On the reconstruction of multivariate exponential sums

Dissertation
zur Erlangung des Doktorgrades (Dr. rer. nat.)

vorgelegt dem
Fachbereich Mathematik/Informatik
der Universität Osnabrück

von

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2017

Acknowledgements

First of all, I deeply thank my advisors, Stefan Kunis and Tim Römer, for accepting me as their student, suggesting the subject of my thesis, and for their invaluable guidance and advice throughout the years. I owe them much.

Hans Michael Möller, through numerous discussions, also had a deep impact on my work. In particular, two key theorems in Chapter 2 are based on his arguments. I thank him for his continual interest in my work and for openly sharing his ideas. I also thank my coauthor Thomas Peter, from whom I learned a lot about Prony's method.

I thank Hans Munthe-Kaas, whom I met at a summer school in Italy, for inviting me to Bergen, for his interest and support, and for sharing his time and insights with me. My thanks go to John Abbott, for inviting me to my first stay in Genova, useful discussions, and keeping in touch ever since, and I thank Aldo Conca and Matteo Varbaro for their warm hospitality during my second stay in Genova, and for their kind interest in my work. I am very grateful towards Annie Cuyt, for invitations to several wonderful workshops and for generously sharing her wisdom.

It is a pleasure to thank Ragnar-Olaf Buchweitz for the good suggestion to move to Osnabrück, and I also wish to thank Julio José Moyano Fernández, for the immediate strengthening of that suggestion. I am grateful for the help I received from Thomas Haarmann and his crew at virtUOS to get settled in Osnabrück.

I thank everyone at the institute of mathematics at Osnabrück University for providing an outstandingly friendly and productive working environment. Particularly, I thank Maria Anna Gausmann and Sabine Schröder for always being helpful and supportive far beyond the call of duty, Lê Xuân Thanh, for sharing many happy moments, and Markus Wageringel, for several fruitful discussions during the last year.

It is clear that I could not have written this thesis without my family and friends. I am glad to have them.

I gratefully acknowledge the support I got during the preparation of this thesis from the DFG Graduate School GRK-1916 "Combinatorial Structures in Geometry".

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September 2017

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0. Introduction

Exponential sums, that is, linear combinations of exponential functions, appear prominently in many areas both within mathematics and also in the applied sciences. For example, a scientist attempting to obtain information about a sound wave might be using an exponential sum as mathematical model. If the the question arises which the constituting frequencies in this exponential sum are, then one is facing a reconstruction problem. The task hereby is to determine, by considering nothing but a finite number of samples, the finitely many non-zero components of the coefficients vector (f_b) of an exponential sum $f = \sum_b f_b \exp_b$ with respect to the vector space basis consisting of all exponential functions $\exp_b$, $b \in \mathbb{C}$.

A classical approach due to Prony [66] proceeds by translating an exponential sum $f\colon \mathbb{N} \to \mathbb{C}$ computationally into a polynomial $p \in \mathbb{C}[\mathrm{x}]$ whose roots $b_1, \dots, b_r \in \mathbb{C}$ correspond to the support of f, i. e., one has $\{b \in \mathbb{C} \mid f_b \neq 0\} = \{b_1, \dots, b_r\}$. After the equation $p = 0$ is solved, possibly by approximate methods, and $b_1, \dots, b_r$ are thus obtained, the problem is thereby reduced to a finite dimensional interpolation problem and one may compute the vector of non-zero coefficients $(f_{b_1}, \dots, f_{b_r})$ by (approximately) solving a system of linear equations.

This thesis is concerned with a multivariate generalization of this classical problem and its solution, where the bases $b \in K^n$ of the exponentials $\exp_b\colon \mathbb{N}^n \to K$ are points in an n-dimensional affine space over a field K. The generalization of Prony's method given here proceeds in an analogous way to the classical case, translating the exponential sum $f\colon \mathbb{N}^n \to K$ with coefficient vector $(f_b)_{b \in K^n}$ into a system of polynomial equations $p_1, \dots, p_k \in K[\mathrm{x}_1, \dots, \mathrm{x}_n]$ such that the support of f is cut out as their zero-locus i. e., one has

$$\{b \in K^n \mid f_b \neq 0\} = \mathrm{Z}(p_1, \dots, p_k) = \{b \in K^n \mid p(b) = 0 \text{ for all } \ell = 1, \dots, k\}.$$

In particular, one has to deal with the difficulty that without additional assumptions $\mathrm{Z}(p_1, \dots, p_k)$ may be an infinite set, a problem that occurs in the univariate case only in the form that $\mathrm{Z}(p) = \mathbb{C}$, i. e., $p = 0$. Finding a small degree for the construction of the polynomials depends, in contrast to the univariate case, on the geometry of the points and is therefore a more delicate problem.

The basis for this thesis is formed by the following articles and preprints.

[55] S. Kunis, T. Peter, T. Römer, and U. von der Ohe. A multivariate generalization of Prony's method. *Linear Algebra Appl.*, 490:31–47, February 2016.

[54] S. Kunis, H. M. Möller, and U. von der Ohe. Prony's method on the sphere. Preprint, `arXiv:1603.02020v1 [math.NA]`, 11 pages, March 2016.

[53] S. Kunis, H. M. Möller, T. Peter, and U. von der Ohe. Prony's method under an almost sharp multivariate Ingham inequality. Accepted for publication in *J. Fourier Anal. Appl.* Preprint available at `arXiv:1705.11017v1 [math.NA]`, 12 pages, May 2017.

The thesis also contains several new results and is structured as follows. In Chapter 1, we list some conventions and notations that are used in the thesis for the convenience of the reader. Some of these are repeated when they are first used. In Chapter 2, we introduce the setting and develop several variants of Prony's method for multivariate exponential sums over an arbitrary field. In Chapter 3, we apply the theory from Chapter 2 to two particular cases: exponential sums supported on the real sphere and exponential sums supported on the complex torus. In Chapter 4, we discuss some out of the many alternative approaches to the subject of exponential sum reconstruction.

1. Preliminaries

In an attempt to avoid ambiguity, in this chapter we list some general conventions we adhere to throughout this thesis. We also recall some commonly used definitions for the reader to pick up as needed. Some of the definitions also appear later in the text but are still included here for easy reference.

1.1. General preliminaries

The following definitions and conventions are used throughout. The symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, integers, rational, real, and complex numbers, respectively, with their usual algebraic or topologic stuctures inferred from the context. Zero is regarded as a natural number. For a set M, $|M|$ denotes the *cardinality* of M and $\mathcal{P}(M) := \{A \mid A \subseteq M\}$ denotes the *power set* of M and we define

$$\mathcal{P}_{\mathrm{f}}(M) := \{A \in \mathcal{P}(M) \mid |A| \in \mathbb{N}\}$$

to be the set of finite subsets of M. For sets M, N, a *function* from M to N is a triple (M, f, N) where $f \subseteq M \times N$ is a relation such that for all $m \in M$ there is exactly one $n \in N$ with $(m, n) \in f$. As usual, n is then denoted by $f(m)$. We write $f \colon M \to N$ as abbreviation of "(M, f, N) is a function". When defining a function $f \colon M \to N$ and for each $m \in M$ an element $a_m \in N$ is defined, the notation $f \colon M \to N$, $m \mapsto a_m$, signifies that $f(m) = a_m$. As usual, we simply write f instead of (M, f, N). In literature related to this thesis, functions in the just described sense are occasionally referred to as "blackbox functions" to distinguish them from, e. g., polynomial functions given by coefficient vectors.

For sets M, N, we denote by

$$N^M := \{f \mid f \colon M \to N\}$$

the set of all functions from M to N. The *image* of $A \subseteq M$ under a function $f \colon M \to N$ is written

$$f[A] := \{f(a) \mid a \in A\}.$$

Similarly, for a subset $B \subseteq N$,

$$f^{-1}[B] := \{a \in M \mid f(a) \in B\}$$

denotes the *preimage* of B under f. If $+\colon M \times N \to G$ is a function, we denote, as usual, $m + n := +(m, n)$ for $m \in M$, $n \in N$, and write simply $A + B$ instead of

$+[A \times B] = \{a + b \mid a \in A,\, b \in B\}$ for $A \subseteq M$, $B \subseteq N$. Furthermore, for $f\colon M \to N$ and $A \subseteq M$,

$$f \restriction A := (A, f \cap (A \times N), N)\colon A \to N$$

denotes the *restriction* of f to A. For any n-tuple $b \in M^n$, $b_j \in M$ denotes the *j-th coordinate projection* of b, unless mentioned otherwise. If $b \in M^n$ and M is regarded as a multiplicative monoid, then for $\alpha \in \mathbb{N}^n$,

$$b^\alpha := \prod_{j=1}^{n} b_j^{\alpha_j} \in M.$$

We follow the convention that the empty product is the neutral element of M. In particular, in any unitary ring A we have

$$0^0 = 1 \in A,$$

where the base is $0 \in A$, the exponent is $0 \in \mathbb{N}$, and 1 is the unit element of A. The empty function $\emptyset\colon \emptyset \to A$ is the unique element of $A^\emptyset$, i. e., for a ring A, $A^\emptyset = \{\emptyset\}$ is regarded as the zero A-algebra.

Rings are always understood to be commutative. Unless a statement is made to the contrary, the symbols x, y, z, x_i ($i \in I$, where I may be an arbitrary set), etc. always denote distinct indeterminates over the considered ring. Let A be an arbitrary ring. In the polynomial algebra

$$S := A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$$

over A in n indeterminates $\mathrm{x}_1, \ldots, \mathrm{x}_n$, we let $\mathrm{x} := (\mathrm{x}_1, \ldots \mathrm{x}_n) \in S^n$ and have $\mathrm{x}^\alpha = \mathrm{x}_1^{\alpha_1} \cdots \mathrm{x}_n^{\alpha_n} \in S$ for $\alpha \in \mathbb{N}^n$ as a special case of the above definition.

By the notation

$$N \leq M$$

we indicate that N is a substructure of the algebraic structure M. For example, if M is an A-module then $N \leq M$ indicates that N is an A-submodule of M. For an A-module M and a subset $E \subseteq M$,

$$\langle E \rangle_A := \bigcap \{N \leq M \mid E \subseteq N\} = \Big\{ \sum_{i=1}^{n} \lambda_i m_i \;\Big|\; n \in \mathbb{N},\, \lambda_i \in A,\, m_i \in E \Big\}$$

denotes the A-submodule of M *generated* by E. In particular, this notation will be used for ideals of A (the A-submodules of A) and vector spaces (K-modules for a field K).

For a ring A and finite sets F_1, F_2, an element of $A^{F_1 \times F_2}$ is called *matrix* over A. For $B = (b_{i,\ell})_{\substack{i \in F_1 \\ \ell \in F_2}} \in A^{F_1 \times F_2}$, $C = (c_{\ell,j})_{\substack{\ell \in F_2 \\ j \in F_3}} \in A^{F_2 \times F_3}$, where F_i are finite sets, the *matrix product* $BC = B \cdot C = (d_{i,j})_{\substack{i \in F_1 \\ j \in F_3}} \in A^{F_1 \times F_3}$ is defined by $d_{i,j} := \sum_{\ell \in F_2} b_{i,\ell} c_{\ell,j}$. Usually we work with the properties that $BC = (Bc_j)_{j \in F_3}$ where $c_j = (c_{\ell,j})_{\ell \in F_2} \in A^{F_2} = A^{F_2 \times \{1\}}$ are the *columns* of C and $Bx = \sum_{\ell \in F_2} x_\ell b_\ell$ for any $x \in A^{F_2}$ where $b_\ell = (b_{i,\ell})_{i \in F_1} \in A^{F_1} = A^{F_1 \times \{1\}}$ are the columns of B.

For a polynomial $p \in S = A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ and $\alpha \in \mathbb{N}^n$, we denote the coefficient of x^α of p by p_α, unless mentioned otherwise. The *support* of p is denoted by

$$\operatorname{supp}(p) := \{\alpha \in \mathbb{N}^n \mid p_\alpha \neq 0\} \in \mathcal{P}_{\mathrm{f}}(\mathbb{N}^n).$$

Thus, $p = \sum_{\alpha \in \operatorname{supp}(p)} p_\alpha \mathrm{x}^\alpha$ holds for any polynomial $p \in S$. A *monomial* is a polynomial $p \in S$ with $|\operatorname{supp}(p)| = 1$ and $p_\alpha = 1$ for the unique $\alpha \in \operatorname{supp}(p)$, that is, a polynomial of the form $\mathrm{x}^\alpha = \mathrm{x}_1^{\alpha_1} \cdots \mathrm{x}_n^{\alpha_n}$ for some $\alpha \in \mathbb{N}^n$. For $D \subseteq \mathbb{N}^n$, we set

$$\mathrm{x}^D := \{\mathrm{x}^\alpha \mid \alpha \in D\}.$$

The set of all monomials in n indeterminates is denoted by

$$\operatorname{Mon}^n := \mathrm{x}^{\mathbb{N}^n} = \{\mathrm{x}^\alpha \mid \alpha \in \mathbb{N}^n\}.$$

Since $\mathrm{x}^\alpha \mathrm{x}^\beta = \mathrm{x}^{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{N}^n$, $(\operatorname{Mon}^n, \cdot)$ is a commutative monoid which is isomorphic to $(\mathbb{N}^n, +)$ via the monoid isomorphism $\mathbb{N}^n \to \operatorname{Mon}^n$, $\alpha \mapsto \mathrm{x}^\alpha$. The *total degree* of a monomial x^α is denoted by

$$\operatorname{tot\,deg}(\mathrm{x}^\alpha) := \sum_{j=1}^{n} \alpha_j \in \mathbb{N}$$

and the *total degree* of a polynomial $p \in S \setminus \{0\}$ is

$$\operatorname{tot\,deg}(p) := \max(\operatorname{tot\,deg}[\mathrm{x}^{\operatorname{supp}(p)}]) = \max\{\operatorname{tot\,deg}(\mathrm{x}^\alpha) \mid \alpha \in \operatorname{supp}(p)\} \in \mathbb{N}.$$

The *maximal degree* of a monomial $\mathrm{x}^\alpha \in \operatorname{Mon}^n$ is denoted by

$$\operatorname{max\,deg}(\mathrm{x}^\alpha) := \max\{\alpha_j \mid j = 1, \ldots, n\} \in \mathbb{N}$$

and the *maximal degree* of a polynomial $p \in S \setminus \{0\}$ is

$$\operatorname{max\,deg}(p) := \max(\operatorname{max\,deg}[\mathrm{x}^{\operatorname{supp}(p)}]) = \max\{\operatorname{max\,deg}(\mathrm{x}^\alpha) \mid \alpha \in \operatorname{supp}(p)\} \in \mathbb{N}.$$

For $\alpha \in \mathbb{N}^n$ we also set

$$\operatorname{tot\,deg}(\alpha) := \operatorname{tot\,deg}(\mathrm{x}^\alpha)$$

and

$$\operatorname{max\,deg}(\alpha) := \operatorname{max\,deg}(\mathrm{x}^\alpha).$$

A polynomial $p \in S$ gives rise to the *polynomial function*

$$\begin{aligned} \mathrm{f}_p \colon A^n &\longrightarrow A, \\ b &\longmapsto \sum_{\alpha \in \operatorname{supp}(p)} p_\alpha b^\alpha. \end{aligned}$$

As usual, for $p \in S$ and $b \in A^n$, we also write $p(b)$ instead of $\mathrm{f}_p(b)$. For an arbitrary subset $M \subseteq A^n$, we regard A^M as an A-algebra with multiplication defined pointwise, and the A-algebra homomorphism

$$\begin{aligned} \operatorname{ev}^M \colon S &\longrightarrow A^M, \\ p &\longmapsto \mathrm{f}_p \restriction M = (p(b))_{b \in M}, \end{aligned}$$

is called *evaluation homomorphism* at M. For a subset $D \subseteq \mathbb{N}^n$ let

$$S_D := \langle \mathrm{x}^D \rangle_A,$$

which is a free A-submodule of S with A-basis x^D, and let

$$\mathrm{ev}_D^M := \mathrm{ev}^M \restriction S_D$$

be the restriction of ev^M to S_D. Clearly, ev_D^M is an A-module homomorphism. Further, the ideal

$$\mathrm{I}(M) := \ker(\mathrm{ev}^M) = \{p \in S \mid \text{for all } b \in M,\, p(b) = 0\}$$

is called *vanishing ideal* of M. For subsets $M \subseteq A^n$ and $D \subseteq \mathbb{N}^n$ let

$$\mathrm{I}_D(M) := \ker(\mathrm{ev}_D^M) = S_D \cap \mathrm{I}(M).$$

For an arbitrary subset $I \subseteq S$,

$$\mathrm{Z}(I) := \{b \in A^n \mid \text{for all } p \in I,\, p(b) = 0\}$$

denotes the *zero locus* of I.

Occasionally, to avoid confusion, we may add an index to the above notations that should signify the ring over which the construction is taken, like $\mathrm{I}_A(M)$ or $\mathrm{Z}_A(I)$ for the ring A.

1.2. Preliminaries for Chapter 2

We give some additional preliminaries needed in Section 2.5 and Section 2.6.

1.2.1. Preliminaries for Section 2.5

In Section 2.5 we need the following variants, which are relative to a fixed variety in A^n, of some definitions given in Section 1.1. They are given here in the same fashion as they are given in Cox-Little-O'Shea [22, Chapter 5, § 4] over fields.

Let $B \subseteq A^n$ be an arbitrary subset. Let

$$S_B := S/\mathrm{I}(B)$$

be the *coordinate algebra*[1] of B and for $D \subseteq \mathbb{N}^n$ let

$$S_{D,B} := S_D/\mathrm{I}_D(B).$$

[1] At least for a field $A = K$, in the literature $A[B]$ is also standard notation for the coordinate algebra. We prefer S_B here for reasons of consistency.

Remark 1.1: The A-algebra S_B is isomorphic to $\{\mathrm{f}_p \restriction B \mid p \in S\} \leq A^B$ via the A-algebra isomorphism $S_B \to \{\mathrm{f}_p \restriction B \mid p \in S\}$, $p + \mathrm{I}(B) \mapsto \mathrm{f}_p \restriction B$. Thus one may identify $p + \mathrm{I}(B)$ with the function $\mathrm{f}_p \restriction B \colon B \to A$. Since for $D_1 \subseteq D_2 \subseteq \mathbb{N}^n$ the A-module homomorphism $S_{D_1} \to S_{D_2,B}$, $p \mapsto p + \mathrm{I}_{D_2}(B)$, has $\mathrm{I}_{D_1}(B)$ as its kernel, $S_{D_1,B} = S_{D_1}/\mathrm{I}_{D_1}(B)$ is embedded into $S_{D_2,B}$ via the embedding

$$\begin{aligned} S_{D_1,B} &\hookrightarrow S_{D_2,B}, \\ p + \mathrm{I}_{D_1}(B) &\longmapsto p + \mathrm{I}_{D_2}(B). \end{aligned}$$

In particular, for $D \subseteq \mathbb{N}^n$, the A-module $S_{D,B}$ is isomorphic to $\{\mathrm{f}_p \restriction B \mid p \in S_D\} \leq A^B$ by mapping $p + \mathrm{I}_D(B)$, $p \in S_D$, to $\mathrm{f}_p \restriction B$.

For $M \subseteq B \subseteq A^n$ let

$$\begin{aligned} \mathrm{ev}_B^M \colon \qquad S_B &\longrightarrow A^M, \\ p + \mathrm{I}(B) &\longmapsto \mathrm{ev}^M(p) = \mathrm{f}_p \restriction M, \end{aligned}$$

which is well-defined by the above Remark 1.1, and for $D \subseteq \mathbb{N}^n$, via the embedding $S_{D,B} \hookrightarrow S_B$, let

$$\mathrm{ev}_{D,B}^M := \mathrm{ev}_B^M \restriction S_{D,B}.$$

Further let

$$\mathrm{I}_B(M) := \ker(\mathrm{ev}_B^M) = \{p + \mathrm{I}(B) \mid p \in S,\, \mathrm{f}_p \restriction M = 0\} = \mathrm{I}(M)/\mathrm{I}(B)$$

(which is an ideal in S_B) and

$$\mathrm{I}_{D,B}(M) := \ker(\mathrm{ev}_{D,B}^M) = S_{D,B} \cap \mathrm{I}_B(M) = \mathrm{I}_D(M)/\mathrm{I}_D(B).$$

Note that by the third isomorphism theorem(s)

$$S_B/\mathrm{I}_B(M) \cong S_M$$

and

$$S_{D,B}/\mathrm{I}_{D,B}(M) \cong S_{D,M}.$$

Furthermore, for a subset $J \subseteq S_B$ let

$$\mathrm{Z}_B(J) := \{b \in B \mid \text{for all } q \in S \text{ with } q + \mathrm{I}(B) \in J,\, q(b) = 0\}$$

be the *zero locus relative to B* of J.

There does not seem to be any confusion possible with I_A, V_A, or Z_A as defined previously, where A denotes the ring of coefficients (and is usually omitted from the notation).

1.2.2. Preliminaries for Section 2.6

In Section 2.6, standard notions from the theory of Gröbner bases will be used, that can be found in any textbook on Gröbner bases or computer algebra, such as Cox-Little-O'Shea [22], Becker-Weispfenning [8], Adams-Loustaunau [1], or Kreuzer-Robbiano [50].

A total order $\leq$ (i. e., $\leq$ is reflexive, transitive, antisymmetric, and connex) on Mon^n is called *monomial order* on Mon^n if $1 \leq u$ for all $u \in \mathrm{Mon}^n$ and $v \leq w$ implies $u \cdot v \leq u \cdot w$ for all $u, v, w \in \mathrm{Mon}^n$. It is a standard fact that all monomial orders on Mon^n are well-orders on Mon^n, i. e., every non-empty subset of Mon^n has a $\leq$-least element. A monomial order $\leq$ on Mon^n is called *degree compatible* if $u \leq v$ implies $\operatorname{tot}\deg(u) \leq \operatorname{tot}\deg(v)$ for all $u, v \in \mathrm{Mon}^n$.

For $p \in S \setminus \{0\}$ let

$$\mathrm{in}_{\leq}(p) := \max_{\leq}(\mathrm{x}^{\mathrm{supp}(p)}) = \max_{\leq}\{\mathrm{x}^\alpha \mid \alpha \in \mathrm{supp}(p)\} \in \mathrm{Mon}^n$$

be the *initial monomial* of p. For an arbitrary subset I of S let

$$\mathrm{in}_{\leq}(I) := \mathrm{in}_{\leq}[I \setminus \{0\}] \subseteq \mathrm{Mon}^n$$

be the *initial set* of I, and we call its complement in Mon^n,

$$\mathrm{N}_{\leq}(I) := \mathrm{Mon}^n \setminus \mathrm{in}_{\leq}(I) \subseteq \mathrm{Mon}^n,$$

the *normal set* of I.

For a field K and an ideal I of $S = K[\mathrm{x}_1, \dots, \mathrm{x}_n]$, a subset $G \subseteq I$ is a *Gröbner basis* of I if G is finite, $0 \notin G$, and $\langle \mathrm{in}_{\leq}(G)\rangle_S = \langle \mathrm{in}_{\leq}(I)\rangle_S$. If G is a Gröbner basis of I, then it follows that $\langle G\rangle_S = I$.

Let $(P, \leq)$ be any partially ordered set (i. e., $\leq$ is reflexive, transitive, and antisymmetric on P) and $M \subseteq P$. A subset $B \subseteq M$ is a *$\leq$-basis* of M if for every $y \in M$ there is an $x \in B$ with $x \leq y$. The partially ordered set $(P, \leq)$ is *Dickson* if every subset of P has a finite $\leq$-basis.

Let $\leq_{\mathrm{p}}$ be the partial order on $\mathbb{N}^n$ defined by $\alpha \leq_{\mathrm{p}} \beta$ if and only if $\alpha_j \leq \beta_j$ for all $j = 1, \dots, n$. Clearly, the divisibility relation $|$ on Mon^n is a partial order on Mon^n and there is an isomorphism of partially ordered monoids

$$(\mathrm{Mon}^n, \cdot, |) \cong (\mathbb{N}^n, +, \leq_{\mathrm{p}})$$

given by $\mathbb{N}^n \to \mathrm{Mon}^n$, $\alpha \mapsto \mathrm{x}^\alpha$.

The following well-known lemma can be found e. g. in Becker-Weispfenning [8, Corollary 4.48].

LEMMA 1.2 (Dickson's lemma): *$(\mathrm{Mon}^n, |)$ is Dickson.*

Furthermore we need the following notions. As usual, the *spectrum* of a ring A is denoted by

$$\mathrm{Spec}(A) := \{P \subseteq A \mid P \text{ prime ideal of } A\},$$

and for an arbitrary set $I \subseteq S = A[\mathrm{x}_1, \dots, \mathrm{x}_n]$,

$$\mathrm{V}(I) := \{P \in \operatorname{Spec}(S) \mid I \subseteq P\}$$

denotes the (*algebraic*) *variety* of I. Furthermore, for a subset $J \subseteq S_B$ let

$$\mathrm{V}_B(J) := \{Q \in \operatorname{Spec}(S_B) \mid J \subseteq Q\}$$

be the (*algebraic*) *variety relative to* B of J.

1.3. Preliminaries for Chapter 3

In Section 3.2, we use the following definitions and results. For any norm $\|\|\colon \mathbb{R}^n \to \mathbb{R}^{\geq 0}$, $\varepsilon \in \mathbb{R}^{>0}$, $x \in \mathbb{R}^n$, we let

$$\widetilde{\mathrm{B}}_\varepsilon^{\|\|}(x) := \{y \in \mathbb{R}^n \mid \|x - y\| \leq \varepsilon\}$$

be the *closed* ε-*ball with center* x (*w. r. t.* $\|\|$).

Definition: The *gamma function* is defined as

$$\begin{aligned} \Gamma\colon\ \mathbb{R}^{>0} &\longrightarrow \mathbb{R}, \\ x &\longmapsto \int_{\mathbb{R}^{>0}} t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t. \end{aligned}$$

In the following Theorem 1.3 we collect results on the gamma function that will be useful in Section 3.2. They can be found in treatments of the gamma function, such as Artin [3, 4].

Theorem 1.3: (a) *We have, for all* $n \in \mathbb{N}$,

$$\Gamma(n+1) = n!.$$

(b) *We have* (*see, e. g. Artin [4, p. 19]*)

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

(c) (Stirling's approximation formula, cf. Artin [3, p. 23] resp. [4, p. 24].[2]) *We have, for all* $x \in \mathbb{R}^{>0}$,

$$\Gamma(x) = \sqrt{\frac{2\pi}{x}} \cdot \left(\frac{x}{\mathrm{e}}\right)^x \cdot \mathrm{e}^{\mu(x)}$$

[2] There is some ambiguity in the statements on the cited pages. In addition, there seems to be a mistranslation in the English version of the proof [[3, p. 20, l. −1], [4, p. 22, l. 11]], where ϑ is claimed to be independent of x. We state here an unambiguous and (hopefully) correct version.

with

$$\begin{aligned}\mu\colon\ \mathbb{R}^{>0} &\longrightarrow \mathbb{R},\\ x &\longmapsto \sum_{k=0}^{\infty}\Big(x+k+\frac{1}{2}\Big)\ln\Big(1+\frac{1}{x+k}\Big)-1.\end{aligned}$$

Furthermore, there is a $\vartheta\colon \mathbb{R}^{>0}\to\,]0,1[$ *such that*

$$\mu(x)=\frac{\vartheta(x)}{12\cdot x}$$

for all $x\in\mathbb{R}^{>0}$*. In particular,* $\mu[\mathbb{R}^{\geq 1}]\subseteq\,]0,1/12[$.

(d) (Legendre duplication formula, cf. Artin [4, p. 24].) *We have, for all* $x\in\mathbb{R}^{>0}$,

$$\Gamma(2x)=\frac{2^{2x-1}}{\sqrt{\pi}}\cdot\Gamma(x)\,\Gamma\Big(x+\frac{1}{2}\Big).$$

(e) (Relationship between Γ and B, cf. Artin [4, p. 19].) *We have, for all* $x,y\in\mathbb{R}^{>0}$,

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)},$$

where

$$\begin{aligned}\mathrm{B}\colon\ \mathbb{R}^{>0}\times\mathbb{R}^{>0} &\longrightarrow \mathbb{R},\\ (x,y) &\longmapsto \int_{]0,1[} t^{x-1}(1-t)^{y-1}\,\mathrm{d}t,\end{aligned}$$

denotes the beta function.

Furthermore we make use of Rodrigues' formula, which states that

$$\frac{1}{2^r r!}\frac{\partial^r}{\partial x^r}((x^2-1)^r)=\mathrm{P}_r(x)$$

where P_r denotes the r-th *Legendre polynomial* which are defined inductively by $\mathrm{P}_0=1$, $\mathrm{P}_1=\mathrm{x}$, and $(r+1)\mathrm{P}_{r+1}=(2r+1)\mathrm{x}\mathrm{P}_r-r\mathrm{P}_{r-1}$ for $r\geq 1$. We also use some standard notions from the theory of weak derivatives which may be found, e. g., in Jost [48].

We make use of standard notions of Fourier analysis. In particular, the following Poisson summation formula is applied in Section 3.2. It can be found in many textbooks on Fourier analysis, such as, e. g., Gröchenig [38, Proposition 1.4.2].

Theorem 1.4 (Poisson summation formula): *Let* $\psi\in\mathrm{L}^1(\mathbb{R}^n)$ *and suppose that for some* $c,\varepsilon\in\mathbb{R}^{>0}$ *we have*

$$|\psi(x)|\leq c\cdot(1+|x|)^{-(n+\varepsilon)}$$

and

$$|\mathcal{F}_n(\psi)(v)|\leq c\cdot(1+|v|)^{-(n+\varepsilon)}$$

for all $x,v\in\mathbb{R}^n$*. Then, for all* $x\in\mathbb{R}^n$ *and with* $b_x:=(\mathrm{e}^{2\pi\mathrm{i}x_1},\ldots,\mathrm{e}^{2\pi\mathrm{i}x_n})^\top\in\mathbb{T}^n$,

$$\sum_{\alpha\in\mathbb{Z}^n}\mathcal{F}_n(\psi)(\alpha)\cdot\exp_{b_x}(\alpha)=\sum_{\alpha\in\mathbb{Z}^n}\psi(x+\alpha)$$

where both ($\mathbb{Z}^n$-)*series are absolutely convergent.*

2. Reconstruction of multivariate exponential sums over an arbitrary field

In this chapter we study multivariate exponential sums in a purely algebraic context with the goal of generalizing Prony's classical reconstruction theory for univariate exponential sums. In comparison to the classical setting, we deal with two directions of generalization at once. The algebraic framework adds only minor difficulty to the proofs. In fact, in Prony's pioneering work [66] no explicit mention is made about the nature of the involved quantities—and it is not needed, for nothing is being exploited but that they are elements of a field. Even if nothing else, the author feels that this level of generality helps to clarify the underlying arguments. Generalization to the multivariate scenario, though in hindsight it may seem straightforward, requires some effort and is the dominating theme of this thesis.

The chapter is divided into seven sections as follows. Section 2.1 contains the fundamental definition of multivariate exponential sums and an elementary discussion of some of their properties. In Section 2.2, the machinery for the reconstruction of multivariate exponential sums is set up, generalizing Prony's work for the univariate case. In Section 2.3, the theory previously developed is illustrated computationally on some explicit exponential sums. Some examples are computed in floating point arithmetic. In Section 2.4, a variant is discussed in which the Hankel-like matrix used in Section 2.2 is replaced by a Toeplitz-like matrix. In Section 2.5, the theory is generalized to exponential sums whose support lives on an algebraic variety. An application is given for a special kind of algebraic variety that allows to extract additional information on the number of samples needed for performing Prony's method. In Section 2.6, it is shown that, when working with the total degree on the polynomial algebra, the polynomials used to cut out the support already generate its vanishing ideal. The chapter ends with Section 2.7 and an attempt to shed some additional light on the algebraic nature of the theory. The generality carried through parts of the theory is motivated by this application.

Unless stated otherwise, throughout this chapter K denotes a field and A is an integral domain that contains K as a subring, with $Q := \mathrm{Quot}(A)$ being the quotient field of A. Rings are always understood to be commutative. Furthermore, $n \in \mathbb{N} \setminus \{0\}$ always denotes a non-zero natural number, which will be the number of variables of the exponentials and the corresponding number of indeterminates of the polynomial algebra $S := A[\mathrm{x}_1, \dots, \mathrm{x}_n]$ over A.

2.1. Multivariate exponential sums

The following definition of multivariate exponential sum and the associated notion of rank is fundamental for this thesis.

Definition: (a) For $b = (b_1, \ldots, b_n) \in A^n$ and $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we use the usual notation

$$b^\alpha := \prod_{j=1}^{n} b_j^{\alpha_j}$$

and we call the function

$$\begin{aligned} \exp_b \colon \mathbb{N}^n &\longrightarrow A, \\ \alpha &\longmapsto b^\alpha, \end{aligned}$$

n-variate exponential over A. For $b = 0 \in A$ and $\alpha = 0 \in \mathbb{N}$, we adhere to the convention

$$b^\alpha = 0^0 := 1 \in A.$$

Since $b_j = \exp_b(\mathrm{u}_j)$, where

$$\mathrm{u}_j := (\delta_{ij})_{i=1,\ldots,n} = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$$

denotes the j-th *unit tuple* in $\mathbb{N}^n$, b is uniquely determined by the exponential $\exp_b$. The n-tuple $b \in A^n$ is called the *base* of $\exp_b$.

(b) Let $B \subseteq A^n$ be an arbitrary subset. The K-subvector space of $A^{\mathbb{N}^n}$ generated by the exponentials $\exp_b$, $b \in B$,

$$\begin{aligned} \operatorname{Exp}_B^n(A) := \langle \exp_b \mid b \in B \rangle_K &= \Big\{ \sum_{b \in M} \lambda_b \exp_b \;\Big|\; M \in \mathcal{P}_{\mathrm{f}}(B) \text{ and } \lambda \in K^M \Big\} \\ &= \Big\{ \sum_{i=1}^{r} \lambda_i \exp_{b_i} \;\Big|\; r \in \mathbb{N},\, \lambda_i \in K,\, b_i \in B \Big\}, \end{aligned}$$

is called *K-vector space of n-variate exponential sums over A supported on B*. Elements of $\operatorname{Exp}_B^n(A)$ are called *n-variate exponential sums over A* (*supported on B*). In case $B = A^n$ we omit the subscript B, i. e., we set

$$\operatorname{Exp}^n(A) := \operatorname{Exp}_{A^n}^n(A).$$

(c) For an exponential sum $f \in \operatorname{Exp}_B^n(A)$, we call

$$\begin{aligned} \operatorname{rank}(f) &:= \min\{|M| \mid M \subseteq B,\, f \in \operatorname{Exp}_M^n(A)\} \\ &= \min\Big\{ |M| \;\Big|\; M \in \mathcal{P}_{\mathrm{f}}(B) \text{ and there is a } \lambda \in K^M \text{ with } f = \sum_{b \in M} \lambda_b \exp_b \Big\} \end{aligned}$$

the *rank* of f. It is not a priori clear that the rank of f is independent of B. However, we will later show that the exponentials $\exp_b$, $b \in A^n$, are linearly independent (cf. Corollary 2.22/Remark 2.23 (b)) and take this in advance as justification for not introducing notation like "$\operatorname{rank}_B(f)$".

Some relevant examples of exponential sums are the following.

EXAMPLE 2.1: (a) Taking $A = K = \mathbb{C}$ as a $\mathbb{C}$-algebra and $n = 1$, Prony computed coefficient vectors of $f \in \mathrm{Exp}^1(\mathbb{C})$ already in 1795, and in fact dealt with the additional difficulty of taking necessarily inaccurate measurements of physical experiments as evaluations. As already mentioned, his method works over any field K.

(b) For $A = K = \mathbb{C}$ and the complex n-torus

$$B = \mathbb{T}^n := \{z \in \mathbb{C}^n \mid |z_j| = 1 \text{ for all } j = 1, \dots, n\} \subseteq \mathbb{C}^n,$$

we obtain

$$\mathrm{Exp}^n_{\mathbb{T}^n}(\mathbb{C}) = \Big\{ \sum_{b \in M} \lambda_b \exp_b \;\Big|\; M \in \mathcal{P}_{\mathrm{f}}(\mathbb{T}^n) \text{ and } \lambda \in \mathbb{C}^M \Big\}.$$

This space is of great importance for applications in signal processing. Note that for $b \in \mathbb{T}^n$ and $\alpha \in \mathbb{N}^n$, we have

$$\exp_b(\alpha) = b^\alpha = b_1^{\alpha_1} \cdots b_n^{\alpha_n} = \mathrm{e}^{\mathrm{i}\alpha_1\varphi_1} \cdots \mathrm{e}^{\mathrm{i}\alpha_n\varphi_n} = \mathrm{e}^{\mathrm{i}\langle\varphi,\alpha\rangle},$$

where

$$\varphi := \arg(b) := (\arg(b_1), \dots, \arg(b_n)) \in [0, 2\pi[^n$$

denotes the argument of $b \in \mathbb{T}^n \subseteq \mathbb{C}^n$ and $\langle\rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the euclidean scalar product. A result concerning this example is given in Section 3.2.

(c) To see a connection to applications in signal processing more clearly, recall that $\cos(\varphi) = \mathrm{Re}(\mathrm{e}^{\mathrm{i}\varphi})$ and $\sin(\varphi) = \mathrm{Im}(\mathrm{e}^{\mathrm{i}\varphi})$ for all $\varphi \in \mathbb{R}$. Since $\mathrm{Re}(z) = 1/2 \cdot (z + \overline{z})$ and $\mathrm{Im}(z) = 1/(2\mathrm{i}) \cdot (z - \overline{z})$ for all $z \in \mathbb{C}$, we have

$$\begin{aligned} c := \mathrm{i}_{\mathbb{R},\mathbb{C}} \circ \cos \restriction \mathbb{N} = \mathrm{i}_{\mathbb{R},\mathbb{C}} \circ \mathrm{Re} \circ \exp_{\mathrm{e}^{\mathrm{i}}} \colon \; \mathbb{N} &\longrightarrow \mathbb{C}, \\ \alpha &\longmapsto 1/2 \cdot \exp_{\mathrm{e}^{\mathrm{i}}}(\alpha) + 1/2 \cdot \exp_{\mathrm{e}^{-\mathrm{i}}}(\alpha), \end{aligned}$$

(where, only for formal reasons, $\mathrm{i}_{\mathbb{R},\mathbb{C}} \colon \mathbb{R} \hookrightarrow \mathbb{C}$ is the inclusion map) and

$$\begin{aligned} s := \mathrm{i}_{\mathbb{R},\mathbb{C}} \circ \sin \restriction \mathbb{N} = \mathrm{i}_{\mathbb{R},\mathbb{C}} \circ \mathrm{Im} \circ \exp_{\mathrm{e}^{\mathrm{i}}} \colon \; \mathbb{N} &\longrightarrow \mathbb{C}, \\ \alpha &\longmapsto 1/(2\mathrm{i}) \cdot \exp_{\mathrm{e}^{\mathrm{i}}}(\alpha) - 1/(2\mathrm{i}) \cdot \exp_{\mathrm{e}^{-\mathrm{i}}}(\alpha). \end{aligned}$$

Hence we see that

$$c, s \in \mathrm{Exp}^1_{\mathbb{T}}(\mathbb{C}).$$

Thus, the reconstruction problem for arbitrary exponential sums specializes to the reconstruction problem for linear combinations of trigonometric sequences, i. e., discrete signals. Clearly, $\mathrm{rank}(c), \mathrm{rank}(s) \leq 2$ and Prony's classical theory implies that $\mathrm{rank}(c) = \mathrm{rank}(s) = 2$.

