mathcal{M}\times\mathcal{M})} \mathrm{Fun}(\Delta^1, \mathcal{M})$$

is an equivalence.

ρ is essentially surjective because every morphism $X \to Y$ in $\mathcal{M}$ with $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ has to lie over the unique non-identity morphism of Δ^1.

To see that ρ is fully faithful, it is enough to see that

$$\beta : (\mathcal{C} \times \mathcal{D}) \times_{(\mathcal{M}\times\mathcal{M})} \mathrm{Fun}(\Delta^1, \mathcal{M}) \to \mathrm{Fun}(\Delta^1, \mathcal{M}),$$

$$\beta \circ \rho : \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M}) \xrightarrow{\rho} (\mathcal{C} \times \mathcal{D}) \times_{(\mathcal{M}\times\mathcal{M})} \mathrm{Fun}(\Delta^1, \mathcal{M}) \xrightarrow{\beta} \mathrm{Fun}(\Delta^1, \mathcal{M})$$

are fully faithful.

But we have pullback squares

$$\begin{array}{ccccccc} \mathcal{C} & \longrightarrow & \mathcal{M} & \mathcal{D} & \longrightarrow & \mathcal{M} & \mathrm{Fun}_{\Delta^1}(\Delta^1, \mathcal{M}) & \xrightarrow{\beta\circ\rho} & \mathrm{Fun}(\Delta^1, \mathcal{M}) \\ \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \Delta^1 & \{1\} & \longrightarrow & \Delta^1 & \{\mathrm{id}_{\Delta^1}\} & \longrightarrow & \mathrm{Fun}(\Delta^1, \Delta^1), \end{array}$$

where the bottom and thus also the top functors are fully faithful.

□

Lemma 6.72. *Let $\mathcal{C}^{\otimes}$ be a monoidal category and* A *an associative algebra of $\mathcal{C}$ and let* M *be a left* A*-module structure on* A*, i.e.* $M \in \{A\} \times_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{C})$.

Denote $A' \in \{A\} \times_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{C})$ the left A*-module structure on* A *that comes from* A*, i.e.* A' *is the composition* $\mathrm{LM}^{\otimes} \to \mathrm{Ass}^{\otimes} \xrightarrow{A} \mathcal{C}^{\otimes}$.

Denote $\mu_M : A \otimes A \to A$ the left action map provided by M *and similar for* A.

Denote $\eta : \mathbb{1}_{\mathcal{C}} \to A$ the unit of A *and* ψ *the composition* $A \simeq A \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{A\otimes\eta} A \otimes A \xrightarrow{\mu_M} A$.

Then there is a canonical equivalence of spaces

$$\{A\} \times_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{C})(A', M) \simeq \mathcal{C}(A, A)(\mathrm{id}_A, \psi).$$

In particular M *is equivalent to* A' *in the category* $\{A\} \times_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{C})$ *if and only if the composition* $\psi : A \simeq A \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{A\otimes\eta} A \otimes A \xrightarrow{\mu_M} A$ *is the identity.*

Especially M *is equivalent to* A' *in the category* $\{A\} \times_{\mathcal{C}} \mathrm{LMod}_A(\mathcal{C})$ *if and only if* μ_M *is equivalent to* $\mu_{A'}$.

Proof. The morphism $A \otimes \mathbb{1}_{\mathcal{C}} \xrightarrow{A\otimes\eta} A \otimes A \xrightarrow{\mu_A} A$ is the canonical equivalence. Thus $\eta : \mathbb{1}_{\mathcal{C}} \to A$ exhibits A as the free left A-module generated by $\mathbb{1}_{\mathcal{C}}$ so that the canonical map

$$\gamma : \mathrm{LMod}_A(\mathcal{C})(A', M) \to \mathcal{C}(A, A) \to \mathcal{C}(\mathbb{1}_{\mathcal{C}}, A)$$

is an equivalence.

Denote β the composition

$$\mathcal{C}(\mathbb{1}_{\mathcal{C}}, A) \to \mathcal{C}(A \otimes \mathbb{1}_{\mathcal{C}}, A \otimes A) \xrightarrow{\mathcal{C}(A\otimes\mathbb{1}_{\mathcal{C}}, \mu_M)} \mathcal{C}(A \otimes \mathbb{1}_{\mathcal{C}}, A) \simeq \mathcal{C}(A, A).$$

The composition $\beta \circ \gamma : \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \to \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}) \to \mathcal{C}(\mathrm{A}, \mathrm{A})$ is the forgetful map $\mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \to \mathcal{C}(\mathrm{A}, \mathrm{A})$.

Thus γ induces an equivalence

$$\gamma' := \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \gamma : \{\mathrm{A}\} \times_{\mathcal{C}} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \simeq$$

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \to \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}).$$

The composition $\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \xrightarrow{\gamma'} \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}) \to \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$ is equivalent to the map

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \to \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \to \mathcal{C}(\mathrm{A}, \mathrm{A}) \to \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$$

and is thus equivalent to the constant map with value $\eta : \mathbb{1}_{\mathcal{C}} \to \mathrm{A}$.

Therefore γ' gives rise to a map

$$\zeta : \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \to (\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})) \times_{\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})} \{\eta\} \simeq$$

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \{\psi\} \simeq \mathcal{C}(\mathrm{A}, \mathrm{A})(\mathrm{id}_{\mathrm{A}}, \psi)$$

such that the composition

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \xrightarrow{\zeta} (\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})) \times_{\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})} \{\eta\} \to$$

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$$

is equivalent to γ'.

Thus ζ admits a left inverse and it is enough to see that the composition

$$(\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})) \times_{\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})} \{\eta\} \to \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}) \xrightarrow{\gamma'^{-1}}$$

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \xrightarrow{\zeta} (\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})) \times_{\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})} \{\eta\}$$

is equivalent to the identity.

This is equivalent to the condition that the composition

$$(\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})) \times_{\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})} \{\eta\} \to \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}) \xrightarrow{\gamma'^{-1}}$$

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \xrightarrow{\gamma'} \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$$

is equivalent over $\mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$ to the canonical map

$$(\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})) \times_{\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})} \{\eta\} \to \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}).$$

Choosing the inverse γ'^{-1} of γ' in $\mathcal{S}_{/\mathcal{C}(\mathbb{1}_{\mathcal{C}},\mathrm{A})}$ the composition

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A}) \xrightarrow{\gamma'^{-1}} \{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathrm{LMod}_{\mathrm{A}}(\mathcal{C})(\mathrm{A}', \mathrm{M}) \xrightarrow{\gamma'}$$

$$\{\mathrm{id}_{\mathrm{A}}\} \times_{\mathcal{C}(\mathrm{A},\mathrm{A})} \mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$$

is equivalent over $\mathcal{C}(\mathbb{1}_{\mathcal{C}}, \mathrm{A})$ to the identity.

□

6.2.6 Enriched adjunctions

We prove some very basic properties of enriched adjunctions.

We show in cor. 6.74 that a $\mathcal{C}$-enriched functor $G : \mathcal{N}^\otimes \to \mathcal{M}^\otimes$ admits a left adjoint relative to $LM^\otimes$ if and only if its underlying functor $\mathcal{N} \to \mathcal{M}$ admits a left adjoint $F : \mathcal{M} \to \mathcal{N}$ and for all objects $M \in \mathcal{M}, N \in \mathcal{N}$ the canonical morphism

$$[F(M), N] \to [G(F(M)), G(N)] \to [M, G(N)]$$

is an equivalence.

Simiarly we show that a $\mathcal{C}$-enriched functor $F : \mathcal{M}^\otimes \to \mathcal{N}^\otimes$ admits a right adjoint relative to $LM^\otimes$ if and only if its underlying functor $\mathcal{M} \to \mathcal{N}$ admits a right adjoint $G : \mathcal{N} \to \mathcal{M}$ and for all objects $M \in \mathcal{M}, N \in \mathcal{N}$ the canonical morphism

$$[M, G(N)] \to [F(M), F(G(N))] \to [F(M), N]$$

is an equivalence.

Especially we obtain that a $\mathcal{C}$-enriched functor $F : \mathcal{M}^\otimes \to \mathcal{N}^\otimes$ is an equivalence of operads over $LM^\otimes$ if and only if its underlying functor $\mathcal{M} \to \mathcal{N}$ is essentially surjective and for all objects $M, M' \in \mathcal{M}$ the canonical morphism

$$[M, M'] \to [F(M), F(M')]$$

is an equivalence (cor. 6.75).

Given a $\mathcal{C}$-enriched functor $G : \mathcal{N}^\otimes \to \mathcal{M}^\otimes$ that admits a left adjoint relative to $LM^\otimes$ we observe that for all objects $N, N' \in \mathcal{N}$ the canonical morphism

$$[N, N'] \to [G(N), G(N')]$$

is an equivalence if and only if the underlying functor $\mathcal{N} \to \mathcal{M}$ of G is fully faithful.

In this case we call the adjunction $\mathcal{M}^\otimes \rightleftarrows \mathcal{N}^\otimes : G$ a $\mathcal{C}$-enriched localization.

Given a $\mathcal{C}$-enriched localization $L : \mathcal{M}^\otimes \to \mathcal{N}^\otimes : \iota$ we observe that an object M of $\mathcal{M}$ belongs to the essential image of ι if and only if for all local equivalences, i.e. for all morphisms $f : A \to B$ of $\mathcal{M}$ such that $L(f)$ is an equivalence, the induced morphism $[B, M] \to [A, M]$ is an equivalence.

We start with the following basic lemma that describes when a map of operads is part of an operadic adjunction:

Lemma 6.73. *Let $\mathcal{O}^\otimes$ be an operad.*

1. *Let $G : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ be a map of operads over $\mathcal{O}^\otimes$.*

   The following conditions are equivalent:

   (a) G admits a left adjoint relative to $\mathcal{O}^\otimes$.
   (b) For every object X of $\mathcal{O}$ the induced functor $G_X : \mathcal{D}_X \to \mathcal{C}_X$ on the fiber over X admits a left adjoint $F_X : \mathcal{C}_X \to \mathcal{D}_X$ and for all $n \in \mathbb{N}$ and objects $X_1, ...X_n, W$ of $\mathcal{O}$ and objects $Y_1 \in \mathcal{C}_{X_1}, ..., Y_n \in \mathcal{C}_{X_n}, Z \in \mathcal{D}_W$ the canonical map

   $$\mathrm{Mul}_{\mathcal{D}}(F_{X_1}(Y_1), ..., F_{X_n}(Y_n), Z) \to$$

$$\mathrm{Mul}_{\mathcal{C}}(G_{X_1}(F_{X_1}(Y_1)),...,G_{X_n}(F_{X_n}(Y_n)),G_W(Z)) \to$$
$$\mathrm{Mul}_{\mathcal{C}}(Y_1,...,Y_n,G_W(Z))$$

is an equivalence.

