Optimal Harvesting of Fish Populations with Age Structure

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BY

SADIQ AL-NASSIR

SUPERVISOR:
PROF. EM. DR. HORST BEHNCKE

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List of notations

$\bar{a}$ = Equilibrium value of variable a.

a, b = The cost parameters.

c = The probability of an egg to develop into a young adult.

c_1 = Number of eggs per kilogram.

$E(t)$ = The number of eggs in year t.

E_o = The scaling parameter which signifies depensation.

F_i = Average fecundity per females of class i, per year.

$H_1(h, s)$ = The total harvesting amount.

h_i = Fishing effort in class i.

h_{opt} = The optimal effort intensity in the static model.

h_{optc} = The optimal effort intensity in the control model.

J = The total yield.

K = The growth rate in Bertanalffy equation.

k = The number of classes of fish species.

k_1 = The percentage of total harvest.

k_2 = Maximum age of fish.

k_3 = Constant.

$L(t), W(t)$ = Length and weight at age t.

L_∞, W_∞ = Maximum length and weight of the fish respectively.

$n(0)$ = The net reproductive rate at zero density.

$n(h, s)$ = the net reproductive rate for harvesting rate h and mesh size parameter.

NH = the net gain yield.

P_0 = The oviposition perturbation factor.

P_1 = The larval and juvenile perturbation factor.

P_2 = The adult perturbation factor.

p_i = Growth factor of the gonads.

q_i = Price of one unit fish in class i.

R = Recruitment function.

$r(E)$ = Depensation function.

s = Mesh size parameter.

s_1 = Time of development of gonads.

s_o = The optimal mesh width which gives optimal net gain.

s_m = The optimal mesh width which gives optimal harvesting amount.

T, F = Transition and fertility matrices.

t_c = No harvest duration.

T_1 = the duration of the juvenile stage.

$x_i(t)$ = Number of fish of age t in class i.

$\mathcal{H}$ = The Hamiltonian.

α = Measure of the density depenpdence in the model.

α_1 = Constant.

μ = Mortality of fish.

τ_i = The survival probability for class i.

σ_i = The fraction of individuals in age class i which survives and remains in class i after one year.

$\tau \frac{1}{T_i}$ = The fraction that survives and moves to the next age class.

$\mathcal{J}$ = Jacobian matrix.

λ_A = Perron-Frobenius eigenvalue of A.

$\mathcal{T}, J$ = Time horizon and the total yield.

Summary of the thesis (Abstract)

In this thesis an age structured model of fish and harvesting of fish are studied. This includes the basic properties as well as the analysis of equilibria and their stability for various recruitment functions. The only nonlinearity of this model lies in the juvenile stage in form of the recruitment function.
Recruitment is considered as recruitment into the first year fertile class. The Ricker recruitment function will mostly be used . But we also discuss the Beverton-Holt recruitment and compare its impact.

The main result is that monotone recruitment functions lead to a globally stable equilibrium. For Ricker recruitment functions periodic solutions are possible, but this model for cod and herring always leads to stable equilibria.
Harvesting occurs in all classes separately with no harvesting in the first class. We describe harvesting in form of harvesting effort, resulting in an increase of mortality. This approach allows to model the interference of natural mortality and fishing mortality. Harvesting cost will be considered as a linear function of effort and the corresponding maxima are determined. This has the advantage to keep the number of parameters low. This makes the model more tractable and easier to apply to other stocks or species of fish. Optimization of the harvest is determined and analyzed. It is based on harvesting intensity and mesh size. We also show that harvesting increases stability of the static model.

This model is extended to a discrete optimal control model in order to determine optimal sustainable policies. The solutions are almost of the fastest approach type to the equilibrium type solution. Periodic or pulse solutions do not arise. Near optimal strategies are analyzed and other feasible strategies are also considered.

In last part of this thesis the problem of extinction of population is investigated. The effect of various factors as well as natural factors on marine fish populations are studied, which are described by the model in order to get some idea on the size of various factors. Most fishing takes place near the break even intensity. It is defined as the fishing mortality at which cost equals profit. As a further aspect recovery scenarios are studied. Concrete numerical studies for optimal solutions are determined for cod and herring. Care is taken to take into account all possible realistic parameters but also to keep the model tractable for other species.

Chapter 1

A Mathematical Model for fishery

1.1 Introduction

Two thirds of the earth are covered by the sea. Therefore marine fish could be the most important source of food for men, if properly managed. This has not been the case so far and many fish species have been driven to near extinction by over exploitation. In[5] Boris Worm, a marine ecologist at Dalhousie University in Nova Scotia,reported that the catches had dropped to less than 10% of the recorded maximum. For currently fished species it had decreased to 29%. If this trend continues the global collapse of all currently fished species will occur by the mid-21st century(2048). In addition only 9% of European fish stocks will have achieved a sustainable levels by 2022 [1].

In a recent article marine ecologist R. Froese gives a moving portrait of this precarious situation [23]. He states that European fisheries has a profit margin of 3-6% compared to New Zealand where reformed fisheries reach profit margins of up to 40%. The reasons for this deplorable situation are mainly political and economical. They are political, because catch quotas will to be agreed upon almost worldwide. Each nation not agreeing to such quotas will have an advantage over the others. So there is little pressure to join into such negotiations, even less to adhere to the recommendations. The fate of the International Whaling Commission illustrates these problems. Even within the EU there is little interest to agree on sensibly low catch quotas. The management of fishery in Europe is poor or totally

absent. According to[1], in 2010,76% of the species commercially caught by European fleets were outside the fishing quotas. For example a complete ban could cost about 20000 work places in the UK alone. For this reason agreements are usually achieved at the cost of sensible solutions. In addition fish factories and fleets have bound a considerable capital. Even the fact that fishing at low densities is economically senseless, because the catch does not pay for the cost is ruled out, because the EU hands out considerable subsidies. In 2009 fishing sector subsidies totalled 3.3 billion Euro, three times more than typically quoted public figures. In addition more than two-thirds of these subsidies are used either to enhance fishing and increase overfishing or can be considered as obscure subsidies [1]. This is clearly another example of shortsightedness of politicians and pressure from lobbyist. In addition to outlandish quotas illegal fishing and the problem of the bycatch seriously undermine the possibility of attaining a sustainable optimal equilibrium.

The idea to exploit the sea as much as possible and as cheaply as possible has led to the employment of ever larger and more sophisticated fishing factories. These hunt the fish with sonar, GPS spotter planes and gigantic fishing gear. That the use of ground nets turns large domains of the sea floor into a waste is now well established.These fishing factories can travel far from their homeland and thus endanger the local fishing industry elsewhere ,e.g. some African coastal waters. Economic realism shows that a species will be harvested to a level, where it is barely profitable.Subsidies distort this and can drive a species near to extinction. Needed is thus a complete stop of subsidies unless it is for the closing down of ships or factories or employment of fishermen in other areas. New forms of negotiations, stricter enforcement of the catch quotas, limits on the fishing gear, control of the bycatch and size restrictions on the fishing vessels are needed. All this belongs to the political aspect of the fishing industry.

The difficulties to be encountered there can easily be envisaged when one considers whaling and more recently the problem of carbon dioxide emission. Here the biological and to a lesser extent the economic aspects are only considered. But even there serious problems arise because of the complexity of the marine ecosystem.

The aim is to develop a strategy that leads to a sustainable fishery with an optimal output. For a single isolated species this problem is well defined. A management of one species, however, will affect others, which in turn influence the species to be caught. Another problem in this connection is the bycatch, because the young fish of the targeted species may appear as the bycatch of another species and conversely the bycatch affects other species. Finally upgrading of a catch is a problem.

The FAO (Food and Agriculture Organization) thus demands to minimize the capture and mortality of species and sizes which are not going to be used [25]. Other suggestions require the bycatch to be counted with the catch. Another optimization goal should thus be to maintain the diversity and structure of the marine ecosystem. In the above article[25]. S.M. Garcia et al argue that this can only achieved by a balanced harvesting, which distributes a moderate mortality from fish across the widest possible range of species stocks and sizes in an ecosystem. The authors support this by a number of simulations. Since only a few species are commercially important such a concept is even less likely to be followed.