(d) (1) Consider the K-algebra

$$A_n := K[\mathrm{y}_{i,j} \mid i \in \mathbb{N},\, j = 1, \ldots, n] = \bigcup_{i \in \mathbb{N}} K[\mathrm{y}_{i,j} \mid j = 1, \ldots, n],$$

and let

$$\mathrm{y}_i := (\mathrm{y}_{i,1}, \ldots, \mathrm{y}_{i,n}) \in (A_n)^n$$

and

$$B_n := \{\mathrm{y}_i \mid i \in \mathbb{N}\} \subseteq (A_n)^n.$$

The K-vector space

$$\begin{aligned} \mathrm{FExp}^n(K) := \mathrm{Exp}^n_{B_n}(A_n) &= \langle \exp_{\mathrm{y}_i} \mid i \in \mathbb{N} \rangle_K \\ &= \Big\{ \sum_{i=1}^{r} \lambda_i \exp_{\mathrm{y}_i} \;\Big|\; r \in \mathbb{N},\, \lambda_i \in K \Big\} \end{aligned}$$

is called *K-vector space of formal exponential sums over K* and its elements are called *formal exponential sums over K*. Note that $\mathrm{FExp}^n(K)$ is countably generated as a K-vector space. (By the later Corollary 2.22 we get $\dim_K(\mathrm{FExp}^n(K)) = |\mathbb{N}|$.)

(2) Let $r \in \mathbb{N}$ and consider the K-algebra

$$A_{n,r} := K[\mathrm{y}_{i,j} \mid i = 1, \ldots, r,\, j = 1, \ldots, n]$$

of polynomials over K in $r \cdot n$ indeterminates. Let

$$\mathrm{y}_i := (\mathrm{y}_{i,1}, \ldots, \mathrm{y}_{i,n}) \in (A_{n,r})^n$$

and

$$B_{n,r} := \{\mathrm{y}_i \mid i = 1, \ldots, r\} \subseteq (A_{n,r})^n.$$

Then

$$\begin{aligned} \mathrm{FExp}^n_r(K) := \mathrm{Exp}^n_{B_{n,r}}(A_{n,r}) &= \langle \exp_{\mathrm{y}_1}, \ldots, \exp_{\mathrm{y}_r} \rangle_K \\ &= \Big\{ \sum_{b \in M} \lambda_b \exp_b \;\Big|\; M \in \mathcal{P}_{\mathrm{f}}(B_{n,r}) \text{ and } \lambda \in K^M \Big\} \\ &= \Big\{ \sum_{i=1}^{r} \lambda_i \exp_{\mathrm{y}_i} \;\Big|\; \lambda_i \in K,\, i = 1, \ldots, r \Big\} \end{aligned}$$

is the *K-vector space of formal rank $\leq r$ exponential sums over K* and its elements are called *formal rank $\leq r$ exponential sums over K*. Note that $\mathrm{FExp}^n_r(K)$ is finite dimensional as a K-vector space. (By the later Corollary 2.22 we get $\dim_K(\mathrm{FExp}^n_r(K)) = r$.)

One should be careful not to confuse the K-vector space $\mathrm{FExp}^n_r(K)$ and the subset $\{F \in \mathrm{FExp}^n(K) \mid \mathrm{rank}(F) \leq r\}$ of $\mathrm{FExp}^n(K)$. The latter is not closed under taking sums. We will return to these examples in Section 2.7.

Remark 2.2: (a) Let R be any integral domain containing A as a subring and let $B \subseteq A^n$. If one considers R as a K-algebra then one has $\mathrm{Exp}_B^n(A) = \mathrm{Exp}_B^n(R)$. Therefore, in situations that require A to be a field one can often replace A by its quotient field Q.

(b) For any subset $B \subseteq A^n$ we have $\dim_K(\mathrm{Exp}_B^n(A)) \leq |B|$ and therefore every exponential sum $f \in \mathrm{Exp}_B^n(A)$ satisfies $\mathrm{rank}(f) \leq |B|$. This is immediate from the definitions. In particular, if the K-algebra A consists of only finitely many elements,[1] then we have the upper bound $|B| \leq |A^n| = |A|^n \in \mathbb{N}$ for $\mathrm{rank}(f)$. By the later Corollary 2.22 it is also true that $\dim_K(\mathrm{Exp}_B^n(A)) = |B|$.

(c) For $b \in A^n$ and $\alpha \in \mathbb{N}^n$ we have $\exp_b(\alpha) = \mathrm{ev}^{\{b\}}(\mathrm{x}^\alpha)$, where $\mathrm{ev}^{\{b\}} \colon A[\mathrm{x}_1, \ldots, \mathrm{x}_n] \to A$ denotes evaluation at b. Thus one may see exponentials as the restrictions $\mathrm{ev}^{\{b\}} \restriction \mathrm{Mon}^n$, identifying $\mathbb{N}^n$ and Mon^n. Considering more generally the evaluations $\mathrm{ev}^{\{b\}}$ instead of exponentials as defined here and exploiting duality theory for polynomial algebras is a perspective taken in some recent works on the subject (see Section 4.6).

In the following remark we explore some elementary algebraic properties of $\mathrm{Exp}_B^n(A)$.

Remark 2.3: (a) For $a, b \in A^n$ we have $\exp_a \cdot \exp_b = \exp_{ab}$, where $\cdot$ denotes componentwise multiplication and also A^n is endowed with the componentwise multiplication.

Proof: For $\alpha \in \mathbb{N}^n$ we have $\exp_a \cdot \exp_b(\alpha) = \exp_a(\alpha)\exp_b(\alpha) = a^\alpha b^\alpha = \prod_{j=1}^n a_j^{\alpha_j} \cdot \prod_{j=1}^n b_j^{\alpha_j} = \prod_{j=1}^n (a_j b_j)^{\alpha_j} = (ab)^\alpha = \exp_{ab}(\alpha)$. «

(b) If $(B, \cdot)$ is a submonoid of $(A^n, \cdot)$, then $\mathrm{Exp}_B^n(A)$ is a K-algebra under componentwise addition and multiplication with unit element $1 = \exp_{(1,\ldots,1)} \in \mathrm{Exp}_B^n(A)$. Furthermore, if $f, g \in \mathrm{Exp}_B^n(A)$, then

$$\mathrm{rank}(fg) \leq \mathrm{rank}(f)\,\mathrm{rank}(g).$$

Equality does not hold in general, see Example 2.4 (d).

Proof: We first show that $fg \in \mathrm{Exp}_B^n(A)$ for all $f, g \in \mathrm{Exp}_B^n(A)$. Let $f = \sum_{b \in M} f_b \exp_b$ and $g = \sum_{c \in N} g_c \exp_c$ with $M, N \in \mathcal{P}_\mathrm{f}(B)$, $f_b, g_c \in K$, $|M| = \mathrm{rank}(f)$, and $|N| = \mathrm{rank}(g)$. Then we have $fg = (\sum_{b \in M} f_b \exp_b) \cdot (\sum_{c \in N} g_c \exp_c) \overset{\text{(a)}}{=} \sum_{b \in M}(\sum_{c \in N} f_b g_c \exp_{bc}) = \sum_{d \in M \times N} f_{d_1} g_{d_2} \exp_{d_1 d_2} \in \mathrm{Exp}_B^n(A)$. As an immediate consequence we obtain $\mathrm{rank}(fg) \leq |M \times N| = |M| \cdot |N| = \mathrm{rank}(f)\,\mathrm{rank}(g)$. Since B is a submonoid of $(A^n, \cdot)$ it follows that $(1, \ldots, 1) \in B$, and we have $\exp_{(1,\ldots,1)}(\alpha) = (1, \ldots, 1)^\alpha = \prod_{j=1}^n 1^{\alpha_j} = \prod_{j=1}^n 1 = 1$ for all $\alpha \in \mathbb{N}^n$. «

[1] If A has only finitely many elements, then, as A is an integral domain, A is a finite field containing the (necessarily finite) field K, so $K = \mathbb{F}_{p^k}$ and $A = \mathbb{F}_{p^\ell}$ for a prime number $p \geq 2$ and $k, \ell \in \mathbb{N}$, $k \mid \ell$.

(c) For all $f = \sum_{b \in M} f_b \exp_b \in \operatorname{Exp}^n(A)$ we have $\exp_0 \cdot f = \left(\sum_{b \in M} f_b\right) \exp_0 = f(0) \cdot \exp_0$. This follows immediately from part (a), since $\exp_0 \exp_b = \exp_{0b} = \exp_0$ for all $b \in A^n$.

(d) The exponential $\exp_0$ is a non-trivial idempotent element of $\operatorname{Exp}^n(A)$. Indeed, by part (c) we have $\exp_0 \cdot \exp_0 = \exp_0$, and the assertion follows from $\exp_0(0) = 1$ and $\exp_0(\mathrm{u}_1) = 0$.

(e) The ring $\operatorname{Exp}^n(A)$ is never an integral domain. This follows immediately from the existence of the non-trivial idempotent element $e := \exp_0 \in \operatorname{Exp}^n(A)$ by part (d): One has $e \cdot (e-1) = 0$ and $e, e-1 \neq 0$.

(f) More generally than in part (d), the idempotent exponentials in $\operatorname{Exp}^n(A)$ are precisely the exponentials $\exp_b$ with base $b \in \{0,1\}^n$.

Proof: Let $b \in A^n$ be such that $\exp_b$ is idempotent. Then $\exp_b(\mathrm{u}_j) = b^{\mathrm{u}_j} = b_j$ is idempotent in A, so $b_j \in \{0,1\}$. Conversely, if $b \in \{0,1\}^n$, then $(\exp_b)^2 = \exp_{b^2} = \exp_b$ by part (a). «

Example 2.4: (a) The space $\operatorname{Exp}^n(A) = \operatorname{Exp}^n_{A^n}(A)$ is always a K-algebra by Remark 2.3 (b).

(b) Since $\mathbb{T}^n$ is multiplicatively closed, $\operatorname{Exp}^n_{\mathbb{T}^n}(\mathbb{C})$ is a $\mathbb{C}$-algebra by Remark 2.3 (b).

(c) If $r \geq 1$, then the K-vector space $\operatorname{FExp}^n_r(K)$ of formal rank $\leq r$ exponential sums is not a K-algebra under componentwise multiplication of formal exponential sums. To see this, note that if $\mathrm{y}_1^\alpha \mathrm{y}_1^\alpha = \exp_{\mathrm{y}_1} \cdot \exp_{\mathrm{y}_1}(\alpha) = \sum_{i=1}^r \lambda_i \exp_{\mathrm{y}_i}(\alpha)$ with $\lambda_i \in K$ for all $\alpha \in \mathbb{N}^n$, this yields $\mathrm{y}_{1,1}^2 = \sum_{i=1}^r \lambda_i \mathrm{y}_{i,1}$ for $\alpha = \mathrm{u}_1$, and by comparison of coefficients one obtains $1 = 0$, a contradiction.

For the same reason $\operatorname{FExp}^n(K)$ is not a K-algebra.

(d) In general it is not true that $\operatorname{rank}(fg) = \operatorname{rank}(f)\operatorname{rank}(g)$ for $f, g \in \operatorname{Exp}^n(A)$. This is clear by Remark 2.3 (e) since for any zero-divisors $f, g \in \operatorname{Exp}^n(A) \setminus \{0\}$, $0 = |\emptyset| = \operatorname{rank}(0) = \operatorname{rank}(fg) < \operatorname{rank}(f)\operatorname{rank}(g)$.

2.2. A generalization of Prony's reconstruction theory

The goal of this section is to develop the foundation for generalizations of Prony's theory that are suitable for the reconstruction of multivariate exponential sums. We begin with a discussion of the bearing of Prony's classical reconstruction method in the following remark. This discussion may also serve as a blueprint for the multivariate generalization that follows. The technique in footnote 2 will also be used later on.

Remark 2.5: The reconstruction problem for $\operatorname{Exp}^1(\mathbb{C})$ consists of the task to compute the coefficient vector of $f \in \operatorname{Exp}^1(\mathbb{C})$ w. r. t. the $\mathbb{C}$-basis $E := \{\exp_b \mid b \in \mathbb{C}\}$ of $\operatorname{Exp}^1(\mathbb{C})$. Since any algorithm can only take into account a finite amount of data, it can only take into account the restriction of f to a finite subset of $\mathbb{N}$. Therefore it is necessary to find a finite subset $F \subseteq \mathbb{N}$ such that the restriction $f \upharpoonright F$ allows to compute the coefficients of f.

There are two distinct approaches: Such a subset F might either be constructed by a reconstruction procedure that would be assumed to be able to evaluate f at (finitely many) arbitrary points (then F could be dependent on f), or it might be a subset F that is independent of f, in which case a reconstruction algorithm can be seen as having the restriction $f \upharpoonright F$ as input.

In either case, the problem as stated above is impossible to solve without further assumptions, since $f \upharpoonright F$ only defines f modulo the non-zero[2] subvector space

$$Z_F := \{g \in \mathrm{Exp}^1(\mathbb{C}) \mid g \upharpoonright F = 0\} = \ker(\upharpoonright F\colon \mathrm{Exp}^1(\mathbb{C}) \to \mathbb{C}^F) \leq \mathrm{Exp}^1(\mathbb{C}).$$

Since being able to reconstruct f implies that one can compute $\mathrm{rank}(f)$, one cannot hope to be able reconstruct f without at least implicit knowledge of $\mathrm{rank}(f)$. Therefore we assume as given also a natural number $d \in \mathbb{N}$ with $r := \mathrm{rank}(f) \leq d$. Under this assumption, the task is to find a finite $F \subseteq \mathbb{N}$ (dependent on d and possibly f) such that

$$Y_F := \{g \in \mathrm{Exp}^1(\mathbb{C}) \mid f - g \in Z_F \text{ and } \mathrm{rank}(g) \leq d\} = \{f\}.$$

This may not immediately appear to be a significant simplification or at all be clear that such a set F exists. Essentially, Prony proved the following in 1795 [66][3]: The set

$$F := \{0, \ldots, 2d\} \subseteq \mathbb{N}$$

solves the problem simultaneously for all $f \in \mathrm{Exp}^1(\mathbb{C})$ with $\mathrm{rank}(f) \leq d$, i. e., for all $f \in \mathrm{Exp}^1(\mathbb{C})$ with $\mathrm{rank}(f) \leq d$ one has $Y_F = \{f\}$. Furthermore, considering the matrix

$$\mathrm{H}_d(f) := (f(\alpha + \beta))_{\substack{\alpha=0,\ldots,d \\ \beta=0,\ldots,d}} \in \mathbb{C}^{(d+1)\times(d+1)},$$

one can construct from $f \upharpoonright F$ a polynomial

$$p \in \mathbb{C}[\mathrm{x}]_d \setminus \{0\},$$

namely p such that the vector of coefficients of p is in

$$\ker(\mathrm{H}_d(f)) \setminus \{0\},$$

and the (finite, in the univariate case considered here) zero locus $\mathrm{Z}(p) \subseteq \mathbb{C}$ of p fulfills

$$f \in \mathrm{Exp}^1_{\mathrm{Z}(p)}(\mathbb{C}).$$

[2] *Proof:* For $k \in \mathbb{N}$ let $f_k\colon \mathbb{N} \to \mathbb{C}$, $\alpha \mapsto (1/\exp_2(k)) \cdot \exp_2(\alpha) - \exp_1(\alpha)$. Then we have $f_k \in \mathrm{Exp}^1(\mathbb{C})$, $f_k(k) = 0$ and $f_k(\alpha) = 2^{\alpha-k} - 1 \geq 1 > 0$ for $\alpha > k$. Therefore, for $F \subseteq \mathbb{N}$ finite, the product $f := \prod_{k\in F} f_k$ satisfies $f \in \mathrm{Exp}^1(\mathbb{C})$ (by Remark 2.3 (b)) and $f \in Z_F \setminus \{0\}$. «

[3] Actually, Prony proved that $F = \{0, \ldots, 2d-1\}$ is sufficient, but in our context $F = \{0, \ldots, 2d\}$ is more appropriate.

Since the set $L := \{\exp_b \mid b \in \mathrm{Z}(p)\} \subseteq E$ is linearly independent, the coefficients $f_b \in \mathbb{C}$ with $f = \sum_{b \in \mathrm{Z}(p)} f_b \exp_b$ are uniquely determined (and, once $\mathrm{Z}(p)$ is known explicitly, they are relatively easy to compute from the entries of $\mathrm{H}_d(f)$, cf. Remark 2.6 for the general case). In this way, the task of computing the coefficients of f w. r. t. the uncountable $\mathbb{C}$-basis E can be reduced to the considerably simpler one of computing the coefficients of f w. r. t. the $\mathbb{C}$-basis $L \subseteq E$ of the $|\mathrm{Z}(p)|$-dimensional subspace $\mathrm{Exp}^1_{\mathrm{Z}(p)}(\mathbb{C}) \leq \mathrm{Exp}^1(\mathbb{C})$.

However, stated like this, the method is not necessarily efficient since there may be $b \in \mathrm{Z}(p)$ such that the coefficient of $\exp_b$ in f is zero. One way to deal with this fact is to compute the greatest common divisor q in $\mathbb{C}[\mathrm{x}]$ of a $\mathbb{C}$-basis of $\ker(\mathrm{H}_d(f)) \hookrightarrow \mathbb{C}[\mathrm{x}]$. Then $\mathrm{Z}(q)$ contains precisely the $b \in \mathbb{C}$ for which the coefficient of $\exp_b$ in f is non-zero.[4] The reason for this lies in the fact that $\ker(\mathrm{H}_d(f))$ generates the vanishing ideal I of the finite set $S := \{b \in \mathbb{C} \mid \text{coefficient of } \exp_b \text{ in } f \text{ is non-zero}\}$. Thus, since $\mathbb{C}[\mathrm{x}]$ is a euclidean domain, one has $\langle \ker \mathrm{H}_d(f) \rangle_{\mathbb{C}[\mathrm{x}]} = I = q \cdot \mathbb{C}[\mathrm{x}]$ with $q \in \mathbb{C}[\mathrm{x}]$ being the greatest common divisor of any ideal basis of I, and then $\mathrm{Z}(q) = \mathrm{Z}(I) = S$, since S is finite.

Of course, the computation via the greatest common divisor and (even more so) the fact that $\mathrm{Z}(p)$ is finite for all non-zero p are specialties of the univariate case.

We will now introduce some basic machinery we will use throughout. For easy reference, the following definitions are also included in the preliminary Section 1.1.

Definition: Let S denote the polynomial algebra over A in n indeterminates,

$$S := A[\mathrm{x}_1, \dots, \mathrm{x}_n].$$

For an arbitrary subset $M \subseteq A^n$, we regard A^M as an A-algebra with multiplication defined pointwise, and the A-algebra homomorphism

$$\begin{aligned} \mathrm{ev}^M \colon S &\longrightarrow A^M, \\ p &\longmapsto \mathrm{f}_p \restriction M = (p(b))_{b \in M}, \end{aligned}$$

is called *evaluation homomorphism* at M. For a subset $D \subseteq \mathbb{N}^n$ let

$$\mathrm{x}^D := \{\mathrm{x}^\alpha \mid \alpha \in D\} \subseteq \mathrm{Mon}^n,$$

let

$$S_D := \langle \mathrm{x}^D \rangle_A$$

be the free A-submodule of S generated by x^D, and let

$$\mathrm{ev}^M_D := \mathrm{ev}^M \restriction S_D$$

[4] For illustration, take $f = 0 \in \mathrm{Exp}^1(\mathbb{C})$ and $d := 1$. Then $\mathrm{H}_d(f) \in \mathbb{C}^{2\times 2}$ is the zero matrix. To reconstruct f under the knowledge that $\mathrm{rank}(f) \leq d = 1$, one may for example take the $\mathbb{C}$-basis $\{\mathrm{x}, 1 - \mathrm{x}\}$ of $\ker(\mathrm{H}_d(f)) \hookrightarrow \mathbb{C}[\mathrm{x}]$ and then compute $q := \gcd(\mathrm{x}, 1 - \mathrm{x}) = 1$ and $\mathrm{Z}(q) = \mathrm{Z}(1) = \emptyset$. None of the basis elements $\mathrm{x}, 1 - \mathrm{x}$ alone cuts out the bases of f as zero locus. On the other hand, computing, e. g., only $\mathrm{x} \in \ker(\mathrm{H}_d(f))$ and then computing $\mathrm{Z}(\mathrm{x}) = \{0\}$, one may see afterwards that $\exp_0(0) = 1 \neq 0 = f(0)$ and deduce that $f = 0$. Note that $f(0)$ is given as an entry of $\mathrm{H}_d(f)$.

be the restriction of ev^M to S_D. For finite $M \subseteq A^n$ and $D \subseteq \mathbb{N}^n$, the evaluation homomorphism ev_D^M is an A-module homomorphism from the finite-dimensional free A-module S_D to the finite-dimensional free A-module A^M. Thus, for computational as well as theoretical purposes, ev_D^M can be identified with a matrix. The transformation matrix of ev_D^M w. r. t. the basis x^D of S_D and the canonical basis

$$\mathrm{U}_M := \{\mathrm{u}_b \mid b \in M\}$$

of A^M, where for $b \in M$,

$$\begin{aligned} \mathrm{u}_b\colon\ M &\longrightarrow A, \\ c &\longmapsto \delta_{bc} := \begin{cases} 1 & \text{if } b = c, \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

denotes the b-th unit vector in A^M, is denoted by

$$\mathrm{V}_D^M \in A^{M \times D}.$$

It is easy to see that

$$\mathrm{V}_D^M = (b^\alpha)_{\substack{b \in M \\ \alpha \in D}}.$$

Indeed, this holds since for all $\alpha \in D$ we have $\mathrm{ev}_D^M(\mathrm{x}^\alpha) = (b^\alpha)_{b \in M} = \sum_{b \in M} b^\alpha \mathrm{u}_b$. In the univariate case $n = 1$ with $D = \{0, \dots, d\} \subseteq \mathbb{N}$, V_D^M is a Vandermonde matrix.

Kernels of these and related homomorphisms will play an important role in the following. These homomorphisms will often be represented by a transformation matrix, such as $\mathrm{V}_D^M \in A^{M \times D}$ for $\mathrm{ev}_D^M\colon S_D \to A^M$. There will also be the need to change the domains (and codomains) of these homomorphisms (for instance, to consider $\mathrm{ev}_{D,Q}^M\colon Q[\mathrm{x}_1, \dots, \mathrm{x}_n]_D \to Q^M$ for the quotient field $Q := \mathrm{Quot}(A)$ of A). Since these changes are not reflected in the transformation matrix, we need a notation to signify over which domain the kernel and image of a matrix is to be considered. To be precise about this while maintaining light-weight notation, we distinguish between the kernels of these homomorphisms by the following definition.

Definition: For any matrix $H \in A^{M \times N}$ and any ring homomorphism from A to R, we define

$$\ker_R(H) := \ker(R^N \to R^M,\, x \mapsto Hx) = \{x \in R^N \mid Hx = 0\}.$$

Similarly we define

$$\mathrm{im}_R(H) := \mathrm{im}(R^N \to R^M,\, x \mapsto Hx) = \{Hx \mid x \in R^N\}.$$

We discuss briefly how to reconstruct the coefficients of an exponential sum with respect to a given set of exponentials.

Remark 2.6: Let $f = \sum_{b \in M} f_b \exp_b \in \mathrm{Exp}^n_M(A)$ with $M \in \mathcal{P}_\mathrm{f}(A^n)$ and $f_b \in K$, and let $D \in \mathcal{P}_\mathrm{f}(\mathbb{N}^n)$ be such that $\mathrm{ev}^M_D \colon S_D \to A^M$ is surjective. One immediately obtains that the tuple $(f_b)_{b\in M} \in K^M$ fulfills

$$
\begin{aligned}
(\mathrm{V}^M_D)^\top \cdot \mathrm{coeff}(f) &= \sum_{b \in M} \mathrm{coeff}(f)_b \cdot (b^\alpha)_{\alpha \in D} \\
&= \Big(\sum_{b \in M} \mathrm{coeff}(f)_b \cdot b^\alpha \Big)_{\alpha \in D} = (f(\alpha))_{\alpha \in D}.
\end{aligned}
$$

Since ev^M_D is surjective, $\ker((\mathrm{V}^M_D)^\top) = \{0\}$, and therefore $\mathrm{coeff}(f)$ is determined as the unique solution of the system of linear equations with system matrix $(\mathrm{V}^M_D)^\top \in A^{D\times M}$ and right-hand side $(f(\alpha))_{\alpha\in D} \in A^D$. This provides a simple method to reconstruct the coefficients of $f \in \mathrm{Exp}^n_M(A)$ w. r. t. a given set $M \in \mathcal{P}_\mathrm{f}(A^n)$, provided ev^M_D is surjective and $f \upharpoonright D$ is given. For this reason, surjectivity conditions on the evaluation homomorphisms ev^M_D are rather natural in the context of reconstructing exponential sums.

In analogy to Prony's work [66] (cf. Remark 2.5), we define a Hankel-like matrix associated to an exponential sum $f \in \mathrm{Exp}^n(A)$. This matrix will play a key role in the reconstruction theory for multivariate exponential sums that follows.

Definition: Let $D \subseteq \mathbb{N}^n$ be an arbitrary subset. For $f \in \mathrm{Exp}^n(A)$ we define the matrix

$$
\mathrm{H}_D(f) := (f(\alpha+\beta))_{\substack{\alpha\in D\\ \beta \in D}} \in A^{D\times D}.
$$

The following Lemma 2.7 is crucial. In part (a) (which is well-known at least in the univariate case $n = 1$) a connection is established between the problem of reconstructing an exponential sum $f \in \mathrm{Exp}^n(A)$ and that of finding a specific factorization of $\mathrm{H}_D(f)$. This connection is deepened in part (b). The proofs are straightforward.

Lemma 2.7: *Let $f = \sum_{b\in M} f_b \exp_b \in \mathrm{Exp}^n(A)$ with $M \in \mathcal{P}_\mathrm{f}(A^n)$ and $(f_b)_{b\in M} \in K^M$. Let $D \in \mathcal{P}_\mathrm{f}(\mathbb{N}^n)$ be arbitrary. Then the following holds.*

(a) *We have*

$$
\mathrm{H}_D(f) = \mathrm{V}^{M\top}_D \cdot C \cdot \mathrm{V}^M_D,
$$

with the diagonal matrix $C := (f_b \mathrm{u}_b)_{b\in M} \in A^{M\times M}$.

(b) *Let $f_b \neq 0$ for all $b \in M$. If $\mathrm{ev}^M_D \colon S_D \to A^M$ is surjective, then*

$$
\ker_A(\mathrm{H}_D(f)) = \ker_A(\mathrm{V}^M_D).
$$

PROOF: For brevity, let $H := \mathrm{H}_D(f)$ and $V := \mathrm{V}_D^M$.

(a) Since $V^\top C = (f_b b^\alpha)_{\substack{\alpha\in D\\ b\in M}}$, we have

$$V^\top C V = (V^\top C (b^\beta)_{b\in M})_{\beta\in D} = \Big(\sum_{b\in M} b^\beta f_b (b^\alpha)_{\alpha\in D}\Big)_{\beta\in D}$$
$$= \Big(\sum_{b\in M} f_b b^{\alpha+\beta}\Big)_{\substack{\alpha\in D\\ \beta\in D}} = (f(\alpha+\beta))_{\substack{\alpha\in D\\ \beta\in D}} = H.$$

(b) By part (a) we always have $\ker_A(V) \subseteq \ker_A(H)$. To show the reverse inclusion, let $C := (f_b \mathrm{u}_b)_{b\in M} \in A^{M\times M}$ as in part (a). We show that $\ker_A(V^\top C) = \{0\}$. Let $Q := \mathrm{Quot}(A)$ be the quotient field of A. Consider $V \in A^{M\times D} \leq Q^{M\times D}$ as a matrix over Q. Since ev_D^M is surjective, the Q-linear map $V\colon Q^D \to Q^M$, $x \mapsto Vx$, is surjective by a trivial argument.[5] Therefore $V^\top\colon Q^M \to Q^D$ is injective by standard linear algebra, which yields $\ker_A(V^\top) = A^M \cap \ker_Q(V^\top) = \{0\}$. Since the coefficients $f_b \in K$, $b \in M$, of f are non-zero and therefore units in A, C is invertible in $A^{M\times M}$, hence $\ker_A(V^\top C) = \ker_A(V^\top) = \{0\}$. Thus, by the trivial fact that for any module homomorphisms $\varphi\colon M \to N$, $\psi\colon N \to P$ with ψ injective, one has $\ker(\psi\circ\varphi) = \ker(\varphi)$, we obtain $\ker_A(H) = \ker_A(V^\top C V) = \ker_A(V)$, as claimed. q. e. d.

REMARK 2.8: More generally than we have done here, for two subsets $D_1, D_2 \subseteq \mathbb{N}^n$ and $f = \sum_{b\in M} f_b \exp_b \in \mathrm{Exp}^n(A)$, one can define the matrix

$$\mathrm{H}_{D_1,D_2}(f) := (f(\alpha+\beta))_{\substack{\alpha\in D_1\\ \beta\in D_2}} \in A^{D_1\times D_2}.$$

Lemma 2.7 then holds in the following analogous form for $\mathrm{H}_{D_1,D_2}(f)$ with identical proofs:

(a) We have
$$\mathrm{H}_{D_1,D_2}(f) = {\mathrm{V}_{D_1}^M}^\top \cdot C \cdot \mathrm{V}_{D_2}^M,$$
with the diagonal matrix $C := (f_b \mathrm{u}_b)_{b\in M} \in A^{M\times M}$.

(b) Let $f_b \neq 0$ for all $b \in M$. If $\mathrm{ev}_{D_1}^M\colon S_{D_1} \to A^M$ is surjective, then
$$\ker_A(\mathrm{V}_{D_2}^M) = \ker_A(\mathrm{H}_{D_1,D_2}(f)).$$

However, we will not dive deeper into this more general setup here.

Subsets $D \subseteq \mathbb{N}^n$ correspond to the sets $\mathrm{x}^D = \{\mathrm{x}^\alpha \mid \alpha \in D\}$ of monomials in n indeterminates with exponents in D. In the following, emphasis will be on subsets $D \subseteq \mathbb{N}^n$ that correspond to sets of monomials bounded by some sort of “degree”. There are two notions of degree that will be in the focus of our attention. We give the definitions here. For easy reference, these are also included in the preliminary Section 1.1.

[5] Let $a \in Q^M$. Then $a = 1/\lambda \cdot b$ for some $\lambda \in A \setminus \{0\}$ and $b \in A^M$. Let $c \in A^D$ such that $V \cdot c = b$. Then $V \cdot (1/\lambda \cdot c) = 1/\lambda \cdot (V \cdot c) = 1/\lambda \cdot b = a$.

Definition: The *total degree* of a monomial $\mathrm{x}^\alpha \in \mathrm{Mon}^n$ is denoted by

$$\operatorname{tot\,deg}(\mathrm{x}^\alpha) := \sum_{j=1}^{n} \alpha_j \in \mathbb{N}$$

and the *total degree* of a polynomial $p \in S \setminus \{0\}$ is

$$\operatorname{tot\,deg}(p) := \max(\operatorname{tot\,deg}[\mathrm{x}^{\operatorname{supp}(p)}]) = \max\{\operatorname{tot\,deg}(\mathrm{x}^\alpha) \mid \alpha \in \operatorname{supp}(p)\} \in \mathbb{N}.$$

The *maximal degree* of a monomial $\mathrm{x}^\alpha \in \mathrm{Mon}^n$ is denoted by

$$\operatorname{max\,deg}(\mathrm{x}^\alpha) := \max\{\alpha_j \mid j = 1, \dots, n\} \in \mathbb{N}$$

and the *maximal degree* of a polynomial $p \in S \setminus \{0\}$ is

$$\operatorname{max\,deg}(p) := \max(\operatorname{max\,deg}[\mathrm{x}^{\operatorname{supp}(p)}]) = \max\{\operatorname{max\,deg}(\mathrm{x}^\alpha) \mid \alpha \in \operatorname{supp}(p)\} \in \mathbb{N}.$$

For $\alpha \in \mathbb{N}^n$ we also set

$$\operatorname{tot\,deg}(\alpha) := \operatorname{tot\,deg}(\mathrm{x}^\alpha)$$

and

$$\operatorname{max\,deg}(\alpha) := \operatorname{max\,deg}(\mathrm{x}^\alpha).$$

In order to provide a theory that includes combinations of these and further notions of degree in a unified way, we introduce an appropriate notion of multi-filtration. Since exponential sums are only defined on $\mathbb{N}^n$, it is appropriate to also define multi-filtrations on $\mathbb{N}^n$. This implies that later on only submodules of $S = A[\mathrm{x}_1, \dots, \mathrm{x}_n]$ occur that are generated by monomials. To avoid confusion, in this context we denote the unit tuples in $\mathbb{N}^t$ by u^t_ℓ, $\ell = 1, \dots, t$. For $\delta \in \mathbb{N}^t$, we will also use the nearby notation

$$\operatorname{min\,deg}(\delta) := \operatorname{min\,deg}(\mathrm{x}^\delta) := \min\{\delta_\ell \mid \ell = 1, \dots, t\}.$$

Definition: Let $t \in \mathbb{N}$ and $(n_0, \dots, n_t) \in \mathbb{N}^{t+1}$ with $0 = n_0 < n_1 < \cdots < n_t = n$. Let $\mathcal{F} \colon \mathbb{N}^t \to \mathcal{P}_\mathrm{f}(\mathbb{N}^n)$. Then $\mathcal{F}$ is a *multi-$\star$-filtration* (or, more precisely, a *t-$\star$-filtration w. r. t.* $(n_0, \dots, n_t)$) on $\mathbb{N}^n$ if the following conditions are satisfied.

(a) $\mathcal{F}_\delta \subseteq \mathcal{F}_\varepsilon$ for all $\delta, \varepsilon \in \mathbb{N}^t$ with $\delta \leq_\mathrm{p} \varepsilon$.[6]

(b) $\mathcal{F}_\delta + \mathcal{F}_\varepsilon \subseteq \mathcal{F}_{\delta+\varepsilon}$ for all $\delta, \varepsilon \in \mathbb{N}^t$.[7]

(c) $\mathcal{F}_0 \neq \emptyset$.

(d) For all $j = 1, \dots, n$, $\mathrm{u}^n_j \in \mathcal{F}_{\mathrm{u}^t_\ell}$ for $\ell \in \{1, \dots, t\}$ with $j \in \{n_{\ell-1} + 1, n_\ell\}$.

[6]The partial order $\leq_\mathrm{p}$ on $\mathbb{N}^t$ is defined by $\delta \leq_\mathrm{p} \varepsilon$ if and only if $\delta_\ell \leq \varepsilon_\ell$ for all $\ell = 1, \dots, t$.

[7]For our purposes we actually only need the property that $\mathcal{F}_\delta + \mathcal{F}_{\mathrm{u}^t_\ell} \subseteq \mathcal{F}_{\delta+\mathrm{u}^t_\ell}$ for all $\delta \in \mathbb{N}^t$, $\ell = 1, \dots, t$.

We will call 1-$\star$-filtrations simply *$\star$-filtrations.*

LEMMA 2.9: *Let $\mathcal{F}^\ell$ be a 1-$\star$-filtration (w. r. t. $(0, n_\ell - n_{\ell-1})$) on $\mathbb{N}^{n_\ell - n_{\ell-1}}$, $\ell = 1, \dots, t$, and let*

$$\begin{aligned} \mathcal{F}\colon\ \mathbb{N}^t &\longrightarrow \mathcal{P}_{\mathrm{f}}(\mathbb{N}^n), \\ \delta &\longmapsto \Big\{\alpha \in \prod_{\ell=1}^t \mathbb{N}^{n_\ell - n_{\ell-1}} \;\Big|\; \alpha_\ell \in \mathcal{F}^\ell_{\delta_\ell} \text{ for all } \ell = 1, \dots, t\Big\}. \end{aligned}$$

Then $\mathcal{F}$ is a t-$\star$-filtration w. r. t. $(n_0, \dots, n_t)$ on $\mathbb{N}^n$.

PROOF: (a) Let $\delta, \varepsilon \in \mathbb{N}^t$ with $\delta \leq_{\mathrm{p}} \varepsilon$ and let $\alpha \in \mathcal{F}_\delta$. By definition we have $\alpha_\ell \in \mathcal{F}^\ell_{\delta_\ell} \subseteq \mathcal{F}^\ell_{\varepsilon_\ell}$ for all $\ell = 1, \dots, t$, and thus $\alpha \in \mathcal{F}_\varepsilon$.

(b) Let $\delta, \varepsilon \in \mathbb{N}^t$, $\alpha \in \mathcal{F}_\delta$ and $\beta \in \mathcal{F}_\varepsilon$. Then $\alpha_\ell \in \mathcal{F}^\ell_{\delta_\ell}$ and $\beta_\ell \in \mathcal{F}^\ell_{\varepsilon_\ell}$ for all $\ell = 1, \dots, n$, and thus $(\alpha + \beta)_\ell = \alpha_\ell + \beta_\ell \in \mathcal{F}^\ell_{\delta_\ell} + \mathcal{F}^\ell_{\varepsilon_\ell} \subseteq \mathcal{F}^\ell_{\delta_\ell + \varepsilon_\ell}$, hence $\alpha + \beta \in \mathcal{F}_{\delta+\varepsilon}$.

(c) For $\ell = 1, \dots, t$ let $\alpha_\ell \in \mathcal{F}^\ell_0$. Then $(\alpha_1, \dots, \alpha_t) \in \mathcal{F}_0$.

(d) Let $j \in \{1, \dots, n\}$ and $\ell \in \{1, \dots, t\}$ with $j \in \{n_{\ell-1} + 1, n_\ell\}$. Let $p \in \{1, \dots, t\}$. We have to show that $(\mathrm{u}^n_j)_p \in \mathcal{F}^p_{(\mathrm{u}^t_\ell)_p}$. *Case 1: $p \neq \ell$.* Then $(\mathrm{u}^n_j)_p = 0 \in \mathcal{F}^p_0 \subseteq \mathcal{F}^p_{(\mathrm{u}^t_\ell)_p}$. *Case 2: $p = \ell$.* Then $(\mathrm{u}^n_j)_p = (\delta_{ij})_{i = n_{\ell-1}+1, \dots, n_\ell} \in \mathbb{N}^{n_\ell - n_{\ell-1}}$, and therefore we have $(\mathrm{u}^n_j)_p \in \mathcal{F}^p_1 = \mathcal{F}^p_{(\mathrm{u}^t_\ell)_\ell} = \mathcal{F}^p_{(\mathrm{u}^t_\ell)_p}$. q. e. d.

DEFINITION: Let $\mathcal{F}^\ell$ be a 1-$\star$-filtration on $\mathbb{N}^{n_\ell - n_{\ell-1}}$, $\ell = 1, \dots, t$. Then the t-$\star$-filtration $\mathcal{F}$ on $\mathbb{N}^n$ constructed from $\mathcal{F}^1, \dots, \mathcal{F}^t$ as in Lemma 2.9 is denoted by

$$\mathcal{F} = \prod_{\ell=1}^t \mathcal{F}^\ell = \mathcal{F}^1 \times \cdots \times \mathcal{F}^t.$$

By the phrase "$\mathcal{F}$ is a t-$\star$-filtration" we will always mean that $\mathcal{F} = \prod_{\ell=1}^t \mathcal{F}^\ell$ for $\star$-filtrations $\mathcal{F}^\ell$.

REMARK 2.10: Let $\mathcal{F}^\ell$ be a $\star$-filtration w. r. t. $(0, n_\ell - n_{\ell-1})$ on $\mathbb{N}^{n_\ell - n_{\ell-1}}$, $\ell = 1, \dots, t$, and let $\mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^t$ be a t-$\star$-filtration on $\mathbb{N}^n$. Then the following holds.