2. *Let* $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ *be a map of operads over* $\mathcal{O}^{\otimes}$.

   The following conditions are equivalent:

   (a) F *admits a left adjoint relative to* $\mathcal{O}^{\otimes}$.

   (b) For every object X *of* $\mathcal{O}$ *the induced functor* $F_X : \mathcal{C}_X \to \mathcal{D}_X$ *on the fiber over* X *admits a right adjoint* $G_X : \mathcal{D}_X \to \mathcal{C}_X$ *and for all* $n \in \mathbb{N}$ *and objects* $X_1,...X_n, W$ *of* $\mathcal{O}$ *and objects* $Y_1 \in \mathcal{C}_{X_1},...,Y_n \in \mathcal{C}_{X_n}, Z \in \mathcal{D}_W$ *the canonical map*

$$\mathrm{Mul}_{\mathcal{C}}(Y_1,...,Y_n,G_W(Z)) \to$$
$$\mathrm{Mul}_{\mathcal{D}}(F_{X_1}(Y_1),...,F_{X_n}(Y_n),F_W(G_W(Z))) \to$$
$$\mathrm{Mul}_{\mathcal{D}}(F_{X_1}(Y_1),...,F_{X_n}(Y_n),Z)$$

   is an equivalence.

Proof. 1. a) implies b): If G admits a left adjoint relative to $\mathcal{O}^{\otimes}$, for every object X of $\mathcal{O}$ the induced functor $G_X : \mathcal{D}_X \to \mathcal{C}_X$ on the fiber over X admits a left adjoint $F_X : \mathcal{C}_X \to \mathcal{D}_X$.

Moreover for every objects $Y \in \mathcal{C}^{\otimes}$ lying over some object X of $\mathcal{O}^{\otimes}$ and all objects $Z \in \mathcal{D}$ lying over some object W of $\mathcal{O}$ the canonical map $\Phi : \mathcal{D}^{\otimes}(F_X(Y),Z) \to \mathcal{C}^{\otimes}(G_X(F_X(Y)),G_W(Z)) \to \mathcal{C}^{\otimes}(Y,G_W(Z))$ over $\mathcal{O}^{\otimes}(X,W)$ is an equivalence.

The pullback of Φ along the full subspace inclusion $\mathrm{Mul}_{\mathcal{O}}(X_1,...,X_n;W) \subset \mathcal{O}^{\otimes}(X,W)$ is equivalent to the canonical map

$$\mathrm{Mul}_{\mathcal{D}}(F_{X_1}(Y_1),...,F_{X_n}(Y_n),Z) \to$$
$$\mathrm{Mul}_{\mathcal{C}}(G_{X_1}(F_{X_1}(Y_1)),...,G_{X_n}(F_{X_n}(Y_n)),G_W(Z)) \to$$
$$\mathrm{Mul}_{\mathcal{C}}(Y_1,...,Y_n,G_W(Z)),$$

where $X_1,...,X_n$ denote the components of X and $Y_1,...,Y_n$ denote the components of Y for some $n \in \mathbb{N}$.

1. b) implies a):

Condition 1. is equivalent to the condition that for all $Y \in \mathcal{C}^{\otimes}$ lying over some object X of $\mathcal{O}^{\otimes}$ there is an object $T \in \mathcal{D}^{\otimes}_X$ and a morphism $\alpha : Y \to G_X(T)$ in $\mathcal{C}^{\otimes}_X$ such that for all objects $Z \in \mathcal{D}^{\otimes}$ lying over some object W of $\mathcal{O}^{\otimes}$ the canonical map $\Psi : \mathcal{D}^{\otimes}(T,Z) \to \mathcal{C}^{\otimes}(G_X(T),G_W(Z)) \to \mathcal{C}^{\otimes}(Y,G_W(Z))$ is an equivalence.

The map Ψ is a map over $\mathcal{O}^{\otimes}(X,W)$ and is thus an equivalence if and only if it induces on the fiber over every morphism $\varphi : X \to W$ of $\mathcal{O}^{\otimes}$ an equivalence.

Using that $\mathcal{O}^{\otimes},\mathcal{C}^{\otimes},\mathcal{D}^{\otimes}$ are operads and $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ is a map of operads over $\mathcal{O}^{\otimes}$ this is equivalent to the condition that Ψ induces an equivalence on the fiber over every active morphism $\varphi : X \to W$ of $\mathcal{O}^{\otimes}$ with $W \in \mathcal{O}$.

Hence Ψ is an equivalence if and only if the pullback Ψ' of Ψ along the full subspace inclusion $\mathrm{Mul}_{\mathcal{O}}(X_1,...,X_n;W) \subset \mathcal{O}^{\otimes}(X,W)$ is an equivalence, where $X_1,...,X_n$ denote the components of X for some $n \in \mathbb{N}$.

But Ψ' is equivalent to the canonical map

$$\mathrm{Mul}_{\mathcal{D}}(\mathrm{T}_1, ..., \mathrm{T}_n, \mathrm{Z}) \to$$

$$\mathrm{Mul}_{\mathcal{C}}(\mathrm{G}_{\mathrm{X}_1}(\mathrm{T}_1), ..., \mathrm{G}_{\mathrm{X}_n}(\mathrm{T}_n), \mathrm{G}_{\mathrm{W}}(\mathrm{Z})) \to \mathrm{Mul}_{\mathcal{C}}(\mathrm{Y}_1, ..., \mathrm{Y}_n, \mathrm{G}_{\mathrm{W}}(\mathrm{Z}))$$

induced by the components $\alpha_i : \mathrm{Y}_i \to \mathrm{G}_{\mathrm{X}_i}(\mathrm{T}_i)$ of α in $\mathcal{D}_{\mathrm{X}_i}$ for $i \in \{1, ..., n\}$, where $\mathrm{Y}_1, ..., \mathrm{Y}_n$ and $\mathrm{T}_1, ..., \mathrm{T}_n$ denote the components of Y respectively T.

2. a) implies b): If F admits a right adjoint relative to $\mathcal{O}^\otimes$, for every object X of $\mathcal{O}$ the induced functor $\mathrm{F}_\mathrm{X} : \mathcal{C}_\mathrm{X} \to \mathcal{D}_\mathrm{X}$ on the fiber over X admits a right adjoint $\mathrm{G}_\mathrm{X} : \mathcal{D}_\mathrm{X} \to \mathcal{C}_\mathrm{X}$.

Moreover for every objects $\mathrm{Y} \in \mathcal{C}^\otimes$ lying over some object X of $\mathcal{O}^\otimes$ and all objects $\mathrm{Z} \in \mathcal{D}$ lying over some object W of $\mathcal{O}$ the canonical map $\Phi : \mathcal{C}^\otimes(\mathrm{Y}, \mathrm{G}_\mathrm{W}(\mathrm{Z})) \to \mathcal{D}^\otimes(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{F}_\mathrm{W}(\mathrm{G}_\mathrm{W}(\mathrm{Z}))) \to \mathcal{D}^\otimes(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{Z})$ over $\mathcal{O}^\otimes(\mathrm{X}, \mathrm{W})$ is an equivalence.

The pullback of Φ along the full subspace inclusion $\mathrm{Mul}_{\mathcal{O}}(\mathrm{X}_1, ..., \mathrm{X}_n; \mathrm{W}) \subset \mathcal{O}^\otimes(\mathrm{X}, \mathrm{W})$ is equivalent to the canonical map

$$\mathrm{Mul}_{\mathcal{C}}(\mathrm{Y}_1, ..., \mathrm{Y}_n, \mathrm{G}_\mathrm{W}(\mathrm{Z})) \to$$

$$\mathrm{Mul}_{\mathcal{D}}(\mathrm{F}_{\mathrm{X}_1}(\mathrm{Y}_1), ..., \mathrm{F}_{\mathrm{X}_n}(\mathrm{Y}_n), \mathrm{F}_\mathrm{W}(\mathrm{G}_\mathrm{W}(\mathrm{Z}))) \to$$
$$\mathrm{Mul}_{\mathcal{D}}(\mathrm{F}_{\mathrm{X}_1}(\mathrm{Y}_1), ..., \mathrm{F}_{\mathrm{X}_n}(\mathrm{Y}_n), \mathrm{Z}),$$

where $\mathrm{X}_1, ..., \mathrm{X}_n$ denote the components of X and $\mathrm{Y}_1, ..., \mathrm{Y}_n$ denote the components of Y for some $n \in \mathbb{N}$.

2. b) implies a): Condition 1. is equivalent to the condition that for all $\mathrm{Z} \in \mathcal{D}$ lying over some object W of $\mathcal{O}$ there is an object $\mathrm{T} \in \mathcal{C}_\mathrm{W}$ and a morphism $\alpha : \mathrm{F}_\mathrm{W}(\mathrm{T}) \to \mathrm{Z}$ in $\mathcal{D}_\mathrm{W}$ such that for all objects $\mathrm{Y} \in \mathcal{C}^\otimes$ lying over some object X of $\mathcal{O}^\otimes$ the canonical map $\Psi : \mathcal{C}^\otimes(\mathrm{Y}, \mathrm{T}) \to \mathcal{D}^\otimes(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{F}_\mathrm{W}(\mathrm{T})) \to \mathcal{D}^\otimes(\mathrm{F}_\mathrm{X}(\mathrm{Y}), \mathrm{Z})$ is an equivalence.

The map Ψ is a map over $\mathcal{O}^\otimes(\mathrm{X}, \mathrm{W})$ and is thus an equivalence if and only if it induces on the fiber over every morphism $\varphi : \mathrm{X} \to \mathrm{W}$ of $\mathcal{O}^\otimes$ an equivalence.

Using that $\mathcal{O}^\otimes, \mathcal{C}^\otimes, \mathcal{D}^\otimes$ are operads and $\mathrm{G} : \mathcal{D}^\otimes \to \mathcal{C}^\otimes$ is a map of operads over $\mathcal{O}^\otimes$ this is equivalent to the condition that Ψ induces an equivalence on the fiber over every active morphism $\varphi : \mathrm{X} \to \mathrm{W}$ of $\mathcal{O}^\otimes$ with $\mathrm{W} \in \mathcal{O}$.