Summing up we see that the ecosystem "sea" or "marine fish" is far too complicated to yield to simple analysis other than simulations. Economic influences make it even more intractable. Topping all this is a completely ambiguous political process. For this reason we shall pursue in this work a much more modest aim. Here a single species will be studied with age structure subject to harvesting, thus extending simple age class models to include age structure. The vital parameters are then tested with respect to their sensitivity. This approach is justified, because most parameters come with fairly large errors. In addition most models are rather incomplete because they neglect the interaction with other species, the influence of the environment or simply the fact that they are not spatial. Therefore it is not surprising that most more detailed models fail in making good predictions. In fact models with more than 10 parameters with each an error of about 10 percent can obviously not perform well. This work will be organized as follows:

In the first chapter, the introduction will be a review the most important papers in this field study. A Leslie- type age class model for marine fish is then developed. This is applied in particular to cod and later to herring. In order to keep the necessary parameters low, many parameters are introduced via relations or well

known laws. It also includes a discussion of the recruitment process as well as the determination of the key parameters. The latter turns out to be more complicated than expected. In particular fecundity, which is one of the key population variables, is difficult to determine and data in this field are rather insecure.

In chapter 2 a general equilibrium model is developed, and its stability is analyzed for various recruitment functions. This is done to fix the parameter ranges, and study their sensitivity on the equilibria. It is shown that a monotone recruitment functions always lead to globally stable equilibria.

In chapter 3 the model is enlarged to include harvesting. Harvesting is described in the form of harvesting effort, resulting in an increase of mortality. This approach allows to model the interference of natural mortality and fishing mortality. Fishing requires a considerable investment in equipment, manpower and production. For this more complex model the effect needs to be analyzed in detail. In a final stage harvesting cost are considered and the corresponding maxima are determined. In addition concrete numerical studies for harvesting models are then applied to concrete fish population for example cod and herring.

In chapter 4, the model is extended to an optimal control system in order to determine optimal sustainable policies. As expected the solutions are almost of the fastest approach to equilibrium type solution. Other strategies are also analyzed. Finally the importance of the results for more general fishery models are studied.

In the final chapter,chapter 5, the problem of extinction of populations is analyzed. For obvious reasons this study is restricted to low reproduction numbers and effect of overfishing as well as natural factors are studied. The environmental factors are

1. Oviposition, which is influenced by temperature and salinity, is modeled by

$$c \longrightarrow ce^{-P_0} \qquad \text{with} \quad 0 \leq P_0 \leq 0.2$$

2. The larval and juvenile phase: In this phase larvae have a high demand for food. They are preyed upon as well as competition, sickness and cannibalism

effect their progression. This phase is modeled by

$$\tau_i = e^{-\mu} \longrightarrow e^{-(\mu+P_1)} \qquad \text{with} \quad 0 \le P_1 \le 0.2$$

3. The adult phase ,in this phase the lack of food is the dominant factor as well as competition with other species. The adult phase is expressed by

$$\tau_i = e^{-\mu} \longrightarrow e^{-(\mu+P_2)} \qquad \text{with} \quad 0 \le P_2 \le 0.2$$

1.2 Mathematical models and previous work

Because of the importance of fish as a food source, many articles have discussed fish harvesting and its optimization. Because of the complexity of the problem most papers discuss only simplistic type optimization models or concentrate on particular aspects. But even these over simplistic papers give valuable insight in the dangers of overfishing, which now in an unfortunate reality for many fish stocks.

The scientific literature abounds in articles deploying this unhappy situation. Yet the recommendation of biologists and fishery scientists are consistently ignored[23]. Even the complete collapse of the cod fishery in Newfoundland twenty years ago has not induced fishermen and politicians to correct their attitude. Despite a complete ban on cod fishing in this region the stock there has not recovered yet. The same can be said about the spring spawning herring in Iceland, which has not rebounded since the sixties. North sea cod is facing a similar peril. Since 1970 the number of spawning cod in the North sea has reached an all time low about 30000 t, which is about 15 % of that in 1970. According to H. Lassen, an International Council for the Exploration of the sea (ICES) advisor, the stock is half of the absolute minimum [19]. Most likely the North Sea cod fishery will then face the same fate as its Newfoundland predecessor. Why then, is it difficult to implement a sustainable harvesting policy?.

Mathematical models of fish have a very long tradition which are ranging from simple biomass -statistical- to rather detailed multi-class models. The fundamental work on quantitative fisheries management was done mainly by Beverton and Holt [3],Ricker[55, 56] and Shaefer [58, 60]. Clark[7, 8], provides a recent comprehensive

account of the field. Previous control models have treated only single species without age structure [7, 8] and there are a number of models with two classes which nonetheless gave valuable insight into optimal fish harvesting [62, 64, 71, 73].

In [26] Getz presented an optimization harvesting model that distinguishes a harvesting season dynamics and the spawning season. All this is modeled by a system of ordinary differential equations and difference equations respectively.His main conclusion, that a two age group harvesting strategy is optimal, is unrealistic because most marine fish are caught with seines or long lines. In [27, 28] the Beverton-Holt theory of harvesting a multi-cohort fish population is formulated in a Leslie matrix frame work and conditions for the ultimate sustainable yield are derived subject to constant recruitment. A Leslie type model was also employed by Levin and Goodyear [40] or Wikan and Eide [71]. They, however, were mostly interested in the appearance of periodic or chaotic solutions. These models hardly reflect biological reality and their conclusion about periodic or chaotic solutions are rather problematic. The history of this development can be found in the surveys of Tahvonen [66] or Wilen [72]. We also refer to the more recent papers by Diekert et al [15, 16, 17],Li and Yakubu [42],and by Tahvonen [65, 67]. Tahvonen et al [68],and Quaas et al [54]. They have developed specific age class models. But these apply only to the Norwegian or Baltic cod stocks,and result in contradictory recommendations. Tahvonen in [69, 67] states a model and gives the equations for an optimal control. However, solutions and equilibria are only discussed and computed for a two age class model favoring periodic or pulse fishing solutions. Also [15, 16, 54] are of the lumped age class form. They give different results regarding the juvenile fish class. Ding et al [14] considered an optimal harvesting with an age-structured model and they derived the necessary conditions and the characterizations for the optimal harvesting strategies. Hjermann et al [36] developed a statistical model. In recent years some further autoregressive statistical model have come up describing the time series of catches. The advantage of our approach is that it can easily adapted to other stocks,whereas these models with very specific data or assumptions can not be applied to other stocks or compared with other data easily.
Biological models typically depend on many parameters like size, growth, fecundity, mortality and environmental parameters. For many of these approximate laws hold, for example allometric laws or other relations which have been found by regression.

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These laws depend on few parameters and will be used as much as possible. In as much as simplifying assumptions are used, these will be tested by sensitivity analysis. Details and comparison with the above mentioned papers will be made as the model is developed. As noted above, the equilibrium model will first be studied. This is done to fix the parameter ranges, and to study the equilibria.

The detailed study will be based mainly on cod, which is the most important species, accounting for the largest part of the world catch. To begin with a few facts on the biology of cod, more specially the north Atlantic cod are stated. These will be used as a guideline for other species. Details can be found in the book by J.A Gulland [31].

1.3   The North Atlantic Cod

Though this species (Gadus morhua) is spread over a vast area in the north Atlantic, and though there are no impenetrable barriers restricting the movement of the individual cod there are a number of distinct stocks. This has been established by genetic analysis and tagging. These stocks are mostly separated by regions of deep water across which there is little movement. Tagging experiments and biochemical analysis have shown that there is a little interchange between these stocks. The ICES (International Council for the Exploration of the Sea distinguish about 20 distinct stocks of cod). Each stock has a definite spawning area from where the eggs drift downstream after spawning early in the year. By June in the south to September further in the north the fish larvae(fingerlings) settle in their nursery grounds in coastal regions. Thereafter the juveniles undertake seasonal migrations until they return to the spawning area as mature fish.