(a) We have $\mathcal{F}_0 = \{0\}$.

Proof: "$\subseteq$": Let $\alpha \in \mathcal{F}_0$ and $k \in \mathbb{N}$ be arbitrary. Then we have $k\alpha = \sum_{i=1}^k \alpha \in \sum_{i=1}^k \mathcal{F}_0 \subseteq \mathcal{F}_{k \cdot 0} = \mathcal{F}_0 \in \mathcal{P}_{\mathrm{f}}(\mathbb{N}^n)$. Therefore $k\alpha = k'\alpha$ for some $k' \in \mathbb{N}$, $k' \neq k$, and hence $(k - k')\alpha = k\alpha - k'\alpha = 0$. Since $k - k' \neq 0$ it follows that $\alpha = 0$. "$\supseteq$": Since $\mathcal{F}$ is a t-$\star$-filtration there is an $\alpha \in \mathcal{F}_0$ and since we have already shown that $\mathcal{F}_0 \subseteq \{0\}$, we have $\alpha = 0$. «

(b) For all $\delta \in \mathbb{N}^t$, if $\alpha \in \mathbb{N}^n$ with $\operatorname{tot\,deg}(\alpha) \leq \min\deg(\delta)$ we have $\alpha \in \mathcal{F}_\delta$.

Proof: We do induction on t. Let $t = 1$ and $\mathcal{F} = \mathcal{F}^1$ be a $\star$-filtration on $\mathbb{N}^n$. We have to show that for all $d \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$ with $\operatorname{tot\,deg}(\alpha) \leq d$ we have $\alpha \in \mathcal{F}_d$. We prove this by induction on d. For $d = 0$ we have $\alpha = 0 \in \mathcal{F}_0$ by part (a). Let $d \geq 1$ and assume inductively that for all $\beta \in \mathbb{N}^n$ with $\operatorname{tot\,deg}(\beta) \leq d-1$ we have $\beta \in \mathcal{F}_{d-1}$. Let w. l. o. g. $\alpha_1 \geq 1$. Then $\beta := \alpha - \mathrm{u}_1^n \in \mathbb{N}^n$ and $\operatorname{tot\,deg}(\beta) = \operatorname{tot\,deg}(\alpha) - 1 \leq d-1$. Thus, by induction hypothesis we have $\beta \in \mathcal{F}_{d-1}$. Therefore we have $\alpha = \beta + \mathrm{u}_1^n \in \mathcal{F}_{d-1} + \mathcal{F}_1 \subseteq \mathcal{F}_d$.

Now let $t \geq 2$, $\delta \in \mathbb{N}^t$ and $\alpha \in \mathbb{N}^n$ with $\operatorname{tot\,deg}(\alpha) \leq \min\deg(\delta)$. Let $\mathcal{F}' := \mathcal{F}^1 \times \cdots \times \mathcal{F}^{t-1}$. Then $\mathcal{F}'$ is a $(t-1)$-$\star$-filtration on $\mathbb{N}^k$ with $k := n_1 + \cdots + n_{t-1}$. Let $\delta' := (\delta_1, \ldots, \delta_{t-1}) \in \mathbb{N}^{t-1}$ and $\alpha' := (\alpha_1, \ldots, \alpha_{t-1}) \in \mathbb{N}^k$. Since $\operatorname{tot\,deg}(\alpha') \leq \operatorname{tot\,deg}(\alpha) \leq \min\deg(\delta) \leq \min\deg(\delta')$, we have by induction hypothesis that $\alpha' \in \mathcal{F}'_{\delta'}$ i. e. $\alpha_\ell = \alpha'_\ell \in \mathcal{F}^\ell_{\delta'_\ell} = \mathcal{F}^\ell_{\delta_\ell}$ for all $\ell = 1, \ldots, t-1$. Furthermore, we have $\alpha_t \leq \operatorname{tot\,deg}(\alpha) \leq \min\deg(\delta) \leq \delta_t$, and therefore $\alpha_t \in \mathcal{F}^t_{\delta_t}$ by induction hypothesis (or by the base case). Thus we have $\alpha_\ell \in \mathcal{F}^\ell_{\delta_\ell}$ for all $\ell = 1, \ldots, t$, that is, $\alpha \in \mathcal{F}_\delta$. «

(c) For all $\alpha \in \mathbb{N}^n$ we have $\alpha \in \mathcal{F}_\delta$ with $\delta := \operatorname{tot\,deg}(\alpha) \cdot (1, \ldots, 1)$. This follows immediately from part (b).

(d) The t-$\star$-filtration $\mathcal{F}$ is exhaustive, i. e., we have $\bigcup_{\delta \in \mathbb{N}^t} \mathcal{F}_\delta = \mathbb{N}^n$. This follows immediately from part (c).

(e) Let $\mathcal{F}^\ell, \mathcal{G}^\ell$ be $\star$-filtrations on $\mathbb{N}^{n_\ell - n_{\ell-1}}$ such that $\mathcal{F}^\ell_d \subseteq \mathcal{G}^\ell_d$ for all $\ell = 1, \ldots, t$ and $d \in \mathbb{N}$. Then we have $(\mathcal{F}^1 \times \cdots \times \mathcal{F}^t)_\delta \subseteq (\mathcal{G}^1 \times \cdots \times \mathcal{G}^t)_\delta$ for all $\delta \in \mathbb{N}^t$. This follows immediately from the definition.

(f) If $\mathcal{F}^\ell_d$ is a lower set[8] in $(\mathbb{N}^{n_\ell - n_{\ell-1}}, \leq_{\mathrm{p}})$ for all $d \in \mathbb{N}$, then $\mathcal{F}_\delta$ is a lower set in $(\mathbb{N}^n, \leq_{\mathrm{p}})$ for all $\delta \in \mathbb{N}^t$. This follows immediately from the definition.

(g) For a norm $\|\|\colon \mathbb{R}^n \to \mathbb{R}^{\geq 0}$, with $\|\mathrm{u}_j\| \leq 1$, $j = 1, \ldots, n$, let

$$\begin{aligned} \mathcal{F}^{\|\|}\colon\ \mathbb{N} &\longrightarrow \mathcal{P}_{\mathrm{f}}(\mathbb{N}^n), \\ d &\longmapsto \mathbb{N}^n \cap \widetilde{\mathrm{B}}^{\|\|}_d(0) = \{\alpha \in \mathbb{N}^n \mid \|\alpha\| \leq d\}. \end{aligned}$$

Then $\mathcal{F}^{\|\|}$ is a 1-$\star$-filtration on $\mathbb{N}^n$, called $\|\|$-*$\star$-filtration* or *$\star$-filtration induced by* $\|\|$. Note that $\mathcal{F}^{\|\|}_d$ is finite by equivalency of norms on $\mathbb{R}^n$, the remaining properties being trivial to check.

Definition: (a) Let $\|\|_1\colon \mathbb{R}^n \to \mathbb{R}^{\geq 0}$ be the 1-norm on $\mathbb{R}^n$ and let

$$\mathcal{T} := \mathcal{T}^n := \mathcal{F}^{\|\|_1}.$$

[8] In any partially ordered set $(X, \leq)$, $L \subseteq X$ is a *lower set* if for all $x \in X$, $y \in L$, $x \leq y$ implies $x \in L$.

The $\star$-filtration $\mathcal{T}^n$ on $\mathbb{N}^n$ is called *total degree $\star$-filtration* or *$\star$-filtration induced by total degree* on $\mathbb{N}^n$. By definition we have

$$\mathcal{T}_d^n = \{\alpha \in \mathbb{N}^n \mid \operatorname{tot}\deg(\alpha) \leq d\}$$

for all $d \in \mathbb{N}$.

(b) Let $\| \|_\infty \colon \mathbb{R}^n \to \mathbb{R}^{\geq 0}$ be the ∞-norm on $\mathbb{R}^n$ and let

$$\mathcal{M} := \mathcal{M}^n := \mathcal{F}^{\| \|_\infty}.$$

The $\star$-filtration $\mathcal{M}^n$ on $\mathbb{N}^n$ is called *maximal degree $\star$-filtration* or *$\star$-filtration induced by maximal degree* on $\mathbb{N}^n$. By definition we have

$$\mathcal{M}_d^n = \{\alpha \in \mathbb{N}^n \mid \max\deg(\alpha) \leq d\}$$

for all $d \in \mathbb{N}$.

Remark 2.11: (a) The exponent in $\mathcal{T}^n$ and $\mathcal{M}^n$ should not be confused with the notations $\prod_{\ell=1}^n \mathcal{T}^1$ or $\prod_{\ell=1}^n \mathcal{M}^1$. The former are 1-$\star$-filtrations on $\mathbb{N}^n$, whereas the latter are n-$\star$-filtration on $\mathbb{N}^n = \mathbb{N}^{1 \cdot n}$.

For illustration, we have $(\mathcal{T}^1 \times \mathcal{T}^2)_\delta = \{\alpha \in \mathbb{N}^3 \mid \alpha_1 \leq \delta_1 \text{ and } \alpha_2 + \alpha_3 \leq \delta_2\}$ for $\delta \in \mathbb{N}^2$.

(b) The total degree $\star$-filtration on $\mathbb{N}^n$ is the least among all $\star$-filtrations on $\mathbb{N}^n$, that is, if $\mathcal{F}$ is an arbitrary $\star$-filtration on $\mathbb{N}^n$, then $\mathcal{T}_d^n \subseteq \mathcal{F}_d$ for all $d \in \mathbb{N}$. This is an immediate consequence of Remark 2.10 (b).

(c) The t-$\star$-filtration $\mathcal{T}^{n_1 - n_0} \times \cdots \times \mathcal{T}^{n_t - n_{t-1}}$ w. r. t. $(n_0, \ldots, n_t)$ on $\mathbb{N}^n$ is the least among all t-$\star$-filtrations $\mathcal{F}^1 \times \cdots \times \mathcal{F}^t$ w. r. t. $(n_0, \ldots, n_t)$ on $\mathbb{N}^n$. This follows immediately from Remark 2.10 (e) and part (b).

(d) In view of part (c), one may note that there is no largest, and not even a maximal, t-$\star$-filtration. Indeed, if $\mathcal{F}$ is any t-$\star$-filtration w. r. t. $(n_0, \ldots, n_t)$ on $\mathbb{N}^n$, then $\mathcal{G} \colon \mathbb{N}^t \to \mathcal{P}_{\mathrm{f}}(\mathbb{N}^n)$, $\delta \mapsto \mathcal{F}_{2\delta}$, clearly is a t-$\star$-filtration w. r. t. $(n_0, \ldots, n_t)$ on $\mathbb{N}^n$ with $\mathcal{F}_\delta \subsetneqq \mathcal{G}_\delta$ for all $\delta \in \mathbb{N}^t \setminus \{0\}$.

(e) Note that we do not require the sets $\mathcal{F}_\delta$ of a t-$\star$-filtration $\mathcal{F}$ on $\mathbb{N}^n$ to be lower sets in $(\mathbb{N}^n, \leq_{\mathrm{p}})$, and indeed this does not have to be the case. For example, let $\mathcal{F}_0 := \{0\}$, $\mathcal{F}_1 := \mathcal{T}_1^n \cup \{3\mathrm{u}_1\}$, and $\mathcal{F}_d := \mathcal{F}_{d-1} + \mathcal{F}_1$ for $d \geq 2$. Clearly, $\mathcal{F}$ is a $\star$-filtration on $\mathbb{N}^n$.

We claim that for all $d \geq 1$, $3d\mathrm{u}_1 \in \mathcal{F}_d$ and $(3d-1)\mathrm{u}_1 \notin \mathcal{F}_d$. In particular, since $(3d-1)\mathrm{u}_1 \leq_{\mathrm{p}} 3d\mathrm{u}_1$, if $d \geq 1$ then $\mathcal{F}_d$ is not a lower set in $(\mathbb{N}^n, \leq_{\mathrm{p}})$.

We split the proof into two parts.

(1) We claim that $3d = \max\{\alpha_1 \mid \alpha \in \mathcal{F}_d\}$. Clearly, this holds for $d = 1$ and since $\mathcal{F}_1 = \mathcal{F}_0 + \mathcal{F}_1$, this can be taken as base case for an inductive argument.

Let $d \geq 2$ and inductively assume that $3(d-1) = \max\{\alpha_1 \mid \alpha \in \mathcal{F}_{d-1}\}$. Since $\mathcal{F}_d = \mathcal{F}_{d-1} + \mathcal{F}_1$, clearly $3d = 3(d-1) + 3 = \max\{\alpha_1 \mid \alpha \in \mathcal{F}_{d-1}\} + \max\{\alpha_1 \mid \alpha \in \mathcal{F}_1\} = \max\{\alpha_1 \mid \alpha \in \mathcal{F}_d\}$.

(2) We prove the main assertion. Note that $3\mathrm{u}_1 \in \mathcal{F}_1 = \mathcal{T}_1^n \cup \{3\mathrm{u}_1\}$ and $2\mathrm{u}_1 \notin \mathcal{F}_1$. Since $\mathcal{F}_1 = \mathcal{F}_0 + \mathcal{F}_1$, this can be taken as base case for an inductive argument. Let $d \geq 2$ and inductively assume that $3(d-1)\mathrm{u}_1 \in \mathcal{F}_{d-1}$ and $(3(d-1)-1)\mathrm{u}_1 \notin \mathcal{F}_{d-1}$. Then we have $3d\mathrm{u}_1 = 3(d-1)\mathrm{u}_1 + 3\mathrm{u}_1 \in \mathcal{F}_{d-1} + \mathcal{F}_1 = \mathcal{F}_d$. Assume that $(3d-1)\mathrm{u}_1 \in \mathcal{F}_d$. Since $\mathcal{F}_d = \mathcal{F}_{d-1} + \mathcal{F}_1$, there are $\alpha \in \mathcal{F}_{d-1}$ and $\beta \in \mathcal{F}_1$ with $(3d-1)\mathrm{u}_1 = \alpha + \beta$. Clearly, $\beta \in \{0, \mathrm{u}_1, 3\mathrm{u}_1\}$. If $\beta = 0$, then $(3d-1)\mathrm{u}_1 = \alpha \in \mathcal{F}_{d-1}$. This is a contradiction to part (1), since $((3d-1)\mathrm{u}_1)_1 = 3d-1 > 3d-3 = 3(d-1) = \max\{\alpha_1 \mid \alpha \in \mathcal{F}_{d-1}\}$. Thus, $\beta \in \{\mathrm{u}_1, 3\mathrm{u}_1\}$. If $\beta = \mathrm{u}_1$, then $(3d-2)\mathrm{u}_1 = (3d-1)\mathrm{u}_1 - \beta = \alpha \in \mathcal{F}_{d-1}$, and similarly to the previous case we arrive at $((3d-2)\mathrm{u}_1)_1 = 3d-2 > 3d-3 = 3(d-1) = \max\{\alpha_1 \mid \alpha \in \mathcal{F}_{d-1}\}$, contradicting part (1). Thus we have $\beta = 3\mathrm{u}_1$, which implies $(3(d-1)-1)\mathrm{u}_1 = 3d\mathrm{u}_1 - 3\mathrm{u}_1 - \mathrm{u}_1 = (3d-1)\mathrm{u}_1 - \beta = \alpha \in \mathcal{F}_{d-1}$, a contradiction to the induction hypothesis. Therefore, $(3d-1)\mathrm{u}_1 \notin \mathcal{F}_d$.

Since we will often work with t-$\star$-filtrations, we introduce some convenient abbreviations in the following definition.

Definition: Let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$. For $\delta \in \mathbb{N}^t$ we have

$$\mathrm{x}^{\mathcal{F}_\delta} = \Big\{ \prod_{\ell=1}^{t} (\mathrm{x}_{n_{\ell-1}+1}, \ldots, \mathrm{x}_{n_\ell})^{\alpha_\ell} \;\Big|\; \alpha \in \mathcal{F}_\delta \Big\} \subseteq \mathrm{Mon}^n$$

and let

$$S_\delta := S_{\mathcal{F}_\delta} = \langle \mathrm{x}^{\mathcal{F}_\delta} \rangle_A \leq S.$$

For $M \subseteq A^n$ let

$$\mathrm{ev}_\delta^M := \mathrm{ev}_{\mathcal{F}_\delta}^M = \mathrm{ev}^M \restriction S_{\mathcal{F}_\delta} : S_{\mathcal{F}_\delta} \to A^M,$$

be the restriction of ev^M to the free A-submodule S_δ of S,

$$\mathrm{I}_\delta(M) := \mathrm{I}_{\mathcal{F}_\delta}(M) = \ker(\mathrm{ev}_{\mathcal{F}_\delta}^M),$$

and

$$\mathrm{V}_\delta^M := \mathrm{V}_{\mathcal{F}_\delta}^M = (b^\alpha)_{\substack{b \in M \\ \alpha \in \mathcal{F}_\delta}} \in A^{M \times \mathcal{F}_\delta}.$$

Finally, for $f \in \mathrm{Exp}^n(A)$ let

$$\mathrm{H}_\delta(f) := \mathrm{H}_{\mathcal{F}_\delta}(f) = (f(\alpha+\beta))_{\substack{\alpha \in \mathcal{F}_\delta \\ \beta \in \mathcal{F}_\delta}} \in A^{\mathcal{F}_\delta \times \mathcal{F}_\delta}$$

The following Lemma 2.12 is well-known for the total degree $\star$-filtration, cf. e. g. Cox-Little-O'Shea [22, Chapter 5, § 3, Proposition 7] for a proof of the only non-trivial implication, which amounts to the base case for our induction.

Lemma 2.12 (Polynomial interpolation): *Let A be any ring with unit element $1 \neq 0$ and let $\mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^t$ be a t-$\star$-filtration w. r. t. $(n_0, \ldots, n_t)$ on $\mathbb{N}^n$. Then the following are equivalent.*

(i) *A is a field.*

(ii) *For all $M \in \mathcal{P}_\mathrm{f}(A^n)$, if $\delta \in \mathbb{N}^t$ and* $\min\deg(\delta) \geq |M| - 1$ *then* $\mathrm{ev}^M_{\mathcal{F}_\delta} \colon S_{\mathcal{F}_\delta} \to A^M$ *is surjective.*

(iii) *For all $M \in \mathcal{P}_\mathrm{f}(A^n)$ there is a $\delta \in \mathbb{N}^t$ such that $\mathrm{ev}^M_{\mathcal{F}_\delta} \colon S_{\mathcal{F}_\delta} \to A^M$ is surjective.*

(iv) *For all $M \in \mathcal{P}_\mathrm{f}(A^n)$, $\mathrm{ev}^M \colon S \to A^M$ is surjective.*

(v) *For all $b \in A \setminus \{0\}$, $\mathrm{ev}^{\{0,(b,\ldots,b)\}} \colon S \to A^2$ is surjective.*

Proof: (i) $\Rightarrow$ (ii): We do induction on t. Let $t = 1$. By Remark 2.11 (b) it is sufficient to prove this for $\mathcal{F}$ being the total degree $\star$-filtration $\mathcal{T} = \mathcal{T}^n$ on $\mathbb{N}^n$. Furthermore, it is sufficient to show that for every $b \in M$ there is a $p \in S_{\mathcal{T}_{|M|-1}}$, such that $\mathrm{f}_p \restriction M = \mathrm{ev}^M_{\mathcal{T}_{|M|-1}}(p) = \mathrm{u}_b$, i. e., such that $p(c) = \delta_{bc}$ for all $c \in M$. Let $b \in M$. Then for every $c \in M \setminus \{b\}$ there is a $j_c \in \{1, \ldots, n\}$ with $b_{j_c} \neq c_{j_c}$. By hypothesis we have that $b_{j_c} - c_{j_c} \in A \setminus \{0\}$ is a unit in the field A. Thus, for $c \in M \setminus \{b\}$ we have

$$q_c := \frac{1}{b_{j_c} - c_{j_c}} \cdot (\mathrm{x}_{j_c} - c_{j_c}) \in S.$$

Clearly we have $\operatorname{tot}\deg(q_c) = 1$, $q_c(b) = 1$, and $q_c(c) = 0$ for all $c \in M \setminus \{b\}$. Thus the product

$$p := \prod_{c \in M \setminus \{b\}} q_c \in S_{\mathcal{T}_{|M|-1}}$$

fulfills $p(c) = \delta_{bc}$ for all $c \in M$. This concludes the (standard) proof of the base case.

Now let $t \geq 2$. By Remark 2.11 (c) we may assume that $\mathcal{F}^\ell = \mathcal{T}^{n_\ell - n_{\ell-1}}$ is the total degree $\star$-filtration on $\mathbb{N}^{n_\ell - n_{\ell-1}}$. Let $b \in M$ and let $\mathcal{F}' := \mathcal{F}^1 \times \cdots \times \mathcal{F}^{t-1}$, $k := n_{t-1}$, and $M' := \{(c_1, \ldots, c_k) \mid c \in M\} \in \mathcal{P}_\mathrm{f}(A^k)$. Clearly $\mathcal{F}'$ is a $(t-1)$-$\star$-filtration w. r. t. $(n_0, \ldots, n_{t-1})$ on $\mathbb{N}^k$. Let $\delta' := (\delta_1, \ldots, \delta_{t-1}) \in \mathbb{N}^{t-1}$. Since $\min\deg(\delta') \geq \min\deg(\delta) \geq |M| - 1 \geq |M'| - 1$, we have by induction hypothesis that $\mathrm{ev}^{M'}_{\mathcal{F}'_{\delta'}}$ is surjective. Therefore there is a $p \in S_{\mathcal{F}'_{\delta'}}$ with $\mathrm{ev}^{M'}_{\mathcal{F}'_{\delta'}}(p) = (b_1, \ldots, b_k)$. Let $M_t := \{(c_{k+1}, \ldots, c_n) \mid c \in M\}$. Since $\delta_t \geq \min\deg(\delta) \geq |M| - 1 \geq |M_t| - 1$, we have by induction hypothesis (or by the base case) that $\mathrm{ev}^{M_t}_{\mathcal{F}^t_{\delta_t}}$ is surjective. Thus there is a $q \in S_{\mathcal{F}^t_{\delta_t}}$ with $\mathrm{ev}^{M_t}_{\mathcal{F}^t_{\delta_t}}(q) = (b_{k+1}, \ldots, b_n)$. Since we have $S_{\mathcal{F}'_{\delta'}} \cap S_{\mathcal{F}^t_{\delta_t}} = \{0\}$ and $M' \cap M_t = \emptyset$, clearly $p + q \in S_{\mathcal{F}_\delta}$ and $\mathrm{ev}^M_{\mathcal{F}_\delta}(p + q) = \mathrm{ev}^M_{\mathcal{F}_\delta}(p) + \mathrm{ev}^M_{\mathcal{F}_\delta}(q) = \mathrm{ev}^M_{\mathcal{F}'_{\delta'}}(p) + \mathrm{ev}^M_{\mathcal{F}^t_{\delta_t}}(q) = \mathrm{ev}^{M'}_{\mathcal{F}'_{\delta'}}(p) + \mathrm{ev}^{M_t}_{\mathcal{F}^t_{\delta_t}}(q) = (b_1, \ldots, b_k) + (b_{k+1}, \ldots, b_n) = b$.

(ii) $\Rightarrow$ (iii): Take $\delta := |M| \cdot (1, \ldots, 1)$.

(iii) $\Rightarrow$ (iv): There is nothing to show here.

(iv) $\Rightarrow$ (v): There is nothing to show here.

(v) $\Rightarrow$ (i): Let $b \in A \setminus \{0\}$. By hypothesis there is a $p \in S$ with $p(0) = 0$ and $p(b, \dots, b) = 1$. Since $p_0 = p(0) = 0$, we have $0 \notin \operatorname{supp}(p)$, i. e., $p = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} p_\alpha \mathrm{x}^\alpha$, thus $1 = p(b, \dots, b) = \sum_{\alpha \in \mathbb{N}^n \setminus \{0\}} p_\alpha b^{\alpha_1} \cdots b^{\alpha_n} \in \langle b \rangle_A$, hence $b \in \mathrm{U}(A)$. q. e. d.

For the case of $A = K$ we get the following corollary.

Corollary 2.13: *Let $\mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^t$ be a t-$\star$-filtration on $\mathbb{N}^n$, and let $f = \sum_{b \in M} f_b \exp_b \in \operatorname{Exp}^n(K)$ with $M \in \mathcal{P}_\mathrm{f}(K^n)$ and $(f_b)_{b \in M} \in K^M$. Let $\delta \in \mathbb{N}^t$ with $\min\deg(\delta) \geq |M| - 1$. Then*

$$\ker_K(\mathrm{V}^M_{\mathcal{F}_\delta}) = \ker_K(\mathrm{H}_{\mathcal{F}_\delta}(f)).$$

Proof: This follows immediately from Lemma 2.7 (b) and Lemma 2.12. q. e. d.

The following Lemma 2.14 provides a tool to prove the surjectivity of $\mathrm{ev}^M_{\mathcal{F}_\delta}$ for a given δ. This criterion will be applied in the proof of Theorem 2.15.

Lemma 2.14: *Let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$, $M \in \mathcal{P}_\mathrm{f}(A^n)$, and $\delta \in \mathbb{N}^t$ be arbitrary. If $\operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta}) = \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_{\delta+\mathrm{u}^t_j}})$ for all $j = 1, \dots, t$ then $\operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\varepsilon}) = \operatorname{im}(\mathrm{ev}^M)$ for all $\varepsilon \in \mathbb{N}^t$ with $\varepsilon \geq_\mathrm{p} \delta$.*

Proof: Clearly, $\operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta}) \subseteq \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\varepsilon}) \subseteq \operatorname{im}(\mathrm{ev}^M)$ holds. Since $S = A[\mathrm{x}_1, \dots, \mathrm{x}_n] = \langle \mathrm{x}^\alpha \mid \alpha \in \mathbb{N}^n \rangle_A$, we have $\operatorname{im}(\mathrm{ev}^M) = \langle (b^\alpha)_{b \in M} \mid \alpha \in \mathbb{N}^n \rangle_A$. We show $(b^\alpha)_{b \in M} \in \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta})$ for all $\alpha \in \mathbb{N}^n$ by induction on $\operatorname{tot\,deg}(\alpha)$. Let $\alpha \in \mathbb{N}^n$. For $\operatorname{tot\,deg}(\alpha) = 0$ we have $\alpha = 0$ and $(b^\alpha)_{b \in M} = (1)_{b \in M} = \mathrm{ev}^M_{\mathcal{F}_\delta}(1) \in \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta})$, since $1 = \mathrm{x}^0 \in \mathrm{x}^{\mathcal{F}_0} \subseteq \mathrm{x}^{\mathcal{F}_\delta}$. Let $\operatorname{tot\,deg}(\alpha) \geq 1$ and assume inductively that $(b^\beta)_{b \in M} \in \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta})$ for all $\beta \in \mathbb{N}^n$ with $\operatorname{tot\,deg}(\beta) = \operatorname{tot\,deg}(\alpha) - 1$. Without loss of generality let $\alpha_1 \geq 1$. Then we have $\beta := \alpha - \mathrm{u}^n_1 \in \mathbb{N}^n$. Since $\operatorname{tot\,deg}(\beta) = \operatorname{tot\,deg}(\alpha) - 1$, by induction hypothesis we have $(b^\beta)_{b \in M} \in \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta})$, hence there is a $p \in S_{\mathcal{F}_\delta}$ with $p(b) = b^\beta$ for all $b \in M$. Let $q := \mathrm{x}_1 \cdot p \in S$. Then $q(b) = b^{\mathrm{u}^n_1} b^\beta = b^\alpha$ for all $b \in M$. Since $\mathcal{F}$ is a t-$\star$-filtration, we have $q \in S_{\mathcal{F}_{\delta+\mathrm{u}^t_\ell}}$ for some $\ell \in \{1, \dots, t\}$. Thus we have $(b^\alpha)_{b \in M} \in \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_{\delta+\mathrm{u}^t_\ell}}) = \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta})$. q. e. d.

The following Theorem 2.15 is crucial for our generalization of Prony's method. Since we apply Lemma 2.12 in the proof, we have to switch to the field case.

Theorem 2.15: *Let $\mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^t$ be a t-$\star$-filtration on $\mathbb{N}^n$ and let $M \in \mathcal{P}_\mathrm{f}(K^n)$. If $\delta \in (\mathbb{N} \setminus \{0\})^t$ is such that $\mathrm{ev}^M_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} : S_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} \to K^M$ is surjective for all $\ell = 1, \dots, t$, then*

$$\mathrm{Z}(\mathrm{I}_{\mathcal{F}_\delta}(M)) = M.$$

Proof: The inclusion "$\supseteq$" is clear. To prove the reverse inclusion, let $b \in \mathrm{Z}(\mathrm{I}_\delta(M))$ and $N := M \cup \{b\}$. We show that $M = N$. We claim that $\mathrm{I}_\varepsilon(N) = \mathrm{I}_\varepsilon(M)$ for all $\varepsilon \leq_\mathrm{p} \delta$. The inclusion "$\subseteq$" is clear, since $M \subseteq N$. Let $p \in \mathrm{I}_\varepsilon(M)$. Since $\varepsilon \leq_\mathrm{p} \delta$, $p \in \mathrm{I}_\delta(M)$, so $p(b) = 0$, and the claim is proven. By the rank-nullity theorem, $\operatorname{rank}(\mathrm{ev}^N_\varepsilon) = \operatorname{rank}(\mathrm{ev}^M_\varepsilon)$

for all $\varepsilon \leq_{\mathrm{p}} \delta$. Therefore $\operatorname{rank}(\mathrm{ev}^N_{\delta-\mathrm{u}^t_\ell}) = \operatorname{rank}(\mathrm{ev}^M_{\delta-\mathrm{u}^t_\ell}) = |M| = \operatorname{rank}(\mathrm{ev}^M_\delta) = \operatorname{rank}(\mathrm{ev}^N_\delta)$, where we use the premise that $\mathrm{ev}^M_{\delta-\mathrm{u}^t_\ell}$ is surjective. By Lemma 2.14 we have $\operatorname{rank}(\mathrm{ev}^N_\varepsilon) = |M|$ for all $\varepsilon \geq \delta - \mathrm{u}^t_\ell$. By Lemma 2.12[9] we have $|N| = \operatorname{rank}(\mathrm{ev}^N_{|N|\cdot(1,\ldots,1)}) \leq |M|$, and thus $M = N$, i. e., $b \in M$. q. e. d.

Remark 2.16: If $A = K$ is a field, $M \in \mathcal{P}_{\mathrm{f}}(K^n)$, and $\mathcal{F} = \mathcal{F}^1 \times \cdots \times \mathcal{F}^t$ is a t-$\star$-filtration on $\mathbb{N}^n$, then $\operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta}) = \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_{\delta+\mathrm{u}^t_\ell}})$ for all $\ell = 1, \ldots, t$ implies that $\mathrm{ev}^M_{\mathcal{F}_\delta} \colon S_{\mathcal{F}_\delta} \to K^M$ is surjective. This is an immediate consequence of Lemma 2.14 and Lemma 2.12. Note further that the condition "$\operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_\delta}) = \operatorname{im}(\mathrm{ev}^M_{\mathcal{F}_{\delta+\mathrm{u}^t_\ell}})$" might be checked computationally by comparing the ranks of the matrices $\mathrm{V}^M_{\mathcal{F}_\delta}$ and $\mathrm{V}^M_{\mathcal{F}_{\delta+\mathrm{u}^t_\ell}}$.

Example 2.17: Let $A = K$ be a field and let $\mathcal{F}$ be a $\star$-filtration on $\mathbb{N}^n$. For an exponential sum $f = \sum_{b\in M} f_b \exp_b \in \operatorname{Exp}^n(K)$ with $M \in \mathcal{P}_{\mathrm{f}}(K^n)$ and $(f_b)_{b\in M} \in K^M$,

$$\operatorname{rank}(\mathrm{H}_d(f)) = \operatorname{rank}(\mathrm{H}_{d+1}(f))$$

does not imply

$$\operatorname{im}(\mathrm{ev}^M_d) = \operatorname{im}(\mathrm{ev}^M_{d+1}),$$

i. e., the surjectivity of $\mathrm{ev}^M_d \colon S_d \to K^M$. An abstract reason for this is clear: This would yield an algorithm to construct a finite subset $F \subseteq \mathbb{N}^n$ such that $f \restriction F$ determined f, contradicting footnote 2 on page 17.

An explicit counterexample is given in Sauer [67, Remark 3]. We give another counterexample that is based on the technique in the cited footnote. Let K be a field and $p \in \mathbb{Z}$ be any prime number with $\operatorname{char}(K) \neq p$ and such that the image of p in K is not a root of unity (e. g. $K = \mathbb{Q}$, $p = 2$), and for $k \in \mathbb{N}$ let

$$\begin{aligned} f_k \colon \mathbb{N} &\longrightarrow K, \\ \alpha &\longmapsto (1/\exp_p(k)) \cdot \exp_p(\alpha) - \exp_1(\alpha). \end{aligned}$$

Then $f_k \in \operatorname{Exp}^1(K)$, $f_k(k) = 1 - 1 = 0$, and $f_k(\alpha) = p^{\alpha-k} - 1 \neq 0$ for $\alpha \in \mathbb{N} \setminus \{k\}$. Let $e \in \mathbb{N}$ be arbitrary. Then the product

$$f_{(e)} := \prod_{k=0}^{2e} f_k \in \operatorname{Exp}^1(K)$$

(cf. Example 2.4 (a)) fulfills $f_{(e)}(\alpha) = 0$ for $\alpha = 0, \ldots, 2e$ and $f_{(e)}(2e+1) \neq 0$. Therefore

$$\mathrm{H}_{\mathcal{T}^1_d}(f_{(e)}) = \big(f_{(e)}(\alpha+\beta)\big)_{\substack{\alpha=0,\ldots,d\\ \beta=0,\ldots,d}} = 0$$

for all $d \leq e$. Clearly $\operatorname{rank}\big(f_{(e)}\big) \geq 2$, so ev^M_0 cannot be surjective for $M \in \mathcal{P}_{\mathrm{f}}(K)$ with $f_{(e)} = \sum_{b\in M} f_b \exp_b$.

[9] This is the only place in the proof where the hypothesis that K is a field and not merely an integral domain is crucial.

Remark 2.18: Under the hypothesis of Theorem 2.15 we have in particular

$$\langle \mathrm{I}_{\mathcal{F}_\delta}(M)\rangle_S \subseteq \mathrm{rad}(\langle \mathrm{I}_{\mathcal{F}_\delta}(M)\rangle_S) \subseteq \mathrm{I}(\mathrm{Z}(\mathrm{I}_{\mathcal{F}_\delta}(M))) = \mathrm{I}(M).$$

If K is algebraically closed, then by Hilbert's Nullstellensatz the second inclusion is actually an equality, that is

$$\mathrm{rad}(\langle \mathrm{I}_{\mathcal{F}_\delta}(M)\rangle_S) = \mathrm{I}(M).$$

Questions that arise here are in particular:

(Q_1) What can be said in the case of non-algebraically closed fields K?

(Q_2) Is $\mathrm{I}(M)$ generated by $\mathrm{I}_{\mathcal{F}_\delta}(M)$, i. e., is $\langle \mathrm{I}_{\mathcal{F}_\delta}(M)\rangle_S$ a radical ideal in S?

(Q_3) Which ideal-theoretic properties does $\langle \mathrm{I}_{\mathcal{F}_\delta}(M)\rangle_S$ have in general?

For the total degree $\star$-filtration $\mathcal{F} = \mathcal{T}^n$ on $\mathbb{N}^n$, questions (Q_1) and (Q_2) are answered in Theorem 2.48, showing that (under the above surjectivity condition on $\mathrm{ev}^M_{\mathcal{T}^n_{d-1}}$) over an arbitrary field K, $\langle \mathrm{I}_{\mathcal{T}^n_d}(M)\rangle_S = \mathrm{I}(M)$.

The following Corollary 2.19 constitutes a generalization of Prony's method.

Corollary 2.19 (Prony's method for $\mathrm{Exp}^n(K)$): *Let K be a field and let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$. Let $f = \sum_{b\in M} f_b \exp_b \in \mathrm{Exp}^n(K)$ with $M \in \mathcal{P}_{\mathrm{f}}(K^n)$ and $f_b \in K \setminus \{0\}$, and let $\delta \in (\mathbb{N}\setminus\{0\})^t$ be such that $\mathrm{ev}^M_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} : S_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} \to K^M$ is surjective for all $\ell = 1, \ldots, t$. Then we have*

$$\mathrm{Z}(\ker_K \mathrm{H}_{\mathcal{F}_\delta}(f)) = M.$$

Proof: Since $\mathrm{ev}^M_{\delta-\mathrm{u}^t_\ell} : S_{\delta-\mathrm{u}^t_\ell} \to K^M$ is surjective, also $\mathrm{ev}^M_\delta : S_\delta \to K^M$ is surjective. Therefore we have $\ker_K(\mathrm{H}_\delta(f)) = \ker_K(\mathrm{V}^M_\delta)$ by Lemma 2.7 (b). Thus, by Theorem 2.15, we arrive at $\mathrm{Z}(\ker_K \mathrm{H}_\delta(f)) = \mathrm{Z}(\ker_K \mathrm{V}^M_\delta) = \mathrm{Z}(\mathrm{I}_\delta(M)) = M$. q. e. d.

Under the hypothesis of Corollary 2.19 it cannot be concluded that $\mathrm{Z}(\ker_K \mathrm{H}_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}}(f)) = M$, even for 1-$\star$-filtrations $\mathcal{F}$, cf. Example 2.27 (b) (1) and Example 2.27 (d).

Corollary 2.20 (Trivial degree bound for Prony's method): *Let K be a field and let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$. Let $f = \sum_{b\in M} f_b \exp_b \in \mathrm{Exp}^n(K)$ with $M \in \mathcal{P}_{\mathrm{f}}(K^n)$ and $\mathrm{rank}(f) = |M|$, and let $\delta := \mathrm{rank}(f)\cdot(1,\ldots,1) \in \mathbb{N}^t$. Then*

$$\mathrm{Z}(\ker_K \mathrm{H}_{\mathcal{F}_\delta}(f)) = M.$$

Proof: If $\mathrm{rank}(f) = 0$ then $f = 0$ and we have $\mathrm{Z}(\ker \mathrm{H}_\delta(f)) = \mathrm{Z}(1) = \emptyset = M$. If $\mathrm{rank}(f) \neq 0$ then $\mathrm{ev}^M_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} : S_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} \to K^M$ is surjective by Lemma 2.12, so the assertion follows from Corollary 2.19. q. e. d.

Remark 2.21: If $B \subseteq A^n$ is finite and $f \in \operatorname{Exp}_B^n(A)$, then $\operatorname{rank}(f) \leq |B|$ (by Remark 2.2 (b)). In particular, if A is a finite field and $f = \sum_{b \in M} f_b \exp_b \in \operatorname{Exp}^n(K)$, then with $\delta := |K^n| \cdot (1, \ldots, 1) = |K|^n \cdot (1, \ldots, 1)$, we have $\mathrm{Z}(\ker \mathrm{H}_{\mathcal{F}_\delta}(f)) = M$ by Corollary 2.20.

Corollary 2.22: *Let $B \subseteq A^n$ be arbitrary. Then the set $\{\exp_b \mid b \in B\}$ is a K-vector space basis of $\operatorname{Exp}_B^n(A)$. In particular, we have*

$$\dim_K(\operatorname{Exp}_B^n(A)) = |B|.$$

Proof: Let $M \in \mathcal{P}_\mathrm{f}(B)$. We have to show that the set $\{\exp_b \mid b \in M\}$ is K-linearly independent. Let $(f_b)_{b \in M} \in K^M$ be such that $f := \sum_{b \in M} f_b \exp_b = 0$ and let $M_0 := \{b \in M \mid f_b \neq 0\}$. Let $Q := \operatorname{Quot}(A)$ and $d := |M_0|$. Then Corollary 2.20 implies that $M_0 = \mathrm{Z}(\ker_Q \mathrm{H}_{\mathcal{T}_d^n}(f)) = \mathrm{Z}(\ker_Q 0) = \mathrm{Z}(1) = \emptyset$. Hence, $f_b = 0$ for all $b \in M$. q. e. d.