Hence Ψ is an equivalence if and only if the pullback Ψ' of Ψ along the full subspace inclusion $\mathrm{Mul}_{\mathcal{O}}(\mathrm{X}_1, ..., \mathrm{X}_n; \mathrm{W}) \subset \mathcal{O}^\otimes(\mathrm{X}, \mathrm{W})$ is an equivalence, where $\mathrm{X}_1, ..., \mathrm{X}_n$ denote the components of X for some $n \in \mathbb{N}$.

But Ψ' is equivalent to the canonical map

$$\mathrm{Mul}_{\mathcal{C}}(\mathrm{Y}_1, ..., \mathrm{Y}_n, \mathrm{T}) \to$$

$$\mathrm{Mul}_{\mathcal{D}}(\mathrm{F}_{\mathrm{X}_1}(\mathrm{Y}_1), ..., \mathrm{F}_{\mathrm{X}_n}(\mathrm{Y}_n), \mathrm{F}_\mathrm{W}(\mathrm{T})) \to \mathrm{Mul}_{\mathcal{D}}(\mathrm{F}_{\mathrm{X}_1}(\mathrm{Y}_1), ..., \mathrm{F}_{\mathrm{X}_n}(\mathrm{Y}_n), \mathrm{Z})$$

induced by $\alpha : \mathrm{F}_\mathrm{W}(\mathrm{T}) \to \mathrm{Z}$, where $\mathrm{Y}_1, ..., \mathrm{Y}_n$ denote the components of Y.
□

Corollary 6.74. *Let* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes, \mathcal{N}^\otimes \to \mathrm{LM}^\otimes$ *be operads over* $\mathrm{LM}^\otimes$ *that exhibit categories* $\mathcal{M}$ *respectively* $\mathcal{N}$ *as pseudo-enriched over a locally cocartesian fibration of operads* $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.

Let $\mathrm{G} : \mathcal{N}^\otimes \to \mathcal{M}^\otimes$ *be a lax* $\mathcal{C}^\otimes$*-linear functor.*

Then G *admits a left adjoint relative to* $LM^{\otimes}$ *if and only if the underlying functor* $\mathcal{N} \to \mathcal{M}$ *admits a left adjoint* $F : \mathcal{M} \to \mathcal{N}$ *and for all objects* $A \in \mathcal{C}, M \in \mathcal{M}, N \in \mathcal{N}$ *the canonical map*

$$Mul_{\mathcal{N}}(A, F(M); N) \to Mul_{\mathcal{M}}(A, G(F(M)); G(N)) \to Mul_{\mathcal{M}}(A, M; G(N))$$

is an equivalence.

Assume that $\mathcal{M}^{\otimes} \to LM^{\otimes}, \mathcal{N}^{\otimes} \to LM^{\otimes}$ *exhibit* $\mathcal{M}$ *respectively* $\mathcal{N}$ *as enriched over the locally cocartesian fibration of operads* $\mathcal{C}^{\otimes} \to Ass^{\otimes}$.

Then G *admits a left adjoint relative to* $LM^{\otimes}$ *if and only if the underlying functor* $\mathcal{N} \to \mathcal{M}$ *admits a left adjoint* $F : \mathcal{M} \to \mathcal{N}$ *and for all objects* $M \in \mathcal{M}, N \in \mathcal{N}$ *the canonical morphism*

$$[F(M), N] \to [G(F(M)), G(N)] \to [M, G(N)]$$

is an equivalence.

Let $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ *be a lax* $\mathcal{C}^{\otimes}$*-linear functor.*

Then F *admits a right adjoint relative to* $LM^{\otimes}$ *if and only if the underlying functor* $\mathcal{M} \to \mathcal{N}$ *admits a right adjoint* $G : \mathcal{N} \to \mathcal{M}$ *and for all objects* $A \in \mathcal{C}, M \in \mathcal{M}, N \in \mathcal{N}$ *the canonical map*

$$Mul_{\mathcal{M}}(A, M; G(N)) \to Mul_{\mathcal{N}}(A, F(M), F(G(N))) \to Mul_{\mathcal{N}}(A, F(M), N)$$

is an equivalence.

Assume that $\mathcal{M}^{\otimes} \to LM^{\otimes}, \mathcal{N}^{\otimes} \to LM^{\otimes}$ *exhibit* $\mathcal{M}$ *respectively* $\mathcal{N}$ *as enriched over the locally cocartesian fibration of operads* $\mathcal{C}^{\otimes} \to Ass^{\otimes}$.

Then F *admits a right adjoint relative to* $LM^{\otimes}$ *if and only if the underlying functor* $\mathcal{M} \to \mathcal{N}$ *admits a right adjoint* $G : \mathcal{N} \to \mathcal{M}$ *and for all objects* $M \in \mathcal{M}, N \in \mathcal{N}$ *the canonical morphism*

$$[M, G(N)] \to [F(M), F(G(N))] \to [F(M), N]$$

is an equivalence.

Proof. Denote $\sigma \in Mul_{LM}(\mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ the unique operation and let $\alpha \in Mul_{Ass}(\mathfrak{a}, ..., \mathfrak{a}; \mathfrak{a})$.

As $\mathcal{M}$ respectively $\mathcal{N}$ are pseudo-enriched over a locally cocartesian fibration of operads $\mathcal{C}^{\otimes} \to Ass^{\otimes}$, for every $A_1, ..., A_n \in \mathcal{C}$ for some $n \in \mathbb{N}$ and $M \in \mathcal{M}, N \in \mathcal{N}$ the pullback of the canonical map

$$Mul_{\mathcal{N}}(A_1, ..., A_n, F(M); N) \to$$

$$Mul_{\mathcal{M}}(A_1, ..., A_n, G(F(M)); G(N)) \to Mul_{\mathcal{M}}(A_1, ..., A_n, M; G(N))$$

over $Mul_{LM}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ to $\{\sigma \circ (\alpha, \mathfrak{m})\} \subset Mul_{LM}(\mathfrak{a}, ..., \mathfrak{a}, \mathfrak{m}; \mathfrak{m})$ is equivalent to the map

$$Mul_{\mathcal{N}}(\otimes_{\alpha}(A_1, ..., A_n), F(M); N) \to$$

$$Mul_{\mathcal{M}}(\otimes_{\alpha}(A_1, ..., A_n), G(F(M)); G(N)) \to Mul_{\mathcal{M}}(\otimes_{\alpha}(A_1, ..., A_n), M; G(N))$$

and the pullback of the canonical map

$$Mul_{\mathcal{M}}(A_1, ..., A_n, M, G(N)) \to$$

$$Mul_{\mathcal{N}}(A_1, ..., A_n, F(M), F(G(N))) \to Mul_{\mathcal{N}}(A_1, ..., A_n, F(M), N)$$

over $\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a},...,\mathfrak{a},\mathfrak{m};\mathfrak{m})$ to $\{\sigma \circ (\alpha,\mathfrak{m})\} \subset \mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a},...,\mathfrak{a},\mathfrak{m};\mathfrak{m})$ is equivalent to the map

$$\mathrm{Mul}_{\mathcal{M}}(\otimes_\alpha(\mathrm{A}_1,...,\mathrm{A}_n),\mathrm{M},\mathrm{G}(\mathrm{N})) \to$$

$$\mathrm{Mul}_{\mathcal{N}}(\otimes_\alpha(\mathrm{A}_1,...,\mathrm{A}_n),\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{G}(\mathrm{N}))) \to \mathrm{Mul}_{\mathcal{N}}(\otimes_\alpha(\mathrm{A}_1,...,\mathrm{A}_n),\mathrm{F}(\mathrm{M}),\mathrm{N}).$$

As all operations of $\mathrm{Mul}_{\mathrm{LM}}(\mathfrak{a},...,\mathfrak{a},\mathfrak{m};\mathfrak{m})$ are of the form $\sigma \circ (\alpha,\mathfrak{m})$ for some $\alpha \in \mathrm{Mul}_{\mathrm{Ass}}(\mathfrak{a},...,\mathfrak{a};\mathfrak{a})$, the statement follows from lemma 6.73.

If $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes, \mathcal{N}^\otimes \to \mathrm{LM}^\otimes$ exhibit $\mathcal{M}$ respectively $\mathcal{N}$ as enriched over the locally cocartesian fibration of operads $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$, for all objects $\mathrm{A} \in \mathcal{C}, \mathrm{M} \in \mathcal{M}, \mathrm{N} \in \mathcal{N}$ the canonical map

$$\mathrm{Mul}_{\mathcal{N}}(\mathrm{A},\mathrm{F}(\mathrm{M});\mathrm{N}) \to \mathrm{Mul}_{\mathcal{M}}(\mathrm{A},\mathrm{G}(\mathrm{F}(\mathrm{M}));\mathrm{G}(\mathrm{N})) \to \mathrm{Mul}_{\mathcal{M}}(\mathrm{A},\mathrm{M};\mathrm{G}(\mathrm{N}))$$

is equivalent to the canonical map

$$\mathcal{C}(\mathrm{A},[\mathrm{F}(\mathrm{M}),\mathrm{N}]) \to \mathcal{C}(\mathrm{A},[\mathrm{G}(\mathrm{F}(\mathrm{M})),\mathrm{G}(\mathrm{N})]) \to \mathcal{C}(\mathrm{A},[\mathrm{M},\mathrm{G}(\mathrm{N})])$$

and the canonical map

$$\mathrm{Mul}_{\mathcal{M}}(\mathrm{A},\mathrm{M};\mathrm{G}(\mathrm{N})) \to \mathrm{Mul}_{\mathcal{N}}(\mathrm{A},\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{G}(\mathrm{N}))) \to \mathrm{Mul}_{\mathcal{N}}(\mathrm{A},\mathrm{F}(\mathrm{M}),\mathrm{N})$$

is equivalent to the canonical map

$$\mathcal{C}(\mathrm{A},[\mathrm{M},\mathrm{G}(\mathrm{N})]) \to \mathcal{C}(\mathrm{A},[\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{G}(\mathrm{N}))]) \to \mathcal{C}(\mathrm{A},[\mathrm{F}(\mathrm{M}),\mathrm{N}]).$$

□

Corollary 6.75. *Let* $\mathcal{M}^\otimes \to \mathrm{LM}^\otimes, \mathcal{N}^\otimes \to \mathrm{LM}^\otimes$ *be operads over* $\mathrm{LM}^\otimes$ *that exhibit categories* $\mathcal{M}$ *respectively* $\mathcal{N}$ *as enriched over a locally cocartesian fibration of operads* $\mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$.