The different stocks can thus be distinguished by their spawning area, nursery grounds and habitat. Cod fishery is known since more than 1000 years. For example the town of Bergen once was once the center for the stock fish in during the Hanse days. Since about 150 years ago the catch was kept fresh on ice. This, however, limited the travel times of the fishing ships to about 2 weeks, and only one stock could be harvested at a time. The new fishing factories, which directly

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turn out fillets and other fish products, are not limited in this way and may therefore harvest several stocks per trip,and travel much farther. With modern gear and equipment fish can be caught more effectively.

The Life Cycle of Marine Fish

Cod, Herring, or plaice and other marine fish follow an annual cycle. Marine fish move from the open sea to the spawning grounds. There lay a very large number of eggs. Most of these marine fish species lay as many as 10^4 to 10^7 eggs in each spawning, cod for e.g, 5.10^5 eggs per kilo [10, 31], see table 1.1 . These are released into the sea , where they are fertilized. After two or three weeks the fish larvae hatch. Many of the larvae die of starvation, sickness or predation. Thus their mortality may be as much as 5 percent per day [10]. After oviposition the adult fish return to the open sea,which offers more food. Thus there is no density effect due to competition among adult fish though cannibalism has been reported. Marine fish have the following life cycle:

Spawning $\overset{\longleftarrow}{\underline{\text{migration}}}$ Adult Fish

$\downarrow$ $\uparrow$

Egg $\longrightarrow$ Larvae $\longrightarrow$ Juvenile Fish

$\longleftarrow$ Recruitment $\longrightarrow$

The Yearly Cycle of Marine Fish
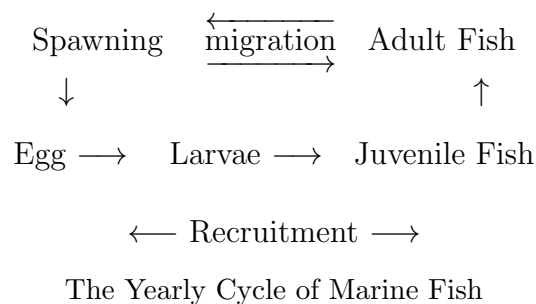
The Growth of Fish

The growth of fish is best described by the Von Bertalanffy growth equation[70], which gives the Length $L(t)$ and the weight $W(t)$ as a function of the age t,

$$L(t) = L_\infty(1 - e^{-K(t-t_1)}) \tag{1.1}$$

$$W(t) = W_\infty(\frac{L(t)}{L_\infty})^3 \tag{1.2}$$

here t is measured in years and L_∞, W_∞ are the maximum length and weight of the fish respectively. This is important because the number of eggs produced by a fish of age t is proportional to $W(t)$. $L(t)$ will also play a role in connection with the size of the meshes of fishing nets, because one can expect the width of fish to follow the same law. For cod we will choose the onset of fertility to be 4 years. The Von Bertalanffy growth parameters of cod are $K = 0.12, W_\infty = 17$, and $t_1 = 0$ [41].

1.4 Fecundity

Fecundity is one of the most important aspects of fish reproductive biology. It gives the key information for successful management of marine fish. In particular it is a measure for the recovery of a population from very low levels[2]. The number of eggs produced by a mature female is most likely proportional to its weight [3]. That means, if the eggs are of the same size, bigger fish are more fecund than smaller ones. If the eggs are mainly generated on the surface of the gonads an allometric exponent of two thirds would be more plausible. But the gonads are heavily convoluted, which again makes a proportionality to the weight more likely. Thus we use proportionality to the weight, as this is commonly done. But one will have to take into account some time for the full development of the gonads. If there are no available data about fecundity, we will represent it as

$$F_i = c_1 p_i W_i \tag{1.3}$$

Here W_i denotes the average weight of fish in class i. The function

$$p_i = \begin{cases} \frac{i-1}{s_1} & : 2 \leq i \leq s_1 \\ 1 & : s_1 < i \leq k_2 \end{cases} \tag{1.4}$$

describes the growth of the gonads in s_1 years. In general we will choose $s_1 \simeq \frac{k_2}{4}$. Thus young fish produce comparatively less eggs than their weight would suggest. This modification is well supported by data. It makes older fish even more valuable for reproduction. This effect can be seen with actual data of striped bass in [40] and cod in Diekert et al [15]. The constant c_1 can be interpreted as the number of eggs in one kilogram weight of fish. It should be added that in general such data are rather insecure, because of the difficulties in measuring them directly. In particular we neglect environmental factors such as temperature and the nutritional state of the female fish. At low densities this might lead to larger and more robust eggs.

Species	K Von Bertalenffy parameter	Max. Age	Max. Weight	Max. length	Onset of fertility	Weight at begin of fertility	Mortality per year	Fecundity number of Eggs/ Gram
Herring	0.35	20-25	1000 gm	40 cm	3-7	125 gm	0.1-0.2	200-360
Cod	0.12	30 years	17kg	1.1m	4 years	3-8 kg	0.18-0.3	500
Plaice	0.095	15 years	2,867 kg	68.5 cm	4-5 years	210 gm	0.13-0.18	200
Haddock	0.20	20 years	1.209 kg	53 cm	2 years	89 gm	0.2	495
Hake	0.8	15 years	3.8 kg	79 cm	3 years	0.3 kg	0.2	–
Anchovy	0.43	5 years	35 gm	16 cm	2 years	24.2 gm	0.2	–
Sole	0.42	—	482	37.7cm	3 years	—	—	500 or 900

Table 1.1: biological parameters for some fish species[3, 13, 25, 41, 55, 56]

The mortality is generally taken to be constant. This is certainly not true and a Gompertz like form would be more realistic for higher ages. Nonetheless a constant mortality is used most commonly. Since the mortality for fish is very large, the precise form is of lesser importance,because the higher age classes are comparatively small. A survival of 0.8 for example implies that only $0.8^{20} . 100 = 1.2\%$ of cod reach the age class of 24. A growth law, which is active only in the last third of the lifetime will only effect the last classes of the cod population. For simplicity, however, we will use a constant mortality μ, this can be extended to a monotonically increasing function. For this reason one can set either $\mu(t) = 0.2$ or $\mu(t) = 0.18 \quad 1 \leq t \leq 20$, $\mu(t) = 0.18 + 0.012(t - 20)$, for cod. A little thought,however, shows that this will

hardly influence the main results. For other fish other similar considerations apply. The more acceptable values of μ in cod fisheries lie between 0.18 and 0.22 per year. It will be advantageous, however, to express survival through the factor $e^{-\mu} = \tau$.

1.5 The Recruitment Process

The most important and most critical role in fish population dynamics is played by the recruitment. This describes the development process from egg to juvenile fish respectively adult fish. This, however leaves much room for interpretation. Thus the definition of recruitment is not clear to begin with and depends on the class into which recruitment leads. So when using recruitment, one has to be careful with the definition. Here recruitment means recruitment into the adult class when fertility begins. Despite its importance the recruitment process is as yet poorly understood. Most of the marine fish species lay a very large number of eggs, as it was mentioned before. These are laid directly into the sea, where they are fertilized. The large number indicates that most of them perish or are eaten. Within several weeks the larvae hatch from these eggs and begin to feed on plankton. So oviposition is closely tied to the plankton cycles which in turn depend on the available sunlight. Since fish larvae depend on plankton, there is a corresponding competition between larvae leading to a depensation in the recruitment process. Cannibalism might also be responsible for this depensation. This process is generally modeled by the recruitment function $R(x)$, where $R(x)$ gives the number of adult fish which develop out of x eggs. The most commonly used forms are

$$R(x) = cx(1 + \frac{x}{E_0})^{-1} \qquad\qquad Beverton - Holt \qquad\qquad (1.5)$$

$$R(x) = cxe^{\frac{-x}{E_0}} \qquad\qquad Ricker \qquad\qquad (1.6)$$

Other forms are also possible for e.g.[59, 33]

$$R(x) = cx(1 + (\frac{x}{E_0})^{\alpha})^{-1} \quad , \quad \alpha \geq 1 \qquad\qquad Shepherd$$

$$R(x) = cx(1 + \frac{x}{E_0})^{-\alpha} \quad , \qquad \alpha \geq 1 \qquad\qquad Hassell$$

The main reason for these functions is their mathematical simplicity. So far there is not yet a proper mathematical description of the recruitment process nor a plausible derivation of these functions. The fit of these function to actual catch-data is likewise inconclusive, because at the moment only low density data are available. There are good indications that R is nearly linear for small x. i.e depensation at low population levels has not been observed [44]. That there is a depensation at higher population levels is plausible and generally accepted. This depensation is due to competition among larvae and cannibalism against eggs or larvae. Even though the study of Cushing and Horwood [9] gives good arguments for a dome shaped curve, this is not conclusive yet, because there are too few high density catch data. Note that the Ricker model leads to a dome shaped recruitment functions. In addition climate variations, piracy and the bycatch make catch data difficult to analyze. In these expressions x usually has the dimension of population or animals, because the number of eggs produced is generally proportional to the body weight.