Corollary 2.22 allows us to introduce the following definition.

Definition: Let $f \in \operatorname{Exp}^n(A)$.

(a) By Corollary 2.22 there is a unique $M \in \mathcal{P}_\mathrm{f}(A^n)$ with $f = \sum_{b \in M} \lambda_b \exp_b$ for some $\lambda \in (K \setminus \{0\})^M$. We call M the *support* of f, denoted by

$$\operatorname{supp}(f).$$

(b) The vector of non-zero coefficients of f w. r. t. the basis $\{\exp_b \mid b \in A^n\}$ of $\operatorname{Exp}^n(A)$ is denoted by

$$\operatorname{coeff}(f) \in K^{\operatorname{supp}(f)}$$

and called the *coefficient vector* of f.

We show by standard arguments that $\operatorname{supp}(f)$ (and thus also $\operatorname{rank}(f)$ and $\operatorname{coeff}(f)$) is independent of $B \subseteq A^n$ with $f \in \operatorname{Exp}_B^n(A)$ in the following remark.

Remark 2.23: (a) Let $B \subseteq A^n$ be a subset. If $f \in \operatorname{Exp}_B^n(A)$, then $\operatorname{supp}(f) \subseteq B$.

Proof: Since $f \in \operatorname{Exp}_B^n(A)$, there is an $M \in \mathcal{P}_\mathrm{f}(B)$ with $f = \sum_{b \in M} \lambda_b \exp_b$ for some $\lambda \in (K \setminus \{0\})^M$. Since $\operatorname{Exp}_B^n(A) \leq \operatorname{Exp}^n(A)$, by definition we have $\operatorname{supp}(f) = M \subseteq B$. «

(b) For all $f \in \operatorname{Exp}^n(A)$ we have

$$\operatorname{rank}(f) = |\operatorname{supp}(f)|.$$

In particular, the rank of f is independent of $B \subseteq A^n$ with $f \in \operatorname{Exp}_B^n(A)$, retroactively justifying the notation.

Proof: "$\leq$": By definition we have $f \in \operatorname{Exp}_{\operatorname{supp}(f)}^n(A)$. Thus we have $\operatorname{rank}(f) \leq |\operatorname{supp}(f)|$ by definition of $\operatorname{rank}(f)$.

"$\geq$": Let $M \subseteq B$ be arbitrary with $f \in \operatorname{Exp}_M^n(A)$. By part (a), we have $\operatorname{supp}(f) \subseteq M$. Therefore we have $|\operatorname{supp}(f)| \leq \operatorname{rank}(f)$. «

Remark 2.24: For an exponential sum $f \in \mathrm{Exp}^n(A)$, there is the following way to reconstruct $\mathrm{supp}(f)$ and $\mathrm{coeff}(f)$ that is justified by the preceding theory.

(1) Choose a t-$\star$-filtration on $\mathbb{N}^n$ and guess a $\delta \in (\mathbb{N} \setminus \{0\})^t$ such that the evaluation homomorphism $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} : S_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} \to A^{\mathrm{supp}(f)}$ is surjective. The choice $\delta = \max\{1, \mathrm{rank}(f)\} \cdot (1, \dots, 1)$ always works.

(2) Compute $f \restriction (\mathcal{F}_\delta + \mathcal{F}_\delta)$ and arrange the values into the matrix $\mathrm{H}_\delta(f) \in A^{\mathcal{F}_\delta \times \mathcal{F}_\delta}$.

(3) Compute a generating set E for $\ker_A(\mathrm{H}_\delta(f))$ (or of $\ker_Q(\mathrm{H}_\delta(f))$, $Q := \mathrm{Quot}(A)$).

(4) Compute the zero locus Z of E (over A or over Q).

(5) Compute the unique solution x of the system of linear equations $\left(\mathrm{V}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}}\right)^\top \cdot x = (f(\alpha))_{\alpha \in \mathcal{F}_{\delta-\mathrm{u}^t_\ell}}$. The computation may be performed over any convenient field containing A as a subring, the unique solution x necessarily being in $K^{\mathrm{supp}(f)}$.

(6) Output: $Z = \mathrm{supp}(f)$ and $x = \mathrm{coeff}(f)$.

Needless to say, computation of the zero locus Z is a major problem in itself, even for univariate exponential sums.

A simple algorithmic formulation is given in Algorithm 2.1.

Data: $f \restriction (\mathcal{F}_\delta + \mathcal{F}_\delta)$ for $f \in \mathrm{Exp}^n(A)$ and $\delta \in (\mathbb{N} \setminus \{0\})^t$ with $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}}$ surjective.
Result: $\mathrm{rank}(f)$, $\mathrm{supp}(f)$, and $\mathrm{coeff}(f)$.
Compute Q-basis E of $\ker(\mathrm{H}_{\mathcal{F}_\delta}(f))$;
Compute $\mathrm{supp}(f) = \mathrm{Z}(E)$;
Compute $\mathrm{rank}(f) = |\mathrm{supp}(f)|$;
Compute unique solution $\mathrm{coeff}(f)$ of $\left(\mathrm{V}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_1}}\right)^\top \cdot \mathrm{coeff}(f) = (f(\alpha))_{\alpha \in \mathcal{F}_{\delta-\mathrm{u}^t_1}}$;

Algorithm 2.1: Prony's method for $\mathrm{Exp}^n(A)$.

Parts (b) and (c) of the following Remark 2.25 are well-known. They are of interest in this context and we give short proofs based on standard facts for the convenience of the reader.

Remark 2.25: Let K be a field. Then the following holds.

(a) Let $f \in \mathrm{Exp}^n(K)$ and $d \in \mathbb{N} \setminus \{0\}$ be such that $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{T}_{d-1}}$ is surjective. Then clearly $\mathrm{rank}(f) \leq |\mathrm{supp}(f)| = |\mathcal{T}_{d-1}| = \binom{n+d}{d}$.

(b) Let K be algebraically closed, $n \geq 2$, and $p_1, \dots, p_k \in S := K[\mathrm{x}_1, \dots, \mathrm{x}_n]$. If $\mathrm{Z}(p_1, \dots, p_k)$ is finite, then $\gcd(p_1, \dots, p_k) = 1$.

Proof: If $\gcd(p_1, \dots, p_k) \neq 1$, then there is a $p \in S \setminus K$ such that $p \mid p_1, \dots, p_k$. Since K is algebraically closed and $n \geq 2$, we have $|\mathrm{Z}(p)| = \infty$, and since $\mathrm{Z}(p) \subseteq \mathrm{Z}(p_1, \dots, p_k)$, we have $|\mathrm{Z}(p_1, \dots, p_k)| = \infty$. «

(c) Let $p_1, \dots, p_k \in S := K[\mathrm{x}, \mathrm{y}]$. If $\gcd(p_1, \dots, p_k) = 1$, then $\mathrm{Z}(p_1, \dots, p_k)$ is finite.

Proof: Let $I := \langle p_1, \dots, p_k \rangle_S$. We denote the Krull dimension of a ring R by $\operatorname{Krull\,dim}(R)$. We show that $\dim(\mathrm{Z}(I)) = \operatorname{Krull\,dim}(S/I) = 0$. By Krull's principal ideal theorem we have $\operatorname{Krull\,dim}(S) = 2$. Since $\operatorname{Krull\,dim}(S/I) \leq \operatorname{Krull\,dim}(S) - \operatorname{ht}(I)$, it is sufficient to show that $\operatorname{ht}(I) \geq 2$. Since $\gcd(p_1, \dots, p_k) = 1$, we have in particular $I \neq \{0\}$, so $\operatorname{ht}(I) \geq 1$. Suppose $\operatorname{ht}(I) = 1$. Then there is a prime ideal P of S with $I \subseteq P$ and $\operatorname{ht}(P) = 1$. Since S is factorial we have $P = \langle p \rangle_S$ for some prime element $p \in S$, and thus $p \mid p_1, \dots, p_k$, a contradiction. Therefore we have $\operatorname{ht}(I) \geq 2$, i. e. $\dim(\mathrm{Z}(I)) = 0$ and thus $\mathrm{Z}(I)$ is finite. «

We conclude the section by mentioning a well-known application to reconstruction of multivariate polynomials.

Remark 2.26: Any reconstruction method for multivariate exponential sums also yields an approach for reconstructing multivariate polynomials. To see this, let $p \in A[\mathrm{x}_1, \dots, \mathrm{x}_n]$ and choose any $b \in A^n$. Then consider the exponential sum

$$\begin{aligned} f_{p,b}\colon \mathbb{N}^n &\longrightarrow A, \\ \alpha &\longmapsto p(b_1^{\alpha_1}, \dots, b_n^{\alpha_n}) = \sum_{\beta \in \operatorname{supp}(p)} p_\beta \cdot (b_1^{\beta_1}, \dots, b_n^{\beta_n})^\alpha. \end{aligned}$$

If b is such that $(b_1^{\beta_1}, \dots, b_n^{\beta_n}) \neq (b_1^{\gamma_1}, \dots, b_n^{\gamma_n})$ for all $\beta, \gamma \in \operatorname{supp}(p)$ with $\beta \neq \gamma$,[10] then $\operatorname{supp}(f_{p,b}) = \{(b_1^{\beta_1}, \dots, b_n^{\beta_n}) \mid \beta \in \operatorname{supp}(p)\}$, and thus also $\operatorname{supp}(p)$, and $\operatorname{coeff}(f_{p,b}) = (p_\beta)_{\beta \in \operatorname{supp}(p)}$ may in principle be recovered with the help of Prony's method.

2.3. Computational examples

In this section, the theory from Section 2.1 is illustrated by means of several computational examples. Of course, in some cases the theory already implies the result, but we also want to assume the perspective of a scientist who applies Prony's method in order to reconstruct exponential sums with unknown support.

Example 2.27: (a) This is the simplest possible case. Let $f := 0 \in \operatorname{Exp}^n(A)$. Then, for any $D \in \mathcal{P}_\mathrm{f}(\mathbb{N}^n)$ with $0 \in D$, we have $\mathrm{H}_D(f) = 0 \in A^{D \times D}$ and $\ker(\mathrm{H}_D(0)) = A^D \cong A[\mathrm{x}_1, \dots, \mathrm{x}_n]_D \ni \mathrm{x}^0 = 1$, and thus $\mathrm{Z}(\ker \mathrm{H}_D(0)) = \mathrm{Z}(1) = \emptyset = \operatorname{supp}(f)$. This confirms by explicit computation what had been proven before, namely that Prony's method works in particular for $f = 0 \in \operatorname{Exp}^n(A)$. Note that for any $D \subseteq \mathbb{N}^n$, $\operatorname{ev}_D^{\operatorname{supp}(f)}\colon S_D \to A^{\operatorname{supp}(f)} = \{0\}$ is surjective.

(b) Let $b \in A^n$, $\lambda \in K \setminus \{0\}$, and consider the exponential sum $f := \lambda \exp_b \in \operatorname{Exp}^n(A)$. Clearly, $\operatorname{supp}(f) = \{b\}$, and we will reconstruct this set from samples of f below in part (2). We work with the total degree $\star$-filtration $\mathcal{F} = \mathcal{T}^n$ on $\mathbb{N}^n$.

[10] Since A is an integral domain, for $\beta_j > \gamma_j$, $b_j^{\beta_j} = b_j^{\gamma_j}$ implies $b_j = 0$ or $b_j^{\beta_j - \gamma_j} = 1$. Thus, for example for $A = \mathbb{C}$, choosing $b_j \neq 0$ to not be a root of unity for all $j = 1, \dots, n$ always works.

First, in part (1) we demonstrate how Prony's method can fail if the degree $d \in \mathbb{N}$ is chosen too small.

Note that by Corollary 2.20, $d = 1$ is sufficient for Prony's reconstruction method and that the theory makes no statement for $d = 0$.

(1) Let $d := 0 \in \mathbb{N}$. We have $\mathrm{H}_0(f) = (f(\alpha+\beta))_{\alpha,\beta\in\mathcal{T}_0^n} = (f(0)) = (\lambda) \in A^{1\times 1}$. Since $\lambda \neq 0$, $\ker(\mathrm{H}_0(f)) = \{0\}$, and we have

$$\mathrm{Z}(\ker \mathrm{H}_0(f)) = \mathrm{Z}(0) = A^n \supsetneqq \operatorname{supp}(f).$$

In particular, this is a case where the zero locus of $\ker(\mathrm{H}_d(f))$ is not a zero-dimensional algebraic variety.

(2) Let $d := 1 \in \mathbb{N}$. Ordering the elements of $\mathcal{T}_1^n$ as $0 < \mathrm{u}_1 < \mathrm{u}_2 < \cdots < \mathrm{u}_n$ and setting $\mathrm{u}_0 := 0 \in \mathbb{N}^n$ and $b_0 := 1 \in A$, we have

$$\mathrm{H}_1(f) = (f(\alpha+\beta))_{\alpha,\beta\in\mathcal{T}_1^n} = (f(\mathrm{u}_i + \mathrm{u}_j))_{\substack{i=0,\dots,n\\ j=0,\dots,n}} = (\lambda b_i b_j)_{\substack{i=0,\dots,n\\ j=0,\dots,n}}.$$

Clearly, row i of $\mathrm{H}_1(f)$ is equal to b_i multiplied by row 0, i. e., $(\lambda b_i b_j)_{j=0,\dots,n} = b_i \cdot (\lambda b_j)_{j=0,\dots,n}$. Therefore it is easy to see that $\ker_Q(\mathrm{H}_1(f))$ is generated by the set $\{(-b_i, \mathrm{u}_i^\top)^\top \mid i = 1,\dots,n\}$. By our choice of ordering the elements of $\mathcal{T}_1^n$, this corresponds to the set of polynomials $\{\mathrm{x}_i - b_i \mid i = 1,\dots,n\}$. Thus the support of f is computed as

$$\operatorname{supp}(f) = \mathrm{Z}(\ker_Q \mathrm{H}_1(f)) = \mathrm{Z}(\{\mathrm{x}_i - b_i \mid i = 1,\dots,n\}) = \{b\}.$$

To compute the coefficient of $\exp_b$ in f, solve the system of linear equations (with the unique solution x being in $K^{\operatorname{rank}(f)} = K$)

$$1 \cdot x = b^0 \cdot x = (\mathrm{V}_0^{\{b\}})^\top \cdot x = (f(\alpha))_{\alpha\in\mathcal{T}_0^n} = f(0) = \lambda,$$

which yields $x = \lambda$. We have thus "verified", and illustrated, by explicit computations, Prony's method for exponential sums of rank 1.

(c) In these examples we have $K = \mathbb{R}$ and $A = \mathbb{R}^2$, with computations being performed in floating point arithmetic. Therefore, in particular, these examples should be (ever so slightly) closer to a real world engineering application of Prony's method. Evaluation of exponential sums and computation of the polynomials is performed by the software `Octave` [34] and we use the polynomial equation solver `Bertini` [7] to compute zero loci.

Let $r \in \mathbb{N}$ and for $i = 1,\dots,r$ let $a_i := 2\pi(i-1)/r \in [0, 2\pi[$. Let

$$b_i := (\cos(a_i), \sin(a_i))^\top \in \mathbb{S}^1 \subseteq \mathbb{R}^2.$$

Consider the exponential sums

$$f_r := \sum_{i=1}^{r} \exp_{b_i} \in \operatorname{Exp}^2_{\mathbb{S}^1}(\mathbb{R})$$

	r	d	success	time (s)	comment
$f_r = g_r$	0	0	yes	0.011 616	$E = \{1\}$
$f_r = g_r$	1	1	yes	0.028 602 1	
f_r	2	2	yes	0.066 974 2	
g_r	2	2	yes	0.052 80 71	
f_r	3	2	yes	0.044 340 8	
g_r	3	2	yes	0.043 985 1	
f_r	10	3	no	8.855 64	7 non-real solutions
g_r	10	3	no	11.612 1	7 non-real solutions
f_r	10	4	no	46.252 6	9 non-real solutions
g_r	10	4	no	280.438	6 non-real solutions
f_r	10	5	yes	2.541 69	
g_r	10	5	yes	1.398 33	
f_r	25	5	no	374.965	11 non-real solutions
f_r	25	6	no	246 1.54	4 non-real solutions
f_r	50	8	no	209 37.2	no solutions found
f_r	100	5	no	727.607	7 non-real solutions
f_r	100	10	no	710 85.2	no solutions found
f_r	100	13	no	349 070	no solutions found

Table 2.1.: Results of Example 2.27 (c).

and

$$g_r := \sum_{i=1}^{r} \frac{1}{i} \exp_{b_i} \in \mathrm{Exp}^2_{\mathbb{S}^1}(\mathbb{R}).$$

Choosing $d \in \mathbb{N}$, evaluating f_r in floating point arithmetic and computing a $\mathbb{C}$-basis E of $\ker_{\mathbb{C}} \mathrm{H}_{\mathcal{T}_d}(f_r)$ using `Octave`, and afterwards computing $\mathrm{Z}(E)$ by giving E as input to `Bertini` (and analogously for g_r) yields the results presented in Table 2.1. It should be noted that the times for `Bertini` and also the produced solutions can vary considerably, presumably due to randomization techniques.

(d) This is an illustration of possible “spurious roots”, i. e., $\mathrm{Z}(\ker \mathrm{H}_d(f))$ is finite and $\mathrm{supp}(f) \subsetneqq \mathrm{Z}(\ker \mathrm{H}_d(f))$, with floating point arithmetic. Consider the exponential sum

$$f := \sum_{i=1}^{8} \exp_{b_i} \in \mathrm{Exp}^2_{\mathbb{T}^2}(\mathbb{C})$$

with

$$b_i := (\mathrm{e}^{\mathrm{i}\pi(\cos(2\pi(i-1)/7)+1)}, \mathrm{e}^{\mathrm{i}\pi(\sin(2\pi(i-1)/7)+1)}),$$

$i = 1, \ldots, 7$, and

$$b_8 := (\mathrm{e}^0, \mathrm{e}^0) = (1, 1).$$

The points $b_1, \dots, b_7$ are in the image of a circle under the map $[0, 2\pi[^2 \to \mathbb{T}^2$, $\varphi \mapsto (\mathrm{e}^{\mathrm{i}\varphi_1}, \mathrm{e}^{\mathrm{i}\varphi_2})$. Numerical computation with `Octave` and `Bertini` yields the following: Start with the floating point approximations

$$\begin{aligned}
\widetilde{b}_1 &:= (\;\;\, 1.000\,000 - 0.000\,000 \cdot \mathrm{i}, \quad -1.000\,000 + 0.000\,000 \cdot \mathrm{i}),\\
\widetilde{b}_2 &:= (\;\;\, 0.378\,296 - 0.925\,685 \cdot \mathrm{i}, \quad\;\;\, 0.774\,168 - 0.632\,980 \cdot \mathrm{i}),\\
\widetilde{b}_3 &:= (-0.765\,441 + 0.643\,506 \cdot \mathrm{i}, \quad\;\;\, 0.996\,900 - 0.078\,685 \cdot \mathrm{i}),\\
\widetilde{b}_4 &:= (\;\;\, 0.951\,993 + 0.306\,121 \cdot \mathrm{i}, \quad -0.206\,220 - 0.978\,506 \cdot \mathrm{i}),\\
\widetilde{b}_5 &:= (\;\;\, 0.951\,993 + 0.306\,121 \cdot \mathrm{i}, \quad -0.206\,220 + 0.978\,506 \cdot \mathrm{i}),\\
\widetilde{b}_6 &:= (-0.765\,441 + 0.643\,506 \cdot \mathrm{i}, \quad\;\;\, 0.996\,900 + 0.078\,685 \cdot \mathrm{i}),\\
\widetilde{b}_7 &:= (\;\;\, 0.378\,296 - 0.925\,685 \cdot \mathrm{i}, \quad\;\;\, 0.774\,168 + 0.632\,980 \cdot \mathrm{i}),\\
\widetilde{b}_8 &:= (\;\;\, 1.000\,000 + 0.000\,000 \cdot \mathrm{i}, \quad\;\;\, 1.000\,000 + 0.000\,000 \cdot \mathrm{i}),\\
\widetilde{f}_i &:= 1.000\,000 + 0.000\,000 \cdot \mathrm{i}, \text{ for } i = 1, \dots, 8.
\end{aligned}$$

We have $\mathrm{rank}(\mathrm{V}_{\mathcal{T}_2^2}^{\mathrm{supp}(f)}) = 6 < 8 = \mathrm{rank}(f)$ (numerically), so the theory makes no statement if Prony's method succeeds with $\mathcal{F} = \mathcal{T}$ and $d = 3$. Working with the matrix $\widetilde{\mathrm{H}}_{\mathcal{T}_3^2}(\widetilde{f})$ (the tilde indicates that all occurring computations are done approximately by `Octave`), one obtains the system of polynomials

$$\begin{aligned}
\widetilde{p}_1 := \;\; &(\;\;\, 0.417\,926\,345\,097\,310\,1 + 0.000\,000\,000\,000\,000\,0 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^0\\
+ &(-0.314\,828\,760\,396\,303\,3 + 0.025\,658\,856\,568\,843\,6 \cdot \mathrm{i}) \cdot \mathrm{x}^1\mathrm{y}^0\\
+ &(-0.071\,016\,862\,045\,126\,7 - 0.072\,711\,423\,202\,379\,6 \cdot \mathrm{i}) \cdot \mathrm{x}^2\mathrm{y}^0\\
+ &(-0.060\,256\,537\,391\,368\,7 + 0.086\,048\,064\,682\,210\,8 \cdot \mathrm{i}) \cdot \mathrm{x}^3\mathrm{y}^0\\
+ &(-0.508\,737\,457\,270\,699\,5 + 0.083\,044\,268\,831\,143\,4 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^1\\
+ &(\;\;\, 0.474\,026\,912\,961\,756\,6 - 0.121\,195\,895\,075\,547\,6 \cdot \mathrm{i}) \cdot \mathrm{x}^1\mathrm{y}^1\\
+ &(\;\;\, 0.002\,233\,370\,725\,909\,7 - 0.080\,488\,055\,591\,724\,5 \cdot \mathrm{i}) \cdot \mathrm{x}^2\mathrm{y}^1\\
+ &(\;\;\, 0.276\,305\,574\,964\,078\,2 - 0.158\,798\,754\,094\,867\,3 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^2\\
+ &(-0.248\,129\,760\,228\,589\,9 + 0.119\,803\,256\,046\,192\,0 \cdot \mathrm{i}) \cdot \mathrm{x}^1\mathrm{y}^2\\
+ &(\;\;\, 0.032\,477\,173\,583\,032\,8 + 0.118\,639\,681\,836\,129\,2 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^3,\\
\widetilde{p}_2 := \;\; &(\;\;\, 0.424\,906\,637\,314\,188\,5 + 0.000\,000\,000\,000\,000\,0 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^0\\
+ &(-0.281\,420\,003\,510\,222\,9 - 0.143\,510\,024\,932\,188\,5 \cdot \mathrm{i}) \cdot \mathrm{x}^1\mathrm{y}^0\\
+ &(\;\;\, 0.093\,630\,092\,557\,297\,7 - 0.291\,015\,945\,100\,425\,6 \cdot \mathrm{i}) \cdot \mathrm{x}^2\mathrm{y}^0\\
+ &(-0.110\,189\,730\,825\,496\,5 + 0.614\,282\,144\,464\,633\,5 \cdot \mathrm{i}) \cdot \mathrm{x}^3\mathrm{y}^0\\
+ &(\;\;\, 0.297\,053\,230\,001\,918\,4 + 0.000\,887\,857\,768\,406\,6 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^1\\
+ &(-0.181\,658\,648\,567\,969\,6 + 0.098\,811\,854\,784\,405\,0 \cdot \mathrm{i}) \cdot \mathrm{x}^1\mathrm{y}^1\\
+ &(-0.158\,669\,931\,345\,573\,1 - 0.054\,731\,974\,994\,114\,2 \cdot \mathrm{i}) \cdot \mathrm{x}^2\mathrm{y}^1
\end{aligned}$$

$$+\,(-0.217\,892\,994\,341\,549\,8 - 0.060\,375\,188\,675\,850\,3 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^2$$
$$+\,(\ \ 0.090\,965\,998\,805\,783\,0 - 0.119\,380\,985\,756\,169\,2 \cdot \mathrm{i}) \cdot \mathrm{x}^1\mathrm{y}^2$$
$$+\,(\ \ 0.043\,275\,349\,911\,623\,9 - 0.044\,967\,737\,558\,697\,2 \cdot \mathrm{i}) \cdot \mathrm{x}^0\mathrm{y}^3.$$

Computing their zero locus with `Bertini` yields the *nine* points

$$\begin{aligned}
\widetilde{b}^{\mathrm{r}}_1 &:= (\ \ 1.000\,000\,000\,000\,000 - 0.000\,000\,000\,000\,002 \cdot \mathrm{i},\\
&\quad -1.000\,000\,000\,000\,003 + 0.000\,000\,000\,000\,004 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_2 &:= (\ \ 0.378\,295\,862\,438\,166 - 0.925\,684\,741\,400\,745 \cdot \mathrm{i},\\
&\quad\ \ 0.774\,168\,060\,530\,330 - 0.632\,980\,105\,575\,766 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_3 &:= (-0.765\,440\,894\,342\,871 + 0.643\,506\,206\,083\,193 \cdot \mathrm{i},\\
&\quad\ \ 0.996\,899\,539\,492\,553 - 0.078\,684\,866\,140\,770 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_4 &:= (\ \ 0.951\,992\,691\,551\,888 + 0.306\,120\,752\,697\,019 \cdot \mathrm{i},\\
&\quad -0.206\,220\,016\,161\,382 - 0.978\,505\,648\,902\,649 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_5 &:= (\ \ 0.951\,992\,691\,551\,889 + 0.306\,120\,752\,697\,019 \cdot \mathrm{i},\\
&\quad -0.206\,220\,016\,161\,387 + 0.978\,505\,648\,902\,652 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_6 &:= (-0.765\,440\,894\,342\,869 + 0.643\,506\,206\,083\,196 \cdot \mathrm{i},\\
&\quad\ \ 0.996\,899\,539\,492\,545 + 0.078\,684\,866\,140\,655 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_7 &:= (\ \ 0.378\,295\,862\,438\,165 - 0.925\,684\,741\,400\,745 \cdot \mathrm{i},\\
&\quad\ \ 0.774\,168\,060\,530\,330 + 0.632\,980\,105\,575\,765 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_8 &:= (\ \ 1.000\,000\,000\,000\,009 + 0.000\,000\,000\,000\,001 \cdot \mathrm{i},\\
&\quad\ \ 0.999\,999\,999\,999\,994 + 0.000\,000\,000\,000\,015 \cdot \mathrm{i}),\\
\widetilde{b}^{\mathrm{r}}_9 &:= (-0.364\,658\,069\,491\,770 + 0.830\,735\,511\,685\,468 \cdot \mathrm{i},\\
&\quad\ \ 0.730\,828\,417\,435\,265 + 3.122\,523\,676\,516\,970 \cdot \mathrm{i}).
\end{aligned}$$

For $i = 1, \ldots, 8$, $\widetilde{b}^{\mathrm{r}}_i$ is a reasonable approximation of $\widetilde{b}_i$ (and of b_i) (the ordering of $\widetilde{b}^{\mathrm{r}}_1, \ldots, \widetilde{b}^{\mathrm{r}}_8$ is chosen to correspond to that of $b_1, \ldots, b_8$), but $\widetilde{b}^{\mathrm{r}}_9$ is "superfluous" in the sense that it is not a reasonable approximation of any base of f or $\widetilde{f}$, and all the bases of f are already approximated by $\widetilde{b}^{\mathrm{r}}_i$, $i = 1, \ldots, 8$.

The coefficients reconstructed from $\widetilde{b}^{\mathrm{r}}_1, \ldots, \widetilde{b}^{\mathrm{r}}_9$ are

$$\begin{aligned}
\widetilde{f}^{\mathrm{r}}_1 &:= 0.998\,862\,782\,218\,374 - 0.005\,023\,213\,472\,438\,09 \cdot \mathrm{i},\\
\widetilde{f}^{\mathrm{r}}_2 &:= 1.001\,231\,852\,418\,05 + 0.001\,525\,370\,791\,624\,65 \cdot \mathrm{i},\\
\widetilde{f}^{\mathrm{r}}_3 &:= 1.011\,398\,959\,176\,53 + 0.001\,212\,061\,167\,716\,01 \cdot \mathrm{i},\\
\widetilde{f}^{\mathrm{r}}_4 &:= 1.004\,535\,415\,051\,13 - 0.002\,102\,409\,175\,448\,83 \cdot \mathrm{i},\\
\widetilde{f}^{\mathrm{r}}_5 &:= 0.995\,936\,855\,649\,441 + 0.001\,610\,128\,046\,869\,79 \cdot \mathrm{i},\\
\widetilde{f}^{\mathrm{r}}_6 &:= 0.988\,394\,613\,143\,294 - 0.001\,067\,122\,386\,579\,17 \cdot \mathrm{i},
\end{aligned}$$

$\widetilde{f}^{\mathrm{r}}_7 := 0.998\,779\,753\,359\,746 - 0.001\,398\,913\,541\,678\,15 \cdot \mathrm{i},$
$\widetilde{f}^{\mathrm{r}}_8 := 1.000\,486\,755\,343\,18 + 0.005\,244\,098\,569\,933\,65 \cdot \mathrm{i},$
$\widetilde{f}^{\mathrm{r}}_9 := 0.000\,373\,013\,640\,265\,429 + 0.000\,000\,000\,000\,004\,107\,825\,191\,113\,08 \cdot \mathrm{i}.$

As it might be expected, the "spurious" exponential $\exp_{\widetilde{b}^{\mathrm{r}}_9}$ plays almost no role in the reconstruction.

Considering the reconstruction

$$\widetilde{f}^{\mathrm{r}} := \sum_{i=1}^{9} \widetilde{f}^{\mathrm{r}}_i \exp_{\widetilde{b}^{\mathrm{r}}_i},$$

one obtains (by `Octave` computations) that

$$\max\{|\widetilde{f}(\alpha) - \widetilde{f}^{\mathrm{r}}(\alpha)| \mid \alpha \in \mathcal{T}^2_6\} = 0.033\,086\,095\,101\,764\,0$$

(which is the largest absolute entrywise difference occurring in the Hankel matrices of $\widetilde{f}$ and its reconstruction $\widetilde{f}^{\mathrm{r}}$) and that

$$\Big(\sum_{\alpha\in\mathcal{T}^2_6} |\widetilde{f}(\alpha) - \widetilde{f}^{\mathrm{r}}(\alpha)|\Big)/|\mathcal{T}^2_6| = 0.007\,196\,645\,889\,261\,97$$

(which is the average absolute entrywise difference occurring in the Hankel matrices of $\widetilde{f}$ and its reconstruction $\widetilde{f}^{\mathrm{r}}$).

On the other hand, we have $\mathrm{rank}(\mathrm{V}^{\mathrm{supp}(f)}_{\mathcal{T}^2_3}) = 8 = \mathrm{rank}(f)$ and Prony's method succeeds with the combination of `Octave` and `Bertini` with $d = 4$, as predicted by the theory.

Remark 2.28: In numerical computations like those in Example 2.27 (c), a possible strategy for improvement is to alter the $\mathbb{C}$-basis E of $\ker(\mathrm{H}_d(f))$ computed by `Octave` before giving it to `Bertini` in order to improve the performance of the latter.

2.4. A Toeplitz variation

In this section we give a variation of Prony's method where the Hankel-like matrix $\mathrm{H}_\delta(f)$ is replaced by a Toeplitz-like matrix $\mathrm{T}_\delta(f)$. Since this does not work with exponential sums as defined previously (their domain is $\mathbb{N}^n$), we first give a modified version of the definition, in which the domain is extended to $\mathbb{Z}^n$. To do this, we have to restrict to bases $b \in A^n$ each of whose components is invertible, i. e., to units of the algebra A^n.

Definition: The group of units of A^n is denoted by

$$\mathrm{U}(A^n) := \{b \in A^n \mid b \text{ unit in } A^n\} = \mathrm{U}(A)^n.$$

For $b \in \mathrm{U}(A^n)$ let

$$\begin{aligned} {}_{\mathbb{Z}}\exp_b\colon \mathbb{Z}^n &\longrightarrow A, \\ \alpha &\longmapsto b^\alpha := \prod_{j=1}^n b_j^{\alpha_j}, \end{aligned}$$

be the *n-variate exponential on $\mathbb{Z}^n$ over A* with *base b* (which is of course uniquely determined by $\exp_b$) and for an arbitrary subset $B \subseteq \mathrm{U}(A^n)$ let

$$ {}_{\mathbb{Z}}\mathrm{Exp}_B^n(A) := \langle {}_{\mathbb{Z}}\exp_b \mid b \in B \rangle_K \leq A^{\mathbb{Z}^n}$$

be the *K-vector space of n-variate exponential sums on $\mathbb{Z}^n$* (*supported on B*). Elements of ${}_{\mathbb{Z}}\mathrm{Exp}_B^n(A)$ are called *n-variate exponential sums on $\mathbb{Z}^n$*, and we set

$$ {}_{\mathbb{Z}}\mathrm{Exp}^n(A) := {}_{\mathbb{Z}}\mathrm{Exp}^n_{\mathrm{U}(A^n)}(A).$$

Furthermore, for $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ and a subset $D \subseteq \mathbb{N}^n$ let

$$\mathrm{T}_D(f) := (f(\beta - \alpha))_{\substack{\alpha \in D \\ \beta \in D}} \in A^{D \times D}.$$

For a t-$\star$-filtration $\mathcal{F}$ on $\mathbb{N}^n$ and $\delta \in \mathbb{N}^t$ we use the abbreviation

$$\mathrm{T}_\delta(f) := \mathrm{T}_{\mathcal{F}_\delta}(f) \in A^{\mathcal{F}_\delta \times \mathcal{F}_\delta}.$$

The point of this section is to provide a variation on the theory given by Corollary 2.19 in order to reconstruct $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ and use the matrix $\mathrm{T}_\delta(f)$, which has a different structure (Toeplitz-like instead of Hankel-like), instead of $\mathrm{H}_\delta(f \restriction \mathbb{N}^n)$.

Remark 2.29: Note that for $f \in {}_{\mathbb{Z}}\mathrm{Exp}_B^n(A)$, certainly $f \restriction \mathbb{N}^n \in \mathrm{Exp}_B^n(A)$. In particular, the method given by Corollary 2.19 for reconstructing $f \in \mathrm{Exp}^n(A)$ also works for ${}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ by considering restrictions to $\mathbb{N}^n$. Since the restrictions ${}_{\mathbb{Z}}\exp_b \restriction \mathbb{N}^n$, $b \in A^n$, are linearly independent by Corollary 2.22, the set $\{{}_{\mathbb{Z}}\exp_b \mid b \in B\}$ is a K-basis of ${}_{\mathbb{Z}}\mathrm{Exp}_B^n(A)$. This justifies the following definition.

Definition: For $f \in {}_{\mathbb{Z}}\mathrm{Exp}_B^n(A)$, let $\mathrm{supp}(f) := \mathrm{supp}(f \restriction \mathbb{N}^n)$ be the *support* of f, let $\mathrm{coeff}(f) := \mathrm{coeff}(f \restriction \mathbb{N}^n)$ be the *coefficient vector* of f, and let $\mathrm{rank}(f) := \mathrm{rank}(f \restriction \mathbb{N}^n)$ be the *rank* of f. All of these notions are independent of the set $B \subseteq \mathrm{U}(A^n)$.

Remark 2.30: In the univariate case with the total degree $\star$-filtration $\mathcal{T} = \mathcal{T}^1 = \mathcal{M}^1$ on $\mathbb{N}$, a variation of Prony's method with $\mathrm{H}_r(f)$ replaced by $\mathrm{T}_r(f)$ follows easily by the following argument. For $f = \sum_{b \in M} f_b \, {}_{\mathbb{Z}}\exp_b \in {}_{\mathbb{Z}}\mathrm{Exp}^1(A)$ with $M \in \mathcal{P}_{\mathrm{f}}(\mathrm{U}(A))$ and $f_b \in K \setminus \{0\}$, $r := \mathrm{rank}(f)$, let

$$g_f := \sum_{b \in M} \frac{f_b}{b^r} \exp_b \in \mathrm{Exp}^1(A)$$

and let $P_r := (\mathrm{u}_r, \ldots, \mathrm{u}_0) \in A^{(r+1)\times(r+1)}$. ($P_r$ is a permutation matrix "reversing the order of the rows" when multiplied from the left.) Then

$$\begin{aligned}\mathrm{T}_r(f) = (f(\beta-\alpha))_{\substack{\alpha\in\mathcal{T}_r\\ \beta\in\mathcal{T}_r}} &= \Big(\sum_{b\in M} f_b b^{\beta-\alpha}\Big)_{\substack{\alpha\in\mathcal{T}_r\\ \beta\in\mathcal{T}_r}} = \Big(\sum_{b\in M} \frac{f_b}{b^r} b^{\beta+r-\alpha}\Big)_{\substack{\alpha\in\mathcal{T}_r\\ \beta\in\mathcal{T}_r}}\\ &= P_r \cdot \Big(\sum_{b\in M} \frac{f_b}{b^r} b^{\alpha+\beta}\Big)_{\substack{\alpha\in\mathcal{T}_r\\ \beta\in\mathcal{T}_r}} = P_r \cdot \mathrm{H}(g_f).\end{aligned}$$

Thus $\ker \mathrm{T}_r(f) = \ker(P_r\,\mathrm{H}_r(g_f)) = \ker(\mathrm{H}_r(g_f))$, and hence by Corollary 2.20 we have

$$\mathrm{Z}(\ker \mathrm{T}_r(f)) = \mathrm{Z}(\ker \mathrm{H}_r(g_f)) = \mathrm{supp}(g_f) = \mathrm{supp}(f).$$

In order to give analogous statements as in Section 2.1 afterwards, we state and prove the following elementary lemma.

Lemma 2.31: *Let $v_1, \ldots, v_r \in K^n$ with $v_{i,j} \neq 0$ for all $i = 1, \ldots, r$, $j = 1, \ldots, n$. Let $1/v_i := (1/v_{i,1}, \ldots, 1/v_{i,n})^\top \in K^n$. Then*

$$\dim_K(\langle v_1, \ldots, v_r\rangle_K) = \dim_K(\langle 1/v_1, \ldots, 1/v_r\rangle_K).$$

Proof: For reasons of symmetry, it is enough to show "$\leq$". Thus, without loss of generality, let $v_1, \ldots, v_r$ be linearly independent. Let $\lambda_i \in K$ with $\sum_{i=1}^r \lambda_i \cdot 1/v_i = 0$. Let $\mu := \prod_{j=1}^n \prod_{i=1}^r v_{i,j} \in K \setminus \{0\}$. Let $\ell_i := i+1$ for $1 \leq i < r$ and $\ell_r := 1$. Since $\ell_i \neq \ell_k$ for $i \neq k$, $v_{\ell_1}, \ldots, v_{\ell_r}$ are linearly independent. We have $\mu \cdot 1/v_i = \eta_i v_{\ell_i}$ for some $\eta_i \in K \setminus \{0\}$. Therefore we have

$$0 = \mu \sum_{i=1}^r \lambda_i \cdot 1/v_i = \sum_{i=1}^r \lambda_i \eta_i v_{\ell_i},$$

hence $\lambda_i\eta_i = 0$ by linear independence of $v_{\ell_1}, \ldots, v_{\ell_r}$, and thus $\lambda_i = 0$. q. e. d.