Let $\mathrm{F} : \mathcal{M}^\otimes \to \mathcal{N}^\otimes$ *be a* $\mathcal{C}$*-enriched functor.*

Then F *is an equivalence of operads over* $\mathrm{LM}^\otimes$ *if and only if the underlying functor* $\mathcal{M} \to \mathcal{N}$ *is essentially surjective and for all objects* $\mathrm{M},\mathrm{M}' \in \mathcal{M}$ *the canonical morphism*

$$[\mathrm{M},\mathrm{M}'] \to [\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{M}')]$$

is an equivalence.

Proof. Assume that the underlying functor $\mathcal{M} \to \mathcal{N}$ of F is essentially surjective.

If for all objects $\mathrm{M},\mathrm{M}' \in \mathcal{M}$ the canonical morphism

$$\alpha : [\mathrm{M},\mathrm{M}'] \to [\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{M}')]$$

is an equivalence, then the underlying functor $\mathcal{M} \to \mathcal{N}$ of F is fully faithful, using the canonical equivalence $\mathcal{C}(\mathbb{1},[\mathrm{A},\mathrm{B}]) \simeq \mathcal{M}(\mathrm{A},\mathrm{B})$ for all $\mathrm{A},\mathrm{B} \in \mathcal{M}$, and is thus an equivalence.

Hence the underlying functor $\mathcal{M} \to \mathcal{N}$ of F admits a right adjoint $\mathrm{G} : \mathcal{N} \to \mathcal{M}$ such that unit and couit of the adjunction are equivalences.

So for all objects $\mathrm{M} \in \mathcal{M}, \mathrm{N} \in \mathcal{N}$ the canonical morphism

$$[\mathrm{M},\mathrm{G}(\mathrm{N})] \to [\mathrm{F}(\mathrm{M}),\mathrm{F}(\mathrm{G}(\mathrm{N}))] \to [\mathrm{F}(\mathrm{M}),\mathrm{N}]$$

is an equivalence.

Thus by corollary 6.74 $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ admits a right adjoint relative to $LM^{\otimes}$.

As unit and couit of the adjunction $\mathcal{M} \rightleftarrows \mathcal{N} : G$ are equivalences, unit and couit of the adjunction $F : \mathcal{M}^{\otimes} \rightleftarrows \mathcal{N}^{\otimes}$ relative to $LM^{\otimes}$ are equivalences so that $F : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes}$ is an equivalence of operads over $LM^{\otimes}$.

□

Observation 6.76. *Let* $\mathcal{M}^{\otimes} \to LM^{\otimes}, \mathcal{N}^{\otimes} \to LM^{\otimes}$ *be operads over* $LM^{\otimes}$ *that exhibit categories* $\mathcal{M}$ *respectively* $\mathcal{N}$ *as enriched over a locally cocartesian fibration of operads* $\mathcal{C}^{\otimes} \to Ass^{\otimes}$.

Let $G : \mathcal{N}^{\otimes} \to \mathcal{M}^{\otimes}$ *be a* $\mathcal{C}$*-enriched functor that admits a left adjoint relative to* $LM^{\otimes}$.

The underlying functor $\mathcal{N} \to \mathcal{M}$ *of* G *is fully faithful if and only if for all objects* $N, N' \in \mathcal{N}$ *the canonical morphism*

$$[N, N'] \to [G(N), G(N')]$$

is an equivalence.

In this case we call the adjunction $\mathcal{M}^{\otimes} \rightleftarrows \mathcal{N}^{\otimes} : G$ *a* $\mathcal{C}$*-enriched localization.*

Proof. The if direction follows from the fact that the canonical map

$$\mathcal{C}(\mathbb{1}, [N, N']) \to \mathcal{C}(\mathbb{1}, [G(N), G(N')])$$

is equivalent to the canonical map $\mathcal{N}(N, N') \to \mathcal{M}(G(N), G(N'))$.

The only if direction follows from the fact that for all objects A of $\mathcal{C}$ the map

$$\mathcal{C}(A, [N, N']) \to \mathcal{C}(A, [G(N), G(N')])$$

is equivalent to the canonical map

$$Mul_{\mathcal{N}}(A, N; N') \to Mul_{\mathcal{M}}(A, G(N); G(N'))$$

that factors as

$$Mul_{\mathcal{N}}(A, N; N') \to Mul_{\mathcal{M}}(A, F(G(N)); N') \simeq Mul_{\mathcal{M}}(A, G(N); G(N'))$$

and the fact that the counit $F(G(N)) \to N$ is an equivalence if the underlying functor $\mathcal{N} \to \mathcal{M}$ of G is fully faithful.

□

Observation 6.77. *Let* $L : \mathcal{M}^{\otimes} \to \mathcal{N}^{\otimes} : \iota$ *be a* $\mathcal{C}$*-enriched localization.*

An object M *of* $\mathcal{M}$ *belongs to the essential image of* ι *if and only if for all local equivalences, i.e. for all morphisms* $f : A \to B$ *of* $\mathcal{M}$ *such that* $L(f)$ *is an equivalence, the induced morphism* $[B, M] \to [A, M]$ *is an equivalence:*

If M *belongs to the essential image of* ι, *i.e.* $M \simeq \iota(N)$ *for some* $N \in \mathcal{N}$, *the induced morphism* $[B, M] \to [A, M]$ *is equivalent to the morphism* $[L(B), N] \to [L(A), N]$ *and is thus an equivalence.*

On the other hand if for all local equivalences $f : A \to B$ *the induced morphism* $[B, M] \to [A, M]$ *is an equivalence, for all local equivalences* $f : A \to B$ *the induced map* $\mathcal{M}(B, M) \to \mathcal{M}(A, M)$ *is an equivalence so that* M *belongs to the essential image of* ι.

6.2.7 Adjunctions in 2-categories

In this subsection we show (prop. 6.78) that a morphism $g : Y \to X$ in a 2-category $\mathcal{C}$ admits a left adjoint if and only if the following two conditions hold:

1. For every object Z of $\mathcal{C}$ the induced functor $g^Z := [Z, g] : [Z, Y] \to [Z, X]$ admits a left adjoint f^Z.
2. For every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced functor $[\varphi, X] : [Z', X] \to [Z, X]$ preserves the unit.

Proposition 6.78. *Let $\mathcal{C}$ be a 2-category.*

Let X, Y be objects of $\mathcal{C}$ and $g : Y \to X$ a morphism of $\mathcal{C}$.

1. *Let $\mathsf{f} : X \to Y$ be a morphism of $\mathcal{C}$ and $\eta : \mathrm{id}_X \to g \circ \mathsf{f}$ a 2-morphism of $\mathcal{C}$.*
   Then there is a 2-morphism $\varepsilon : \mathsf{f} \circ g \to \mathrm{id}_Y$ of $\mathcal{C}$ satisfying the triangular identities $(\varepsilon \circ \mathsf{f}) \circ (\mathsf{f} \circ \eta) = \mathrm{id}_{\mathsf{f}}$ and $(g \circ \varepsilon) \circ (\eta \circ g) = \mathrm{id}_g$ if and only if the following condition holds:
   For every object Z of $\mathcal{C}$ the induced natural transformation $\eta^Z := [Z, \eta] : \mathrm{id}_{[Z,X]} \to [Z, g] \circ [Z, \mathsf{f}]$ exhibits the functor $\mathsf{f}^Z := [Z, \mathsf{f}] : [Z, X] \to [Z, Y]$ as left adjoint to the functor $g^Z := [Z, g] : [Z, Y] \to [Z, X]$.
2. *Let $\mathcal{C}$ be a closed and cotensored left module over Cat_∞.*
   Then the morphism $g : Y \to X$ of $\mathcal{C}$ admits a left adjoint $\mathsf{f} : X \to Y$ if and only if the following two conditions hold:
   (a) For every object Z of $\mathcal{C}$ the induced functor $g^Z := [Z, g] : [Z, Y] \to [Z, X]$ admits a left adjoint f^Z.
   (b) For every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced natural transformation $\mathsf{f}^Z \circ [\varphi, X] \to \mathsf{f}^Z \circ [\varphi, X] \circ g^{Z'} \circ \mathsf{f}^{Z'} \simeq \mathsf{f}^Z \circ g^Z \circ [\varphi, Y] \circ \mathsf{f}^{Z'} \to [\varphi, Y] \circ \mathsf{f}^{Z'}$ is an equivalence.

Remark 6.79. *The compatibility condition of (b) is equivalent to the condition that for every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced natural transformation*

$$[\varphi, X] \circ g^{Z'} \to g^Z \circ \mathsf{f}^Z \circ [\varphi, X] \circ g^{Z'} \simeq g^Z \circ [\varphi, Y] \circ \mathsf{f}^{Z'} \circ g^{Z'} \to g^Z \circ [\varphi, Y]$$

is an equivalence and is equivalent to the condition that for every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$ the induced functor $[\varphi, X] : [Z', X] \to [Z, X]$ preserves the unit of the adjunction in the following sense:

Let $\eta : \mathrm{id} \to g^{Z'} \circ \mathsf{f}^{Z'}$ be a unit of the adjunction $\mathsf{f}^{Z'} : [Z', X] \to [Z', Y] : g^{Z'}$ and $H : Z' \to X$ be a morphism of $\mathcal{C}$.

Then the composition $H \circ \varphi \xrightarrow{\eta(H) \circ \varphi} g^{Z'}(\mathsf{f}^{Z'}(H)) \circ \varphi \simeq g^Z(\mathsf{f}^{Z'}(H) \circ \varphi)$ yields for every morphism $T : Z \to Y$ of $\mathcal{C}$ an equivalence

$$[Z, Y](\mathsf{f}^{Z'}(H) \circ \varphi, T) \to [Z, X](g^Z(\mathsf{f}^{Z'}(H) \circ \varphi), g^Z(T)) \to$$

$$[Z, X](H \circ \varphi, g^Z(T)).$$

Proof. We show 1:

Denote $\varepsilon^Z : f^Z \circ g^Z \to \mathrm{id}$ the counit of the adjunction $f^Z = [Z, f] : [Z, X] \rightleftarrows [Z, Y] : g^Z = [Z, g]$ and set $\varepsilon := \varepsilon^Y(\mathrm{id}_Y) : f \circ g \to \mathrm{id}_Y$.