Thus, E_0, will have to be interpreted as the population size or scaling parameter, which measures the onset of depensation at higher densities. It has the dimension of population and may vary from stock to stock. E_0 appears as the maximum of R for the Ricker recruitment and in concrete cases this would be the means to determine it. This form of the Ricker recruitment thus separates the two properties -fecundity at low density with the measure $n(0)$, the net reproductive rate at zero density, and depensation at high densities expressed through E_0 in a product form. Since we will concentrate in our analysis in general on a single stock, E_0 acts as a scaling parameter, roughly proportional to the size of stock. Thus its value has no effect upon stability as we will see later. Even though E_0 acts as a scaling parameter, it is an important invariant for a particular stock. A small E_0 will imply bad survival conditions or much competition among the larvae, whereas a large E_0 describes good environmental conditions for the larvae. In as much as there is a large variation of $n(0)$ one can also expect a similar large variation in E_0. Ultimately E_0 can only be determined from the maximum of catches. (1.5) and (1.6) have been used mainly because of their simplicity. However, they differ greatly in their effect on the dynamics of fish population as will be seen later. (1.6) and more generally monotone recruitment functions will always lead to locally stable equilibria.

Levin and Goodyear have shown, that Ricker recruitment with a high fertility and high mortality may lead to oscillations and possibly even chaos[40]. The parameters in that model are unrealistic, however list them . This will be illustrated by applying various recruitment functions to the Levin Goodyear paper. This also applies to the model of Wikan and Eide [71]. It is generally believed, that the recruitment function is almost linear for low densities, with no depensation.

Depensation at higher levels is accepted likewise. If one writes $R(x) = cxr(x)$ the term c gives the probability that an egg or certain number of eggs will be turn into a viable adult at low densities. So far there is no plausible explanation for form of the recruitment function because data on landings of fish have a large error and are difficult to interpret. In particular data for large x, where depensation is dominant, are missing.

1.6 The Usher Model

The yearly cycle of marine fish suggests to describe their dynamics as a discrete age class model.This can also be interpreted as a size or stage class model because of (1.1). For this assume that we have the age classes $1, 2, ..., k$ of length $T_1, T_2, ..., T_k$ years where, T_1 is the duration of juvenile stage. We also set $T_k = 1$. Assume that fertility begins with class 2. Let $x_i(t)$ denote the number of fish in age class i at time t, with $x(t) = (x_1(t), x_2(t), ..., x_k(t))$, the age distribution of the population at time t. For convenience we also assume that the sex ratio is 1:1 in the eggs. Thus the focus is on the females only. The number of eggs in year t is given by

$$E(t) = \sum_{i=2}^{k} F_i x_i(t) \tag{1.7}$$

where F_i denotes the average fecundity (number of eggs) per females of class i. Note that all eggs are treated in the same way independent of their parents age. With this the number of recruits becomes

$$R(E) = cEr(E) \tag{1.8}$$

While, c measures the probability of an egg to develop into a young adult, the monotonically decreasing function $r(E)$ describes the depensation at higher level

densities due to competition, sickness, starvation and possibly cannibalism. The factor r(E) is normalized to 1 for low densities[46], where these depensation effects are not active. This can be achieved by a proper choice of c. Thus $r(0) = 1$. Then

$$x_1(t+1) = \tau_1(1 - \frac{1}{T_1})x_1(t) + R(E(t)) \tag{1.9}$$

$$x_i(t+1) = \tau_{i-1}\frac{1}{T_{i-1}}x_{i-1}(t) + \tau_i(1 - \frac{1}{T_i})x_i(t) \quad i = 2, ..., k$$

Here, $\tau_i = e^{-\mu_i}$, where μ_i is the approximate mortality in class i. The term $\tau_i(1 - \frac{1}{T_i})$ is the fraction of individuals in age class i which survive and remain in class i after one year, and $\tau_i\frac{1}{T_i}$ is the fraction that survives and moves to the next age class. Note that the equation for $x_k(t)$ with $T_k = 1$ implies that all fish in this class die within the next year. This assumption is not a serious restriction, because the higher age classes are very thinly occupied. To see this note that the probability of a 3 year old cod to reach the age of 30 is only $0.8^{27} \approx 24 \times 10^{-4}$. This model is thus of the generalized Leslie form. The only nonlinearity of these models arises in the juvenile stage, because density dependence in the life history of adult marine fish has not been established yet. This is probably due to the vastness of the sea and the fact that the fish migrate. Then the model (1.9) can be written as

$$x(t+1) = A(x)x(t) \tag{1.10}$$

with

$$A(x) = \begin{bmatrix} \sigma_1 & r_2(x) & r_3(x) & \cdots & r_k(x) \\ \tilde{\tau}_1 & \sigma_2 & 0 & \cdots & 0 \\ 0 & \tilde{\tau}_2 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{\tau}_{k-1} & \sigma_k \end{bmatrix} \tag{1.11}$$

with $\sigma_i = \tau_i(1 - \frac{1}{T_i}), \tilde{\tau}_i = \frac{\tau_i}{T_i}, r_{i+1} = cF_{i+1}r(E), i = 1, 2, ..., k-1$ and $\sigma_k = \tau_k(1 - \frac{1}{T_k})$. Models like (1.10),(1.11) have be discussed on detail in [6] or [11]. For the moment assume the r_i to be constant. The matrix A is nonnegative, so that the Perron-Frobenius theory can be applied[38]. A is primitive if $r_k > 0$ and some $\sigma_i > 0$ or if $r_k, r_{k-1} > 0$ [6]. In this case the spectral radius λ_A of A is an eigenvalue, which dominates all other eigenvalues λ, $\lambda_A > | \lambda |$. The associated eigenvector x_A has positive components only. If the r_i would not depend on E the population would

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eventually grow with the factor λ_A and the population profile would be given by x_A. Since λ_A is difficult to compute, Cushing and Yicang [12], have introduced the concept of net reproduction number n. To define it, one writes $A = T + F$, where T is the transition part and F the fertility part, all $r'_i s$ of the matrix. Then $N = (I - T)^{-1}F$ is well defined, because $0 \leq \sigma_i < 1$. Moreover it is nonnegative and of rank one. The following lemma was first shown by Cushing and Yicang[12], and it appeared in [50].

Lemma 1.6.1. *The eigenvalue n of $N = (I-T)^{-1}F$ satisfies $1 < \lambda_A < n$ iff $n > 1$, and $\lambda_A = 1$ iff $n = 1$. For $\lambda_A < 1$ one has $n < \lambda_A < 1$.*

Proof. Let x be the normalized Perron Frobenius eigenvector for A i.e $(x)_1 = 1$ and let x^* be the corresponding vector for $N^* = ((I-T)^{-1}F)^*$.
Then $Ax = (T+F)x = \lambda_A x$ shows $(\lambda_A - 1)x + (I-T)x = Fx$. Since $(I-T)$ is invertible, this gives $(\lambda_A - 1)(I-T)^{-1}x + x = (I-T)^{-1}Fx = Nx$.
Multiply by x^* from right to get

$$< (\lambda_A - 1)[(I-T)^{-1} - I]x + (\lambda_A - 1)x + x, x^* > = < x, N^*x^* > .$$

Now $(I-T)^{-1} - I = T + T^2 + T^3 + ...$ is nonnegative. Since also x and x^* are nonnegative we get with $N^*x^* = nx^*$ the inequality $\lambda_A < x, x^* > \leq n < x, x^* >$ if $(\lambda_A - 1) \geq 0$. The other relations follow similarly. $\qquad \square$

The importance of the net reproductive value n lies in the fact, that it can be computed directly. For this let $y = (1, y_2, ..., y_k)^t$ be an eigenvector of N. Then $Fy = n(I-T)y$, yields,
$y_i = \frac{\tau_{i-1}......\tau_1}{(1-\sigma_i)......(1-\sigma_2)} = L_i, i = 2, 3, ..., k$, so that

$$n(E) = (1 - \sigma_1)^{-1} \sum_{i=2}^{k} r_i L_i, \quad r_i = cF_i r(E). \tag{1.12}$$

Thus $n = n(E)$ gives the average number of fish a typical fish will produce in its life time.