There is the following analogue to Lemma 2.7. The proof is identical to the proof of Lemma 2.7, with appropriate changing of V_D^M into $\mathrm{V}_D^{1/M}$ and corresponding application of Lemma 2.31. It is included here merely for completeness.

Lemma 2.32: *Let $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ and $M := \mathrm{supp}(f)$. Let $D \in \mathcal{P}_\mathrm{f}(\mathbb{N}^n)$ be arbitrary. Then the following holds.*

(a) *We have*

$$\mathrm{T}_D(f) = {\mathrm{V}_D^{1/M}}^\top \cdot C \cdot \mathrm{V}_D^M,$$

with

$$1/M := \{1/b \mid b \in M\}$$

and the diagonal matrix $C := (\mathrm{coeff}(f)_b \mathrm{u}_b)_{b\in M} \in A^{M\times M}$.

(b) *Let $f_b \neq 0$ for all $b \in M$. If $\mathrm{ev}_D^M : S_D \to A^M$ is surjective, then*

$$\ker_A(\mathrm{T}_D(f)) = \ker_A(\mathrm{V}_D^M).$$

PROOF: For brevity, let $T := \mathrm{T}_D(f)$, $V := \mathrm{V}_D^M$, and $W := \mathrm{V}_D^{1/M}$.

(a) Since $W^\top C = (f_b/b^\alpha)_{\substack{\alpha\in D\\ b\in M}}$, we have

$$\begin{aligned} V^\top C V &= (W^\top C(b^\beta)_{b\in M})_{\beta\in D} = \Big(\sum_{b\in M} b^\beta (f_b/b^\alpha)_{\alpha\in D}\Big)_{\beta\in D} \\ &= \Big(\sum_{b\in M} f_b b^{\beta-\alpha}\Big)_{\substack{\alpha\in D\\ \beta\in D}} = (f(\beta-\alpha))_{\substack{\alpha\in D\\ \beta\in D}} = T. \end{aligned}$$

(b) By part (a) we always have $\ker_A(V) \subseteq \ker_A(T)$. To show the reverse inclusion let $C \in A^{M\times M}$ be as in part (a). We show that $\ker_A(W^\top C) = \{0\}$. Let $Q := \mathrm{Quot}(A)$ be the quotient field of A. Consider $V, W \in A^{M\times D} \leq Q^{M\times D}$ as matrices over Q. Since ev_D^M is surjective, the Q-linear map $V\colon Q^D \to Q^M$, $x \mapsto Vx$, is surjective by an easy argument (see footnote 5 on page 21). By Lemma 2.31 we have $\mathrm{rank}(W) = \mathrm{rank}(V) = |M|$, thus $W\colon Q^D \to Q^M$, $x \mapsto Wx$, is surjective. Therefore $W^\top\colon Q^M \to Q^D$ is injective by standard linear algebra, which yields $\ker_A(W^\top) = A^M \cap \ker_Q(W^\top) = \{0\}$. Since the coefficients of f are non-zero and therefore units in A, C is invertible in $A^{M\times M}$, hence $\ker_A(W^\top C) = \ker_A(W^\top) = \{0\}$. Thus, by the same argument as in Lemma 2.7 we obtain $\ker_A(T) = \ker_A(W^\top C V) = \ker_A(V)$, as claimed. q. e. d.

COROLLARY 2.33: *Let $D \in \mathcal{P}_\mathrm{f}(\mathbb{N}^n)$ and $f \in {}_\mathbb{Z}\mathrm{Exp}^n(A)$. If $\mathrm{ev}_D^{\mathrm{supp}(f)}\colon S_D \to A^{\mathrm{supp}(f)}$ is surjective, then*

$$\ker_A(\mathrm{T}_D(f)) = \ker_A(\mathrm{H}_D(f \restriction \mathbb{N}^n)).$$

PROOF: By Lemma 2.32 (b) and Lemma 2.7 (b) we immediately get $\ker_A(\mathrm{T}_D(f)) = \ker_A(\mathrm{V}_D^{\mathrm{supp}(f)}) = \ker_A(\mathrm{H}_D(f \restriction \mathbb{N}^n))$. q. e. d.

EXAMPLE 2.34: Note that in general $\mathrm{rank}(\mathrm{T}_D(f)) \neq \mathrm{rank}(\mathrm{H}_D(f \restriction \mathbb{N}^n))$ for $f \in {}_\mathbb{Z}\mathrm{Exp}^n(K)$, and therefore in Corollary 2.33 the condition "$\mathrm{ev}_D^{\mathrm{supp}(f)}$ surjective" cannot be omitted. For example, let

$$f := 2\exp_6 + 4\exp_4 - \exp_3 - 8\exp_2 + 3\exp_1 \in {}_\mathbb{Z}\mathrm{Exp}^1(\mathbb{Q}).$$

Then we have

$$\mathrm{H}_{\mathcal{T}_1}(f \restriction \mathbb{N}) = \begin{pmatrix} f(0) & f(1) \\ f(1) & f(2) \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 12 & 98 \end{pmatrix}$$

and

$$\mathrm{T}_{\mathcal{T}_1}(f) = \begin{pmatrix} f(0) & f(1) \\ f(-1) & f(0) \end{pmatrix} = \begin{pmatrix} 0 & 12 \\ 0 & 0 \end{pmatrix}.$$

Clearly, we have $\mathrm{rank}(\mathrm{T}_{\mathcal{T}_1}(f)) = 1 < 2 = \mathrm{rank}(\mathrm{H}_{\mathcal{T}_1}(f \restriction \mathbb{N}))$.

The following Corollary 2.35 is an analogue for ${}_\mathbb{Z}\mathrm{Exp}^n(K)$ of Corollary 2.19.

Corollary 2.35 (Prony's method for ${}_{\mathbb{Z}}\mathrm{Exp}^n(K)$): *Let $A = K$ be a field and let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$. Let $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(K)$ and let $\delta \in (\mathbb{N} \setminus \{0\})^t$ be such that $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} : S_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}} \to K^{\mathrm{supp}(f)}$ is surjective for all $\ell = 1, \ldots, t$. Then we have*

$$\mathrm{Z}(\ker_K \mathrm{T}_{\mathcal{F}_\delta}(f)) = \mathrm{supp}(f).$$

Proof: By Corollary 2.33 and Corollary 2.19 we obtain that $\mathrm{Z}(\ker_K(\mathrm{T}_\delta(f))) = \mathrm{Z}(\ker_K(\mathrm{H}_\delta(f \restriction \mathbb{N}^n))) = \mathrm{supp}(f \restriction \mathbb{N}^n) = \mathrm{supp}(f)$. q. e. d.

Remark 2.36: In the style of Lemma 2.7/2.32, one can try to find further matrices $X_D(f)$ (that can be computed solely from the restriction $f \restriction L$ to some $L \in \mathcal{P}_{\mathrm{f}}(\mathbb{N}^n)$ dependent only on $D \subseteq \mathbb{N}^n$) such that $\ker_A(X_D(f)) = \ker_A(\mathrm{V}^{\mathrm{supp}(f)}_D)$. To the best of the author's knowledge this is an open problem.

As in Section 2.2, we give a simple algorithmic formulation for the reconstruction of $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ in Algorithm 2.2.

Data: $f \restriction (\mathcal{F}_\delta - \mathcal{F}_\delta)$ for $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ and $\delta \in (\mathbb{N} \setminus \{0\})^t$ with $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_\ell}}$ surjective.
Result: $\mathrm{rank}(f)$, $\mathrm{supp}(f)$, and $\mathrm{coeff}(f)$.
Compute Q-basis E of $\ker(\mathrm{T}_{\mathcal{F}_\delta}(f))$;
Compute $\mathrm{supp}(f) = \mathrm{Z}(E)$;
Compute $\mathrm{rank}(f) = |\mathrm{supp}(f)|$;
Compute unique solution $\mathrm{coeff}(f)$ of $\left(\mathrm{V}^{\mathrm{supp}(f)}_{\mathcal{F}_{\delta-\mathrm{u}^t_1}}\right)^\top \cdot \mathrm{coeff}(f) = (f(\alpha))_{\alpha \in \mathcal{F}_{\delta-\mathrm{u}^t_1}}$;

Algorithm 2.2: Prony's method, Toeplitz variation, for ${}_{\mathbb{Z}}\mathrm{Exp}^n(A)$.

2.5. Exponential sums supported on algebraic varieties

In the previous Sections 2.2 and 2.4, we have developed a theory for reconstruction of exponential sums $f \in \mathrm{Exp}^n(K) \cup {}_{\mathbb{Z}}\mathrm{Exp}^n(K)$. However, so far we did not consider the case that a subset $B \subseteq K^n$ is given with $f \in \mathrm{Exp}^n_B(K)$ (or $B \subseteq \mathrm{U}(K^n)$ and $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n_B(K)$) to improve the method. In Section 2.5.1, an adaptation is made for algebraic varieties $B \subseteq K^n$. In Section 2.5.2 we develop this further in order to reduce the computational cost of Prony's method for certain algebraic hypersurfaces $B \subseteq K^n$ with given equations.

As before, let A be an integral domain containing the field K, $n \in \mathbb{N} \setminus \{0\}$, and $S = A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$. The following definitions are also given in the preliminary Section 1.2.1.

Let $B \subseteq A^n$ be an arbitrary subset. Let

$$S_B := S/\mathrm{I}(B)$$

be the *coordinate algebra* of B and for $D \subseteq \mathbb{N}^n$ let

$$S_{D,B} := S_D / \mathrm{I}_D(B).$$

For $M \subseteq B \subseteq A^n$ let

$$\begin{aligned} \mathrm{ev}_B^M\colon \qquad S_B &\longrightarrow A^M, \\ p + \mathrm{I}(B) &\longmapsto \mathrm{ev}^M(p) = \mathrm{f}_p \restriction M, \end{aligned}$$

which is well-defined by Remark 1.1 in the preliminary section, and for $D \subseteq \mathbb{N}^n$, via the embedding

$$\begin{aligned} S_{D,B} &\hookrightarrow S_B, \\ p + \mathrm{I}_D(B) &\longmapsto p + \mathrm{I}(B), \end{aligned}$$

let

$$\mathrm{ev}_{D,B}^M := \mathrm{ev}_B^M \restriction S_{D,B}.$$

Further let

$$\mathrm{I}_B(M) := \ker(\mathrm{ev}_B^M) = \{p + \mathrm{I}(B) \mid p \in S,\, \mathrm{f}_p \restriction M = 0\} = \mathrm{I}(M)/\mathrm{I}(B)$$

(which is an ideal in S_B) and

$$\mathrm{I}_{D,B}(M) := \ker(\mathrm{ev}_{D,B}^M) = S_{D,B} \cap \mathrm{I}_B(M) = \mathrm{I}_D(M)/\mathrm{I}_D(B).$$

Note that by the third isomorphism theorem(s)

$$S_B/\mathrm{I}_B(M) \cong S_M$$

and

$$S_{D,B}/\mathrm{I}_{D,B}(M) \cong S_{D,M}.$$

Furthermore, for a subset $J \subseteq S_B$ let

$$\mathrm{Z}_B(J) := \{b \in B \mid \text{for all } q \in S \text{ with } q + \mathrm{I}(B) \in J,\, q(b) = 0\}$$

be the *zero locus relative to* B of J.

There does not seem to be any confusion possible with I_A, V_A, or Z_A as defined previously, where A denotes the ring of coefficients (and is usually omitted from the notation).

By the following simple remark surjectivity conditions on the evaluation homomorphisms in this section are equivalent. The two parts make essentially the same statement, with part (b) being the matrix version of part (a).

REMARK 2.37: Let $B \subseteq A^n$, $M \subseteq B$, and $D \subseteq \mathbb{N}^n$. Then the following holds.

(a) The following are equivalent:

   (i) $\mathrm{ev}_D^M\colon S_D \to A^M$ is surjective.

   (ii) $\mathrm{ev}_{D,B}^M\colon S_{D,B} \to A^M$ is surjective.

Proof: (i) $\Rightarrow$ (ii): Let $a \in A^M$. By hypothesis there is a $p \in S_D$ with $\mathrm{ev}_D^M(p) = a$. We have $p + \mathrm{I}_D(B) \in S_{D,B}$ and $\mathrm{ev}_{D,B}^M(p + \mathrm{I}_D(B)) = \mathrm{ev}_D^M(p) = a$.

(ii) $\Rightarrow$ (i): Let $a \in A^M$. By hypothesis there is a $p \in S_D$ with $a = \mathrm{ev}_{D,B}^M(p + \mathrm{I}_D(B)) = \mathrm{ev}_D^M(p)$. «

(b) Let $L \subseteq D$ be such that $\overline{\mathrm{x}^L} \subseteq \overline{\mathrm{x}^D}$ is an A-basis of $S_{D,B}$. Then the transformation matrix of $\mathrm{ev}_{D,B}^M$ w. r. t. $\overline{\mathrm{x}^L}$ and the canonical basis U_M of A^M is V_L^M. Indeed, for every $\alpha \in L$ one has $\mathrm{ev}_{D,B}^M(\overline{\mathrm{x}^\alpha}) = (b^\alpha)_{b\in M} = \sum_{b\in M} b^\alpha \mathrm{u}_b$. Of course, at least in the case of $A = K$ being a field, this provides a further proof of part (a).

2.5.1. General algebraic varieties

In the following Lemma 2.38, for subsets $M \subseteq B \subseteq A^n$, a connection is established between the Zariski closure $\mathrm{Z}(\mathrm{I}(M))$ of M and the relative Zariski closure $\mathrm{Z}_B(\mathrm{I}_{D,B}(M))$ of M w. r. t. $D \subseteq \mathbb{N}^n$ and B. The proof is straightforward and likely to be well-known.

Lemma 2.38: *Let $M \subseteq B \subseteq A^n$ and $D \subseteq \mathbb{N}^n$. Then we have*

$$M \subseteq B \cap \mathrm{Z}(\mathrm{I}(M)) \subseteq \mathrm{Z}_B(\mathrm{I}_{D,B}(M)) \subseteq \mathrm{Z}(\mathrm{I}_D(M)).$$

(*Of course, if A is a field and M is finite, then $M = \mathrm{Z}(\mathrm{I}(M)) = B \cap \mathrm{Z}(\mathrm{I}(M))$.*)

Proof: The first inclusion is clear.

To prove the middle inclusion, let $q \in S$ with $q + \mathrm{I}(B) \in \mathrm{I}_{D,B}(M) \subseteq S_B$. Then there is a $p \in \mathrm{I}_D(M)$ with $q + \mathrm{I}(B) = p + \mathrm{I}(B)$. Since $b \in \mathrm{Z}(\mathrm{I}(M))$, we have $p(b) = 0$, and since $b \in B$, we have $q(b) = p(b) = 0$.

To prove the remaining inclusion, let $b \in \mathrm{Z}_B(\mathrm{I}_{D,B}(M))$ and $p \in \mathrm{I}_D(M)$. We have to show that $p(b) = 0$. Since $p \in S_D$, we have $p + \mathrm{I}_D(B) \in S_D/\mathrm{I}_D(B) = S_{D,B}$ and we have $\mathrm{ev}_{D,B}^M(p + \mathrm{I}_D(B)) = \mathrm{ev}_B^M(p + \mathrm{I}(B)) = \mathrm{ev}^M(p) = \mathrm{ev}_D^M(p) = 0$, i. e., $p + \mathrm{I}_D(B) \in \ker(\mathrm{ev}_{D,B}^M) = \mathrm{I}_{D,B}(M)$. Since $b \in \mathrm{Z}_B(\mathrm{I}_{D,B}(M))$, it follows that $p(b) = 0$. q. e. d.

Since we apply Theorem 2.15 in the following corollary, we drop the extra generality of A being an integral domain and switch to a field $A = K$.

Corollary 2.39: *Let $B \subseteq K^n$ and $M \in \mathcal{P}_\mathrm{f}(B)$ and let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$. Let $\delta \in (\mathbb{N} \setminus \{0\})^t$ be such that $\mathrm{ev}_{\mathcal{F}_{\delta-\mathrm{u}_\ell^t}}^M$ is surjective for all $\ell = 1, \ldots, t$. Let $\mathcal{B}_\delta \subseteq \mathcal{F}_\delta$ be such that $S_{\mathcal{B}_\delta,B} = S_{\mathcal{F}_\delta,B}$. Then we have*

$$\mathrm{Z}_B(\mathrm{I}_{\mathcal{B}_\delta,B}(M)) = M.$$

Proof: Since $\mathrm{ev}_{\mathcal{F}_{\delta-\mathrm{u}_\ell^t}}^M$ is surjective, we have $\mathrm{Z}(\mathrm{I}_{\mathcal{F}_\delta}(M)) = M$ by Theorem 2.15. Since $S_{\mathcal{B}_\delta,B} = S_{\mathcal{F}_\delta,B}$, by Remark 2.37 we have $\mathrm{I}_{\mathcal{B}_\delta,B}(M) = \mathrm{I}_{\mathcal{F}_\delta,B}(M)$. Thus, on account of Lemma 2.38 we obtain $M \subseteq \mathrm{Z}_B(\mathrm{I}_{\mathcal{B}_\delta,B}(M)) = \mathrm{Z}_B(\mathrm{I}_{\mathcal{F}_\delta,B}(M)) \subseteq \mathrm{Z}(\mathrm{I}_{\mathcal{F}_\delta}(M)) = M$. q. e. d.

Combining Remark 2.37 and Lemma 2.52 with Corollary 2.19 yields the following corollary. Part (a) describes the version with a Hankel-like matrix and part (b) provides analogous statements with a Toeplitz-like matrix. Note that by Remark 2.37 (a), the surjectivity of the evaluation homomorphisms $\mathrm{ev}_{\mathcal{F}_\delta}^{\mathrm{supp}(f)}$ on $S_{\mathcal{F}_\delta}$ is equivalent to the surjectivity of the evaluation homomorphisms $\mathrm{ev}_{\mathcal{B}_\delta,B}^{\mathrm{supp}(f)}$ on $S_{\mathcal{B}_\delta,B}$.

COROLLARY 2.40 (Prony's method on algebraic varieties): *Let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$.*

(a) *Let $B \subseteq K^n$ be a subset and $f \in \mathrm{Exp}_B^n(K)$. Let $\delta \in (\mathbb{N}\setminus\{0\})^t$ be such that $\mathrm{ev}_{\mathcal{F}_{\delta-\mathrm{u}_\ell^t}}^{\mathrm{supp}(f)}$ is surjective for all $\ell = 1, \dots, t$. Let $\mathcal{B}_\delta \subseteq \mathcal{F}_\delta$ be such that $S_{\mathcal{B}_\delta,B} = S_{\mathcal{F}_\delta,B}$. Then we have*
$$\mathrm{Z}_B(\ker \mathrm{H}_{\mathcal{B}_\delta}(f)) = \mathrm{supp}(f).$$

(b) *Let $B \subseteq \mathrm{U}(K^n)$ be a subset and $f \in {}_{\mathbb{Z}}\mathrm{Exp}_B^n(K)$. Let $\delta \in (\mathbb{N}\setminus\{0\})^t$ be such that $\mathrm{ev}_{\mathcal{F}_{\delta-\mathrm{u}_\ell^t}}^{\mathrm{supp}(f)}$ is surjective for all $\ell = 1, \dots, t$. Let $\mathcal{B}_\delta \subseteq \mathcal{F}_\delta$ be such that $S_{\mathcal{B}_\delta,B} = S_{\mathcal{F}_\delta,B}$. Then we have*
$$\mathrm{Z}_B(\ker \mathrm{T}_{\mathcal{B}_\delta}(f)) = \mathrm{supp}(f).$$

PROOF: (a) By Lemma 2.7 (b), Remark 2.37, and Corollary 2.39, we have
$$\begin{aligned}\mathrm{Z}_B(\ker \mathrm{H}_{\mathcal{B}_\delta}(f)) = \mathrm{Z}_B(\ker \mathrm{V}_{\mathcal{B}_\delta}^{\mathrm{supp}(f)}) &= \mathrm{Z}_B(\ker \mathrm{V}_{\mathcal{F}_\delta}^{\mathrm{supp}(f)})\\ &= \mathrm{Z}_B(\mathrm{I}_{\mathcal{B}_\delta,B}(\mathrm{supp}(f))) = \mathrm{supp}(f).\end{aligned}$$

(b) Since $\mathrm{ev}_{\mathcal{B}_\delta}^{\mathrm{supp}(f)}$ is surjective, by Corollary 2.33 and part (a) we have
$$\mathrm{Z}_B(\ker \mathrm{T}_{\mathcal{B}_\delta}(f)) = \mathrm{Z}_B(\ker \mathrm{H}_{\mathcal{B}_\delta}(f \restriction \mathbb{N}^n)) = \mathrm{supp}(f \restriction \mathbb{N}^n) = \mathrm{supp}(f).$$ q. e. d.

REMARK 2.41: Let $B \subseteq K^n$, $f \in \mathrm{Exp}_B^n(K)$ and $\mathcal{B}_\delta \subseteq \mathcal{F}_\delta$ be as in Corollary 2.40 (a).

(a) By the preceeding results we have
$$\begin{aligned}\mathrm{supp}(f) &= \mathrm{Z}_B(\ker \mathrm{H}_{\mathcal{B}_\delta}(f))\\ &= B \cap \mathrm{Z}(\ker \mathrm{H}_{\mathcal{B}_\delta}(f))\\ &= \mathrm{Z}(\mathrm{I}(B) \cup \ker \mathrm{H}_{\mathcal{B}_\delta}(f)).\end{aligned}$$

Therefore, if polynomials $p_1, \dots, p_k \in S$ with $\langle p_1, \dots, p_k\rangle_S = \mathrm{I}(B)$ are given, then $\mathrm{supp}(f)$ may be computed by solving the system of polynomial equations $\{p_1, \dots, p_k, q_1, \dots, q_\ell\}$ for an ideal basis $\{q_1, \dots, q_\ell\} \subseteq S$ of $I := \langle \ker \mathrm{H}_{\mathcal{B}_\delta}(f)\rangle_S$. Of course, a K-basis of $\ker \mathrm{H}_{\mathcal{B}_\delta}(f)$ is also an ideal basis of I.

(b) Let $\mathcal{B}_\delta$ be chosen such that $|\mathcal{B}_\delta| = |\overline{\mathrm{x}^{\mathcal{B}_\delta}}|$ and $\overline{\mathrm{x}^{\mathcal{B}_\delta}}$ is a K-basis of $S_{\mathcal{F}_\delta,B}$. Then one has
$$|\mathcal{B}_\delta| = \dim_K(S_{\mathcal{F}_\delta,B}) = \dim_K(S_{\mathcal{F}_\delta}/\mathrm{I}_{\mathcal{F}_\delta}(B)) = |\mathcal{F}_\delta| - \dim_K(\mathrm{I}_{\mathcal{F}_\delta}(B))$$

Therefore, except for the situation $\mathrm{Z}(\mathrm{I}(B)) = K^n$ (which is the setting of previous sections) then the matrix $\mathrm{H}_{\mathcal{B}_\delta}(f) \in K^{\mathcal{B}_\delta \times \mathcal{B}_\delta} = K^{\dim_K(S_{\mathcal{F}_\delta,B}) \times \dim_K(S_{\mathcal{F}_\delta,B})}$ is a strict submatrix of the matrix $\mathrm{H}_{\mathcal{F}_\delta}(f) \in K^{\mathcal{F}_\delta \times \mathcal{F}_\delta}$ that one would use without having equations for $\mathrm{I}(B)$. Therefore one may work with only the restriction $f \upharpoonright (\mathcal{B}_\delta + \mathcal{B}_\delta)$ instead of $f \upharpoonright (\mathcal{F}_\delta + \mathcal{F}_\delta)$. However, it is not clear if the "order" of the problem is hereby reduced. A sufficient condition is given in the following section.

2.5.2. Order reducing algebraic varieties

We give a suitable definition that leads to a more efficient method.

Definition: Let $B \subseteq A^n$, $I \subseteq S$, $D \subseteq \mathbb{N}^n$, $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$, $j_0 \in \{1, \ldots, n\}$, and $k_0 \in \mathbb{N}$.

(a) (B, D, j_0, k_0) is *order reducing* if with $D' := \{\alpha \in D \mid \alpha_{j_0} < k_0\}$,

$$S_{D,B} \subseteq S_{D',B}.$$

(Of course, then $S_{D,B} = S_{D',B}$.)

(b) (I, D, j_0, k_0) is *order reducing* if $(\mathrm{Z}(I), D, j_0, k_0)$ is order reducing

(c) $(B, \mathcal{F}, j_0, k_0)$ is *order reducing* if for all $\delta \in \mathbb{N}^t$, $(B, \mathcal{F}_\delta, j_0, k_0)$ is order reducing.

(d) $(I, \mathcal{F}, j_0, k_0)$ is *order reducing* if $(\mathrm{Z}(I), \mathcal{F}, j_0, k_0)$ is order reducing.

Remark 2.42: Let $j_0 \in \{1, \ldots, n\}$ and $k_0 \in \mathbb{N}$. Assume that in the situation of Corollary 2.40 (a), $(B, \mathcal{F}, j_0, k_0)$ is order reducing. Reconstructing f naively using $\mathrm{H}_{\mathcal{F}_\delta}(f) \in K^{\mathcal{F}_\delta \times \mathcal{F}_\delta}$, one needs $|\mathcal{F}_\delta + \mathcal{F}_\delta| \leq |\mathcal{F}_{2\delta}|$ "samples" of f, where equality may hold for all $\delta \in \mathbb{N}^t$ (e. g., for $\mathcal{F}$ being the total or maximal degree $\star$-filtration on $\mathbb{N}^n$). Working instead with the submatrix $\mathrm{H}_{\mathcal{B}_\delta}(f) \in K^{\mathcal{B}_\delta \times \mathcal{B}_\delta}$ of $\mathrm{H}_{\mathcal{F}_\delta}(f)$, one needs only

$$|\mathcal{B}_\delta + \mathcal{B}_\delta| \leq |\mathcal{F}'_\delta + \mathcal{F}'_\delta| \leq |\{\alpha \in \mathcal{F}_{2\delta} \mid \alpha_{j_0} < 2k_0\}|$$

samples of f, the set $\mathcal{B}_\delta + \mathcal{B}_\delta$ being a subset of $\mathcal{F}_\delta + \mathcal{F}_\delta$ that is bounded by the constant $2k_0$ in one coordinate direction.

In the following we show in particular that for $p = 1 - \sum_{j=1}^n \mathrm{x}_j^2$, i. e., the sphere, $(p, \mathcal{T}^n, j_0, k_0)$, $j_0 = 1, \ldots, n$, $k_0 = 2$, is order reducing.

Lemma 2.43: *Let $p \in S := A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, $D \subseteq \mathbb{N}^n$, $j_0 \in \{1, \ldots, n\}$ and $k_0 \in \mathbb{N}$, $\alpha_0 := k_0 \mathrm{u}_{j_0} \in \mathbb{N}^n$, such that the following conditions hold.*

(a) $\alpha_0 \in \mathrm{supp}(p)$.

(b) *For all $\gamma \in \mathrm{supp}(p) \setminus \{\alpha_0\}$, $\gamma_{j_0} < k_0$.*

(c) *D is a lower set in $(\mathbb{N}^n, \leq_{\mathrm{p}})$.*

(d) *For all $\gamma \in \mathrm{supp}(p) \setminus \{\alpha_0\}$, all $\alpha \in D$, $\alpha_{j_0} \geq k_0$ implies $\alpha + \gamma - \alpha_0 \in D$.*

Then (p, D, j_0, k_0) is order reducing.

PROOF: Without loss of generality let $j_0 = n$. Let $D' := \{\alpha \in D \mid \alpha_n < k_0\}$. We have to show $\overline{\mathrm{x}^D} \subseteq \langle \overline{\mathrm{x}^{D'}} \rangle_A$. Let $\alpha \in D$ and assume inductively that for all $\beta \in D$ with $\beta_n < \alpha_n$, $\overline{\mathrm{x}^\beta} \in \langle \overline{\mathrm{x}^{D'}} \rangle_A$. If $\alpha_n < k_0$ then $\alpha \in D'$ and $\overline{\mathrm{x}^\alpha} \in \overline{\mathrm{x}^{D'}} \subseteq \langle \overline{\mathrm{x}^{D'}} \rangle_A$. Thus let $\alpha_n \geq k_0$. Then $\beta := \alpha - \alpha_0 \in \mathbb{N}^n$ and since D is a lower set, $\beta \in D$. By hypothesis we have $\beta + \gamma = \alpha + \gamma - \alpha_0 \in D$ for all $\gamma \in \operatorname{supp}(p) \setminus \{\alpha_0\}$, and $(\beta + \gamma)_n = \beta_n + \gamma_n = \alpha_n + \gamma_n - k_0 < \alpha_n$. Hence, by induction hypothesis, $\overline{\mathrm{x}^{\beta+\gamma}} \in \langle \overline{\mathrm{x}^{D'}} \rangle_A$ for all $\gamma \in \operatorname{supp}(p) \setminus \{\alpha_0\}$. Therefore we arrive at

$$\overline{\mathrm{x}^\alpha} = \overline{\mathrm{x}^\beta} \cdot \overline{\mathrm{x}^{\alpha_0}} = \overline{\mathrm{x}^\beta} \cdot \Big(- \sum_{\gamma \in \operatorname{supp}(p) \setminus \{\alpha_0\}} p_\gamma \overline{\mathrm{x}^\gamma} \Big) = - \sum_{\gamma \in \operatorname{supp}(p) \setminus \{\alpha_0\}} p_\gamma \overline{\mathrm{x}^{\beta+\gamma}} \in \langle \overline{\mathrm{x}^{D'}} \rangle_A. \quad \text{q. e. d.}$$

The following theorem provides a family of examples that includes in particular the sphere as order reducing.

THEOREM 2.44: *Let $p := 1 - \sum_{j=1}^n \mathrm{x}_j^{k_j} \in A[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ with $k_j \in \mathbb{N}$ and let $j_0 \in \{1, \ldots, n\}$ with $k_{j_0} = \max\{k_1, \ldots, k_n\} \geq 1$. Then $(p, \mathcal{T}^n, j_0, k_{j_0})$ is order reducing. In particular, for all $j_0 \in \{1, \ldots, n\}$, $(1 - \sum_{j=1}^n \mathrm{x}_j^2, \mathcal{T}^n, j_0, 2)$ is order reducing.*

PROOF: Let $k_0 := k_{j_0}$ and $\alpha_0 := k_0 \mathrm{u}_{j_0}$ and let $d \in \mathbb{N}$. We have to show that $(p, \mathcal{T}_d^n, j_0, k_0)$ is order reducing. We go through the conditions (a)–(d) in Lemma 2.43.

(a) Since $k_{j_0} \geq 1$ we have $\alpha_0 = k_{j_0} \mathrm{u}_{j_0} \in \{0\} \cup \{k_j \mathrm{u}_j \mid k_j \geq 1\} = \operatorname{supp}(p)$.

(b) Let $\gamma \in \operatorname{supp}(p) \setminus \{\alpha_0\}$. Then $\gamma = 0$ or $\gamma = k_j \mathrm{u}_j$ for some $j \neq j_0$. In either case, $\gamma_{j_0} = 0 < k_{j_0} = k_0$.

(c) This holds for any $\star$-filtration $\mathcal{F}$ on $\mathbb{N}^n$ induced by a norm $\|\|$ on $\mathbb{R}^n$ with $\|\mathrm{u}_j\| \leq 1$.

(d) Let $\gamma \in \operatorname{supp}(p) \setminus \{\alpha_0\}$ and $\alpha \in \mathcal{T}_d^n$ with $\alpha_{j_0} \geq k_0$. Clearly we have $\alpha + \gamma - \alpha_0 \in \mathbb{N}^n$ (this always holds under the premise of condition (d) in Lemma 2.43). If $\gamma = 0$, clearly we have $\alpha + \gamma - \alpha_0 = \alpha - \alpha_0 \in \mathcal{T}_d^n$. If $\gamma \neq 0$ then $\gamma = k_j \mathrm{u}_j$ for some $j \neq j_0$ and $\operatorname{tot deg}(\alpha + \gamma - \alpha_0) = \operatorname{tot deg}(\alpha + k_j \mathrm{u}_j - k_{j_0} \mathrm{u}_{j_0}) \leq \operatorname{tot deg}(\alpha) \leq d$, so $\alpha + \gamma - \alpha_0 \in \mathcal{T}_d^n$.

Thus, $(p, \mathcal{T}_d^n, j_0, k_0)$ is order reducing by Lemma 2.43. q. e. d.

2.6. A stronger result for the total degree $\star$-filtration

In Corollary 2.19 we proved that for an exponential sum $f \in \operatorname{Exp}^n(K)$, under a polynomial interpolation condition, we have $\mathrm{Z}(\ker \mathrm{H}_{\mathcal{F}_\delta}(f)) = \operatorname{supp}(f)$. As mentioned in Remark 2.18, one may ask the question if the vanishing ideal $\mathrm{I}(\operatorname{supp}(f))$ is generated by $\ker(\mathrm{H}_{\mathcal{F}_\delta}(f))$, which is clearly a stronger statement. In this section, we prove that this is the case, over an arbitrary field, if $\mathcal{F}$ is the total degree $\star$-filtration. Complete and gratefully given credit for the arguments to prove the crucial Theorem 2.48 belongs to H. Michael Möller.

Preliminary material from the theory of Gröbner bases that is needed in this section has been collected in Section 1.2.2.

We begin with two statements that will be used in the proof of Theorem 2.48.

Lemma 2.45 (cf. Fassino-Möller [35, Proposition 2]): *Let A be an arbitrary ring, let $S := A[\mathrm{x}_1, \dots, \mathrm{x}_n]$, let $\leq$ be a monomial order on Mon^n, I be an ideal in S and $\pi_I \colon S \to S/I$, $p \mapsto p + I$, be the canonical epimorphism onto S/I. Let $u \in \mathrm{N}_\leq(I)$ and let*

$$M_u := \mathrm{in}_\leq(\pi_I^{-1}[\{u + I\}]) \subseteq \mathrm{Mon}^n.$$

Then u is the $\leq$-least element of M_u.

Proof: Since $u = \mathrm{in}_\leq(u)$, it is clear that $u \in M_u$. Let $p \in \pi_I^{-1}[\{u + I\}]$. We have to show that $u \leq \mathrm{in}_\leq(p)$. If $p = u$ we are done, so let $u - p \neq 0$, $u - p = \sum_\alpha c_\alpha \mathrm{x}^\alpha$ with $c_\alpha \in A$. *Case 1:* For all $\alpha \in \mathbb{N}^n$, $u \neq c_\alpha \mathrm{x}^\alpha$. Then the coefficient of u in $p = u - \sum_\alpha c_\alpha \mathrm{x}^\alpha$ is non-zero, so $u \leq \mathrm{in}_\leq(p)$ and we are done. *Case 2:* $u = c_\alpha \mathrm{x}^\alpha$ for some $\alpha \in \mathbb{N}^n$. Since u is a monomial, we have $c_\alpha = 1$ and $u = \mathrm{x}^\alpha \leq u' := \mathrm{in}_\leq(u - p)$. Since $u \in \mathrm{N}_\leq(I) = \mathrm{Mon}^n \setminus \mathrm{in}_\leq(I)$ and $u' \in \mathrm{in}_\leq(I)$, we certainly have $u \neq u'$, and thus $u < u' = \mathrm{in}_\leq(u - \sum_\alpha c_\alpha \mathrm{x}^\alpha) = \mathrm{in}_\leq(p)$. q. e. d.

Lemma 2.46: *Let A be an arbitrary ring, $S := A[\mathrm{x}_1, \dots, \mathrm{x}_n]$, let $\leq$ be a degree compatible monomial order on Mon^n, let $M \in \mathcal{P}_\mathrm{f}(A^n)$, and $d \in \mathbb{N}$ such that $\mathrm{ev}^M_{\mathcal{T}^n_d} \colon S_{\mathcal{T}^n_d} \to A^M$ is surjective. Then*

$$\mathrm{N}_\leq(\mathrm{I}_A(M)) \subseteq S_{\mathcal{T}^n_d}.$$

Proof: For brevity, let $I := \mathrm{I}_A(M)$. Let $u \in \mathrm{N}_\leq(I) \subseteq S$ and let $\overline{u} := u + I \in S/I$ be the image of u in S/I. We have to show that $\operatorname{tot deg}(u) \leq d$. Since $\mathrm{ev}^M_d \colon S_d \to A^M$ is surjective, there is a $p \in S_d$ with $\mathrm{ev}^M(p) = \mathrm{ev}^M_d(p) = \mathrm{ev}^M(u)$. Since $I = \ker(\mathrm{ev}^M)$, the map $\mathrm{ev}^M_M \colon S/I \to A^M$, $p + I \mapsto \mathrm{ev}^M(p)$, is injective (in fact an isomorphism). By the above we have $\mathrm{ev}^M_M(\overline{p}) = \mathrm{ev}^M_M(\overline{u})$, hence $\overline{p} = \overline{u}$ in S/I, i. e., $p \in \pi_I^{-1}[\{u + I\}]$. Since u is the $\leq$-least element of $\mathrm{in}_\leq[\pi_I^{-1}[\{u + I\}]]$ by Lemma 2.45, we have $u \leq p$, and since $\leq$ is degree compatible, we have that $\operatorname{tot deg}(u) \leq \operatorname{tot deg}(p) \leq d$. q. e. d.

Remark 2.47: Under the hypothesis in Lemma 2.46 we always have $\mathrm{I}_A(M) \neq \{0\}$, since otherwise Lemma 2.46 would imply $\mathrm{Mon}^n = \mathrm{N}_\leq(\mathrm{I}_A(M)) \subseteq S_{\mathcal{T}^n_d}$, which is clearly false (for $n \geq 1$, which we assume throughout) since $\mathrm{x}_1^{d+1} \in \mathrm{Mon}^n \setminus S_{\mathcal{T}^n_d}$.

Theorem 2.48: *Let A be an arbitrary integral domain, $Q := \mathrm{Quot}(A)$ be the quotient field of A, let $S := A[\mathrm{x}_1, \dots, \mathrm{x}_n]$, and $T := Q[\mathrm{x}_1, \dots, \mathrm{x}_n]$. Let $M \in \mathcal{P}_\mathrm{f}(A^n) \setminus \{\emptyset\}$ and $d \in \mathbb{N} \setminus \{0\}$ be such that $\mathrm{ev}^M_{\mathcal{T}^n_{d-1}} \colon S_{\mathcal{T}^n_{d-1}} \to A^M$ is surjective. Then we have*

$$\langle \mathrm{I}_{\mathcal{T}^n_d}(M) \rangle_T = \mathrm{I}_Q(M).$$

Proof: For brevity, let $I := \mathrm{I}_Q(M)$.

The part "$\subseteq$" of the claim clearly holds, since $\mathrm{I}_d(M) \subseteq \mathrm{I}_A(M) \subseteq \mathrm{I}_Q(M) = I$.

In order to prove the reverse inclusion, we construct a Gröbner basis G of I with $G \subseteq \mathrm{I}_d(M)$. To this end, let $\leq$ be a degree compatible monomial order on Mon^n.

Let L be the partially ordered set $\mathrm{in}_\leq(I) \subseteq \mathrm{Mon}^n$ with the divisibility relation $\mid$ as partial order. By Dickson's lemma (Lemma 1.2 in the preliminary Section 1.2.2) there is a finite $|$-basis D of L, i. e., D is finite, $D \subseteq L$, and for every $u \in L$ there is a $u' \in D$ with $u' \mid u$. Let (by well-ordering of $(\mathbb{N}, \leq)$) D be of least cardinality among all finite $|$-bases of L.