In the following we will see that η and ε are related by the triangular identities.

The triangular identities of the adjunctions

$$f^X = [X, f] : [X, X] \rightleftarrows [X, Y] : g^X = [X, g]$$

$$f^Y = [Y, f] : [Y, X] \rightleftarrows [Y, Y] : g^Y = [Y, g]$$

imply that both compositions $(\varepsilon^X \circ f^X) \circ (f^X \circ \eta^X)$ and $(g^Y \circ \varepsilon^Y) \circ (\eta^Y \circ g^Y)$ of natural transformations of functors $[X, X] \to [X, Y]$ respectively $[Y, Y] \to [Y, X]$ are homotopic to the identity.

Evaluating at id_X respectively id_Y we see that the compositions $\varepsilon^X(f) \circ (f \circ \eta)$ and $(g \circ \varepsilon) \circ (\eta \circ g)$ are homotopic to the identity.

So it remains to show that $\varepsilon^X(f) : f^X(g^X(f)) = f \circ g \circ f \to f$ is homotopic to $\varepsilon \circ f : f \circ g \circ f \to f$.

This is equivalent to the condition that $\varepsilon \circ f : f^X(g \circ f) = f \circ g \circ f \to f$ is adjoint to the identity of $g^X(f) = g \circ f$ with respect to the adjunction $f^X = [X, f] : [X, X] \rightleftarrows [X, Y] : g^X = [X, g]$, in other words that

$$g^X(\varepsilon \circ f) \circ \eta^X(g \circ f) = (g \circ \varepsilon \circ f) \circ (\eta \circ g \circ f) = ((g \circ \varepsilon) \circ (\eta \circ g)) \circ f$$

is homotopic to the identity of $g \circ f$.

But we have already seen that $(g \circ \varepsilon) \circ (\eta \circ g)$ is homotopic to the identity of g.

As next we prove 2.

Denote $(-)^{\simeq} : \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \to \mathcal{R}_{\mathcal{C}}, \ \widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}} \to \widehat{\mathcal{R}}_{\mathsf{Cat}_\infty \times \mathcal{C}}$ the right adjoints of the full subcategory inclusions $\mathcal{R}_{\mathcal{C}} \subset \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ respectively $\widehat{\mathcal{R}}_{\mathsf{Cat}_\infty \times \mathcal{C}} \subset \widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}}$ that take fiberwise the maximal subspace.

Denote $\theta : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Cat}_\infty) \simeq \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ the functor adjoint to the functor $[-,-] : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Cat}_\infty$ so that the composition $\theta : \mathcal{C} \to \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \xrightarrow{(-)^{\simeq}} \mathcal{R}_{\mathcal{C}} \simeq \mathcal{P}(\mathcal{C})$ is the Yoneda-embedding.

We have a canonical equivalence $\theta(Z^K) \simeq \theta(Z)^K$ of cartesian fibrations over $\mathcal{C}$ natural in $K \in \mathsf{Cat}_\infty$ and $Z \in \mathcal{C}$ classified by the canonical equivalence $[-, Z^K] \simeq \mathrm{Fun}(K, -) \circ [-, Z]$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}_\infty$ represented by the natural equivalence

$$\mathsf{Cat}_\infty(W, [T, Z^K]) \simeq \mathcal{C}((K \times W) \otimes T, Z) \simeq \mathsf{Cat}_\infty(W, \mathrm{Fun}(K, [T, Z]))$$

for $W \in \mathsf{Cat}_\infty$ and $T \in \mathcal{C}$.

The functor $(\Delta^1)^{\mathrm{op}} \to \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$ corresponding to $\theta(g) : \theta(Y) \to \theta(X)$ is classified by a map $\mathcal{Z} \to \mathcal{C} \times \Delta^1$ of cartesian fibrations over Δ^1 that is a cartesian fibration.
By corollary 6.42 condition (a) of 2. implies that $\mathcal{Z} \to \mathcal{C} \times \Delta^1$ is a map of bicartesian fibrations over Δ^1 encoding an adjunction $F : \theta(X) \rightleftarrows \theta(Y) : \theta(g)$ relative to $\mathcal{C}$, which we view as an adjunction in the 2-category $\mathsf{Cat}_{\infty/\mathcal{C}}$.

Condition (b) of 2. implies that the left adjoint $F : \theta(X) \to \theta(Y)$ of $\theta(g)$ is a map of cartesian fibrations over $\mathcal{C}$ so that the adjunction $F : \theta(X) \rightleftarrows \theta(Y) : \theta(g)$ in the 2-category $\mathsf{Cat}_{\infty/\mathcal{C}}$ is an adjunction in the 2-category $\mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$.

Let $\lambda : \mathrm{id}_{\theta(X)} \to \theta(g) \circ F$ be the unit of this adjunction relative to $\mathcal{C}$ and $\Phi : \theta(X) \to \theta(X)^{\Delta^1} \simeq \theta(X^{\Delta^1})$ the corresponding map of cartesian fibrations over $\mathcal{C}$ under the canonical equivalence
$\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}(\theta(X), \theta(X)^{\Delta^1}) \simeq \mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{C}}(\theta(X), \theta(X)))$.

Set $\mathsf{f} := F_X(\mathrm{id}_X) : X \to Y, \phi := \Phi_X(\mathrm{id}_X) : X \to X^{\Delta^1}$ and $\eta := \lambda(\mathrm{id}_X) : \mathrm{id}_X \to g \circ \mathsf{f}$ so that the morphism η of $[X, X]$ is adjoint to $\phi : X \to X^{\Delta^1}$.

By the Yoneda-lemma the induced maps $F^{\simeq} : \theta(X)^{\simeq} \to \theta(Y)^{\simeq}$ and $\Phi^{\simeq} : \theta(X)^{\simeq} \to \theta(X^{\Delta^1})^{\simeq}$ of right fibrations over $\mathcal{C}$ are equivalent over $\mathcal{C}$ to $\theta(\mathsf{f})^{\simeq} : \theta(X)^{\simeq} \to \theta(Y)^{\simeq}$ respectively $\theta(\phi)^{\simeq} : \theta(X)^{\simeq} \to \theta(X^{\Delta^1})^{\simeq}$.

By the first part of the lemma it is enough to show that there is an equivalence $\alpha : F \to \theta(\mathsf{f})$ of maps $\theta(X) \to \theta(Y)$ of cartesian fibrations over $\mathcal{C}$ and a commutative square

$$\begin{array}{ccc} \mathrm{id}_{\theta(X)} & \xrightarrow{\lambda} & \theta(g) \circ F \\ \downarrow{\scriptstyle\simeq} & & \downarrow{\scriptstyle \theta(g)\circ\alpha} \\ \mathrm{id}_{\theta(X)} & \xrightarrow{\theta(\eta)} & \theta(g) \circ \theta(\mathsf{f}) \end{array}$$

in $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}$.

This commutative square considered as an equivalence in the category $\mathrm{Fun}(\Delta^1, \mathrm{Fun}^{\mathrm{cart}}_{\mathcal{C}}(\theta(X), \theta(X))) \simeq \mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}}(\theta(X), \theta(X)^{\Delta^1})$ between λ and $\theta(\eta)$ corresponds to an equivalence $\beta : \Phi \to \theta(\phi)$ of maps $\theta(X) \to \theta(X^{\Delta^1}) \simeq \theta(X)^{\Delta^1}$ of cartesian fibrations over $\mathcal{C}$ that is sent by the map $\theta(X^{\Delta^1}) \to \theta(X^{\{1\}})$ of cartesian fibrations over $\mathcal{C}$ to the equivalence $\theta(g) \circ \alpha$.

Denote γ the composition

$$\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Cat}_\infty) \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}}, \mathsf{Cat}_\infty)) \simeq$$

$$\mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \mathsf{Cat}_\infty) \subset \mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \widehat{\mathsf{Cat}_\infty}) \simeq \widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}},$$

where the functor $\mathsf{Cat}_\infty \to \mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}}, \mathsf{Cat}_\infty)$ is adjoint to the functor $\mathrm{Fun}(-,-) : \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$.

The composition $\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \xrightarrow{\gamma} \widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}} \xrightarrow{(-)^{\simeq}} \widehat{\mathcal{R}}_{\mathsf{Cat}_\infty \times \mathcal{C}}$ is equivalent to the embedding

$$\mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \simeq \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Cat}_\infty) \subset \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}}, \widehat{\mathcal{S}})) \simeq$$

$$\mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}}, \widehat{\mathcal{S}}) \simeq \widehat{\mathcal{R}}_{\mathsf{Cat}_\infty \times \mathcal{C}}$$

induced by the Yoneda-embedding $\mathsf{Cat}_\infty \subset \mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}}, \widehat{\mathcal{S}})$.

Consequently it is enough to show that there is an equivalence $\alpha' : \gamma(F)^{\simeq} \to \gamma(\theta(\mathsf{f}))^{\simeq}$ of maps $\gamma(\theta(X))^{\simeq} \to \gamma(\theta(Y))^{\simeq}$ of right fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ and an equivalence $\beta' : \gamma(\Phi)^{\simeq} \to \gamma(\theta(\phi))^{\simeq}$ of maps $\gamma(\theta(X))^{\simeq} \to \gamma(\theta(X^{\Delta^1}))^{\simeq}$ of right fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ that is sent by the map $\gamma(\theta(X^{\Delta^1}))^{\simeq} \to \gamma(\theta(X^{\{1\}}))^{\simeq}$ of right fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ to the equivalence $\gamma(\theta(g))^{\simeq} \circ \alpha' : \gamma(\theta(g))^{\simeq} \circ \gamma(F)^{\simeq} \to \gamma(\theta(g))^{\simeq} \circ \gamma(\theta(\mathsf{f}))^{\simeq}$ of maps $\gamma(\theta(X))^{\simeq} \to \gamma(\theta(X))^{\simeq}$ of right fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$.

By the canonical equivalence $[- \otimes -, -] \simeq \mathrm{Fun}(-, [-, -])$ of functors $\mathsf{Cat}_\infty^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Cat}_\infty$ the functor $\gamma \circ \theta : \mathcal{C} \to \mathsf{Cat}^{\mathrm{cart}}_{\infty/\mathcal{C}} \to \widehat{\mathsf{Cat}}^{\mathrm{cart}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}}$ is equivalent via an equivalence ζ to the functor $\mathcal{C} \xrightarrow{\theta} \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \subset \widehat{\mathsf{Cat}}_{\infty/\mathcal{C}}^{\mathrm{cart}} \xrightarrow{\mu^*} \widehat{\mathsf{Cat}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}}^{\mathrm{cart}}$, where μ^* takes the pullback along the left action functor $\mu : \mathsf{Cat}_\infty \times \mathcal{C} \to \mathcal{C}$.