Remark 1.6.2. In the remainder I shall specifically choose $T_1 = a$, a is a duration of juvenile stage and $T_2 = T_3 = ... = T_k = 1$. So that the matrix A in model (1.9) becomes

$$A(x) = \begin{bmatrix} \sigma_1 & r_2(x) & r_3(x) & \cdots & r_k(x) \\ \tau_1 & 0 & 0 & \cdots & 0 \\ 0 & \tau_2 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tau_{k-1} & 0 \end{bmatrix} \qquad (1.13)$$

with $\sigma_1 = \tau_1(1-\frac{1}{T_1}), \sigma_i = 0, \tau_1 = \frac{e^{-\mu_1}}{T_1}, \tau_i = e^{-\mu_i}$ and $r_i(x) = cF_i r(E), \quad i = 2, ..., k.$

1.7 Determining the parameters

In this thesis I will exemplarily treat the case of cod mainly. Other species of fish will be dealt later in the same manner. Data on the size and age of fish can be obtained by direct measurement. Much more difficult to estimate are the fecundity,mortality and other developmental parameters of fish larvae. Here every thing is based on individual fish. Most often authors use biomass or other units. Since the fish have different weight the transition may not be easy. The widespread choice of the Ricker recruitment functions is largely due to its mathematical simplicity. The constant mortality ≈ 0.2 for cod is likewise not in accordance with general biological mortality laws. Parameters for marine fish are generally inferred from catch data and landings. Illegal fishing, misreporting and the by catch, however, make these data rather insecure. From these catch data the population parameters are then inferred via models. Another difficulty arises from the fact that different authors use different units, for example, biomass, number of fish, fecundity index..., which makes a comparison difficult. More often the terms used are not even defined properly. Thus large errors are to be expected.

In this thesis $x_i(t)$ will always denote the number of fish in age class i at time t. Despite some personal misgiving $\mu \approx 0.2$ will be used as a mortality for cod. A more plausible linear function $\mu_i = 0.18 + i0.004$ has a little effects on the results, because a mortality of 0.2 already implies that the higher age classes are occupied rather thinly. Actual data suggest that the fecundity of female cod is proportional to the weight. At lower age the egg production is not quite up to that, because the gonads may not be fully developed. This will be described by the function p

introduced above. The number of young adult fish produced by x_i fish in class i at low density, where the depensation function $r(E)$ is almost 1, is given by $cF_i x_i$. Here c gives the probability that an egg or certain number of eggs will be turn into a viable adult. Note that this implies that all eggs are treated alike. Since different fish stocks show considerable variations [31]. c will vary with the fish stock under considerations. In this case c is a measure for the quality of the breeding area.

c is proportional to $n(0)$, the net reproductive value at low density. $n(0)$ is also a measure how fast a population can recover from a precarious low density.

In this thesis $n(0)$ is the key vital parameter, because it has a direct biological meaning. It has been determined by R.Myers et al([29],[34],[45],[46],[47],[48],[49]), for various fish stocks. It is more meaningful, since there exists an explicit formula based on the matrix elements of A. In [48] R. Myers has determined the net reproductive number $n(0)$ at zero density for a number of different stocks of cod he found a range of $n(0)$ between 5 and 30, except a stock in the Irish sea with $n = 115$. Values as high as that, however, seem to be statistical outliers. The difficulty in estimating $n(0)$ ($\hat{\alpha}$ there), comes from the fact that all natural cod stocks are subject to harvesting. The paper of R.Mayer also contains data on other species. These,however,are rather uncertain, caused by external natural factors like climate or location and man induced factors like harvesting misreporting of the catch, piracy together with ill reporting of landings. In general the data are obtained from estimate of landings, rarely from direct counts. The most plausible values thus will lie between 5 and 15,rarely 20. This allows to determine c from (1.12) because $r(E)$ is normalized to $r(0) = 1$.

For cod population the important parameter c will be determined from $n(0)$, namely $c = \frac{n(0)(1-\sigma_1)}{\sum_{i=2}^{28} F_i L_i}$. Later the optimal h, and optimal mesh width s will be determined as well as the maximum sustainable yield for various values of $n(0)$. According to [41], the maximum age of cod is 30 years, the onset fertility is at age 4. The weight is approximately given by $W(t) = 17(1 - e^{-0.12t})^3$ kg. The mortality rate μ will be approximately to 0.2. The author in [41], also gave $E_0 = 4.1 \times 10^{14}$ for cod. So throughout these parameters will be used. But note that E_0 is a scaling parameter.

If one wants to discuss the influence of mortality , one can consider either constant mortality or variable mortality. For a constant mortality in all classes one gets for $c \times 10^{-6}$.

Mortality		$\mu \approx 0.18$	$\mu \approx 0.19$	$\mu \approx 0.20$	$\mu \approx 0.21$	$\mu \approx 0.22$
n(0)=3	c:	0.9384	1.0748	1.2267	1.3999	1.5908
n(0)=5	c:	1.5640	1.7913	2.0446	2.3333	2.6514
n(0)=7	c:	2.1910	2.5078	2.8624	3.2666	3.7119
n(0)=10	c:	3.1280	3.5825	4.0891	4.6665	5.3027
n(0)=15	c:	4.6921	5.3738	6.1337	6.9998	7.9541

Table 1.2: The constant c for cod population for different values of n(0)and μ.

For a variable mortality

$$\mu(t) = \begin{cases} \mu & : 0 \leq t \leq 20 \\ \mu + \frac{0.3-\mu}{10} * (t-20) & : 20 < t \leq 30 \end{cases}$$

One gets for $c \times 10^{-6}$.

Mortality		$\mu \approx 0.18$	$\mu \approx 0.19$	$\mu \approx 0.20$	$\mu \approx 0.21$	$\mu \approx 0.22$
n(0)=3	c:	0.9509	1.0867	1.2390	1.4093	1.5995
n(0)=5	c:	1.5849	1.8112	2.0650	2.3489	2.6659
n(0)=7	c:	2.2189	2.5357	2.8910	3.2885	3.7322
n(0)=10	c:	3.1698	3.6225	4.1300	4.6968	5.3318
n(0)=15	c:	4.7547	5.4337	6.1950	7.0467	7.9976

Table 1.3: The constant c for cod population for different values of n(0) by using a variable mortality

Myers also gave the maximum reproductive rate for Atlantic herring [45]. It is less than that for cod. Thus by the same method above one can also compute c for

several values of $n(0)$. Mortality is constant in all classes then one gets for $c \times 10^{-5}$.

Mortality		$\mu \approx 0.18$	$\mu \approx 0.19$	$\mu \approx 0.20$	$\mu \approx 0.21$	$\mu \approx 0.22$
n(0)=3	c:	0.9647	1.0789	1.2027	1.3414	1.4916
n(0)=5	c:	1.6079	1.7981	2.0046	2.2357	2.4859
n(0)=7	c:	2.2510	2.5174	2.8064	3.1300	3.4803
n(0)=10	c:	3.2158	3.5962	4.0092	4.4714	4.9719
n(0)=13	c:	4.1805	4.6751	5.2119	5.8128	6.4635

Table 1.4: The constant c for herring population for different values of n(0)and μ.

According to[50], the maximum age of herring is 20-25 years and the weight of herring can be approximated it by the Bertalanffy equation with $K = 0.35, t_1 = 1.01971$ and $W_\infty = 1$ kg. $E_0 = 10^8$. The mortality rate $\mu \approx 0.2$ will also be used, and set the onset fertility at 3. The maximum age is assumed to be 25 years.