We show that $D \subseteq S_d$. Let $u \in D$. Assume that $u = 1$ $(= \mathrm{x}^0)$. Let $p \in I \setminus \{0\}$ with $\mathrm{in}_{\leq}(p) = u = 1$. Since $\leq$ is degree compatible, we have that $\operatorname{tot}\deg(p) = \operatorname{tot}\deg(\mathrm{in}_{\leq}(p)) = \operatorname{tot}\deg(u) = \operatorname{tot}\deg(1) = 0$, i. e., $p \in A \setminus \{0\}$. Since $p(a) = 0$ for $a \in M$ $(\neq \emptyset)$, we have $p = 0$. Since $p \neq 0$, this is a contradiction. Therefore $u \neq 1$, that is, $u = \mathrm{x}_j u'$ for some $j \in \{1, \dots, n\}$ and $u' \in \mathrm{Mon}^n$. Assume that $u' \in L$. Since D is a $|$-basis of L, there is a $u'' \in D$ with $u'' \mid u' \mid u$. Therefore $D' := D \setminus \{u\}$ is a finite $|$-basis of L with $|D'| = |D| - 1 < |D|$, contradicting the choice of D. Thus $u' \in \mathrm{Mon}^n \setminus L = \mathrm{N}_{\leq}(I)$, so by Lemma 2.46 we have $\operatorname{tot}\deg(u') \leq d - 1$, hence $\operatorname{tot}\deg(u) = \operatorname{tot}\deg(\mathrm{x}_j u') = \operatorname{tot}\deg(u') + 1 \leq d$, as claimed.

For $u \in D$ let $g_u \in I \setminus \{0\}$ with $\mathrm{in}_{\leq}(g_u) = u$. By multiplying with an appropriate element of A, we can assume that $g_u \in S$. Let

$$G := \{g_u \mid u \in D\}.$$

Then G is finite, $G \subseteq I \setminus \{0\}$, and $\langle \mathrm{in}_{\leq}(G) \rangle_T = \langle \mathrm{in}_{\leq}(I) \rangle_T$, so G is a Gröbner basis of I. Therefore it follows that $\langle G \rangle_T = I$. Since $\leq$ is degree compatible, we have by the above that $\operatorname{tot}\deg(g_u) = \operatorname{tot}\deg(\mathrm{in}_{\leq}(g_u)) = \operatorname{tot}\deg(u) \leq d$ for all $u \in D$, and thus we arrive at $G \subseteq \mathrm{I}_d(M)$. q. e. d.

Remark 2.49: It is tempting to attempt to prove a version of Theorem 2.48 for more general "degree functions" $deg\colon A[\mathrm{x}_1, \dots, \mathrm{x}_n] \to \mathbb{N}$. A naive approach by transferring the definition literally and considering "*deg*-compatible" monomial orders works with all arguments unchanged, but does not include the relevant notion of maximal degree, since for $n \geq 2$ there is no "max deg-compatible" monomial order. To see this, let $\leq$ be any monomial order on Mon^n, $n \geq 2$, and w. l. o. g. let $\mathrm{x}_2 \leq \mathrm{x}_1$. Then one has $\mathrm{x}_2^2 \leq \mathrm{x}_1\mathrm{x}_2$ and $\max\deg(\mathrm{x}_2^2) = 2 \not\leq 1 = \max\deg(\mathrm{x}_1\mathrm{x}_2)$.

For the rest of the section, A denotes an integral domain that contains the field K as a subring.

Recall that the *spectrum* of a ring A is denoted by

$$\mathrm{Spec}(A) := \{P \subseteq A \mid P \text{ prime ideal of } A\},$$

and for an arbitrary set $I \subseteq S = A[\mathrm{x}_1, \dots, \mathrm{x}_n]$,

$$\mathrm{V}(I) := \{P \in \mathrm{Spec}(S) \mid I \subseteq P\}$$

denotes the (*algebraic*) *variety* of I.

The following Corollary 2.50 constitutes a strengthening of Corollary 2.19 and Corollary 2.35 for the total degree case that is based on Theorem 2.48.

Corollary 2.50 (Prony's method for $\mathrm{Exp}^n(A)$, total degree version):
Let $Q := \mathrm{Quot}(A)$ be the quotient field of A and $T := Q[\mathrm{x}_1, \dots, \mathrm{x}_n]$. Then the following holds.

(a) *Let* $f \in \operatorname{Exp}^n(A)$ *and* $d \in \mathbb{N} \setminus \{0\}$ *be such that* $\operatorname{ev}^{\operatorname{supp}(f)}_{\mathcal{T}^n_{d-1}} \colon S_{\mathcal{T}^n_{d-1}} \to A^{\operatorname{supp}(f)}$ *is surjective. Then we have*

$$\langle \ker_A \operatorname{H}_{\mathcal{T}^n_d}(f) \rangle_T = \operatorname{I}_Q(\operatorname{supp}(f)).$$

In particular, we have

$$\operatorname{V}_Q(\ker_A \operatorname{H}_{\mathcal{T}^n_d}(f)) = \operatorname{V}_Q(\operatorname{I}_Q(\operatorname{supp}(f))) = \operatorname{Spec}(T_{\operatorname{supp}(f)})$$

and

$$\operatorname{Z}_A(\ker_A \operatorname{H}_{\mathcal{T}^n_d}(f)) = \operatorname{Z}_Q(\ker_A \operatorname{H}_{\mathcal{T}^n_d}(f)) = \operatorname{supp}(f).$$

(b) *Let* $f \in {}_{\mathbb{Z}}\operatorname{Exp}^n(A)$ *and* $d \in \mathbb{N} \setminus \{0\}$ *be such that* $\operatorname{ev}^{\operatorname{supp}(f)}_{\mathcal{T}^n_{d-1}} \colon S_{\mathcal{T}^n_{d-1}} \to A^{\operatorname{supp}(f)}$ *is surjective. Then we have*

$$\langle \ker_A \operatorname{T}_{\mathcal{T}^n_d}(f) \rangle_T = \operatorname{I}_Q(\operatorname{supp}(f)).$$

In particular, we have

$$\operatorname{V}_Q(\ker_A \operatorname{T}_{\mathcal{T}^n_d}(f)) = \operatorname{V}_Q(\operatorname{I}_Q(\operatorname{supp}(f))) = \operatorname{Spec}(T_{\operatorname{supp}(f)})$$

and

$$\operatorname{Z}_A(\ker_A \operatorname{T}_{\mathcal{T}^n_d}(f)) = \operatorname{Z}_Q(\ker_A \operatorname{T}_{\mathcal{T}^n_d}(f)) = \operatorname{supp}(f).$$

PROOF: (a) By Theorem 2.48 we have $\langle \operatorname{I}_d(\operatorname{supp}(f)) \rangle_T = \operatorname{I}_Q(\operatorname{supp}(f))$. Since the evaluation homomorphism in degree d, $\operatorname{ev}_d^{\operatorname{supp}(f)} \colon S_d \to A^{\operatorname{supp}(f)}$, is also surjective, we obtain $\ker_A(\operatorname{V}_d^{\operatorname{supp}(f)}) = \ker_A(\operatorname{H}_d(f))$ by Lemma 2.7 (b).

The second statement is a direct consequence of this.

By the first part we have $\operatorname{Z}_Q(\ker_A \operatorname{H}_d(f)) = \operatorname{Z}_Q(\operatorname{I}_Q(\operatorname{supp}(f))) = \operatorname{supp}(f)$. Since $\operatorname{supp}(f) \subseteq A^n$, we have $\operatorname{supp}(f) = A^n \cap \operatorname{Z}_Q(\ker_A \operatorname{H}_d(f)) = \operatorname{Z}_A(\ker_A \operatorname{H}_d(f))$.

(b) This follows immediately from Corollary 2.33 and part (a). q. e. d.

REMARK 2.51: Note that we have shown in this section that, under the assumption of surjectivity of $\operatorname{ev}^{\operatorname{supp}(f)}_{\mathcal{T}^n_{d-1}}$, that $\langle \ker \operatorname{H}_{\mathcal{T}^n_d}(f) \rangle_T$ is a radical ideal in T. This is the difference to the previous sections. If $\langle \ker \operatorname{H}_{\mathcal{F}_d}(f) \rangle_T$ is a radical ideal for a general $\star$-filtration $\mathcal{F}$ on $\mathbb{N}^n$ or for $\mathcal{F} = \mathcal{M}^n$ is not known to the author.

The remainder of this section is devoted to a "relative" version of Corollary 2.50 for exponential sums supported on an algebraic variety.

For $B \subseteq A^n$ and a subset $J \subseteq S_B$ let

$$\operatorname{V}_B(J) := \{Q \in \operatorname{Spec}(S_B) \mid J \subseteq Q\}$$

be the (*algebraic*) *variety relative to* B of J.

We have the following analogue to Lemma 2.38.

LEMMA 2.52: *Let $B \subseteq A^n$, $M \subseteq B$, and $D \subseteq \mathbb{N}^n$. Identifying corresponding prime ideals of $S = A[\mathrm{x}_1, \dots, \mathrm{x}_n]$ and $S_B = S/\mathrm{I}(B)$, we then have*

$$\mathrm{Spec}(S_M) = \mathrm{V}(\mathrm{I}(M)) \subseteq \mathrm{V}_B(\mathrm{I}_{D,B}(M)) \subseteq \mathrm{V}(\mathrm{I}_D(M)).$$

PROOF: The equality $\mathrm{Spec}(S_M) = \mathrm{V}(\mathrm{I}(M))$ is clear by the definitions.

To prove the middle inclusion, let $Q \in \mathrm{V}(\mathrm{I}(M))$. Then $Q = P/\mathrm{I}(M)$ for some $P \in \mathrm{Spec}(S)$ with $\mathrm{I}(M) \subseteq P$. We have to show that $Q \in \mathrm{Spec}(S_B)$ and $\mathrm{I}_{D,B}(M) \subseteq Q$. Since $M \subseteq B$, we have $\mathrm{I}(B) \subseteq \mathrm{I}(M) \subseteq P$ and therefore $P/\mathrm{I}(B) \in \mathrm{Spec}(S_B)$. Via the embedding $S_{D,B} \hookrightarrow S_B$, $p + \mathrm{I}_D(B) \mapsto p + \mathrm{I}(B)$, we have $\mathrm{I}_{D,B}(M) \subseteq \mathrm{I}_B(M) = \mathrm{I}(M)/\mathrm{I}(B) \subseteq P/\mathrm{I}(B)$. Identifying $Q = P/\mathrm{I}(M)$ and $P/\mathrm{I}(B)$, we arrive at $Q = P/\mathrm{I}(B) \in \mathrm{V}_B(\mathrm{I}_B(M)) \subseteq \mathrm{V}_B(\mathrm{I}_{D,B}(M))$.

To prove the remaining inclusion,[11] let $Q \in \mathrm{V}_B(\mathrm{I}_{D,B}(M))$. Then $Q = P/\mathrm{I}(B)$ for some $P \in \mathrm{Spec}(S)$ with $\mathrm{I}(B) \subseteq P$. We have to show that $P \in \mathrm{V}(\mathrm{I}_D(M))$, i. e., $\mathrm{I}_D(M) \subseteq P$. Via the embedding $S_{D,B} \hookrightarrow S_B$, $p + \mathrm{I}_D(B) \mapsto p + \mathrm{I}(B)$, $\mathrm{I}_D(M)/\mathrm{I}_D(B) = \mathrm{I}_{D,B}(M)$ is a subset of $Q = P/\mathrm{I}(B)$. Therefore, $P \in \mathrm{V}(\mathrm{I}_D(M))$ follows from the correspondence theorem for submodules of factor modules. q. e. d.

The following corollary is an analogue to Corollary 2.39.

COROLLARY 2.53: *Let $B \subseteq K^n$ and $M \in \mathcal{P}_\mathrm{f}(B)$. Let $d \in \mathbb{N} \setminus \{0\}$ and $\mathrm{ev}^M_{\mathcal{T}_{d-1}}$ be surjective. Let $\mathcal{B}_d \subseteq \mathcal{T}_d$ be such that $S_{\mathcal{B}_d,B} = S_{\mathcal{T}_d,B}$. Then we have*

$$\mathrm{V}_B(\mathrm{I}_{\mathcal{B}_d,B}(M)) = \mathrm{Spec}(S_M).$$

PROOF: Since $\mathrm{ev}^M_{\mathcal{T}_{d-1}}$ is surjective, we have $\langle \mathrm{I}_{\mathcal{T}_d}(M)\rangle_S = \mathrm{I}(M)$ by Theorem 2.48. Since $S_{\mathcal{B}_d,B} = S_{\mathcal{T}_d,B}$, by Remark 2.37 we have $\mathrm{I}_{\mathcal{B}_d,B}(M) = \mathrm{I}_{\mathcal{T}_d,B}(M)$. Thus, on account of Lemma 2.52 we obtain $\mathrm{Spec}(S_M) \subseteq \mathrm{V}(\mathrm{I}_{\mathcal{B}_d}(M)) = \mathrm{V}(\mathrm{I}_{\mathcal{T}_d}(M)) = \mathrm{V}(\mathrm{I}(M)) = \mathrm{Spec}(S_M)$. q. e. d.

The following Corollary 2.54 constitutes a strengthening of Corollary 2.40 for the total degree case that is based on Theorem 2.48. Again, it comes in Hankel and in Toeplitz versions.

COROLLARY 2.54 (Prony's method on algebraic varieties, total degree version):

(a) *Let $B \subseteq K^n$ be a subset and $f \in \mathrm{Exp}^n_B(K)$. Let $d \in \mathbb{N} \setminus \{0\}$ and $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{T}_{d-1}}$ be surjective. Let $\mathcal{B}_d \subseteq \mathcal{T}_d$ be such that $S_{\mathcal{B}_d,B} = S_{\mathcal{T}_d,B}$. Then we have*

$$\mathrm{V}_B(\ker \mathrm{H}_{\mathcal{B}_d}(f)) = \mathrm{Spec}(S_{\mathrm{supp}(f)}).$$

(b) *Let $B \subseteq \mathrm{U}(K^n)$ be a subset and $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n_B(K)$. Let $d \in \mathbb{N} \setminus \{0\}$ and $\mathrm{ev}^{\mathrm{supp}(f)}_{\mathcal{T}_{d-1}}$ be surjective. Let $\mathcal{B}_d \subseteq \mathcal{T}_d$ be such that $S_{\mathcal{B}_d,B} = S_{\mathcal{T}_d,B}$. Then we have*

$$\mathrm{V}_B(\ker \mathrm{T}_{\mathcal{B}_d}(f)) = \mathrm{Spec}(S_{\mathrm{supp}(f)}).$$

[11] A proof that makes the identification more explicit goes as follows. Let $Q \in \mathrm{V}_B(\mathrm{I}_{D,B}(M))$. Then $Q = P/\mathrm{I}(B)$ for some $P \in \mathrm{Spec}(S)$ with $\mathrm{I}(B) \subseteq P$. We have to show that $P \in \mathrm{V}(\mathrm{I}_D(M))$, i. e., $\mathrm{I}_D(M) \subseteq P$. Let $p \in \mathrm{I}_D(M) = \ker(\mathrm{ev}^M_D)$. Then we have $\mathrm{ev}^M_{D,B}(p + \mathrm{I}_D(B)) = \mathrm{ev}^M_B(p + \mathrm{I}(B)) = \mathrm{ev}^M(p) = \mathrm{ev}^M_D(p) = 0$, i. e., $p + \mathrm{I}_D(B) \in \ker(\mathrm{ev}^M_{D,B}) = \mathrm{I}_{D,B}(M)$. Therefore $p + \mathrm{I}(B) \in Q = P/\mathrm{I}(B)$. Hence we have $p + \mathrm{I}(B) = q + \mathrm{I}(B)$ for some $q \in P$, so $r := p - q \in \mathrm{I}(B) \subseteq P$, and thus $p = r + q \in P$. Therefore $\mathrm{I}_D(M) \subseteq P$.

PROOF: (a) By Lemma 2.7 (b), Remark 2.37, and Corollary 2.53, we have

$$\begin{aligned}\mathrm{V}_B(\ker \mathrm{H}_{\mathcal{B}_d}(f)) = \mathrm{V}_B(\ker \mathrm{V}_{\mathcal{B}_d}^{\mathrm{supp}(f)}) &= \mathrm{V}_B(\ker \mathrm{V}_{\mathcal{T}_d}^{\mathrm{supp}(f)}) \\ &= \mathrm{V}_B(\mathrm{I}_{\mathcal{T}_d,B}(\mathrm{supp}(f))) = \mathrm{Spec}(S_{\mathrm{supp}(f)}).\end{aligned}$$

(b) Since $\mathrm{ev}_{\mathcal{B}_d}^{\mathrm{supp}(f)}$ is surjective, by Corollary 2.33 and part (a) we have

$$\mathrm{V}_B(\ker \mathrm{T}_{\mathcal{B}_d}(f)) = \mathrm{V}_B(\ker \mathrm{H}_{\mathcal{B}_d}(f \restriction \mathbb{N}^n)) = \mathrm{Spec}(S_{\mathrm{supp}(f\restriction\mathbb{N}^n)}) = \mathrm{Spec}(S_{\mathrm{supp}(f)}).$$

q. e. d.

2.7. An application to formal exponential sums

In this section, we apply the theory previously developed to the formal exponential sums introduced in Example 2.1 (d).

REMARK 2.55: Let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$.

(a) For $\delta \in \mathbb{N}^t$, let

$$\mathrm{P}_\delta^n := \mathrm{Z} \circ \ker_K \circ \mathrm{H}_\delta \colon \mathrm{Exp}^n(K) \to \mathcal{P}(K^n).$$

Consider the function

$$\begin{aligned}\mathrm{supp}\colon \mathrm{Exp}^n(K) &\longrightarrow \mathcal{P}(K^n), \\ f &\longmapsto \mathrm{supp}(f).\end{aligned}$$

Let $M \in \mathcal{P}_{\mathrm{f}}(K^n)$. If $\delta \in (\mathbb{N} \setminus \{0\})^t$ and for all $\ell = 1, \ldots, t$, $\mathrm{ev}_{\mathcal{F}_{\delta-\mathrm{u}_\ell^t}}^M \colon S_{\mathcal{F}_{\delta-\mathrm{u}_\ell^t}} \to K^M$ is surjective, then

$$\mathrm{P}_\delta^n \restriction \mathrm{Exp}_M^n(K) = \mathrm{supp} \restriction \mathrm{Exp}_M^n(K).$$

This is an immediate consequence of Corollary 2.19.

(b) There is a "non-constructive" version of the above considerations. Let

$$\begin{aligned}\mathrm{P}^n\colon \mathrm{Exp}^n(K) &\longrightarrow \mathcal{P}(K^n), \\ f &\longmapsto \bigcap_{\delta\in\mathbb{N}^t} \mathrm{P}_\delta^n(f).\end{aligned}$$

We have shown that $\mathrm{P}^n = \mathrm{supp}$. The definition of P^n is non-constructive in the sense that there is an intersection over an infinite set involved.

Recall that

$$\mathrm{FExp}^n(K) = \mathrm{Exp}_{B_n}^n(A_n) = \langle \exp_{\mathrm{y}_i} \mid i \in \mathbb{N}\rangle_K,$$

where A_n is the K-algebra

$$A_n := K[\mathrm{y}_{i,j} \mid i \in \mathbb{N},\, j = 1, \ldots, n],$$

let $\mathrm{y}_i := (\mathrm{y}_{i,1}, \ldots, \mathrm{y}_{i,n}) \in (A_n)^n$ for $i \in \mathbb{N}$ and $B_n := \{\mathrm{y}_i \mid i \in \mathbb{N}\} \subseteq (A_n)^n$.
Let $r \in \mathbb{N}$, and for pairwise distinct $b_1, \ldots, b_r \in K^n$ let

$$b := (b_{1,1}, \ldots, b_{1,n}, \ldots, b_{i,1}, \ldots, b_{i,n}, \ldots, b_{r,1}, \ldots, b_{r,n}) \in K^{r \cdot n},$$

and let $\mathrm{ev}^b \colon A_n \to K$ be the evaluation homomorphism at b, i. e., the unique ring homomorphism $A_n \to K$ with $a \mapsto a$ for $a \in K$ and $\mathrm{y}_{i,j} \mapsto b_{i,j}$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$, and $\mathrm{y}_{i,j} \mapsto 0$ for $i > r$ and $j = 1, \ldots, n$. For a formal exponential sum $F \colon \mathbb{N}^n \to A_n$ let

$$\mathrm{ev}^b(F) := \mathrm{ev}^b \circ F \colon \mathbb{N}^n \to K.$$

Note that $\mathrm{ev}^b(F) \in \mathrm{Exp}^n(K)$. For a subset $P \subseteq (A_n)^n$ let

$$\mathrm{ev}^b(P) := \mathrm{ev}^b[P] = \{\mathrm{ev}^b(p) \mid p \in P\}.$$

Let $\mathcal{F}$ be a t-$\star$-filtration on $\mathbb{N}^n$. As in Remark 2.55 (a) let $Q_n := \mathrm{Quot}(A_n)$ be the quotient field of A_n and

$$\begin{aligned} \mathrm{FP}^n_\delta &= \mathrm{Z} \circ \ker_Q \circ \mathrm{H}_\delta \colon \mathrm{FExp}^n(K) \to \mathcal{P}((A_n)^n), \\ \mathrm{P}^n_\delta &= \mathrm{Z} \circ \ker_K \circ \mathrm{H}_\delta \colon \mathrm{Exp}^n(K) \to \mathcal{P}(K^n), \end{aligned}$$

and as in Remark 2.55 (b) let

$$\begin{aligned} \mathrm{FP}^n \colon \mathrm{FExp}^n(K) &\longrightarrow \mathcal{P}_\mathrm{f}(B_n), \\ F &\longmapsto \bigcap_{\delta \in \mathbb{N}^t} \mathrm{FP}^n_\delta(F), \end{aligned}$$

and

$$\begin{aligned} \mathrm{P}^n \colon \mathrm{Exp}^n(K) &\longrightarrow \mathcal{P}_\mathrm{f}(K^n), \\ f &\longmapsto \bigcap_{\delta \in \mathbb{N}^t} \mathrm{P}^n_\delta(f). \end{aligned}$$

Remark 2.56: (a) Under the assumptions and with the notation above, consider the following diagram.

$$\begin{tikzcd} \mathrm{FExp}^n(K) \arrow[r, "\mathrm{FP}^n"] \arrow[d, "\mathrm{ev}^b"'] & \mathcal{P}_\mathrm{f}(B_n) \arrow[d, "\mathrm{ev}^b"] \\ \mathrm{Exp}^n(K) \arrow[r, "\mathrm{P}^n"'] & \mathcal{P}_\mathrm{f}(K^n) \end{tikzcd}$$

This diagram is commutative, since for $F = \sum_{i \in N} F_i \exp_{\mathrm{y}_i} \in \mathrm{FExp}^n(K)$, $F_i \in K \setminus \{0\}$, $M := \mathrm{supp}(\mathrm{ev}^b(F)) = \{b_i \mid i \in N\}$, we have

$$\begin{aligned} \mathrm{ev}^b \circ \mathrm{FP}^n(F) &= \mathrm{ev}^b \circ \mathrm{supp}(F) = \mathrm{ev}^b(\mathrm{supp}(F)) = \mathrm{ev}^b(\{\mathrm{y}_i \mid i \in N\}) = \{b_i \mid i \in N\} \\ &= M = \mathrm{supp}(\mathrm{ev}^b(F)) = \mathrm{P}^n \circ \mathrm{ev}^b(F). \end{aligned}$$

(b) With the above notations adapted to the case of $\mathrm{FExp}_r^n(K)$, consider the following diagram.

$$\begin{tikzcd}
\mathrm{FExp}_r^n(K) \arrow[r, "\mathrm{FP}^n"] \arrow[d, "\mathrm{ev}^b"'] & \mathcal{P}_\mathrm{f}(B_{n,r}) \arrow[d, "\mathrm{ev}^b"] \\
\mathrm{Exp}^n(K) \arrow[r, "\mathrm{P}^n"'] & \mathcal{P}_\mathrm{f}(K^n)
\end{tikzcd}$$

This diagram is commutative, since for $F = \sum_{i\in N} F_i \exp_{\mathrm{y}_i} \in \mathrm{FExp}_r^n(K)$, $F_i \in K \setminus \{0\}$, $M := \mathrm{supp}(\mathrm{ev}^b(F)) = \{b_i \mid i \in N\}$, we have

$$\begin{aligned}
\mathrm{ev}^b \circ \mathrm{FP}^n(F) &= \mathrm{ev}^b \circ \mathrm{supp}(F) = \mathrm{ev}^b(\mathrm{supp}(F)) = \mathrm{ev}^b(\{\mathrm{y}_i \mid i \in N\}) = \{b_i \mid i \in N\} \\
&= M = \mathrm{supp}(\mathrm{ev}^b(F)) = \mathrm{P}^n \circ \mathrm{ev}^b(F).
\end{aligned}$$

(c) In an attempt to break the above diagrams down into computational steps, one might consider the following diagram.

$$\begin{tikzcd}
\mathrm{FExp}_r^n(K) \arrow[r, "\mathrm{H}_\delta"] \arrow[d, "\mathrm{ev}^b"'] \arrow[dr, phantom, "(\mathrm{I})"] & A_{n,r}^{\mathcal{F}_\delta \times \mathcal{F}_\delta} \arrow[r, "\ker_{A_{n,r}}"] \arrow[d, "\mathrm{ev}^b"'] \arrow[dr, phantom, "(\mathrm{II})"] & \mathrm{G}(A_{n,r}^{\;n}) \arrow[r, "\langle\rangle"] \arrow[d, "\mathrm{ev}^b"] \arrow[dr, phantom, "(\mathrm{III})"] & \mathrm{Id}(A_{n,r}[\mathrm{x}_1, \dots, \mathrm{x}_n]) \arrow[r, "\mathrm{Z}"] \arrow[d, "\mathrm{ev}^b"] \arrow[dr, phantom, "(\mathrm{IV})"] & \mathcal{P}(A_{n,r}^{\;n}) \arrow[d, "\mathrm{ev}^b"] \\
\mathrm{Exp}^n(K) \arrow[r, "\mathrm{H}_\delta"'] & K^{\mathcal{F}_\delta \times \mathcal{F}_\delta} \arrow[r, "\ker_K"'] & \mathrm{G}(K^n) \arrow[r, "\langle\rangle"'] & \mathrm{Id}(K[\mathrm{x}_1, \dots, \mathrm{x}_n]) \arrow[r, "\mathrm{Z}"'] & \mathcal{P}(K^n)
\end{tikzcd}$$

Here $\mathrm{G}(A_{n,r}^{\;n})$ denotes the set of $A_{n,r}$-submodules of $(A_{n,r})^n$, $\mathrm{Id}(A_{n,r}[\mathrm{x}_1, \dots, \mathrm{x}_n])$ the set of ideals of $A_{n,r}[\mathrm{x}_1, \dots, \mathrm{x}_n]$, and we extend the definition of ev^b to other domains in nearby ways, e. g. for a polynomial $p = \sum_\alpha p_\alpha \mathrm{x}^\alpha \in A_{n,r}[\mathrm{x}_1, \dots, \mathrm{x}_n]$ let

$$\mathrm{ev}^b(p) := \sum_\alpha \mathrm{ev}^b(p_\alpha)\mathrm{x}^\alpha \in K[\mathrm{x}_1, \dots, \mathrm{x}_n].$$

Of course, there is considerable amount of choice in the individual steps. However, it is not clear if the individual subdiagrams are commutative.

3. Reconstruction of multivariate exponential sums over $\mathbb{R}$ and $\mathbb{C}$

In the previous chapter we developed a reconstruction theory for multivariate exponential sums over an arbitrary field. The present chapter concerns the case over the fields of real and complex numbers and exponential sums supported on the real sphere resp. the complex torus.

In Section 3.1 we study the case of multivariate exponential sums over the real numbers supported on the real sphere as an application of a theorem of Kunis and the theory developed in Chapter 2. In Section 3.2 we prove a theorem similar to the theorem of Kunis and give a corresponding application to the reconstruction of complex exponential sums supported on the complex torus.

As in the previous chapter, $n \in \mathbb{N} \setminus \{0\}$ always denotes a non-zero natural number.

3.1. Exponential sums supported on the real $(n-1)$-sphere

Definition: Let

$$\mathbb{S}^{n-1} := \mathrm{Z}_{\mathbb{R}}\Big(1 - \sum_{j=1}^{n} \mathrm{x}_j^2\Big) = \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\} \subseteq \mathbb{R}^n$$

be the (*real*) $(n-1)$-*sphere*. Furthermore, for $d \in \mathbb{N}$ and with $S := \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, let

$$\mathrm{SH}_d^n := S_{\mathcal{T}_d, \mathbb{S}^{n-1}} = S_{\mathcal{T}_d} / \mathrm{I}_{\mathcal{T}_d}(\mathbb{S}^{n-1}).$$

The $\mathbb{R}$-vector space SH_d^n is called *space of* (*real*) *spherical harmonics of degree at most d* and its elements are called (*real*) *spherical harmonics* (*of degree at most d*).

As recalled in Remark 1.1 in the preliminary section, SH_d^n can be identified with the space of restrictions of polynomial functions $\mathbb{R}^n \to \mathbb{R}$ of total degree at most d to the sphere $\mathbb{S}^{n-1}$. Since $\mathbb{S}^{n-1}$ is by definition the zero locus of the polynomial $p := 1 - \sum_{j=1}^n \mathrm{x}_j^2 \in S = \mathbb{R}[\mathrm{x}_1, \ldots, \mathrm{x}_n]$, we have that $(\mathbb{S}^{n-1}, \mathcal{T}^n, j_0, 2)$ is order reducing by Theorem 2.44. Therefore, the reconstruction problem for $\mathrm{Exp}^n_{\mathbb{S}^{n-1}}(\mathbb{R})$ can be solved efficiently by the theory of Chapter 2.

In the following we apply a criterion for the surjectivity of $\mathrm{ev}^M_{\mathcal{T}_d, \mathbb{S}^{n-1}}$ in terms of a separation property of $M \in \mathcal{P}_{\mathrm{f}}(\mathbb{S}^{n-1})$. The following is an appropriate notion of distance on the sphere. Recall that by the Cauchy-Schwarz inequality, for $b, c \in \mathbb{S}^{n-1}$ one has $\langle b, c \rangle \in [-1, 1]$, where $\langle\rangle \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denotes the euclidean scalar product on $\mathbb{R}^n$.

Definition: For $M \in \mathcal{P}_{\mathrm{f}}(\mathbb{S}^{n-1})$, the *spherical separation* of M is defined as

$$\mathrm{sep}_{\mathrm{s}}(M) := \inf\{\arccos(\langle b, c\rangle) \mid b, c \in M,\, b \neq c\} \in \mathbb{R} \cup \{\infty\},$$

where $\inf \emptyset = \infty$.

Kunis provided a quantitative analysis of the condition number of the transformation matrix of $\mathrm{ev}^{M}_{\mathcal{T}_d, \mathbb{S}^{n-1}} \colon \mathrm{SH}^n_d \to \mathbb{R}^M$ for a finite set $M \subseteq \mathbb{S}^{n-1}$ and a specific $\mathbb{R}$-basis of SH^n_d in terms of the spherical separation of M [52, Theorem 1]. We will use the following qualitative version that follows from Kunis' theorem. Similar theorems have also been provided in Marzo-Pridhnani [58].

Theorem 3.1 (Polynomial interpolation on the real sphere): *Let $n \geq 2$ and $M \in \mathcal{P}_{\mathrm{f}}(\mathbb{S}^{n-1})$. If $d \in \mathbb{N}$ is such that*

$$d > \frac{5\pi n}{2\,\mathrm{sep}_{\mathrm{s}}(M)},$$

then $\mathrm{ev}^{M}_{\mathcal{T}_d, \mathbb{S}^{n-1}} \colon \mathrm{SH}^n_d \to \mathbb{R}^M$ is surjective.

Combining Theorem 3.1 with results from Chapter 2 yields the following corollary. Part (b) is the Toeplitz version of part (a) for exponential sums supported on $\mathbb{S}^{n-1} \cap (\mathbb{R} \setminus \{0\})^n$.

Corollary 3.2: *Let $n \geq 2$ and for $d \in \mathbb{N}$ let $\mathcal{B}_d \subseteq \mathcal{T}_d$ be such that*

$$\overline{\mathrm{x}^{\mathcal{B}_d}} := \{\mathrm{x}^\alpha + \mathrm{I}_d(\mathbb{S}^{n-1}) \mid \alpha \in \mathcal{B}_d\}$$

generates SH^n_d. Then the following holds.

(a) *Let $f \in \mathrm{Exp}^n_{\mathbb{S}^{n-1}}(\mathbb{R})$ and $d \in \mathbb{N}$ such that*

$$d > \frac{5\pi n}{2\,\mathrm{sep}_{\mathrm{s}}(\mathrm{supp}(f))} + 1.$$

Then we have

$$\mathrm{V}_{\mathbb{S}^{n-1}}(\ker \mathrm{H}_{\mathcal{B}_d}(f)) = \mathrm{Spec}(S_{\mathrm{supp}(f)})$$

and

$$\mathrm{Z}_{\mathbb{S}^{n-1}}(\ker \mathrm{H}_{\mathcal{B}_d}(f)) = \mathrm{supp}(f).$$

(b) *Let $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n_{\mathbb{S}^{n-1}}(\mathbb{R})$ and $d \in \mathbb{N}$ such that*

$$d > \frac{5\pi n}{2\,\mathrm{sep}_{\mathrm{s}}(\mathrm{supp}(f))} + 1.$$

Then we have

$$\mathrm{V}_{\mathbb{S}^{n-1}}(\ker \mathrm{T}_{\mathcal{B}_d}(f)) = \mathrm{Spec}(S_{\mathrm{supp}(f)})$$

and

$$\mathrm{Z}_{\mathbb{S}^{n-1}}(\ker \mathrm{T}_{\mathcal{B}_d}(f)) = \mathrm{supp}(f).$$

For any $j_0 \in \{1, \dots, n\}$, the choice $\mathcal{B}_d = \{\alpha \in \mathcal{T}_d \mid \alpha_{j_0} < 2\}$ works.

Proof: These are immediate consequences of Theorem 3.1, Corollary 2.54, and Theorem 2.44. q. e. d.

Theorem 3.1 has the following corollary.

Corollary 3.3: *Let $M \in \prod_{n \geq 2} \mathcal{P}_{\mathrm{f}}(\mathbb{S}^{n-1})$ such that there is a $q \in \mathbb{R}$ with*

$$0 < q < \operatorname{sep}_{\mathrm{s}}(M_n) \text{ for all } n \geq 2,$$

and for $n \geq 2$ let

$$d_n := \min\{d \in \mathbb{N} \mid \operatorname{ev}^{M_n}_{\mathbb{S}^{n-1},d} \colon \mathrm{SH}^n_d \to \mathbb{R}^{M_n} \text{ is surjective}\}.$$

Then we have $d_n \in \mathrm{O}(n)$.

Proof: By Theorem 3.1, $\operatorname{ev}^{M_n}_{\mathbb{S}^{n-1},e_n}$ is surjective for $e_n := \left\lceil \frac{5\pi n}{2q} \right\rceil$. Thus $d_n \leq e_n$ for all $n \geq 2$, and hence $d_n \in \mathrm{O}(e_n) = \mathrm{O}(n)$. q. e. d.

Corollary 3.4: *Let $f \in \prod_{n \geq 2} \operatorname{Exp}^n_{\mathbb{S}^{n-1}}(\mathbb{R})$ such that there is a $q \in \mathbb{R}$ with*

$$0 < q < \operatorname{sep}_{\mathrm{s}}(\operatorname{supp}(f_n)) \text{ for all } n \geq 2,$$

and for $n \geq 2$ let

$$d_n := \min\{d \in \mathbb{N} \mid \mathrm{V}_{\mathbb{S}^{n-1}}(\ker \mathrm{H}_{\mathcal{T}_d}(f_n)) = \operatorname{Spec}(S_{\operatorname{supp}(f_n)})\}.$$

Then we have $d_n \in \mathrm{O}(n)$.

Proof: This follows immediately from Corollary 3.2 and Corollary 3.3. q. e. d.

3.2. Exponential sums supported on the complex n-torus

In this section we prove a theorem similar to Theorem 3.1 and apply it to the reconstruction problem for exponential sums. We prove the following technical lemma in order to state a slightly weakened form in Theorem 3.6 afterwards.

Lemma 3.5: *Let $n, p \in \mathbb{N} \setminus \{0\}$, p even, let*

$$c_p := \left(\frac{\Gamma(\frac{p}{2}+1) \cdot \Gamma(p+\frac{3}{2})}{\pi^p \cdot \Gamma(\frac{p+3}{2})} \right)^{1/p} \in \mathbb{R}^{>0},$$

where Γ denotes the gamma function (cf. Section 1.3), and let $d, q \in \mathbb{R}^{>0}$ with

$$d > \frac{c_p}{q} \cdot \sqrt[p]{n}.$$

Then there is a function $\psi\colon \mathbb{R}^n \to \mathbb{R}$ with the following properties.

(a) $\operatorname{supp}(\psi) \subseteq \widetilde{\mathrm{B}}_q^{\|\|_\infty}(0)$.

(b) *The Fourier transform $\mathcal{F}_n(\psi)\colon \mathbb{R}^n \to \mathbb{R}$ of ψ exists.*

(c) *$\mathcal{F}_n(\psi)$ is bounded.*

(d) *$\mathcal{F}_n(\psi)(v) \geq 0$ for all $v \in \mathbb{R}^n$ with $\|v\|_p \leq d$.*

(e) *$\mathcal{F}_n(\psi)(v) \leq 0$ for all $v \in \mathbb{R}^n$ with $\|v\|_p > d$.*

(f) $\psi(0) > 0$.

Proof: Let $r := p/2 \in \mathbb{N}$ and let

$$\begin{aligned}\varphi_r\colon\ \mathbb{R} &\longrightarrow \mathbb{R},\\ x &\longmapsto \begin{cases}\left(1-\left(\frac{2x}{q}\right)^2\right)^r & \text{if } x \in \left]-\frac{q}{2},\frac{q}{2}\right[,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Clearly we have $\varphi_r \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. We claim that $h_r \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ defined by

$$\begin{aligned}h_r\colon\ \mathbb{R} &\longrightarrow \mathbb{R},\\ x &\longmapsto \begin{cases}(-1)^r q^{-r} 4^r r! \cdot \mathrm{P}_r\left(\frac{2x}{q}\right) & \text{if } x \in \left]-\frac{q}{2},\frac{q}{2}\right[,\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

where P_r denotes the r-th Legendre polynomial, normalized such that $\mathrm{P}_r(1) = 1$, is an r-th weak derivative of φ_r (cf. Section 1.3 or Jost [48, Chapter V, Section 20]). For this we have to show that

$$\int_{\mathbb{R}} t \cdot h_r = (-1)^r \int_{\mathbb{R}} \varphi_r \cdot \partial^r(t)$$

for all $t \in \mathrm{C}_0^r(\mathbb{R})$. Let $t \in \mathrm{C}_0^r(\mathbb{R})$. Then we have

$$\begin{aligned}(-1)^r \int_{\mathbb{R}} \varphi_r \cdot \partial^r(t) &= (-1)^r \int_{\left]-\frac{q}{2},\frac{q}{2}\right[} \varphi_r \cdot \partial^r(t)\\ &\overset{\text{integration by parts}}{=} (-1)^r \underbrace{\left(\varphi_r\left(\frac{q}{2}\right) \cdot \partial^{r-1}(t)\left(\frac{q}{2}\right) - \varphi_r\left(-\frac{q}{2}\right) \cdot \partial^{r-1}(t)\left(-\frac{q}{2}\right)\right)}_{=0}\\ &\quad -(-1)^r \int_{\left]-\frac{q}{2},\frac{q}{2}\right[} \frac{\partial}{\partial x}\varphi_r(x) \cdot \frac{\partial^{r-1}}{\partial x^{r-1}} t(x)\,\mathrm{d}x\\ &= \cdots = (-1)^{2r} \int_{\left]-\frac{q}{2},\frac{q}{2}\right[} \frac{\partial^r}{\partial x^r}\varphi_r(x) \cdot t(x)\,\mathrm{d}x\\ &\overset{\text{chain rule}}{=} \left(\frac{2}{q}\right)^r \cdot \int_{\left]-\frac{q}{2},\frac{q}{2}\right[} \frac{\partial^r}{\partial y^r}\Big|_{y=\frac{2x}{q}} \left((1-y^2)^r\right) \cdot t(x)\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&= \Big(-\frac{2}{q}\Big)^r \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \frac{\partial^r}{\partial y^r}_{\big|y=\frac{2x}{q}} \Big((y^2-1)^r\Big) \cdot t(x)\,\mathrm{d}x \\
&\overset{\text{Rodrigues}}{\underset{\text{Section 1.3}}{=}} \Big(-\frac{2}{q}\Big)^r 2^r r! \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \mathrm{P}_r\Big(\frac{2x}{q}\Big) \cdot t(x)\,\mathrm{d}x \\
&= \int_{]-\frac{q}{2},\frac{q}{2}[} (-1)^r q^{-r} 4^r r! \cdot \mathrm{P}_r\Big(\frac{2x}{q}\Big) \cdot t(x)\,\mathrm{d}x \\
&= \int_{\mathbb{R}} h_r \cdot t,
\end{aligned}
$$

proving the claim.