Especially the maps $\gamma(\theta(\mathrm{g})) : \gamma(\theta(\mathrm{Y})) \to \gamma(\theta(\mathrm{X}))$ and $(\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{g}) : (\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{Y}) \to (\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{X})$ of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ are canonically equivalent.

The functor $\mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \subset \widehat{\mathsf{Cat}}_{\infty/\mathcal{C}}^{\mathrm{cart}} \xrightarrow{\mu^*} \widehat{\mathsf{Cat}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}}^{\mathrm{cart}}$ is a symmetric monoidal functor under Cat_∞ with respect to the cartesian structures and thus especially Cat_∞-linear.

By ... the functor $\mathsf{Cat}_\infty \to \mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}}, \mathsf{Cat}_\infty)$ adjoint to the functor $\mathrm{Fun}(-,-) : \mathsf{Cat}_\infty^{\mathrm{op}} \times \mathsf{Cat}_\infty \to \mathsf{Cat}_\infty$ is lax Cat_∞-linear, where $\mathrm{Fun}(\mathsf{Cat}_\infty^{\mathrm{op}}, \mathsf{Cat}_\infty)$ is endowed with the diagonal action of Cat_∞.

Hence the functor $\gamma : \mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}} \to \widehat{\mathsf{Cat}}_{\infty/\mathsf{Cat}_\infty \times \mathcal{C}}^{\mathrm{cart}}$ is lax $\mathsf{Cat}_{\infty/\mathcal{C}}^{\mathrm{cart}}$-linear and thus especially Cat_∞-linear.

Thus the natural transformation over $\mathsf{Cat}_\infty \times \mathcal{C}$

$$(\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \lambda : \mathrm{id}_{(\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{X})} \to ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{g})) \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{F})$$

exhibits the map $(\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{F} : (\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{X}) \to (\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{Y})$ of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ as left adjoint to $(\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{g}) : (\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{Y}) \to (\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{X})$ relative to $\mathsf{Cat}_\infty \times \mathcal{C}$ and

$$\gamma(\lambda) : \mathrm{id}_{\gamma(\theta(\mathrm{X}))} \to \gamma(\theta(\mathrm{g})) \circ \gamma(\mathrm{F})$$

exhibits the map $\gamma(\mathrm{F}) : \gamma(\theta(\mathrm{X})) \to \gamma(\theta(\mathrm{Y}))$ of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ as left adjoint to $\gamma(\theta(\mathrm{g})) : \gamma(\theta(\mathrm{Y})) \to \gamma(\theta(\mathrm{X}))$ relative to $\mathsf{Cat}_\infty \times \mathcal{C}$.

Consequently there is an equivalence

$$\sigma : \gamma(\mathrm{F}) \to \zeta_{\mathrm{Y}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{F}) \circ (\zeta_{\mathrm{X}})^{-1}$$

of maps $\gamma(\theta(\mathrm{X})) \to \gamma(\theta(\mathrm{Y}))$ of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ and an equivalence

$$\chi : \gamma(\Phi) \to \zeta_{\mathrm{X}^{\Delta^1}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \Phi) \circ (\zeta_{\mathrm{X}})^{-1}$$

of maps $\gamma(\theta(\mathrm{X})) \to \gamma(\theta(\mathrm{X}^{\Delta^1}))$ of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ that is sent by the map $\gamma(\theta(\mathrm{X}^{\Delta^1})) \to \gamma(\theta(\mathrm{X}^{\{1\}}))$ of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ to the equivalence

$$\gamma(\theta(\mathrm{g})) \circ \gamma(\mathrm{F}) \xrightarrow{\gamma(\theta(\mathrm{g}))\circ\sigma} \gamma(\theta(\mathrm{g})) \circ \zeta_{\mathrm{Y}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{F}) \circ (\zeta_{\mathrm{X}})^{-1}$$
$$\simeq \zeta_{\mathrm{X}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathrm{g})) \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{F}) \circ (\zeta_{\mathrm{X}})^{-1}$$

of maps of cartesian fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$.

We define the equivalence β' as the composition

$$\gamma(\Phi)^{\simeq} \xrightarrow{\chi^{\simeq}} \zeta^{\simeq}_{\mathrm{X}^{\Delta^1}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \Phi^{\simeq}) \circ (\zeta^{\simeq}_{\mathrm{X}})^{-1}$$
$$\simeq \zeta^{\simeq}_{\mathrm{X}^{\Delta^1}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\phi)^{\simeq}) \circ (\zeta^{\simeq}_{\mathrm{X}})^{-1} \simeq \gamma(\theta(\phi))^{\simeq}$$

and the equivalence α' as the composition

$$\gamma(\mathrm{F})^{\simeq} \xrightarrow{\sigma^{\simeq}} \zeta^{\simeq}_{\mathrm{Y}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \mathrm{F}^{\simeq}) \circ (\zeta^{\simeq}_{\mathrm{X}})^{-1} \simeq \zeta^{\simeq}_{\mathrm{Y}} \circ ((\mathsf{Cat}_\infty \times \mathcal{C}) \times_{\mathcal{C}} \theta(\mathsf{f})^{\simeq}) \circ (\zeta^{\simeq}_{\mathrm{X}})^{-1}$$
$$\simeq \gamma(\theta(\mathsf{f}))^{\simeq}.$$

Then β' is sent by the map $\gamma(\theta(X^{\Delta^1}))^{\simeq} \to \gamma(\theta(X^{\{1\}}))^{\simeq}$ of right fibrations over $\mathsf{Cat}_\infty \times \mathcal{C}$ to the equivalence

$$\gamma(\theta(g))^{\simeq} \circ \alpha' : \gamma(\theta(g))^{\simeq} \circ \gamma(F)^{\simeq} \to \gamma(\theta(g))^{\simeq} \circ \gamma(\theta(\mathsf{f}))^{\simeq}.$$

□

Remark 6.80. *Let $\mathcal{C}$ be a closed and cotensored left module over Cat_∞.*

Let X, Y be objects of $\mathcal{C}$ and $\mathsf{f} : X \to Y$, $g : Y \to X$ morphisms of $\mathcal{C}$.

Denote $\mathcal{F} : \mathcal{X} \to \mathcal{Y}, \mathcal{G} : \mathcal{Y} \to \mathcal{X}$ the maps of cartesian fibrations over $\mathcal{C}$ classifying the natural transformations $[-, \mathsf{f}] : [-, X] \to [-, Y]$ respectively $[-, g] : [-, Y] \to [-, X]$ of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}_\infty$.

Then f is left adjoint to g if and only if $\mathcal{F}$ is left adjoint to $\mathcal{G}$.

Let $\mathcal{C}$ be a 2-category. We call a morphism $g : Y \to X$ of $\mathcal{C}$ a localisation if there is a morphism $\mathsf{f} : X \to Y$ in $\mathcal{C}$ and 2-morphisms $\eta : \mathrm{id}_X \to g \circ \mathsf{f}, \varepsilon : \mathsf{f} \circ g \to \mathrm{id}_Y$ in $\mathcal{C}$ with ε an equivalence satisfying the triangular identities $(\varepsilon \circ \mathsf{f}) \circ (\mathsf{f} \circ \eta) = \mathrm{id}_{\mathsf{f}}$ and $(g \circ \varepsilon) \circ (\eta \circ g) = \mathrm{id}_g$.

Corollary 6.81. *Let $\mathcal{C}$ be a closed and cotensored left module over Cat_∞ and $g : Y \to X$ a morphism of $\mathcal{C}$.*

Then g is a localisation if and only if the following two conditions are satisfied:

1. *For every object Z of $\mathcal{C}$ the induced functor $[Z, g] : [Z, Y] \to [Z, X]$ is a localisation.*
2. *For every morphism $\varphi : Z \to Z'$ of $\mathcal{C}$, the induced functor $[\varphi, X] : [Z', X] \to [Z, X]$ preserves local equivalences.*

Proof. If g is a localisation or if condition 1. and 2. hold, g admits a left adjoint $\mathsf{f} : X \to Y$ in $\mathcal{C}$ according to proposition 6.78 with counit $\varepsilon : \mathsf{f} \circ g \to \mathrm{id}_Y$.

As $[Z, -] : \mathcal{C} \to \mathsf{Cat}_\infty$ is a 2-functor, the natural transformation $[Z, \varepsilon] : [Z, \mathsf{f}] \circ [Z, g] \to \mathrm{id}_{[Z,Y]}$ is the counit of the induced adjunction $[Z, \mathsf{f}] : [Z, X] \rightleftarrows [Z, Y] : [Z, g]$.

Consequently $\varepsilon : \mathsf{f} \circ g \to \mathrm{id}_Y$ is an equivalence if and only if for every object Z of $\mathcal{C}$ the counit of the adjunction $[Z, \mathsf{f}] : [Z, X] \rightleftarrows [Z, Y] : [Z, g]$ is an equivalence.

□

6.3 Appendix C: General Appendix

Proposition 6.82.

The categories Op_∞ *and* $\mathrm{Calg}(\mathsf{Cat}_\infty)$ *admit closed symmetric monoidal structures and the functor* $\mathrm{Env}(-)^\otimes : \mathrm{Op}_\infty \to \mathrm{Calg}(\mathsf{Cat}_\infty)$ *is symmetric monoidal.*

Especially the free functor $\mathsf{Cat}_\infty \to \mathrm{Calg}(\mathsf{Cat}_\infty)$ is symmetric monoidal being the restriction of $\mathrm{Env}(-)^\otimes : \mathrm{Op}_\infty \to \mathrm{Calg}(\mathsf{Cat}_\infty)$ to the full symmetric monoidal subcategory $\mathsf{Cat}_\infty \subset \mathrm{Op}_\infty$.

Proof. Recall that for every (symmetric) monoidal category $\mathcal{C}$ and (commutative) algebra A in $\mathcal{C}$ the category $\mathcal{C}_{/A}$ admits an induced (symmetric) monoidal structure.
Especially the symmetric monoidal structure on $\mathcal{F}\mathrm{in}_*$ given by the smash product $\wedge$ gives rise to a symmetric monoidal structure on the category $\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ encoded by a cocartesian fibration of operads $(\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*})^\otimes \to \mathcal{F}\mathrm{in}_*$.