In this model we have so far neglected the influence of the climate. Temperature and salinity will act in two ways directly or through the prey. Its influence through food abundance in the larval stage could be modeled through a time dependent depensation function. Food availability in the adult stage would result in a variable E_0. The influence of the prey or competitors would also be important factors.

Chapter 2

Nonlinear Matrix models and their equilibria

2.1 The stability

The most important concept for an equilibrium $\bar{x}$ in a dynamical system is its stability. In general global stability can only be determined by means of Lyapunov functions. From a practical point of view and for applications this is even more important because we are looking for sustainable solutions, which may be interpreted as an equilibrium. These equilibria will have to be stable or locally stable at least, because of fluctuations from the environment. Thus we are not only looking for equilibria, but for stable equilibria.

So far no Lyapnuov functions are known, even though this model looks rather innocent. In fact the existence of periodic solutions in the models of Levin and Goodyear[40] and Wikan [71] show that such functions may be difficult to come by. For the local stability one has to show that all eigenvalues of the Jacobian at the equilibrium lie inside the unit disc. A general nonlinear dynamical system can be written as

$$x(t+1) = g(x(t)) \tag{2.1}$$

where g is a function $\;g{:}R^k \longrightarrow R^k$. Equilibrium solutions of equation (2.1) are defined by

$$\bar{x} = g(\bar{x}) \tag{2.2}$$

Then one defines :

Definition 2.1.1. An equilibrium vector $\bar{x}$ is said to be

1. Stable if given $\epsilon > 0$ there exists $\delta > 0$ such that $|x - \bar{x}| < \delta$ implies $\left| g^k(x) - \bar{x} \right| < \epsilon$ for all $k \in Z^+$.

2. Attracting if there exists $\epsilon > 0$ such that $|x - \bar{x}| < \epsilon$ implies $\lim_{k \to +\infty} g^k(x) = \bar{x}$. It is globally attracting if ϵ is unrestricted.

3. Asymptotically stable if it is both stable and attracting. It is globally asymptotically stable if it is both stable and globally attracting.

4. Unstable if it not stable.

Local stability of a dynamical system (2.1), can be determined from the Jacobian $\mathcal{J}(\bar{x})$ of the system. $\mathcal{J}_{ij} = \partial_{x_j} g_i(\bar{x})$. If all eigenvalues λ of $\mathcal{J}$ satisfy $|\lambda| < 1$, the system is locally stable at $\bar{x}$ [43]. Since the system (1.9) is linear in all but the first coordinates. $\mathcal{J}$ is given by

$$
\mathcal{J} =
\begin{bmatrix}
\sigma_1 & V_2 & V_3 & \cdots & V_k \\
\tau_1 & \sigma_2 & 0 & \cdots & 0 \\
0 & \tau_2 & \ddots & \vdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \tau_{k-1} & \sigma_k
\end{bmatrix}
$$

with $V_i = \frac{dR}{dE}\frac{dE}{dx_i}$ where $R = cEr(E) = \sum_{i=2}^{k} cx_i F_i r(E)$.

It is advantageous to replace the above model by an equivalent one, which is more transparent and easier to handle. Let C denote the diagonal matrix $C = diag(1, c_2, c_3, ..., c_k)$ $c_i > 0$. Then $M = C^{-1}AC$ and A are matrices with identical spectra. Let

$$
c_i \;=\; L_i \tag{2.3}
$$

Then

$$M = \begin{bmatrix} \sigma_1 & \tilde{r}_2 & \tilde{r}_3 & \cdots & \tilde{r}_k \\ \tilde{\tau}_1 & \sigma_2 & 0 & \cdots & 0 \\ 0 & \tilde{\tau}_2 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{\tau}_{k-1} & \sigma_k \end{bmatrix}$$

is equivalent to system (1.10), with

$$\tilde{\tau}_{i-1} = L_i^{-1} \tau_{i-1} L_{i-1} = (\frac{1 - \sigma_i}{\tau_{i-1}}) = 1 - \sigma_i \quad \text{and} \quad \tilde{\tau}_i = r_i L_i$$

Thus

$$\tilde{\tau}_{i-1} + \sigma_i = 1 \quad \text{for} \quad i = 2, 3, ..., k.$$

The model, which we are using, has the properties $T_1 = a$, where a is the duration of juvenile stage, and $T_2 = ... = T_k = 1$, thus $\sigma_2 = \sigma_3 = ... = \sigma_k = 0$.

In this case the matrix M becomes:

$$M = \begin{bmatrix} \sigma_1 & \tilde{r}_2 & \tilde{r}_3 & \cdots & \tilde{r}_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{2.4}$$

with

$$\tilde{r}_i = cF_i L_i r(E) \geq 0 \quad i = 2, ..., k.$$

So $\tilde{r}_i$ gives the expected number of eggs, a female is going to produce in the year i, because L_i is just the probability to reach that age.

This model is completely equivalent to the original model, because A and $M = C^{-1}AC$ describe the same dynamics.

$$y(t + 1) = C^{-1}AC(C^{-1}x(t)) = (C^{-1}AC)y(t) = My(t) \tag{2.5}$$

Thus it remains to express E in terms of $y_i(t)$. Now $L_i^{-1}x_i(t) = y_i(t)$ or $L_iy_i(t) = x_i(t)$. Thus

$$E(t) = \sum_{i=2}^{k} F_ix_i(t) = \sum_{i=2}^{k} F_iL_iy_i(t) \tag{2.6}$$

is the expectation value of eggs a female produces in its lifetime.

To determined the equilibrium vector $y = (y_1, y_2, ..., y_k)$ of system (2.5), one can solve $y = C^{-1}ACy = My$. Clearly $y = (0, 0, ..., 0)$ is always an equilibrium vector. In the case $y = (y_1, y_2, ..., y_k) \neq 0$, we get $y_1 = \sigma_1 y_1 + \sum_{i=2}^{k} cF_iL_iy_ir(\bar{E})$, and $(1 - \sigma_1) = \sum_{i=2}^{k} cF_iL_ir(\bar{E})$ with $y_i = y_{i-1}$ for $i = 2, ..., k$.
Hence $y_1 = y_2 = ... = y_k = \bar{y}$, and $\bar{E}(t) = \sum_{i=2}^{k} F_iL_i\bar{y}(t)$.
Thus the net reproductive rates at zero density $n(0)$ and $\bar{y}$ are related by

$$\frac{1}{n(0)} = r\left(\sum_{i=2}^{k} F_iL_i\bar{y}\right)$$

This shows

$$r^{-1}\left(\frac{1}{n(0)}\right) = \sum_{i=2}^{k} F_iL_i\bar{y} \quad or \quad \bar{y} = \frac{cr^{-1}\left(\frac{1}{n(0)}\right)}{n(0)(1 - \sigma_1)} \tag{2.7}$$

Let us note that a positive equilibrium value exists if $r^{-1}\left(\frac{1}{n(0)}\right) > 0$.

Below $\bar{y}$ is expressed as a function of $n(0)$ for various recruitment functions.

Ricker model:	$r(x) = e^{-\frac{x}{E_0}}$	$r^{-1}(x) = -E_0 \ln(x)$	$\bar{y} = \frac{c \ln(n(0))E_0}{n(0)(1-\sigma_1)}$
Beverton-Holt. model:	$r(x) = (1 + \frac{x}{E_0})^{-1}$	$r^{-1}(x) = \frac{(1-x)E_0}{x}$	$\bar{y} = \frac{cE_0(n(0)-1)}{n(0)(1-\sigma_1)}$
Shepherd model:	$r(x) = (1 + (\frac{x}{E_0})^\alpha)^{-1}$	$r^{-1}(x) = E_0(\frac{1-x}{x})^{\frac{1}{\alpha}}$	$\bar{y} = \frac{cE_0(n(0)-1)^{\frac{1}{\alpha}}}{n(0)(1-\sigma_1)}$
Hassell model:	$r(x) = (1 + \frac{x}{E_0})^{-\alpha}$	$r^{-1}(x) = E_0(x^{-\frac{1}{\alpha}} - 1)$	$\bar{y} = \frac{cE_0(n(0)^{\frac{1}{\alpha}}-1)}{n(0)(1-\sigma_1)}$

Table 2.1: The equilibrium value for different recruitment functions

Now the Jacobian matrix of system (2.5) is as above:

$$
\mathcal{J} = \begin{bmatrix}
\sigma_1 & V_2 & V_3 & \cdots & V_k \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}
$$

with $V_i = \frac{dR}{dE}\frac{dE}{dy_i}$ and $R = cEr(E) = \sum_{i=2}^{k} cF_i L_i y_i r(E)$.