Let

$$
\begin{aligned}
\psi_r\colon\ &\mathbb{R}^n \longrightarrow \mathbb{R}, \\
&x \longmapsto (2\pi d)^p \cdot \bigotimes_{j=1}^n (\varphi_r * \varphi_r)(x) - (-1)^r \sum_{k=1}^n \frac{\partial^p}{\partial x_k^p} \bigotimes_{j=1}^n (\varphi_r * \varphi_r)(x).
\end{aligned}
$$

We show that $\psi = \psi_r$ has the desired properties.

Clearly,

$$
\operatorname{supp}(\varphi_r) = \operatorname{supp}(h_r) = [-\frac{q}{2}, \frac{q}{2}],
$$

which implies

$$
\operatorname{supp}(\varphi_r * \varphi_r), \operatorname{supp}(h_r * h_r) \subseteq \overline{[-\frac{q}{2}, \frac{q}{2}] + [-\frac{q}{2}, \frac{q}{2}]}^{||} = \overline{[-q,q]}^{||} = [-q,q],
$$

and thus

$$
\operatorname{supp}(\psi_r) \subseteq [-q,q]^n = \widetilde{\mathrm{B}}_q^{\|\|_\infty}(0),
$$

i. e., ψ_r fulfills property (a).

Clearly we have $\psi_r \in \mathrm{L}^1(\mathbb{R}^n)$, hence the Fourier transform $\mathcal{F}_n(\psi_r)\colon \mathbb{R}^n \to \mathbb{R}$ of ψ_r exists, i. e., ψ_r fulfills property (b).

We show that ψ_r satisfies property (c). Let $\mathcal{F}_1(\varphi_r)$ denote the Fourier transform of φ_r. Since the total variation of the r-th weak derivative h_r of φ_r is finite, there is a $c \in \mathbb{R}^{>0}$ such that $|\mathcal{F}_1(\varphi_r)(v)| \leq c \cdot (1+|v|)^{-(r+1)}$ for all $v \in \mathbb{R}$. Note that for all $v \in \mathbb{R}^n$ we have

$$
\begin{aligned}
\mathcal{F}_n(\psi_r)(v) &= (2\pi d)^p \cdot \mathcal{F}_n\Big(\bigotimes_{j=1}^n (\varphi_r * \varphi_r)\Big)(v) - (-1)^r \sum_{k=1}^n \mathcal{F}_n\Big(\frac{\partial^p}{\partial v_k} \bigotimes_{j=1}^n (\varphi_r * \varphi_r)\Big)(v) \\
&= (2\pi d)^p \cdot \mathcal{F}_n\Big(\bigotimes_{j=1}^n (\varphi_r * \varphi_r)\Big)(v) - (-1)^r \sum_{k=1}^n (2\pi \mathrm{i} v_k)^p \bigotimes_{j=1}^n \mathcal{F}_1(\varphi_r * \varphi_r)(v) \\
&= (2\pi d)^p \cdot \bigotimes_{j=1}^n \mathcal{F}_1(\varphi_r * \varphi_r)(v) - (-1)^r \sum_{k=1}^n (2\pi \mathrm{i} v_k)^p \bigotimes_{j=1}^n \mathcal{F}_1(\varphi_r * \varphi_r)(v) \\
&= (2\pi d)^p \cdot \Big(\Big(\bigotimes_{j=1}^n \mathcal{F}_1(\varphi_r)\Big)(v)\Big)^2 - (-1)^r \Big(\sum_{k=1}^n (2\pi \mathrm{i} v_k)^p\Big)\Big(\Big(\bigotimes_{j=1}^n \mathcal{F}_1(\varphi_r)\Big)(v)\Big)^2
\end{aligned}
$$

$$= \Big((2\pi d)^p - (-1)^r \sum_{k=1}^{n} (2\pi \mathrm{i} v_k)^p\Big) \cdot \Big(\Big(\bigotimes_{j=1}^{n} \mathcal{F}_1(\varphi_r)\Big)(v)\Big)^2$$

$$= ((2\pi d)^p - (-1)^r (2\pi \mathrm{i} \|v\|_p)^p) \cdot \prod_{j=1}^{n} (\mathcal{F}_1(\varphi_r)(v_j))^2$$

$$= ((2\pi d)^p - (2\pi \|v\|_p)^p) \cdot \prod_{j=1}^{n} (\mathcal{F}_1(\varphi_r)(v_j))^2,$$

so $\mathcal{F}_n(\psi_r)$ is clearly bounded, i. e., ψ_r fulfills property (c).

Let $v \in \widetilde{\mathrm{B}}_d^{\|\|_p}(0)$. Then, by the above,

$$\begin{aligned}\mathcal{F}_n(\psi_r)(v) &= ((2\pi d)^p - (2\pi \|v\|_p)^p) \cdot \prod_{j=1}^{n} (\mathcal{F}_1(\varphi_r)(v_j))^2 \\ &\geq ((2\pi d)^p - (2\pi d)^p) \cdot \prod_{j=1}^{n} (\mathcal{F}_1(\varphi_r)(v_j))^2 = 0,\end{aligned}$$

i. e., ψ_r fulfills property (d).

If $v \in \mathbb{R}^n \setminus \widetilde{\mathrm{B}}_d^{\|\|_p}(0)$, then

$$\begin{aligned}\mathcal{F}_n(\psi_r)(v) &= ((2\pi d)^p - (2\pi \|v\|_p)^p) \cdot \prod_{j=1}^{n} (\mathcal{F}_1(\varphi_r)(v_j))^2 \\ &\leq ((2\pi d)^p - (2\pi d)^p) \cdot \prod_{j=1}^{n} (\mathcal{F}_1(\varphi_r)(v_j))^2 = 0,\end{aligned}$$

i. e., ψ_r fulfills property (e).

It remains to show that ψ_r fulfills property (f). Note that, with B denoting the beta function (cf. Section 1.3),

$$\begin{aligned}\varphi_r * \varphi_r(0) &= \int_{\mathbb{R}} \varphi_r(t)\varphi_r(-t)\,\mathrm{d}t = \int_{]-\frac{q}{2},\frac{q}{2}[} \varphi_r(t)\varphi_r(-t)\,\mathrm{d}t \\ &= \int_{]-\frac{q}{2},\frac{q}{2}[} \left(1-\left(\frac{2t}{q}\right)^2\right)^r \left(1-\left(\frac{-2t}{q}\right)^2\right)^r \mathrm{d}t = \int_{]-\frac{q}{2},\frac{q}{2}[} \left(1-\left(\frac{2t}{q}\right)^2\right)^{2r} \mathrm{d}t \\ &= \int_{]-\frac{q}{2},\frac{q}{2}[} \left(1-\left(\frac{2t}{q}\right)^2\right)^p \mathrm{d}t = \frac{q}{2} \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \left(1-\left(\frac{2t}{q}\right)^2\right)^p \cdot \frac{2}{q}\,\mathrm{d}t \\ &= \frac{q}{2} \cdot \int_{]-1,1[} (1-t^2)^p\,\mathrm{d}t \\ &= \frac{q}{2} \cdot 2 \cdot \int_{]0,1[} (1-t^2)^p\,\mathrm{d}t \\ &\underset{t \mapsto \sqrt{t}}{\overset{\text{substitution}}{=}} \frac{q}{2} \cdot 2 \cdot \int_{]0,1[} (1-t)^p \cdot \frac{1}{2} t^{-1/2}\,\mathrm{d}t \\ &= \frac{q}{2} \cdot \int_{]0,1[} t^{-1/2}(1-t)^p\,\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \frac{q}{2} \cdot \mathrm{B}\left(\frac{1}{2}, p+1\right) \\
&\overset{\text{Theorem 1.3 (e)}}{=} \frac{q}{2} \cdot \frac{\Gamma(\frac{1}{2})\,\Gamma(p+1)}{\Gamma(\frac{1}{2}+p+q)} \\
&\underset{\text{Theorem 1.3 (b)}}{\overset{\text{Theorem 1.3 (a)}}{=}} \frac{q\sqrt{\pi}p!}{2\,\Gamma(p+\frac{3}{2})}.
\end{aligned}
$$

Further note that

$$
\begin{aligned}
h_r * h_r(0) &= \int_{\mathbb{R}} h_r(t) h_r(-t)\,\mathrm{d}t \\
&= (-1)^{2r} q^{-2r} 4^{2r} (r!)^2 \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \mathrm{P}_r\left(\frac{2t}{q}\right) \cdot \mathrm{P}_r\left(-\frac{2t}{q}\right) \mathrm{d}t \\
&= \left(\frac{4}{q}\right)^p (r!)^2 \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \mathrm{P}_r\left(\frac{2t}{q}\right) \cdot \mathrm{P}_r\left(-\frac{2t}{q}\right) \mathrm{d}t \\
&= (-1)^r \left(\frac{4}{q}\right)^p (r!)^2 \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \left(\mathrm{P}_r\left(\frac{2t}{q}\right)\right)^2 \mathrm{d}t \\
&= (-1)^r \left(\frac{4}{q}\right)^p (r!)^2 \cdot \frac{q}{2} \cdot \int_{]-\frac{q}{2},\frac{q}{2}[} \left(\mathrm{P}_r\left(\frac{2t}{q}\right)\right)^2 \cdot \frac{2}{q}\,\mathrm{d}t \\
&= (-1)^r \left(\frac{4}{q}\right)^p (r!)^2 \cdot \frac{q}{2} \cdot \int_{]-1,1[} (\mathrm{P}_r(t))^2\,\mathrm{d}t \\
&= (-1)^r \left(\frac{4}{q}\right)^p (r!)^2 \cdot \frac{q}{2} \cdot \frac{2}{2r+1} \\
&= (-1)^r \cdot \frac{4^p (r!)^2}{(p+1)q^{p-1}}.
\end{aligned}
$$

Observe that by the Legendre duplication formula (Theorem 1.3 (d)) we have

$$
\Gamma(p+2) = \Gamma(2\cdot(r+1)) = \frac{2^{p+1}}{\sqrt{\pi}} \cdot \Gamma(r+1)\,\Gamma\left(r+\frac{3}{2}\right).
$$

Since, by hypothesis, we have

$$
dq > c_p \cdot \sqrt[p]{n},
$$

we get

$$
\begin{aligned}
1 &< \frac{(dq)^p}{n \cdot c_p^p} = \frac{(dq)^p \cdot \pi^p \cdot \Gamma(\frac{p+3}{2})}{n \cdot \Gamma(r+1) \cdot \Gamma(p+\frac{3}{2})} = \frac{(dq)^p \cdot \pi^{p+1/2} \cdot 2^{p+1} \cdot \Gamma(r+1) \cdot \Gamma(r+\frac{3}{2})}{n \cdot \sqrt{\pi} \cdot 2^{p+1} \cdot (r!)^2 \cdot \Gamma(p+\frac{3}{2})} \\
&\overset{\text{Legendre}}{=} \frac{(dq)^p \cdot \pi^{p+1/2} \cdot \Gamma(p+2)}{n \cdot 2^{p+1} \cdot (r!)^2 \cdot \Gamma(p+\frac{3}{2})} = \frac{(2\pi dq)^p \cdot \sqrt{\pi} \cdot (p+1)!}{n \cdot 4^p \cdot (r!)^2 \cdot 2\,\Gamma(p+\frac{3}{2})},
\end{aligned}
$$

hence

$$
(2\pi d)^p \cdot \frac{q\sqrt{\pi}p!}{2\,\Gamma(p+\frac{3}{2})} - \frac{n \cdot 4^p \cdot (r!)^2}{(p+1)q^{p-1}} > 0.
$$

Therefore we obtain

$$\begin{aligned}
\psi_r(0) &= (2\pi d)^p \cdot \bigotimes_{j=1}^{n} (\varphi_r * \varphi_r)(0) - (-1)^r \sum_{k=1}^{n} \frac{\partial^p}{\partial x_k^p}_{|x_k=0} \bigotimes_{j=1}^{n} (\varphi_r * \varphi_r)(x) \\
&= (2\pi d)^p \cdot \prod_{j=1}^{n} (\varphi_r * \varphi_r)(0) - (-1)^r \sum_{k=1}^{n} \frac{\partial^p}{\partial x_k^p}_{|x_k=0} \prod_{j=1}^{n} (\varphi_r * \varphi_r)(x_k) \\
&= (2\pi d)^p \cdot ((\varphi_r * \varphi_r)(0))^n - (-1)^r ((\varphi_r * \varphi_r)(0))^{n-1} \cdot \sum_{k=1}^{n} \frac{\partial^p}{\partial x_k^p}_{|x_k=0} (\varphi_r * \varphi_r)(x_k) \\
&= ((2\pi d)^p \cdot ((\varphi_r * \varphi_r)(0))^n \\
&\quad -(-1)^r ((\varphi_r * \varphi_r)(0))^{n-1} \cdot \sum_{k=1}^{n} \frac{\partial^r}{\partial x_k^r}_{|x_k=0} \varphi_r(x_k) * \frac{\partial^r}{\partial x_k^r}_{|x_k=0} \varphi_r(x_k)) \\
&= ((\varphi_r * \varphi_r)(0))^{n-1} \cdot ((2\pi d)^p \cdot (\varphi_r * \varphi_r)(0) - (-1)^r \cdot \sum_{k=1}^{n} (h_r * h_r)(0)) \\
&= ((\varphi_r * \varphi_r)(0))^{n-1} \cdot ((2\pi d)^p \cdot (\varphi_r * \varphi_r)(0) - (-1)^r \cdot n \cdot (h_r * h_r)(0)) \\
&= \underbrace{((\varphi_r * \varphi_r)(0))^{n-1}}_{>0,\text{ see above}} \cdot \underbrace{((2\pi d)^p \cdot \frac{q\sqrt{\pi}p!}{2\,\Gamma(p+\frac{3}{2})} - \frac{n \cdot 4^p \cdot (r!)^2}{(p+1)q^{p-1}})}_{>0,\text{ see above}} \\
&> 0,
\end{aligned}$$

that is, ψ fulfills property (f). Thus $\psi = \psi_r$ works, and the proof is done. q. e. d.

THEOREM 3.6: *Let $n, p \in \mathbb{N} \setminus \{0\}$, p even, and let $d, q \in \mathbb{R}^{>0}$ with*

$$d > \frac{2p+3}{\mathrm{e}\pi q} \cdot \sqrt[p]{n}.$$

Then there is a function $\psi \colon \mathbb{R}^n \to \mathbb{R}$ with the properties (a)–(f) *from Lemma* 3.5.

PROOF: By Lemma 3.5 it is sufficient to show that

$$c_p := \left(\frac{\Gamma(\frac{p}{2}+1) \cdot \Gamma(p+\frac{3}{2})}{\pi^p \cdot \Gamma(\frac{p+3}{2})} \right)^{1/p} \leq \frac{2p+3}{\mathrm{e}\pi}.$$

By Stirling's approximation formula (Theorem 1.3 (c)) we have

$$\begin{aligned}
\Gamma\Big(\frac{2p+3}{2}\Big) &\leq \sqrt{\frac{4\pi}{2p+3}} \cdot \Big(\frac{2p+3}{2\mathrm{e}}\Big)^{(2p+3)/2} \cdot \mathrm{e}^{2/(12(2p+3))} \\
&\leq \sqrt{\frac{4\pi}{2p+3}} \cdot \Big(\frac{2p+3}{2\mathrm{e}}\Big)^{(2p+3)/2} \cdot \mathrm{e}^{1/18} \\
&= \Big(\frac{2p+3}{\mathrm{e}}\Big)^p \cdot \frac{1}{2^p} \cdot \Big(\frac{(2p+3)}{2\mathrm{e}}\Big)^{3/2} \cdot \Big(\frac{4\pi}{2p+3}\Big)^{1/2} \cdot \mathrm{e}^{1/18} \\
&= \Big(\frac{2p+3}{\mathrm{e}}\Big)^p \cdot \frac{(2p+3) \cdot \sqrt{\pi} \cdot \mathrm{e}^{1/18}}{2^{p-1} \cdot (2\mathrm{e})^{3/2}},
\end{aligned}$$

and we claim that
$$\frac{(2p+3)\cdot\sqrt{\pi}\cdot \mathrm{e}^{1/18}}{2^{p-1}\cdot(2\mathrm{e})^{3/2}} \le 1.$$
To prove this, it is sufficient to consider the case $p = 2$. Using the estimates
$$\begin{aligned}
\sqrt{\pi} &\le 2,\\
\mathrm{e}^{1/18} &\le \frac{11}{10} \qquad \left(\text{since } \left(\frac{11}{10}\right)^{18} = \frac{5\,559\,917\,313\,492\,231\,481}{1\,000\,000\,000\,000\,000\,000} \ge 5 \ge \mathrm{e}\right),\\
(2\mathrm{e})^{3/2} &\ge 5^{3/2} = 5\cdot\sqrt{5} \ge 5\cdot 2 = 10,
\end{aligned}$$
one obtains
$$\frac{(2\cdot 2+3)\cdot\sqrt{\pi}\cdot \mathrm{e}^{1/18}}{2^{1}\cdot(2\mathrm{e})^{3/2}} \le \frac{7\cdot 2\cdot\frac{11}{10}}{2\cdot 10} = \frac{7\cdot 11}{10\cdot 10} = \frac{77}{100} \le 1.$$
Therefore, for arbitrary p,
$$\Gamma\left(\frac{2p+3}{2}\right) \le \left(\frac{2p+3}{\mathrm{e}}\right)^{p},$$
and thus
$$\begin{aligned}
c_p = \left(\frac{\Gamma(\frac{p}{2}+1)\cdot\Gamma(p+\frac{3}{2})}{\pi^p\cdot\Gamma(\frac{p+3}{2})}\right)^{1/p} &\le \left(\frac{\Gamma(\frac{p}{2}+1)\cdot(2p+3)^p}{(\mathrm{e}\pi)^p\cdot\Gamma(\frac{p+3}{2})}\right)^{1/p}\\
&= \frac{2p+3}{\mathrm{e}\pi}\cdot\underbrace{\left(\frac{\Gamma(\frac{p+2}{2})}{\Gamma(\frac{p+3}{2})}\right)^{1/p}}_{\le 1}\\
&\le \frac{2p+3}{\mathrm{e}\pi},
\end{aligned}$$
and the proof is done. q. e. d.

In this context, the distance on the n-torus $\mathbb{T}^n \subseteq \mathbb{C}^n$ is measured not by the metric induced by $(\mathbb{C}^n, \|\|)$, but instead by the metric introduced in the following definition.

Definition: Let
$$\begin{aligned}
\mathrm{w}_1\colon\ \mathbb{T}^1\times\mathbb{T}^1 &\longrightarrow \mathbb{R}^{\ge 0},\\
(b,c) &\longmapsto \min\left\{ \left|\frac{1}{2\pi}(\arg(b)-\arg(c))+\alpha\right| \;\middle|\; \alpha\in\mathbb{Z}\right\},
\end{aligned}$$
and more generally for $n\in\mathbb{N}$,
$$\begin{aligned}
\mathrm{w}_n\colon\ \mathbb{T}^n\times\mathbb{T}^n &\longrightarrow \mathbb{R}^{\ge 0},\\
(b,c) &\longmapsto \max\{\mathrm{w}_1(b_j,c_j)\mid j=1,\dots,n\}.
\end{aligned}$$
w_n is called *wrap-around metric* on $\mathbb{T}^n$.

Remark 3.7: (a) $(\mathbb{T}^n, \mathrm{w}_n)$ is a metric space.

(b) Let τ_{w_n} be the topology on $\mathbb{T}^n$ induced by w_n and let τ_{s} be the subspace topology on $\mathbb{T}^n$ induced by $(\mathbb{C}^n, \|\|)$. Then the identity $\mathrm{id}\colon (\mathbb{T}^n, \tau_{\mathrm{s}}) \to (\mathbb{T}^n, \tau_{\mathrm{w}_n})$ is a homeomorphism. In particular, $(\mathbb{T}^n, \mathrm{w}_n)$ is compact.

Definition: Let $M \subseteq \mathbb{T}^n$. The *toroidal separation* of M is defined as

$$\mathrm{sep}_{\mathrm{t}}(M) := \inf\{\mathrm{w}_n(b,c) \mid b, c \in M,\, b \neq c\} \in \mathbb{R}^{\geq 0} \cup \{\infty\},$$

where $\inf \emptyset = \infty$.

Remark 3.8: (a) Let $M \subseteq \mathbb{T}^n$. Then the following are equivalent.

(i) $|M| \leq 1$.

(ii) $\mathrm{sep}_{\mathrm{t}}(M) = \infty$.

Indeed, if $|M| \leq 1$, then $\mathrm{sep}_{\mathrm{t}}(M) = \inf \emptyset = \infty$. On the other hand, if $|M| \geq 2$, then there are $b, c \in M$ with $b \neq c$ and thus $\mathrm{sep}_{\mathrm{t}}(M) \leq \mathrm{w}_n(b,c) \in \mathbb{R}$.

(b) Let $M \subseteq \mathbb{T}^n$. Then we have

$$\mathrm{sep}_{\mathrm{t}}(M) \in [0, 1/2] \cup \{\infty\}.$$

To see this, observe the following. Clearly we have $\mathrm{sep}_{\mathrm{t}}(M) \geq 0$ by definition. Let $\mathrm{sep}_{\mathrm{t}}(M) \neq \infty$. By part (a), there are $b, c \in M$ with $b \neq c$. Let $x := 1/(2\pi) \cdot (\arg(b) - \arg(c))$. Since $\arg(b), \arg(c) \in [0, 2\pi[^n$, clearly $x \in [-1, 1[^n$. Let $\alpha \in \mathbb{Z}^n$ such that $|\alpha_j + x_j| \leq 1/2$ for all $j = 1, \ldots, n$ (if $x_j \in [-1, -1/2[$, set $\alpha_j := 1$; if $x_j \in [-1/2, 1/2[$, set $\alpha_j := 0$; if $x_j \in [1/2, 1[$, set $\alpha_j := -1$). Then $\mathrm{sep}_{\mathrm{t}}(M) \leq \mathrm{w}_n(b,c) \leq \|x + \alpha\|_\infty \leq 1/2$.

(c) Let $M \subseteq \mathbb{T}^n$. Then we have

$$\mathrm{sep}_{\mathrm{t}}(M) = \inf\left\{ \left\| \frac{1}{2\pi}(\arg(b) - \arg(c)) + \alpha \right\|_\infty \;\middle|\; \begin{array}{c} b, c \in M,\, b \neq c, \\ \alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}_1^{\|\|_\infty}(0) \end{array} \right\}.$$

This follows in the same fashion as in part (b).

(d) Let $M \subseteq \mathbb{T}^n$. Then the following are equivalent.

(i) M is finite.

(ii) $\mathrm{sep}_{\mathrm{t}}(M) > 0$.

To see this, observe the following. If M is finite, clearly $\mathrm{sep}_{\mathrm{t}}(M) > 0$. Let M be not finite. Then there is an injective sequence $(b_k)_{k \in \mathbb{N}} \in M^{\mathbb{N}}$. Since $(\mathbb{T}^n, \mathrm{w}_n)$ is compact by Remark 3.7 (b), $(b_k)_k$ has a w_n-Cauchy-subsequence, hence $\mathrm{sep}_{\mathrm{t}}(M) \leq \inf\{\mathrm{w}_n(b_k, b_\ell) \mid k, \ell \in \mathbb{N},\, k \neq \ell\} = 0$.

The following theorem provides a multivariate version of the classical Ingham inequality.

Theorem 3.9 (Multivariate Ingham inequality): *Let* $n, p \in \mathbb{N}\backslash\{0\}$*, p even, and* $d, q \in \mathbb{R}^{>0}$ *with*

$$d > \frac{2p+3}{\mathrm{e}\pi q} \cdot \sqrt[p]{n}.$$

Then there is a $c \in \mathbb{R}^{>0}$ *such that for all* $M \in \mathcal{P}_{\mathrm{f}}(\mathbb{T}^n)$ *with*

$$\operatorname{sep}_{\mathrm{t}}(M) > q$$

and for all $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n_M(\mathbb{C})$*, we have*

$$\sum_{\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|_p}_d(0)} |f(\alpha)|^2 \geq c \cdot \|\operatorname{coeff}(f)\|_2^2.$$

Proof: If $|M| \leq 1$, the assertion holds trivially. Thus let $|M| \geq 2$. By Remark 3.8 (b), (a), and (d), we have $\operatorname{sep}_{\mathrm{t}}(M) \in]0, 1/2]$. Let $\psi \colon \mathbb{R}^n \to \mathbb{R}$ with the properties (a)–(f) in Lemma 3.5/Corollary 3.6. Let $(f_b)_{b \in M} \in \mathbb{C}^M$ with $f = \sum_{b \in M} f_b \exp_b$. Then we have

$$\begin{aligned}
&\max\{\mathcal{F}_n(\psi)(v) \mid v \in \mathbb{R}^n\} \cdot \sum_{\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|_p}_d(0)} |f(\alpha)|^2 \\
&= \sum_{\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|_p}_d(0)} \max\{\mathcal{F}_n(\psi)(v) \mid v \in \mathbb{R}^n\} \cdot |f(\alpha)|^2 \\
&\geq \sum_{\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|_p}_d(0)} \mathcal{F}_n(\psi)(\alpha) \cdot |f(\alpha)|^2 \\
&\geq \sum_{\alpha \in \mathbb{Z}^n} \mathcal{F}_n(\psi)(\alpha) \cdot |f(\alpha)|^2 \\
&= \sum_{\alpha \in \mathbb{Z}^n} \mathcal{F}_n(\psi)(\alpha) \sum_{b,c \in M} f_b \overline{f_c} \exp_b(\alpha) \exp_{\overline{c}}(\alpha) \\
&= \sum_{b,c \in M} f_b \overline{f_c} \sum_{\alpha \in \mathbb{Z}^n} \mathcal{F}_n(\psi)(\alpha) \exp_{b\overline{c}}(\alpha) \\
&\overset{\substack{\text{Poisson summation formula} \\ \text{Section 1.3}}}{=} \sum_{b,c \in M} f_b \overline{f_c} \sum_{\alpha \in \mathbb{Z}^n} \psi\Big(\underbrace{\frac{1}{2\pi}(\arg(b) - \arg(c)) + \alpha}_{\notin \widetilde{\mathrm{B}}^{\|\|_\infty}_q(0) \supseteq \operatorname{supp}(\psi) \text{ if } b \neq c}\Big) \\
&= \sum_{b \in M} \Big(f_b \overline{f_b} \sum_{\alpha \in \mathbb{Z}^n} \psi(\alpha) \Big) \\
&= \Big(\sum_{b \in M} f_b \overline{f_b} \Big) \cdot \Big(\sum_{\alpha \in \mathbb{Z}^n} \psi(\alpha) \Big) \\
&= \|\operatorname{coeff}(f)\|_2^2 \cdot \sum_{\alpha \in \mathbb{Z}^n} \psi(\alpha) \\
&\overset{\substack{q < \operatorname{sep}_{\mathrm{t}}(M) \leq 1/2 \\ \operatorname{supp}(\psi) \cap \mathbb{Z}^n = \{0\}}}{=} \psi(0) \cdot \|\operatorname{coeff}(f)\|_2^2.
\end{aligned}$$

Thus the assertion follows with $c := \psi(0)/\max\{\mathcal{F}_n(\psi)(v) \mid v \in \mathbb{R}^n\} > 0$. q. e. d.

Recall the $\star$-filtration $\mathcal{M}$ induced by maximal degree from Chapter 2,

$$\mathcal{M}_d = \{\alpha \in \mathbb{N}^n \mid \max\deg(\alpha) \le d\}.$$

The following corollary provides a sufficient condition on $d \in \mathbb{N}$ for the surjectivity of the evaluation homomorphism $\mathrm{ev}^M_{\mathcal{M}_d} \colon \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n]_{\mathcal{M}_d} \to \mathbb{C}^M$ in terms of the toroidal separation of $M \subseteq \mathbb{T}^n$.

Corollary 3.10 (Polynomial interpolation on the complex torus): *Let $n \in \mathbb{N} \setminus \{0\}$, $q \in \mathbb{R}^{>0}$ and $d \in \mathbb{N}$ with*

$$d > \frac{2\ln(n) + 3}{q}.$$

Then for all $M \in \mathcal{P}_\mathrm{f}(\mathbb{T}^n)$ with $\mathrm{sep}_\mathrm{t}(M) > q$, $\mathrm{ev}^M_{\mathcal{M}^n_d} \colon \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n]_{\mathcal{M}^n_d} \to \mathbb{C}^M$ is surjective.

Proof: Let $V := \mathrm{V}^M_{\mathcal{M}^n_d} \in \mathbb{C}^{M \times \mathcal{M}^n_d}$ be the transformation matrix of $\mathrm{ev}^M_{\mathcal{M}^n_d}$. We show that $\ker(V^\top) = \{0\}$. Let $(g_b)_{b\in M} \in \mathbb{C}^M \setminus \{0\}$, $g := \sum_{b \in M} g_b \exp_b \in {}_\mathbb{Z}\mathrm{Exp}^n_M(\mathbb{C})$, $\beta := \lfloor d/2 \rfloor \cdot (1, \ldots, 1)^\top \in \mathbb{Z}^n$, $f_b := g_b b^\beta \in \mathbb{C}$, and $f := \sum_{b\in M} f_b \exp_b \in {}_\mathbb{Z}\mathrm{Exp}^n_M(\mathbb{C})$, and let $p := 2\lceil \ln(n) \rceil$. Then we have

$$\begin{aligned}\frac{2p+3}{\mathrm{e}\cdot\pi} \cdot \sqrt[p]{n} &\le \frac{2p+3}{\mathrm{e}\cdot\pi} \cdot \sqrt[p]{\mathrm{e}^{p/2}} = \frac{2p+3}{\mathrm{e}\cdot\pi} \cdot \sqrt{\mathrm{e}} = \frac{2p+3}{\pi\cdot\sqrt{\mathrm{e}}} = \frac{4\lceil\ln(n)\rceil + 3}{\pi\cdot\sqrt{\mathrm{e}}} \le \frac{4\ln(n)+7}{\pi\cdot\sqrt{\mathrm{e}}} \\ &= \frac{4}{\pi\cdot\sqrt{\mathrm{e}}}\cdot\ln(n) + \frac{7}{\pi\cdot\sqrt{\mathrm{e}}} \le \ln(n) + \frac{3}{2} < \frac{d}{2}\cdot q,\end{aligned}$$

and therefore

$$\frac{d}{2} > \frac{2p+3}{\mathrm{e}\cdot\pi\cdot q} \cdot \sqrt[p]{n}.$$

Note that

$$\mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|\|_\infty}_{d/2}(0) \subseteq \mathcal{M}^n_d - \beta.$$

since, if $\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|\|_\infty}_{d/2}(0)$, then $\alpha_j + \beta_j \ge -\lfloor d/2 \rfloor + \lfloor d/2 \rfloor = 0$, i. e., $\alpha + \beta \in \mathbb{N}^n$ and $\|\alpha+\beta\|_\infty = \|\alpha + \lfloor d/2\rfloor \cdot (1,\ldots,1)^\top\|_\infty \le \|\alpha\|_\infty + \lfloor d/2 \rfloor \cdot \|(1,\ldots,1)^\top\|_\infty \le d/2 + \lfloor d/2 \rfloor \le d/2 + d/2 = d$, i. e., $\alpha + \beta \in \mathcal{M}^n_d$, so $\alpha \in \mathcal{M}^n_d - \beta$.

Thus, by Theorem 3.9,

$$\begin{aligned}\|V^\top \cdot (g_b)_{b\in M}\|_2^2 = \sum_{\alpha\in\mathcal{M}^n_d} |g(\alpha)|^2 = \sum_{\alpha\in\mathcal{M}^n_d} |f(\alpha-\beta)|^2 &\ge \sum_{\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|\|_\infty}_{d/2}(0)} |f(\alpha)|^2 \\ &\ge \sum_{\alpha \in \mathbb{Z}^n \cap \widetilde{\mathrm{B}}^{\|\|\|_p}_{d/2}(0)} |f(\alpha)|^2 > 0,\end{aligned}$$

hence $V^\top \cdot (g_b)_{b\in M} \ne 0$ and consequently $\ker(V^\top) = \{0\}$. q. e. d.

In the following we combine Corollary 3.10 with the theory from Chapter 2 in an analogous way to Theorem 3.1 in Section 3.1. Since we are working with the maximal degree $\star$-filtration now, we can only draw conclusions about the zero loci, and since the torus $\mathbb{T}^n$ is not contained in a proper zero locus, we are not able to discard equations.

Since we have $\mathbb{T}^n \subseteq (\mathbb{C} \setminus \{0\})^n$, there is no need to distinguish between $\mathrm{Exp}^n_{\mathbb{T}^n}(\mathbb{C})$ and its counterpart ${}_{\mathbb{Z}}\mathrm{Exp}^n_{\mathbb{T}^n}(\mathbb{C})$ and for the remainder of the section we will not do so.

Corollary 3.11: *Let $n \in \mathbb{N} \setminus \{0\}$ and $f \in \mathrm{Exp}^n_{\mathbb{T}^n}(\mathbb{C})$. If $d \in \mathbb{N}$ is such that*

$$d > \frac{2\ln(n) + 3}{q} + 1.$$

then

$$\mathrm{Z}(\ker \mathrm{T}_{\mathcal{M}_d}(f)) = \mathrm{Z}(\ker \mathrm{H}_{\mathcal{M}_d}(f)) = \mathrm{supp}(f).$$

Proof: This follows immediately from Corollary 3.10 and Corollary 2.19/2.35. q. e. d.

Corollary 3.12: *Let $M \in \prod_{n\in\mathbb{N}\setminus\{0\}} \mathcal{P}_\mathrm{f}(\mathbb{T}^n)$ be such that there is a $q \in \mathbb{R}$ with*

$$0 < q < \mathrm{sep}_\mathrm{t}(M_n) \text{ for all } n \in \mathbb{N},$$

and for $n \in \mathbb{N} \setminus \{0\}$ let

$$d_n := \min\{d \in \mathbb{N} \mid \mathrm{ev}^{M_n}_{\mathcal{M}_d} \colon \mathbb{C}[\mathrm{x}_1, \ldots, \mathrm{x}_n]_{\mathcal{M}_d} \to \mathbb{C}^{M_n} \text{ is surjective}\}.$$

Then we have $d_n \in \mathrm{O}(\ln(n))$.

Proof: By Corollary 3.10, $\mathrm{ev}^{M_n}_{\mathcal{M}_{e_n}}$ is surjective for $e_n := \left\lceil \frac{2\ln(n)+3}{q} \right\rceil$. Thus $d_n \leq e_n$ for all $n \in \mathbb{N} \setminus \{0\}$, and hence $d_n \in \mathrm{O}(e_n) = \mathrm{O}(\ln(n))$. q. e. d.

Remark 3.13: It seems to be an open problem if under the conditions in Corollary 3.12 one has $\mathrm{O}(d_n) \subsetneq \mathrm{O}(\ln(n))$.

Corollary 3.14: *Let $f \in \prod_{n\in\mathbb{N}\setminus\{0\}} \mathrm{Exp}^n_{\mathbb{T}^n}(\mathbb{C})$ be such that there is a $q \in \mathbb{R}$ with*

$$0 < q < \mathrm{sep}_\mathrm{t}(\mathrm{supp}(f_n)) \text{ for all } n \in \mathbb{N},$$

and for $n \in \mathbb{N} \setminus \{0\}$ let

$$d_n := \min\{d \in \mathbb{N} \mid \mathrm{Z}(\ker \mathrm{T}_{\mathcal{M}_d}(f_n)) = \mathrm{supp}(f_n)\}.$$

Then we have $d_n \in \mathrm{O}(\ln(n))$.

Proof: This follows immediately from Corollary 3.11 and Corollary 3.12. q. e. d.

4. Classical and recent approaches to the reconstruction of exponential sums

Prony's article dates back to 1795, and since then there has been a steady flow of research on the reconstruction of exponential sums and related problems in many different areas of mathematics and other fields, such as signal analysis, measure theory, number theory, algebraic geometry, numerical analysis, functional analysis, optimization theory, physics, and quantum chemistry. Some of these approaches employ vastly different methods in order to solve their specific problems. This chapter is concerned with a few out of the many alternative approaches that have been developed to attack the reconstruction of exponential sums and related problems. We also try to present a bit of the history behind the methods in this thesis. However, we will not push ourselves towards a doomed attempt at providing deep insight into all or any of those approaches, nor at a detailed comparison with our work, and confine ourselves to discuss basic principles of a few of them and give some pointers to additional related literature. We proceed in roughly chronological order.

4.1. Prony's original version

In 1795, Prony[1] published his *essai expérimental et analytique* [66] on the physical behavior of some fluids and gases under different temperatures. We already discussed the principles behind the univariate case in Remark 2.5. In his article, Prony assumes that certain physical dependencies could be modeled well by univariate exponential sums, which led him to develop a univariate method to reconstruct these exponential sums which we took as inspiration and generalized in this thesis, cf. Prony [66, *Première partie*].[2]

In contrast to much of the modern focus, Prony was not particularly concerned with exponential sums of large rank, as evidenced by his statement [66, p. 29]

> *«Il n'arrivera presque jamais qu'on ait huit ou neuf résultats à faire entrer dans la formule, et on pourra, sans sortir des limites dans lesquelles on a des*

[1] Gaspard Clair François Marie Riche, baron de Prony, 1755–1839. Among the many honors he has received are: Secretary of mathematical sciences at the French Academy of Sciences; Member of the Royal Swedish Academy of Sciences, the Royal Society of London, the Royal Society of Edinburgh, and an inscription of his name on the Eiffel tower.

[2] An English translation of this part of Prony's article can be found in Auton-Van Blaricum [5, Section 2.0], which also contains a large body of references to literature before 1980.

méthodes pour la solution des équations numériques, traiter tous les cas que la physique présente ordinairement.»[3]

It may be noteworthy that Prony treats the cases of an even and an odd number of samples separately, in order to be slightly more efficient in the case of odd rank r; only $2r-1$ measurements are needed then. In his experiments, Prony interpolates functions modeled by exponential sums of rank three using five (real world) measurements in this way. More on the remarkably history of Prony and his achievements, also besides those that are relevant here, can be found in the book and article by Bradley [13, 14] on "Prony the bridge-builder", a denomination that seems appropriate literally as well as metaphorically.