Given a subcategory $\mathcal{B} \subset \mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ denote $\mathcal{B}^\otimes \subset (\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*})^\otimes$ the suboperad with objects the objects of $\mathcal{B}$ and with morphisms the multimorphisms $X_1, ..., X_n \to Y$ of $(\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*})^\otimes$ corresponding to a commutative square

$$\begin{array}{ccc} X_1 \times ... \times X_n & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{F}\mathrm{in}_*^{\times n} & \xrightarrow{\wedge} & \mathcal{F}\mathrm{in}_* \end{array}$$

such that for every $1 \leq i \leq n$ and objects $Z_j \in \{\langle 1 \rangle\} \times_{\mathcal{F}\mathrm{in}_*} X_j$ for $1 \leq j \leq n$ with $j \neq i$ the induced functor $X_i \simeq \{Z_1\} \times ... \times \{Z_{i-1}\} \times X_i \times \{Z_{i+1}\} \times ... \times \{Z_n\} \to X_1 \times ... \times X_n \to Y$ over $\mathcal{F}\mathrm{in}_*$ is a morphism of $\mathcal{B}$.

We apply this notation to the subcategories $\mathrm{Calg}(\mathsf{Cat}_\infty) \subset \mathrm{Op}_\infty \subset \mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ to get embeddings of operads $\mathrm{Calg}(\mathsf{Cat}_\infty)^\otimes \subset \mathrm{Op}_\infty^\otimes \subset (\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*})^\otimes$.

We will show the following two facts:

- The operads $\mathrm{Calg}(\mathsf{Cat}_\infty)^\otimes, \mathrm{Op}_\infty^\otimes$ are closed symmetric monoidal categories.
  This implies that the lax symmetric monoidal embedding $\mathrm{Calg}(\mathsf{Cat}_\infty)^\otimes \subset \mathrm{Op}_\infty^\otimes$ admits an oplax symmetric monoidal left adjoint lifting the functor $\mathrm{Env}(-)^\otimes$.
- This oplax symmetric monoidal left adjoint is symmetric monoidal.

By [18] proposition 4.1.1.20. an operad $\mathcal{C}^\otimes \to \mathcal{F}\mathrm{in}_*$ is a symmetric monoidal category if the pullback $\mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathcal{C}^\otimes \to \mathrm{Ass}^\otimes$ is a monoidal category.
Moreover an oplax symmetric monoidal functor is symmetric monoidal if its underlying oplax monoidal functor is monoidal.
Hence it is enough to show that the pullbacks

$$\mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Calg}(\mathsf{Cat}_\infty)^\otimes, \ \mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Op}_\infty^\otimes$$

are closed monoidal categories and the oplax monoidal left adjoint of the lax monoidal embedding $\mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Calg}(\mathsf{Cat}_\infty)^\otimes \subset \mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Op}_\infty^\otimes$ is monoidal.

Denote $\mathcal{F}\mathrm{in}_*^\flat$, $(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}$ and $(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}$ the categorical pattern on $\mathcal{F}\mathrm{in}_*$, with respect to which the fibered objects are the categories over $\mathcal{F}\mathrm{in}_*$, the operads respectively the symmetric monoidal categories.

Denote sSet^+ the category of marked simplicial sets and $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat}$, $\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}}, \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}}$ the corresponding simplical and combinatorial model categories modeling $\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}, \mathrm{Op}_\infty$ respectively $\mathrm{Calg}(\mathsf{Cat}_\infty)$.

The maps $\mathcal{F}\mathrm{in}_*^\flat \to (\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}$, $(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}} \to (\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}$ of categorical pattern yield Quillen adjunctions

$$\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat} \rightleftarrows \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}},\ \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}} \rightleftarrows \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}}$$

that model the adjunctions

$$\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*} \rightleftarrows \mathrm{Op}_\infty, \mathrm{Env}(-)^\otimes : \mathrm{Op}_\infty \rightleftarrows \mathrm{Calg}(\mathsf{Cat}_\infty),$$

where the right adjoints are the canonical subcategory inclusions.

Endowed with the smash product $\mathcal{F}\mathrm{in}_*$ is a strict monoidal category (but not a strict symmetric monoidal category) that gives rise to monoid structures in sSet^+ on $\mathcal{F}\mathrm{in}_*^\flat, (\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}$ and $(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}$ via the nerve.

These monoid structures yield symmetric monoidal structures on the categories $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat}$, $\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}}$ and $\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}}$ encoded by cocartesian fibrations of operads $(\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat})^\otimes \to \mathcal{F}\mathrm{in}_*, (\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}})^\otimes \to \mathcal{F}\mathrm{in}_*$ and $(\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}})^\otimes \to \mathcal{F}\mathrm{in}_*$.

The tensorproduct of $(\mathrm{X}, \mathcal{E}), (\mathrm{X}', \mathcal{E}')$ is given by $(\mathrm{X}, \mathcal{E}) \otimes (\mathrm{X}', \mathcal{E}') := (\mathrm{X} \times \mathrm{X}', \mathcal{E} \times \mathcal{E}')$, where $\mathrm{X} \times \mathrm{X}'$ is considered over $\mathcal{F}\mathrm{in}_*$ via the composition $\mathrm{X} \times \mathrm{X}' \to \mathcal{F}\mathrm{in}_* \times \mathcal{F}\mathrm{in}_* \xrightarrow{\wedge} \mathcal{F}\mathrm{in}_*$.

By [18] remark B.2.5. the model categories $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat}$, $\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}}$ and $\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}}$ are monoidal model categories.

Especially the Quillen adjunctions

$$\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat} \rightleftarrows \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}},\ \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}} \rightleftarrows \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}}$$

are monoidal and so model monoidal functors that admit a lax monoidal right adjoint that by construction is an inclusion of planar operads.

The model category sSet^+ models Cat_∞ so that we have a universal functor $\mathsf{sSet}^+ \to \mathsf{Cat}_\infty$ inverting the weak equivalences.

The induced functor $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat} \to \mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ is the universal functor inverting the weak equivalences, in other words $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat}$ models $\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$.

The functor $\mathsf{sSet}^+ \to \mathsf{Cat}_\infty$ preserves finite products and so lifts to a symmetric monoidal functor on cartesian structures, whose underlying monoidal functor makes $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat} \to \mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ to a monoidal functor.

This implies that the monoidal model category $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat}$ models the monoidal category $\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*}$ so that the monoidal Quillen adjunctions $\mathsf{sSet}^+_{/\mathcal{F}\mathrm{in}_*^\flat} \rightleftarrows \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}}$, $\mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Op}}} \rightleftarrows \mathsf{sSet}^+_{/(\mathcal{F}\mathrm{in}_*)_{\mathrm{Sym}}}$ model adjunctions

$$\mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} (\mathsf{Cat}_{\infty/\mathcal{F}\mathrm{in}_*})^\otimes \rightleftarrows \mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Op}_\infty^\otimes,$$

$$\mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Op}_\infty^\otimes \rightleftarrows \mathrm{Ass}^\otimes \times_{\mathcal{F}\mathrm{in}_*} \mathrm{Calg}(\mathsf{Cat}_\infty)^\otimes$$

relative to $\mathrm{Ass}^\otimes$ between closed monoidal categories.

□

As next we show that the category of algebras over an accessible monad on a presentable category is presentable (proposition 6.84) and that the category of coalgebras over a small operad $\mathcal{O}^\otimes$ in a presentably $\mathcal{O}^\otimes$-monoidal category is presentable (proposition 6.83).

From this we deduce that the category of bialgebras over a Hopf operad on a presentably symmetric monoidal category is presentable.

Proposition 6.83.
Let $\mathcal{O}'^\otimes \to \mathcal{O}^\otimes$ be a map of small operads, $\mathcal{D}^\otimes \to \mathcal{O}^\otimes$ a small $\mathcal{O}^\otimes$-monoidal category and $\mathcal{C}^\otimes \to \mathcal{O}^\otimes$ an accessible $\mathcal{O}^\otimes$-monoidal category.

1. *The categories*

$$\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}),\quad \mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}),\quad \mathrm{Fun}^\otimes_{\mathcal{O}}(\mathcal{D},\mathcal{C})$$

are accessible.

For every $Y \in \mathcal{O}'$ lying over some $X \in \mathcal{O}$ the forgetful functors $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ and $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ are accessible.

2. *Assume that for every $X \in \mathcal{O}$ the category $\mathcal{C}_X$ is presentable.*

Then the categories $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ and $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$ are presentable.

So for every $Y \in \mathcal{O}'$ lying over some $X \in \mathcal{O}$ the forgetful functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ admits a left adjoint and the the forgetful functor $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ admits a right adjoint.

Proof. 2. follows from 1. and remark 6.85 and remark 2.1.

1: By lemma 6.86 the categories $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C})$, $\mathrm{Fun}^\otimes_{\mathcal{O}}(\mathcal{D},\mathcal{C})$ are accessible.

Moreover for every $Y \in \mathcal{O}'$ lying over some $X \in \mathcal{O}$ the forgetful functor $\mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ is accessible. By remark 2.1 the forgetful functor $\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \to \mathcal{C}_X$ is accessible.

By [18] proposition 2.2.4.9. the subcategory inclusion $\mathrm{Mon}_{\mathcal{O}}(\mathsf{Cat}_\infty) \subset \mathrm{Op}_{\infty/\mathcal{O}^\otimes}$ from $\mathcal{O}^\otimes$-monoidal categories to operads over $\mathcal{O}^\otimes$ admits a left adjoint $\mathrm{Env}_{\mathcal{O}}(-)^\otimes$, which assigns to an operad over $\mathcal{O}^\otimes$ its enveloping $\mathcal{O}^\otimes$-monoidal category.

We have a canonical equivalence

$$\mathrm{Coalg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathcal{O}'/\mathcal{O}}(\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} \simeq \mathrm{Fun}^\otimes_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{O}'),\mathcal{C}^{\mathrm{rev}})^{\mathrm{op}} \simeq$$

$$\mathrm{Fun}^\otimes_{\mathcal{O}}(\mathrm{Env}_{\mathcal{O}}(\mathcal{O}')^{\mathrm{rev}},\mathcal{C}).$$

So it is enough to see that $(\mathrm{Env}_{\mathcal{O}}(\mathcal{O}')^\otimes)^{\mathrm{rev}} \to \mathcal{O}^\otimes$ or equivalently $\mathrm{Env}_{\mathcal{O}}(\mathcal{O}')^\otimes \to \mathcal{O}^\otimes$ are small $\mathcal{O}^\otimes$-monoidal categories.