Thus $V_i = cF_i L_i r(E) + \sum_{i=2}^{k} cF_i L_i y_i \frac{dr(E)}{dE}\frac{dE}{dy_i} = (r(E) + E\frac{dr(E)}{dE})cF_i L_i, i = 2, ..., k$.
The eigenvalues of $\mathcal{J}$ are determined by $\mathcal{J}x = \lambda x$. Assume $x = (1, x_2, x_3, .., x_k)$, is a λ eigenvector then

$$
\mathcal{J}x = \begin{bmatrix}
\sigma_1 & V_2 & V_3 & \cdots & V_k \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 \\ x_2 \\ \vdots \\ \vdots \\ x_k
\end{bmatrix}
= \lambda
\begin{bmatrix}
1 \\ x_2 \\ \vdots \\ \vdots \\ x_k
\end{bmatrix}
$$

this gives $\lambda x_2 = 1, \lambda x_3 = x_2, ..., x_{k-1} = \lambda x_k$ and $\sigma_1 + \sum_{i=2}^{k} V_i x_i = \lambda$.
If with set $x = \lambda^{-1}$ this becomes a polynomial equation in x

$$
\sigma_1 x + \sum_{i=2}^{k} V_i x^i - 1 = 0 \tag{2.8}
$$

So that the eigenvalues λ of $\mathcal{J}$ are determined by the roots of the characteristic equation (2.8) through $\lambda = \frac{1}{x}$.

It is well known that any operator norm dominates the spectral radius ρ. A particularly useful operator norm can be derived from the sup-norm on $\mathbb{C}^k$. It is then given by $\|A\|_\infty = \max_i \sum_{j=1}^{k} |a_{i,j}|$, which is the maximum of the ℓ^1 norms of the rows

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Lemma 2.1.2. *Let*

$$
B = \begin{bmatrix}
b_1 & b_2 & \cdots & \cdots & b_k \\
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & \ddots & \vdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & \cdots & 0 & 1 & 0
\end{bmatrix}
$$

if $\sum_{i=1}^{k} |b_i| < 1$, then $B^n \to 0$ and $\rho(B) < 1$, $\rho(B)$ is the spectral radius of B.

Proof. Let

$$
C = \begin{bmatrix}
1 & 0 & \cdots & \cdots & 0 \\
0 & a & 0 & \cdots & 0 \\
0 & 0 & a^2 & \vdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & a^{k-1}
\end{bmatrix}
$$

with $a > 1$

$$
C^{-1}BC = \begin{bmatrix}
b_1 & b_2 a & b_3 a^2 & \cdots & b_k a^{k-1} \\
a^{-1} & 0 & 0 & \cdots & 0 \\
0 & a^{-1} & \ddots & \vdots & 0 \\
\vdots & 0 & \ddots & 0 & \vdots \\
0 & \cdots & 0 & a^{-1} & 0
\end{bmatrix}
$$

Now let $f(a) = \sum_{i=1}^{k} |b_i a^{i-1}|$. It is clear that $f(a)$ is continuous increasing function of a and $f(1) < 1$, by assumption then for some a we get $f(a) = \sum_{i=1}^{k} |b_i a^{i-1}| < 1$ so that $\|C^{-1}BC\|_\infty < 1$, thus $\|C^{-1}B^n C\|_\infty \to 0$. This implies $B^n \to 0$ and $\rho(B) < 1$. $\qquad\square$

Lemma 2.1.3. *Assume the recruitment function $R(x) = cxr(x)$ in (1.13) is bounded, continuous and $r(x)$ converges to 0 monotonically. Then all solutions are bounded and there is at most one nontrivial equilibrium.*

Proof. Let $P(t) = \sum_{i=1}^{k} x_i(t)$ denote the total population size at time t.

Thus the system (1.13) can be written as a delay difference equation as follows:

$x_i(t) = L_i x_1(t - (i-1))$ for $i = 2, 3, ..., k,\ t > k$. Then $x_1(t+1) = \sigma_1 x_1(t) +$ M_1 where $M_1 = \sup R(E)$. This implies $x_1(t) \leq \frac{M_1}{1 - \sigma_1}$. Since $x_i(t+1) = \tau_i x_i(t)$ any solution is uniformly bounded. For $\sigma_1 = 0$ the proof is obvious. The second claim follows from the fact that for an equilibrium vector $\bar{y}$ the sum of the first row in $M(2.4)$ is 1. Since r is monotonically decreasing there can be at most one equilibrium. $\qquad\square$

It follows that the equilibrium vector $\bar{y}$ is given by $\bar{y} = \bar{y}_1(1, ..., 1)$ so that $1 = \sigma_1 + \sum_{i=2}^{k} cF_i L_i r(\bar{E})$ where $\bar{E} = \sum_{i=2}^{k} F_i L_i \bar{y}_1$. With $B_1 = \sum_{i=2}^{k} cF_i L_i$ this can be rewritten as

$$1 = \sigma_1 + B_1 r(\bar{E}) \quad \text{and} \quad \bar{E} = \frac{B_1 \bar{y}_1}{c} \tag{2.9}$$

At this point one has to fix the recruitment function. In most cases the Ricker recruitment function or Beverton-Holt recruitment function will be used with

$$r(E) = e^{-\frac{E}{E_0}} \tag{2.10}$$

$$r(E) = (1 + \frac{E}{E_0})^{-1} \tag{2.11}$$

Here E_0 is a scaling parameter, which measures the onset of depensation at higher level densities. Note that stability is independent of E_0. It has the dimension of population and may vary from stock to stock. E_0 appears as the maximum of R in the Ricker function and in concrete cases this would be the means to determine it. This is form of recruitment thus separates the two properties, fecundity at low density with the measure $n(0)$ and depensation at high densities expressed through E_0 in a product form. With $r(E)$ as in (2.10) or (2.11) one can get for the equilibrium value $r(\bar{E}) = \frac{(1-\sigma_1)}{B_1}$ or

$$\bar{E} = E_0 \ln(\frac{B_1}{1 - \sigma_1}), \quad \bar{y}_1 = \frac{cE_0}{B_1} \ln(\frac{B_1}{1 - \sigma_1}) \tag{2.12}$$

and for Beverton-Holt recruitment

$$\bar{E} = E_0(\frac{B_1}{1 - \sigma_1} - 1), \quad \bar{y}_1 = \frac{cE_0}{B_1}(\frac{B_1}{1 - \sigma_1} - 1) \tag{2.13}$$

The function for $\bar{y}_1$ (2.12), has its maximum at $\frac{B_1}{1-\sigma_1} = e$. For Beverton-Holt there

is no maximum. Let Ω be an ω-limit set of a given state. Then Ω is compact. Thus the function

$$E(y) = \sum_{i=2}^{k} F_i L_i y_i \qquad (2.14)$$

for $y = (y_1, y_2, \ldots, y_k) \in \Omega$, is continuous and attains its maximum $E_{max}(\Omega)$ and minimum $E_{min}(\Omega)$.