4.2. Sylvester and the Waring problem

Around 1850, Sylvester[4] [75, 76] was working on the *Waring problem for binary forms.*[5] This incarnation of the problem asks, given a homogeneous polynomial $p \in K[\mathrm{x},\mathrm{y}]$ of (total) degree d, for homogeneous polynomials $\ell_i \in K[\mathrm{x},\mathrm{y}]$ of (total) degree one, and $\lambda_i \in K$, $i = 1,\ldots,r$, such that

$$p = \sum_{i=1}^{r} \lambda_i \ell_i^d$$

with $r \in \mathbb{N}$ being minimal for such a decomposition to exist. Note that the ℓ_i can only be unique up to a non-zero factor, i. e., they correspond to points in the projective space $\mathbb{P}^1_{\mathbb{C}}$. In particular, Sylvester proved that a general binary form $p \in \mathbb{C}[\mathrm{x},\mathrm{y}]$ of odd degree d admits a *unique* (up to non-zero factors) minimal decomposition $p = \sum_{i=1}^{r} \ell_i^d$ with $r = (d+1)/2$. Consider a homogeneous polynomial

$$p = \sum_{i=0}^{d} p_i \mathrm{x}^i \mathrm{y}^{d-1} \in \mathbb{C}[\mathrm{x},\mathrm{y}]$$

in the indeterminates x and y with $d = \operatorname{tot}\deg(p)$. For $i = 0,\ldots,d$ let $c_i := p_i/\binom{d}{i}$ and for $r = 0,\ldots,d$ let

$$\mathrm{C}_r(p) := (c_{i+j})_{\substack{i=0,\ldots,d-r\\ j=0,\ldots,r}} = \begin{pmatrix} c_0 & c_1 & \cdots & c_{r-1} & c_r \\ c_1 & c_2 & \cdots & c_r & c_{r+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{d-r-1} & c_{d-r} & \cdots & c_{d-2} & c_{d-1} \\ c_{d-r} & c_{d-r+1} & \cdots & c_{d-1} & c_d \end{pmatrix} \in \mathbb{C}^{(d-r+1)\times(r+1)}$$

[3] "It will almost never happen that there are eight or nine results to be included in the formula, and without exceeding the limits in which methods for the solution of numerical equations exist, we may treat all the cases ordinarily presented by physics." Note that having eight "results" (Prony is referring to measurements in a physical experiment) leads to a polynomial equation of degree four. Formulas expressing the roots of such polynomials by radicals of the coefficients were known since the 16th century.

[4] James Joseph Sylvester, 1814–1897.

[5] Informal history of Sylvester's method in Iarrobino-Kanev [46, Introduction].

be the r*-th catalecticant* of p. Sylvester has the following theorem, the statement here coming from Brachat-Comon-Mourrain-Tsigaridas [12].

THEOREM 4.1 (Sylvester): *Let* $p = \sum_{i=0}^{d} p_i \mathrm{x}^i \mathrm{y}^{d-1} \in \mathbb{C}[\mathrm{x},\mathrm{y}]$ *and* $r \in \mathbb{N}$. *Then the following are equivalent:*

(i) *There are* $\lambda_i \in \mathbb{C} \setminus \{0\}$ *and* $\ell_i \in K[\mathrm{x},\mathrm{y}]$ *with* $\operatorname{tot}\deg(\ell_i) = 1$ *such that*

$$p = \sum_{i=1}^{r} \lambda_i (a_i \mathrm{x} + b_i \mathrm{y})^d.$$

(ii) *There is an* $f \in \ker \mathrm{C}_r(p) \setminus \{0\}$ *such that the homogeneous polynomial*

$$\sum_{i=0}^{r} f_i \mathrm{x}^i \mathrm{y}^{r-i}$$

has r *distinct roots in* $\mathbb{P}^1_{\mathbb{C}}$.

If these conditions are fulfilled, then $p = \sum_{i=1}^{r} \lambda_i (a_i \mathrm{x} + b_i \mathrm{y})^d$ *with* $\lambda_i, a_i, b_i \in \mathbb{C}$ *given by the following:*

(a) $(b_i, -a_i)$, $i = 1, \ldots, r$, *are the roots of* $\sum_{i=1}^{r} f_i \mathrm{x}^i \mathrm{y}^{r-i}$ *with* $\|(b_i, a_i)\|_2 = 1$.

(b) $(\lambda_1, \ldots, \lambda_r)$ *is the unique* $\lambda \in \mathbb{C}^r$ *with*

$$\begin{pmatrix} b_1^d & \cdots & b_r^d \\ a_1 b_1^{d-1} & \cdots & a_r b_r^{d-1} \\ a_1^2 b_1^{d-2} & \cdots & a_r^2 b_r^{d-2} \\ \vdots & \vdots & \vdots \\ a_1^{d-2} b_1^2 & \cdots & a_r^{d-2} b_r^2 \\ a_1^{d-1} b_1 & \cdots & a_r^{d-1} b_r \\ a_1^d & \cdots & a_r^d \end{pmatrix} \lambda = \begin{pmatrix} p_d \\ p_{d-1} \\ p_{d-2} \\ \vdots \\ p_2 \\ p_1 \\ p_0 \end{pmatrix}.$$

There is a connection (well-known by now, but it seems that Sylvester may not have been aware of this link to Prony's earlier work) between this problem and Sylvester's solution and the reconstruction problem for exponential sums and Prony's method that we explain next. Consider a homogeneous polynomial

$$p = \sum_{i=0}^{d} p_i \mathrm{x}^i \mathrm{y}^{d-1} \in \mathbb{C}[\mathrm{x},\mathrm{y}]$$

in the indeterminates x and y with $d = \operatorname{tot}\deg(p)$, and let

$$p = \sum_{k=1}^{r} \lambda_k (a_k \mathrm{x} + b_k \mathrm{y})^d = \sum_{k=1}^{r} \lambda_k b_k^d \cdot (\frac{a_k}{b_k} \mathrm{x} + \mathrm{y})^d$$

be a Waring decomposition of p with $b_k \neq 0$ for all $k = 1, \dots, r$. Let

$$f_p := \sum_{k=1}^{r} \lambda_k b_k^d \exp_{a_k/b_k} \in \operatorname{Exp}^1(\mathbb{C}).$$

Clearly,

$$\mathrm{C}_{d-r}(p) = \mathrm{H}_r(f_p) \in \mathbb{C}^{(r+1)\times(r+1)},$$

and the interpretation of the kernel $\ker \mathrm{C}_{d-r}(f_p) = \langle q \rangle_{\mathbb{C}}$ as $q = \sum_{i=0}^{r} q_i \mathrm{x}^i \mathrm{y}^{r-i} \in \mathbb{C}[\mathrm{x}, \mathrm{y}]$ is the homogenization of the polynomial $\sum_{i=0}^{r} q_i \mathrm{x}^i$ occurring in Prony's method and $\{(a_k/b_k, 1) \mid k = 1, \dots, r\} = \mathrm{Z}(q) \cap (\mathbb{C} \times \{1\}) = \operatorname{supp}(f_p) \times \{1\}$. Sylvester's method can thus be seen as a projective variant of Prony's method.

A distinction that may be drawn between this problem and the reconstruction problem that Prony considered is that reconstruction of a signal calls for a method using a "small", but sufficiently large number of samples out of an infinite set in order to reconstruct an exponential sum f uniquely, whereas computation of a Waring decomposition has to come up with *any* minimal decomposition from an a priori given finite set (the coefficients of the polynomial p).

Sylvester's approach has recently been extended to the case of more than two indeterminates, see Section 4.6 below.

4.3. Padé approximation

The main idea of Padé[6] approximation goes back at least to Frobenius[7] [36] in 1879. An account of its history can be found in Brezinski [15, 16]. It is by now well-known known that Prony's method is related to this theory. We will recall this relationship in the following along the lines of Weiss-McDonough [79]. The relevant standard material concerning complex functions can be found in any textbook on complex analysis, e. g. the textbook by Remmert [65].

If $s \in \mathbb{C}[\![\mathrm{z}]\!]$ is a power series and $d, e \in \mathbb{N}$, any rational function $R = p/q \in \mathbb{C}(\mathrm{z})$ with $p \in \mathbb{C}[\mathrm{z}]_d$, $q \in \mathbb{C}[\mathrm{z}]_e$, and $q_0 = q(0) \neq 0$ such that the first $d + e$ coefficients of the Taylor series of R in 0 coincide with the respective coefficients of s, i. e., such that

$$R^{(\alpha)}(0) = \alpha! s_\alpha \text{ for all } \alpha = 0, \dots, d + e - 1,$$

is called a *Padé approximant* (*of order* (d, e)) of s.

In order to build the bridge to the univariate case of Prony's method, recall the $\mathbb{C}$-linear map

$$\begin{aligned} \mathcal{Z}\colon \operatorname{Exp}^1(\mathbb{C}) &\longrightarrow \mathbb{C}[\![\mathrm{z}]\!], \\ f &\longmapsto \sum_{\alpha=0}^{\infty} f(\alpha) \mathrm{z}^\alpha, \end{aligned}$$

which is called *z-transformation* on $\operatorname{Exp}^1(\mathbb{C})$.

[6]Henri Eugène Padé, 1863–1953.
[7]Ferdinand Georg Frobenius, 1849–1917.

Let $f = \sum_{b \in M} f_b \exp_b \in \operatorname{Exp}^1(\mathbb{C})$ with $M = \operatorname{supp}(f) \in \mathcal{P}_{\mathrm{f}}(\mathbb{C} \setminus \{0\})$, $f_b \in \mathbb{C} \setminus \{0\}$, and $r := \operatorname{rank}(f) = |M|$. Interpreting power series $s \in \mathbb{C}[\![\mathrm{z}]\!]$ as complex functions $s\colon D_s \to \mathbb{C}$, $z \mapsto \sum_{\alpha=0}^{\infty} s_\alpha z^\alpha$, on the set $D_s := \{z \in \mathbb{C} \mid \sum_{\alpha=0}^{\infty} s_\alpha z^\alpha \text{ is convergent}\}$, note that for any $b \in \mathbb{C} \setminus \{0\}$ and $z \in \mathbb{C}$ the series

$$\mathcal{Z}(\exp_b)(z) = \sum_{\alpha=0}^{\infty} b^\alpha z^\alpha = \sum_{\alpha=0}^{\infty} (bz)^\alpha$$

is convergent if and only if $z \in \mathrm{B}^{||}_{1/|b|}(0) = \{z \in \mathbb{C} \mid |z| < 1/|b|\}$, so $D_{\mathcal{Z}(\exp_b)} = \mathrm{B}^{||}_{1/|b|}(0)$, and that for $z \in D_{\mathcal{Z}(\exp_b)}$ one has

$$\mathcal{Z}(\exp_b)(z) = \frac{1}{1 - bz} = \frac{-1/b}{z - 1/b}.$$

Thus, by linearity of $\mathcal{Z}$, for all $z \in D := \bigcap_{b \in M} \mathrm{B}^{||}_{1/|b|}(0) = \mathrm{B}^{||}_{\min\{1/|b| \mid b \in M\}}(0) \subseteq D_{\mathcal{Z}(f)}$ one has

$$\begin{aligned}\mathcal{Z}(f)(z) &= \sum_{b \in M} f_b\, \mathcal{Z}(\exp_b)(z) = -\sum_{b \in M} \frac{f_b/b}{z - 1/b} = \frac{-\sum_{b \in M} f_b/b \cdot \prod_{c \in M \setminus \{b\}} (z - 1/c)}{\prod_{b \in M} (z - 1/b)} \\ &= \frac{p(z)}{q(z)},\end{aligned}$$

where $q \in \mathbb{C}[\mathrm{z}]_r$ is the monic polynomial of degree r with $\mathrm{Z}(q) = 1/M$ and $p \in \mathbb{C}[\mathrm{z}]_{r-1}$. In particular, $R := \mathcal{Z}(f) \restriction D = \mathcal{Z}(f) \restriction D\colon D \to \mathbb{C}$ is a rational function. Therefore, R is holomorphic on the complex domain D and since $0 \in D$, one has $\mathcal{Z}(f)(z) = R(z) = \sum_{\alpha=0}^{\infty} 1/\alpha! \cdot R^{(\alpha)}(0) z^\alpha$ for all $z \in D$, and thus $f(\alpha) = \mathcal{Z}(f)_\alpha = 1/\alpha! \cdot R^{(\alpha)}(0)$ for all $\alpha \in \mathbb{N}$. In particular, $R = p/q$ is a Padé approximant of $\mathcal{Z}(f)$ (of order $(r-1, r)$). Hence, for all $z \in D$ one has

$$p(z) = q(z) \cdot \mathcal{Z}(f)(z) = q(z) \cdot \sum_{\alpha=0}^{\infty} f(\alpha) z^\alpha$$

and the polynomial $p \in \mathbb{C}[\mathrm{z}]_{r-1}$ may be obtained computationally by a comparison of coefficients, i. e.,

$$p_\alpha = \sum_{\beta=0}^{\alpha} q_\beta f(\alpha - \beta).$$

For a polynomial $q \in \mathbb{C}[\mathrm{z}]$, let $p^* := \mathrm{z}^{\deg(p)} \cdot p(1/\mathrm{z}) = \sum_{\alpha=0}^{\deg(p)} p_{\deg(p)-\alpha} \mathrm{z}^\alpha \in \mathbb{C}[\mathrm{z}]$. Then one has $\mathrm{Z}(p^*) \setminus \{0\} = (1/\mathrm{Z}(p)) \setminus \{0\}$ and therefore, computing $h \in \mathbb{C}[\mathrm{z}]_r$ with $\mathrm{Z}(h) = M$ by applying Prony's method to the exponential sum f one easily obtains the denominator $q = h^*$ in the above Padé approximant $R = p/q$ of $\mathcal{Z}(f)$ by "reversing" the coefficient vector of h.

The nominator $p \in \mathbb{C}[\mathrm{z}]_{r-1}$ may then be computed from a system of linear equations arising from the above coefficient comparison.

Conversely, let $R = p/q$ be a Padé approximant of $\mathcal{Z}(f)$ with $p \in \mathbb{C}[\mathrm{z}]_{r-1}$ and $q \in \mathbb{C}[\mathrm{z}]_r$ with $\gcd(p,q) = 1$. Then by the uniqueness of Padé approximants, $R \restriction U = \mathcal{Z}(f) \restriction U$ for some neighborhood U of 0 and one has

$$p(z) = q(z) \cdot \mathcal{Z}(f)(z) = q(z) \cdot \sum_{\alpha=0}^{\infty} f(\alpha) z^{\alpha}$$

for all $z \in U$, allowing a comparison of coefficients which yields that for $\alpha = r, \ldots, 2r$ one has

$$0 \overset{\deg(p) \leq r-1}{=} p_{\alpha} \overset{\deg(q) \leq r}{=} \sum_{\beta=0}^{r} q_{\beta} f(\alpha - \beta),$$

which is nothing but the statement

$$q \in \ker(\mathrm{H}_r(h_f))$$

used to compute the polynomial q via Prony's method for the exponential sum

$$h_f := \sum_{b \in M} f_b \exp_{1/b} \in \mathrm{Exp}^1(\mathbb{C}).$$

Again, once q is obtained it is a simple matter to form the polynomial q^* which fulfills $\mathrm{Z}(q^*) = M = \mathrm{supp}(f)$. This establishes a tight connection between the univariate Prony problem over $\mathbb{C}$ and univariate Padé approximation.

One particular advantage of this perspective on the reconstruction problem for exponential sums is that for Padé approximants some convergence theorems are known, which provide an explanation for the behavior of the singular values of $\mathrm{H}_d(f) + \varepsilon_d$ if d is increased beyond $\mathrm{rank}(f)$ in the case that ε_d represents noise on the samples of f. For practical applications this is clearly a highly relevant problem.

It is thus natural to ask for multivariate versions of Padé's theory and their relation to multivariate Prony's method. This is a major theme of current research, and among the relevant works in this are Cuyt [28, 29], Brezinski [15], Guillaume-Huard [39], and Cuyt-Brevik Petersen-Verdonk-Waadeland-Jones [30].

4.4. The measure theoretic moment problem

A new step in this study motivated by the emerging measure theory of the 19th century was the study of so-called *moment problems*. For a measure $\mu\colon \mathcal{A} \to \mathbb{R}^{\geq 0} \cup \{\infty\}$ on a σ-algebra $\mathcal{A}$ on some set X, the *k-th moment* of μ is defined as

$$\mathrm{m}_k(\mu) := \int \mathrm{f}_{\mathrm{x}^k} \, \mathrm{d}\mu = \int x^k \, \mathrm{d}\mu(x),$$

and

$$\begin{aligned} \mathrm{m}\colon \mathcal{M}(X, \mathcal{A}) &\longrightarrow \mathbb{R}^{\mathbb{N}}, \\ \mu &\longmapsto (\mathrm{m}_k(\mu))_{k \in \mathbb{N}}, \end{aligned}$$

is called the *moment operator*. It is a natural question to ask for a description of im(m), that is, for a characterization of those sequences that arise as sequence of moments of some measure. Special instances ask for such characterizations for specific measure spaces $(X, \mathcal{A}, \mu)$, such as the Borel measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda \restriction \mathcal{B}(\mathbb{R}))$, where λ denotes the Lebesgue measure. This has led to several variants such as the *Stieltjes moment problem* [72],[8] the *Hausdorff moment problem* [43, 44, 45],[9], the *trigonometric* (or *Toeplitz*) *moment problem*[10] and the *Hamburger moment problem* [40, 41, 42][11], questions of uniqueness of such a measure, and also how to obtain such measures constructively from given truncated sequences of moments.

To see the connection to Prony's method, note that, for an exponential sum $f = \sum_{b \in \mathrm{supp}(f)} f_b \exp_b \in \mathrm{Exp}^n(\mathbb{C})$,

$$\begin{aligned} \mu_f \colon \mathcal{P}(\mathbb{C}^n) &\longrightarrow \mathbb{C}, \\ A &\longmapsto \sum_{b \in \mathrm{supp}(f)} f_b \delta_b(A), \end{aligned}$$

where

$$\begin{aligned} \delta_b \colon \mathcal{P}(\mathbb{C}^n) &\longrightarrow \mathbb{R}^{\geq 0}, \\ A &\longmapsto \begin{cases} 1 & \text{if } b \in A, \\ 0 & \text{otherwise}, \end{cases} \end{aligned}$$

denotes the *Dirac measure* in b, is a complex measure. Furthermore, for $\alpha \in \mathbb{N}^n$, one has

$$f(\alpha) = \sum_{b \in \mathrm{supp}(f)} f_b b^\alpha = \int_{\mathbb{C}^n} \mathrm{f}_{\mathrm{x}^\alpha} \, \mathrm{d}\mu_f$$

is the α-th *moment* of μ_f. For this reason, matrices occurring in these methods are often referred to as *matrices of moments* in the literature. There are many variants of reconstruction methods for multivariate exponential sums and polynomials that are in some way based on this idea, such as in Ben-Or-Tiwari [9], Curto-Fialkow [23, 24, 25, 26, 27], Giesbrecht-Labahn-Lee [37], Laurent [56], Laurent-Mourrain [57], Andersson-Carlsson-de Hoop [2], Collowald-Hubert [21], Peter-Plonka-Schaback [62], Mourrain [60], and Sauer [67, 69, 68], and and also the theory developed here can be seen as being in this line.

4.5. Projection methods

A family of alternative approaches to the reconstruction of multivariate exponential sums may be subsumed under the label *projection methods*. The basic strategy is as follows.

[8] Thomas Joannes Stieltjes, 1856–1894; measures supported on $\mathbb{R}^{\geq 0}$.
[9] Felix Hausdorff, 1868–1942; measures supported on a bounded interval.
[10] Otto Toeplitz, 1881–1940; measures supported on $\mathbb{T}^1$.
[11] Hans Ludwig Hamburger, 1889–1956; measures supported on $\mathbb{R}$.

Let $f = \sum_{b\in M} f_b \exp_b \in \operatorname{Exp}^n(A)$ be an exponential sum. Note that for an arbitrary $\alpha \in \mathbb{N}^n$, the function

$$\begin{aligned} f_\alpha\colon \mathbb{N} &\longrightarrow A,\\ k &\longmapsto f(k\cdot\alpha), \end{aligned}$$

is an exponential sum in $\operatorname{Exp}^1(A)$ with

$$\operatorname{supp}(f_\alpha) = \{b^\alpha \mid b \in M\} \subseteq A,$$

since for $k \in \mathbb{N}$, $f_\alpha(k) = \sum_{b\in M} f_b b^{k\alpha} = \sum_{b\in M} f_b (b^\alpha)^k$. (Of course, $\operatorname{rank}(f_\alpha)$ may be smaller than $\operatorname{rank}(f)$, i. e., $b^\alpha = c^\alpha$ for some distinct $b, c \in M$.) Thus f_α may be reconstructed classically with Prony's method. This is done for several distinct $\alpha_1, \dots, \alpha_t \in \mathbb{N}^n$, resulting in a reconstruction of each $f_{\alpha_\ell} \in \operatorname{Exp}^1(A)$, $\ell = 1, \dots, t$, and in a second step this information gets assembled into a reconstruction of f.

The denomination "projection methods" stems from the following. For $A = K = \mathbb{C}$ and $b \in \mathbb{T}^n$, let $\varphi := \arg(b) := (\arg(b_1), \dots, \arg(b_n))^\top \in [0, 2\pi[^n$ and let $\langle\rangle\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ denote the euclidean scalar product on $\mathbb{R}^n$. In this context (e. g., for applications in signal processing) it makes sense to consider arbitrary exponents in $\mathbb{R}^n$ instead of $\mathbb{N}^n$, i. e., one usually considers

$$\begin{aligned} \exp_b\colon \mathbb{R}^n &\longrightarrow \mathbb{C},\\ \alpha &\longmapsto b^\alpha = b_1^{\alpha_1}\cdots b_n^{\alpha_n}. \end{aligned}$$

Then one clearly has

$$\exp_b(\alpha) = \mathrm{e}^{\mathrm{i}\varphi_1\alpha_1}\cdots \mathrm{e}^{\mathrm{i}\varphi_n\alpha_n} = \mathrm{e}^{\mathrm{i}(\varphi_1\alpha_1+\cdots+\varphi_n\alpha_n)} = \mathrm{e}^{\mathrm{i}\langle\varphi,\alpha\rangle}$$

for all $\alpha \in \mathbb{R}^n$. Application of the projection strategy above for $\alpha \in \mathbb{R}^n$ with $\|\alpha\|_2 = 1$ then leads to $\exp_b(k\alpha) = (\mathrm{e}^{\mathrm{i}\langle\varphi,\alpha\rangle})^k = (\mathrm{e}^{\mathrm{i}\,\operatorname{proj}_{\mathbb{R}\alpha}(\varphi)})^k$, i. e., the frequencies of the bases of $f_\alpha\colon \mathbb{R} \to \mathbb{C}$, $k \mapsto f(k\alpha)$, are the projections of the frequencies of the bases of $f = \sum_b f_b \exp_b$ onto the subspace $\mathbb{R}\alpha$.

Among the references on projection methods are Jiang-Sidiropoulos-ten Berge [47], Giesbrecht-Labahn-Lee [37, Section 4.2], Potts-Tasche [64], Plonka-Wischerhoff [63], Diederichs-Iske [32], and Cuyt-Lee [31].

4.6. Tensor decomposition, renaissance of the Waring problem

The Waring problem for homogeneous forms presented in the binary case in Section 4.2 gained renewed interest around 1990. One additional motivation is that computers made it possible to produce large sets of experimental data, e. g., in physics or biology. These data often have the structure of a tensor, i. e., they are multiarrays of values $T = (T_{i_1,\dots,i_d})_{i_1,\dots,i_d=0,\dots,n} \in K^{n\times\cdots\times n}$. Often these tensors are symmetric in the sense that $T_{i_1,\dots,i_d} = T_{\pi(i_1,\dots,i_d)}$ for every permutation $\pi \in \mathrm{S}_d$. The problem of decomposing a symmetric tensor into a sum of symmetric tensors of rank 1 with least number of summands is equivalent to the decomposition of a homogeneous polynomial over K of degree d in n indeterminates into a sum of powers of linear forms with least number of summands, i. e., to the Waring problem discussed in Section 4.2 for $n = 2$.

An extension of Sylvester's method to the case of an (in principle) arbitrary tensor is expounded in the 1999 monograph by Iarrobino and Kanev [46] and has been developed further, e. g. in Brachat-Comon-Mourrain-Tsigaridas [12]. However, being a generalization of Sylvester's classical method, this approach for decomposing symmetric tensors has the same major problem that a given tensor often does not provide all the coefficients of the catalecticant.

In the way that Sylvester's method is related to Prony's method, this extension also provides an approach to reconstruct multivariate exponential sums. In a preprint from 2016 [60], Mourrain gives such a variant of the theory developed for the Waring problem. As this is very closely related to the approach developed in this thesis, Mourrain's method and its relation to our method is presented in the following.

Mourrain considers *polynomial-exponential functions*, that is, the elements of the K-vector space

$$\mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n) := \Big\{ \sum_{i=1}^{r} \omega_i \exp_{\xi_i} \in K^{\mathbb{N}^n} \;\Big|\; r \in \mathbb{N},\, \omega_i \in K[\mathrm{y}],\, \xi_i \in K^n \Big\} \leq K^{\mathbb{N}^n}.$$

This is more general than our setting in the sense that we only consider polynomial-exponential functions with constant coefficients. Note however that we allow the bases ξ_i to be in the K-algebra A^n instead of only K^n. The intersection of both settings is the space of polynomial-exponential functions with constant coefficients, which is our $\mathrm{Exp}^n(K)$. Another difference is that we also consider the case that the bases ξ_i to be elements of a given subset B of A^n.

For a polynomial-exponential function $f = \sum_{i=1}^{r} \omega_i \exp_{\xi_i} \in \mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n)$ the problem is to reconstruct the points $\xi_i \in K^n$ and $\omega_i \in K[\mathrm{y}_1, \ldots, \mathrm{y}_n]$ from $f \restriction F$ with some finite set $F \subseteq \mathbb{N}^n$. A subtle difference here to our approach is that Mourrain determines a suitable set F for the given f while performing his method, whereas we try to give an F such that Prony's method works for all exponential sums f with some additional properties (i. e., sufficient separation distance of the bases in the toroidal and spherical cases).

For simplicity, let K be a field of characteristic zero. Mourrain identifies f and the power series

$$f = \sum_{\alpha \in \mathbb{N}^n} f(\alpha) \frac{1}{\alpha!} \mathrm{y}^\alpha \in T := K[\![\mathrm{y}_1, \ldots, \mathrm{y}_n]\!],$$

where $\alpha! := \prod_{j=1}^{n} (\alpha_j!)$ for $\alpha \in \mathbb{N}^n$.

Since the dual space $S^* = \mathrm{Hom}_K(S, K)$ of $S := K[\mathrm{x}_1, \ldots, \mathrm{x}_n]$ is (K-vector space isomorphic to) T, one can view $\mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n)$ as a subspace of S^*, identifying f and

$$\begin{aligned} f\colon\; S &\longrightarrow K, \\ p &\longmapsto \sum_{\alpha \in \mathrm{supp}(p)} f(\alpha) p_\alpha. \end{aligned}$$

In particular, one then has $f(\mathrm{x}^\alpha) = f(\alpha)$ for all $\alpha \in \mathbb{N}^n$, and the exponential $\exp_\xi \in \mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n)$ corresponds to the evaluation homomorphism $\mathrm{ev}^\xi \in S^*$. This point of view has the particular advantage that it allows to endow the K-module S^* with a useful S-module structure via

$$\begin{aligned} \star\colon\ S \times S^* &\longrightarrow S^*, \\ (p, f) &\longmapsto p \star f\colon\ S \longrightarrow K, \\ &\qquad\qquad\quad q \longmapsto f(pq). \end{aligned}$$

Viewed as power series, one has

$$p \star q = p(\partial_1, \ldots, \partial_n)(f)$$

where $p(\partial_1, \ldots, \partial_n)$ is the differential operator

$$\begin{aligned} \sum_{\alpha \in \mathrm{supp}(p)} p_\alpha \partial_1^{\alpha_1} \circ \cdots \circ \partial_n^{\alpha_n}\colon\ T &\longrightarrow T, \\ q &\longmapsto \sum_{\beta \in \mathbb{N}^n} q_\beta \sum_{\alpha \in \mathrm{supp}(p)} p_\alpha \underbrace{\mathrm{y}^{\beta-\alpha}}_{:= 0 \text{ if } \alpha > \beta}. \end{aligned}$$

For each $f \in T$ one has the *Hankel operator*

$$\begin{aligned} \mathrm{H}_f\colon\ S &\longrightarrow S^*, \\ p &\longmapsto p \star f. \end{aligned}$$

Mourrain has the following generalization of a theorem of Kronecker[12] from 1881 [51], which concerns the univariate case $n = 1$. Part (a) characterizes those power series f that correspond to polynomial-exponential sequences as those with $\mathrm{rank}(\mathrm{H}_f)$ finite. Therefore, $\mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n)$ is a very natural generalization of $\mathrm{Exp}^n(K)$. Part (b) gives a connection between $\mathrm{rank}(\mathrm{H}_f)$ and the *Macaulay inverse system* $\langle \partial^\alpha \omega_i \mid \alpha \in \mathbb{N}^n \rangle_K$ which generalizes the fact that $\mathrm{rank}(\mathrm{H}_f) = \mathrm{rank}(f)$ if the coefficients of f are constants.

Theorem 4.2 (Mourrain [60, Theorem 3.1]): *Let $f \in K[\![\mathrm{y}_1, \ldots, \mathrm{y}_n]\!]$. Then the following holds.*

(a) *The following are equivalent:*
   (i) $\mathrm{rank}(\mathrm{H}_f) \in \mathbb{N}$.
   (ii) $f \in \mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n)$.

(b) *Let*

$$f = \sum_{i=1}^r \omega_i \exp_{\xi_i} \in \mathcal{P}oly\mathcal{E}xp(\mathrm{y}_1, \ldots, \mathrm{y}_n)$$

with $\omega_i \in K[\mathrm{y}]$ and pairwise distinct $\xi_i \in K^n$, and let

$$\mu(\omega_i) := \dim_K(\langle \partial^\alpha \omega_i \mid \alpha \in \mathbb{N}^n \rangle_K).$$

Then

$$\mathrm{rank}(\mathrm{H}_f) = \sum_{i=1}^r \mu(\omega_i).$$

[12]Leopold Kronecker, 1823–1891.

For computational purposes, for K-subvector spaces $V, W \leq S$ the following *restricted Hankel operator* will be used,

$$\begin{aligned} \mathrm{H}_f^{V,W} \colon\ V &\longrightarrow W^*, \\ p &\longmapsto \mathrm{H}_f(p) \restriction W. \end{aligned}$$

If $f \in \mathrm{Exp}^n(K)$ and $V = W = S_D$ for some $D \subseteq \mathbb{N}^n$, then the transformation matrix of $\mathrm{H}_f^{V,W}$ w. r. t. the bases x^D and its dual is our matrix $\mathrm{H}_D(f)$.

Mourrain also provides a method to compute the zero locus of $\ker(\mathrm{H}_f^{V,W})$ for finite dimensional V, W that is based on arguments similar to Gram-Schmidt orthogonalization w. r. t. the inner product given by $\langle\rangle_f \colon S \times S \to K$, $(p, q) \mapsto f(pq) = (p \star f)(q)$, to compute appropriate K-bases of $K[\xi_i \mid i = 1, \ldots, r]$ in conjunction with the eigenvector method of Auzinger, Möller, and Stetter [6, 71, 59] applied to a matrix pencil with generically chosen coefficients and the flat extension principle of Curto and Fialkow. More on algorithms to compute bases of $K[\xi_i \mid i = 1, \ldots, r]$ can be found in Mourrain [61].

4.7. Further approaches

There is a vast body of literature concerning the reconstruction of exponential sums or related problems in general or Prony's method in particular which we cannot discuss in detail.

In particular, there are approaches using optimization theory, cf. e. g., Candès-Fernandez-Granda [17, 18] and Bendory-Dekel-Feuer [10, 11]. A recent article on the relationship between optimization based approaches and Prony's method is provided by Josz, Lasserre, and Mourrain [49].

Coming from the field of signal analysis, there are approaches under the label annihilating filter methods, cf. e. g. Stoica-Moses [73, 74], Vetterli-Marziliano-Blu [78], Dumitrescu [33], and Shukla-Dragotti [70].

Furthermore, moment problems over certain matrix rings have been considered, e. g. in Choque Rivero-Dyukarev-Fritzsche-Kirstein [20] and Choque Rivero [19].

Index of definitions and theorems

Index of symbols

Symbol	Description	Page
$\mathrm{I}_{D,B}(M)$	vanishing A-module relative to $B \subseteq A^n$ of $D \subseteq \mathbb{N}^n$, $M \subseteq B$	7, 43
$\mathrm{Z}_B(J)$	zero locus relative to $B \subseteq A^n$ of $J \subseteq S_B$	7, 43
$\mathrm{in}_{\leq}(p)$	initial monomial w. r. t. $\leq$ of $p \in A[\mathrm{x}_1, \dots, \mathrm{x}_n] \setminus \{0\}$	8
$\mathrm{in}_{\leq}(I)$	initial set w. r. t. $\leq$ of $I \subseteq A[\mathrm{x}_1, \dots, \mathrm{x}_n]$	8
$\mathrm{N}_{\leq}(I)$	normal set w. r. t. $\leq$ of $I \subseteq A[\mathrm{x}_1, \dots, \mathrm{x}_n]$	8
$\mathrm{Spec}(A)$	spectrum of ring A	8, 49
$\mathrm{V}(I)$	algebraic variety of $I \subseteq A[\mathrm{x}_1, \dots, \mathrm{x}_n]$	9, 49
$\mathrm{V}_B(J)$	algebraic variety relative to $B \subseteq A^n$ of $J \subseteq S_B$	9, 50
$\widetilde{\mathrm{B}}^{\|\|\|}_{\varepsilon}(x)$	closed ε-ball w. r. t. $\|\|\|$ with center $x \in \mathbb{R}^n$	9
Γ	gamma function	9
P_r	r-th Legendre polynomial	10
K	field	11
A	integral domain containing K	11
Q	quotient field of A	11
n	non-zero natural number (number of variables)	11
$\exp_b$	multivariate exponential with base $b \in A^n$	12
u_j	j-th unit tuple in $\mathbb{N}^n$	12
$\mathrm{Exp}^n_B(A)$	K-vector space of n-variate exponential sums supported on $B \subseteq A^n$	12
$\mathrm{Exp}^n(A)$	$= \mathrm{Exp}^n_{A^n}(A)$	12
$\mathrm{rank}(f)$	rank of exponential sum f	12, 39
$\mathbb{T}^n$	complex n-torus	13
$\mathrm{FExp}^n(K)$	K-vector space of formal exponential sums	14
$\mathrm{FExp}^n_r(K)$	K-vector space of formal rank $\leq r$ exponential sums	14
U_M	canonical basis of A^M	19
u_b	b-th unit vector in A^M, $b \in M$	19
V_D^M	transformation matrix of $\mathrm{ev}_D^M \colon S_D \to A^M$	19
$\ker_R(H)$	$= \{x \in R^n \mid Hx = 0\}$, $H \in A^{m \times n} \leq R^{m \times n}$	19
$\mathrm{im}_R(H)$	$= \{Hx \mid x \in R^n\}$, $H \in A^{m \times n} \leq R^{m \times n}$	19
$\mathrm{H}_D(f)$	Hankel-like matrix of $f \in \mathrm{Exp}^n(A)$ w. r. t. $D \subseteq \mathbb{N}^n$	20
u^t_ℓ	ℓ-th unit tuple in $\mathbb{N}^t$	22
$\min\deg(\delta)$	minimal degree of $\delta \in \mathbb{N}^t$	22
$\mathcal{F}^{\|\|\|}$	$\star$-filtration induced by norm $\|\|\| \colon \mathbb{R}^n \to \mathbb{R}^{\geq 0}$, $\|\mathrm{u}_j\| \leq 1$	24
$\mathcal{T}^n$	total degree $\star$-filtration on $\mathbb{N}^n$	24
$\mathcal{M}^n$	maximal degree $\star$-filtration on $\mathbb{N}^n$	25
S_δ	$= S_{\mathcal{F}_\delta}$, $\mathcal{F}$ t-$\star$-filtration on $\mathbb{N}^n$, $\delta \in \mathbb{N}^t$	26
ev_δ^M	$= \mathrm{ev}_{\mathcal{F}_\delta}^M$, $\mathcal{F}$ t-$\star$-filtration on $\mathbb{N}^n$, $\delta \in \mathbb{N}^t$, $M \subseteq A^n$	26
$\mathrm{I}_\delta(M)$	$= \mathrm{I}_{\mathcal{F}_\delta}(M)$, $\mathcal{F}$ t-$\star$-filtration, $\delta \in \mathbb{N}^t$, $M \subseteq A^n$	26
V_δ^M	$= \mathrm{V}_{\mathcal{F}_\delta}^M$, $\mathcal{F}$ t-$\star$-filtration on $\mathbb{N}^n$, $\delta \in \mathbb{N}^t$, $M \subseteq A^n$	26
$\mathrm{H}_\delta(f)$	$= \mathrm{H}_{\mathcal{F}_\delta}(f)$, $\mathcal{F}$ t-$\star$-filtration on $\mathbb{N}^n$, $\delta \in \mathbb{N}^t$, $f \in \mathrm{Exp}^n(A)$	26

Symbol	Description	Page
$\operatorname{supp}(f)$	support of exponential sum f	31, 39
$\operatorname{coeff}(f)$	coefficient vector of exponential sum f	31, 39
${}_{\mathbb{Z}}\exp_b$	n-variate exponential on $\mathbb{Z}^n$ with base $b \in \mathrm{U}(A^n)$	39
${}_{\mathbb{Z}}\mathrm{Exp}_B^n(A)$	K-vector space of n-variate exponential sums on $\mathbb{Z}^n$ supported on $B \subseteq \mathrm{U}(A^n)$	39
${}_{\mathbb{Z}}\mathrm{Exp}^n(A)$	$= {}_{\mathbb{Z}}\mathrm{Exp}^n_{\mathrm{U}(A^n)}(A)$	39
$\mathrm{T}_D(f)$	Toeplitz-like matrix of $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$ w. r. t. $D \subseteq \mathbb{N}^n$	39
$\mathrm{T}_\delta(f)$	$= \mathrm{T}_{\mathcal{F}_\delta}(f)$, $\mathcal{F}$ t-$\star$-filtration, $\delta \in \mathbb{N}^t$, $f \in {}_{\mathbb{Z}}\mathrm{Exp}^n(A)$	39
P_δ^n	n-variate Prony in $\mathcal{F}_\delta$, $\mathcal{F}$ t-$\star$-filtration on $\mathbb{N}^n$, $\delta \in \mathbb{N}^t$	52
P^n	n-variate Prony	52
$\mathbb{S}^{n-1}$	real $(n-1)$-sphere	55
SH_d^n	$\mathbb{R}$-vector space of n-variate real spherical harmonics of degree at most d	55
$\mathrm{sep}_{\mathrm{s}}(M)$	spherical separation of $M \in \mathcal{P}_{\mathrm{f}}(\mathbb{S}^{n-1})$	56
w_n	wrap-around metric on $\mathbb{T}^n$	63
$\mathrm{sep}_{\mathrm{t}}(M)$	toroidal separation of $M \subseteq \mathbb{T}^n$	64
$\mathrm{C}_r(p)$	r-th catalecticant of binary form p	70
$\mathcal{Z}$	z-transformation	72
$\mathrm{m}_k(\mu)$	k-th moment of measure μ	74
μ_f	complex measure associated to $f \in \mathrm{Exp}^n(\mathbb{C})$	75

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