But there is a canonical equivalence

$$\mathrm{Env}_{\mathcal{O}}(\mathcal{O}')^\otimes \simeq \mathrm{Act}(\mathcal{O}^\otimes) \times_{\mathrm{Fun}(\{0\},\mathcal{O}^\otimes)} \mathcal{O}'^\otimes$$

over $\mathrm{Fun}(\{1\},\mathcal{O}^\otimes)$, where $\mathrm{Act}(\mathcal{O}^\otimes) \subset \mathrm{Fun}(\Delta^1,\mathcal{O}^\otimes)$ denotes the full subcategory spanned by the active morphisms of $\mathcal{O}^\otimes$ ([18] construction 2.2.4.1. and proposition 2.2.4.9.).

□

Proposition 6.84. *Let $\mathcal{T}$ be an accessible monad and $\mathcal{Q}$ an accessible comonad (i.e. a monad on $\mathcal{C}^{\mathrm{op}}$ such that $\mathcal{Q}^{\mathrm{op}}: \mathcal{C} \to \mathcal{C}$ is accessible) on an accessible category $\mathcal{C}$.*

1. *The category $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C})$ is accessible and the forgetful functor $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C}) \to \mathcal{C}$ is accessible.*
2. *The category $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}) := \mathrm{Alg}_{\mathcal{Q}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ is accessible and the forgetful functor $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}) \to \mathcal{C}$ is accessible.*
3. *If $\mathcal{C}$ is presentable, then $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C})$ and $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C})$ are presentable.*

Proof. Let κ, κ' be regular cardinals such that $\mathcal{C}$ is κ-accessible and $\mathcal{T}: \mathcal{C} \to \mathcal{C}$ preserves κ'-filtered colimits. Then there is a regular cardinal $\lambda > \kappa'$ such that $\mathcal{C}$ is λ-accessible, i.e. $\mathcal{C} \simeq \mathrm{Ind}_{\lambda}(\mathcal{B})$ for some small category $\mathcal{B}$. As $\lambda > \kappa'$, the functor $\mathcal{T}$ preserves λ-filtered colimits.

Denote $\mathrm{Fun}(\mathcal{C},\mathcal{C})' \subset \mathrm{Fun}(\mathcal{C},\mathcal{C})$ the full subcategory spanned by the λ-accessible functors.

By the universal property of $\mathrm{Ind}_{\lambda}(\mathcal{B})$ composition with the Yoneda-embedding $\mathcal{B} \subset \mathrm{Ind}_{\lambda}(\mathcal{B}) \simeq \mathcal{C}$ yields an equivalence $\mathrm{Fun}(\mathcal{C},\mathcal{C})' \simeq \mathrm{Fun}(\mathcal{B},\mathcal{C})$.

As $\mathcal{C}$ is λ-accessible and $\mathcal{B}$ is small, by lemma 6.86 the category $\mathrm{Fun}(\mathcal{C},\mathcal{C})' \simeq \mathrm{Fun}(\mathcal{B},\mathcal{C}) \simeq \mathrm{Fun}_{\mathcal{B}}(\mathcal{B},\mathcal{B}\times\mathcal{C})$ is accessible.

The monoidal structure on $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ restricts to an accessible monoidal structure on $\mathrm{Fun}(\mathcal{C},\mathcal{C})'$. Pulling back the endomorphism $\mathrm{Fun}(\mathcal{C},\mathcal{C})$-left module structure on $\mathcal{C}$ along the full subcategory inclusion $\mathrm{Fun}(\mathcal{C},\mathcal{C})' \subset \mathrm{Fun}(\mathcal{C},\mathcal{C})$ we obtain an accessible $\mathrm{Fun}(\mathcal{C},\mathcal{C})'$-left module structure on $\mathcal{C}$.

1. and 2. : The category $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C})$ is the category of left modules over $\mathcal{T}$ with respect to the restricted left action of $\mathrm{Fun}(\mathcal{C},\mathcal{C})'$ on $\mathcal{C}$ and $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C})$ is the category of left co-modules over $\mathcal{Q}$ with respect to the restricted left action of $\mathrm{Fun}(\mathcal{C},\mathcal{C})'$ on $\mathcal{C}$.

So by lemma 6.86 the categories

$$\mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})'),\ \mathrm{Coalg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})'),\ \mathrm{LMod}(\mathcal{C}) = \mathrm{Alg}_{/\mathrm{LM}}(\mathcal{C}),\ \mathrm{coLMod}(\mathcal{C}) = \mathrm{Coalg}_{/\mathrm{LM}}(\mathcal{C})$$

are accessible and the forgetful functors

$$\mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})') \to \mathrm{Fun}(\mathcal{C},\mathcal{C})', \quad \mathrm{LMod}(\mathcal{C}) \to \mathcal{C},$$

$$\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})') \to \mathrm{Fun}(\mathcal{C},\mathcal{C})'$$

and

$$\mathrm{Coalg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})') \to \mathrm{Fun}(\mathcal{C},\mathcal{C})', \quad \mathrm{coLMod}(\mathcal{C}) \to \mathcal{C},$$

$$\mathrm{coLMod}(\mathcal{C}) \to \mathrm{Coalg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})') \to \mathrm{Fun}(\mathcal{C},\mathcal{C})'$$

are accessible.

Hence the forgetful functors

$$\mathrm{LMod}(\mathcal{C}) \to \mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})'), \quad \mathrm{coLMod} \to \mathrm{Coalg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})')$$

are accessible so that the categories

$$\mathrm{Alg}_{\mathcal{T}}(\mathcal{C}) = \mathrm{LMod}_{\mathcal{T}}(\mathcal{C}) = \{\mathcal{T}\} \times_{\mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})')} \mathrm{LMod}(\mathcal{C}),$$

$$\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}) = \mathrm{coLMod}_{\mathcal{Q}}(\mathcal{C}) = \{\mathcal{Q}\} \times_{\mathrm{Coalg}(\mathrm{Fun}(\mathcal{C},\mathcal{C})')} \mathrm{coLMod}(\mathcal{C})$$

are accessible and the forgetful functors $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C}) \to \mathcal{C}$ and $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}) \to \mathcal{C}$ are accessible.

3. If $\mathcal{C}$ admits small limits, then by [18] corollary 4.2.3.3. the category $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C}) = \mathrm{LMod}_{\mathcal{T}}(\mathcal{C})$ admits small limits that are preserved by the forgetful functor $\mathrm{Alg}_{\mathcal{T}}(\mathcal{C}) = \mathrm{LMod}_{\mathcal{T}}(\mathcal{C}) \to \mathcal{C}$.

If $\mathcal{C}$ admits small colimits, then by [18] corollary 4.2.3.3. the category $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}) = \mathrm{Alg}_{\mathcal{Q}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathrm{LMod}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$ admits small colimits that are preserved by the forgetful functor $\mathrm{Coalg}_{\mathcal{Q}}(\mathcal{C}) = \mathrm{LMod}_{\mathcal{T}}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \to \mathcal{C}$.
So 3. follows from 1., 2. and remark 6.85.

□

Remark 6.85.
Let $\mathcal{C}$ be an accessible category.

Then the following conditions are equivalent:

- *$\mathcal{C}$ admits small colimits, i.e. $\mathcal{C}$ is presentable.*
- *$\mathcal{C}$ admits small limits.*

Proof. Assume that $\mathcal{C}$ admits small colimits.

Then by [19] proposition 5.5.2.2. a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ is representable if and only if it preserves small limits.

This implies that a functor $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ admits a right adjoint if it preserves small colimits:
If $\mathrm{F} : \mathcal{C} \to \mathcal{D}$ preserves small colimits, for every $\mathrm{X} \in \mathcal{D}$ the presheaf $\mathcal{D}(\mathrm{F}(-), \mathrm{X}) : \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$ preserves small limits and is thus representable so that F admits a right adjoint.

So for every small category K the small colimits preserving diagonal functor $\mathcal{C} \to \mathrm{Fun}(\mathrm{K}, \mathcal{C})$ admits a right adjoint. Hence $\mathcal{C}$ admits small limits.

Assume on the other hand that $\mathcal{C}$ admits small limits.

Then by the proof of [19] proposition 5.5.2.7. a functor $\mathcal{C} \to \mathrm{S}$ is corepresentable if and only if it preserves small limits and is accessible
(In the proof of [19] proposition 5.5.2.7. it is only needed that $\mathcal{C}$ is accessible and admits small limits to deduce the statement).

This implies that a functor $\mathrm{G} : \mathcal{C} \to \mathcal{D}$ admits a left adjoint if it preserves small limits and is accessible:
If $\mathrm{G} : \mathcal{C} \to \mathcal{D}$ preserves small limits and is accessible, for every $\mathrm{X} \in \mathcal{D}$ the presheaf $\mathcal{D}(\mathrm{X}, \mathrm{G}(-)) : \mathcal{C} \to \mathcal{S}$ preserves small limits and is accessible and is thus corepresentable so that G admits a left adjoint.

So for every small category K the small limits preserving and accessible diagonal functor $\mathcal{C} \to \mathrm{Fun}(\mathrm{K}, \mathcal{C})$ admits a left adjoint. Hence $\mathcal{C}$ admits small colimits.

□

Lemma 6.86.
Let S *be a small category,* $\phi : \mathrm{X} \to \mathrm{S}$ *a locally cocartesian fibration and* $\mathcal{E} \subset \mathrm{S}$ *a subcategory.*
Denote $\mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{X})' \subset \mathrm{Fun}_{\mathrm{S}}(\mathrm{S}, \mathrm{X})$ *the full subcategory spanned by the sections of* $\mathrm{X} \to \mathrm{S}$ *that send morphisms of* $\mathcal{E}$ *to locally* ϕ*-cocartesian morphisms.*

If the fibers of $X \to S$ *are accessible and for every morphism* $\Delta^1 \to S$ *the pullback* $\Delta^1 \times_S X \to \Delta^1$ *classifies an accessible functor, then the category* $\mathrm{Fun}_S(S, X)'$ *is accessible and for every object* $s \in S$ *the induced evaluation-functor* $\mathrm{Fun}_S(S, X)' \to X_s$ *is accessible.*

Proof. This follows immediately from proposition [19] 5.4.7.11. and remark 5.4.7.13.

□

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