Lemma 2.1.4. *If $n(0) = (1 - \sigma_1)^{-1} \sum_{i=2}^{k} c F_i L_i$, the low density net reproductive value, satisfies $n(0) \leq 1$, then the trivial equilibrium is globally stable.*

Proof. We have $n(0) = (1 - \sigma_1)^{-1} \sum_{i=2}^{k} c F_i L_i$. Thus $\sum_{i=2}^{k} c F_i L_i \leq (1 - \sigma_1)$.
Let $|.|_\infty$ be the sup.norm on $\mathbb{C}^n$ and let $\|.\|$ be the corresponding operator norm. Let $y(0)$ be an arbitrary initial state and let Ω be its ω -limit set. Ω is compact and E is a continuous function on Ω. Let E_m be the minimal value on Ω. $E_m = 0$ would imply $y_2 = \ldots = y_k = 0$. Such states, however, are not in the range of M in system(2.5). Thus $E_m > 0$ and the first row in M, $\sigma_1 + \sum_{i=2}^{k} c F_i L_i < 1$. Now lemma(2.1.2) shows $M^n \to 0$. This implies $\Omega = 0$, because Ω is M invariant. $\qquad \square$

Lemma 2.1.5. *For model (2.5) we have:*

1. *If $\frac{dR}{dE}(\bar{E}) \geq 0$, then the equilibrium value is locally stable.*

2. *If $\frac{dR}{dE}(\bar{E}) \leq 0, -\sum_{i=2}^{k}(2r(\bar{E}) + \bar{E}\frac{dr(\bar{E})}{dE})c F_i L_i < 0$, then the equilibrium value is locally stable.*

Proof. 1. One has for $i = 2, 3, \ldots, k$ that $V_i = \frac{dR}{dE}\frac{dE}{dy_i} = (r(E) + E\frac{dr(E)}{dE})c F_i L_i$, then the term in the brackets is nonnegative. The equilibrium condition gives $(1 - \sigma_1) = \sum_{i=2}^{k} c F_i L_i r(\bar{E})$.
So that $\sigma_1 + \sum_{i=2}^{k} \frac{dR}{dE}\frac{dE}{dy_i} < \sigma_1 + \sum_{i=2}^{k} c F_i L_i r(\bar{E}) = 1$

2. Let $-\sum_{i=2}^{k}(2r(\bar{E}) + \bar{E}\frac{dr(\bar{E})}{dE})c F_i L_i < 0$, with the equilibrium condition

$$1 = \sigma_1 + \sum_{i=2}^{k} c F_i L_i r(\bar{E})$$

Thus $-\sum_{i=2}^{k} \bar{E}\frac{dr(\bar{E})}{dE})cF_iL_i < 2(1-\sigma_1)$, and $2\sigma_1 - \sum_{i=2}^{k} \bar{E}\frac{dr(\bar{E})}{dE}cF_iL_i < 2$, then

$$\sigma_1 - \sum_{i=2}^{k} cF_iL_ir(\bar{E}) - \sum_{i=2}^{k} \bar{E}\frac{dr(\bar{E})}{dE}cF_iL_i < 1$$

so that

$$\sigma_1 - (r(\bar{E}) + \bar{E}\frac{dr(\bar{E})}{dE})\sum_{i=2}^{k} cF_iL_i < 1$$

but $\quad \frac{dR}{dE}(\bar{E}) = c(r(\bar{E}) + \bar{E}\frac{dr(\bar{E})}{dE}) \leq 0$, hence $\sigma_1 + \sum_{i=2}^{k} |V_i| < 1$, and the result follows from lemma 2.1.2.

$\square$

Proposition 2.1.6. *Consider a general fishery model(1.9) with a smooth monotone increasing bounded recruitment function $R(x) = xr(x)$ such that its depensation part r converges monotonically to 0, and if $n(0) > 1$ then the unique nontrivial equilibrium is globally stable.*

The proof is based on a result of Hautus and Bolis as presented on the book of Elaydi[20, Theorem 5.17].

Proof. We use the standard representation and have
$x_1(t + 1) = \sigma_1 x_1(t) + R(E(t))$ with $E(t) = \sum_{i=2}^{k} F_iL_ix_i(t)$ and
$x_i(t + 1) = x_{i-1}(t), \quad i = 2, ..., k$. For $t > k$ this can be written as
$x_i(t + 1) = x_{i-1}(t) = ... = x_1(t - i + 1)$ and with $y(t) = x_1(t)$ this gives

$$y(t + 1) = f(y(t), y(t - 1), ..., y(t - k + 1)) = \sigma_1 y(t) + cE(t)r(E(t)) \quad (2.15)$$

where $E(t) = \sum_{i=2}^{k} L_iF_iy(t - i + 1)$. By assumption we have $\sigma_1 + c\sum_{i=2}^{k} L_iF_i > 1$ and the equilibrium $\bar{x} = \bar{x}_1(1, 1, ..., 1)$ is define uniquely by $1 - \sigma_1 = c\sum_{i=2}^{k} L_iF_ir(\bar{E})$ with $\bar{E} = \bar{x}_1\sum_{i=2}^{k} L_iF_i = \bar{y}(\sum_{i=2}^{k} L_iF_i)$. The function defined by (2.15) is monotone, because $R(E(t))$ is. Now let $u > \bar{y}$ then with $E_u = u\sum_{i=2}^{k} L_iF_i$ we have $f(u, u, ..., u) - u = (\sigma_1 - 1)u + u(c\sum_{i=2}^{k} L_iF_i)r(E_u) < (\sigma_1 - 1)u + (1 - \sigma_1)u = 0$. This shows(ii)in the theorem for $u > \bar{y}$. The same proof also works for $u < \bar{y}$. Now apply the theorem. $\square$

This proof and a continuity argument indicate that recruitment functions with a very large curvature for which r decays slowly will also be globally stable. This is borne out by a numerical study of the Levin-Goodyear[40] data analyzed with Shepherd and Hassell recruitment functions with weaker domes, even though the Ricker function leads to periodic solutions. The following plot demonstrates this effect. The data correspond to those given by Levin-Goodyear with natural mortality $\mu = 0.7769$ and $n(0) = 15$.
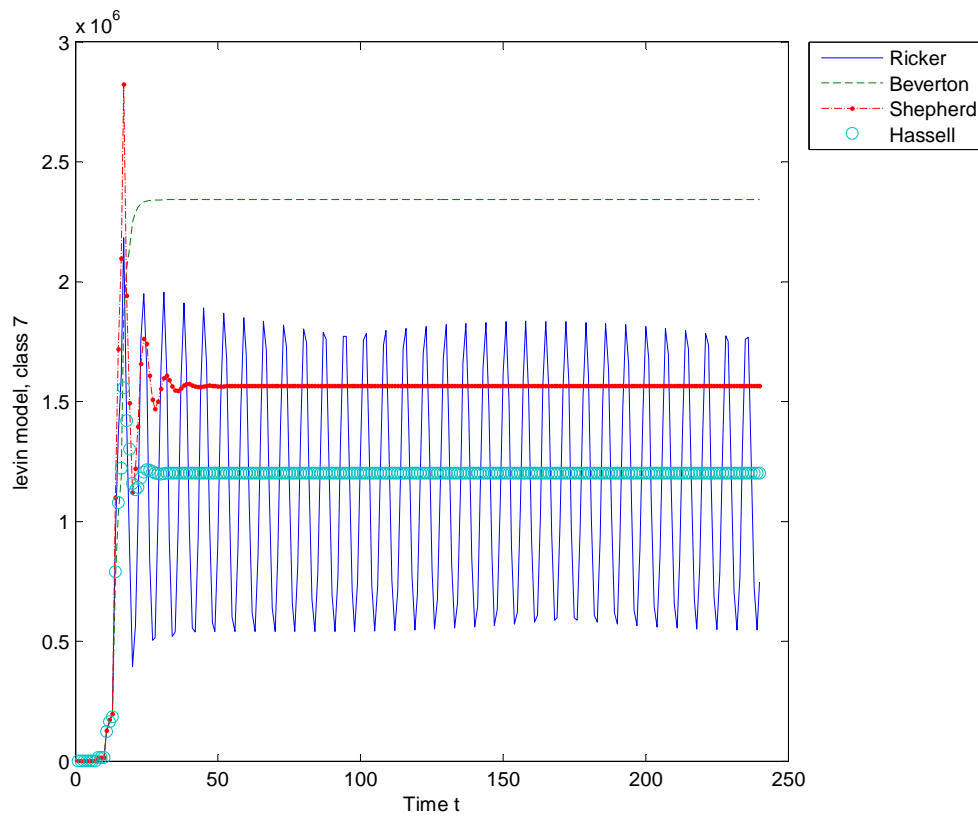
Figure 2.1: The striped bass model for different recruitment functions, $n(0) = 15$. Hassell and Shepherd recruitment functions with $\alpha = 2$. $E_0 = 400$ (B.H),$E_0 = 1000$ (R,H,S) and its value has no effect upon stability.
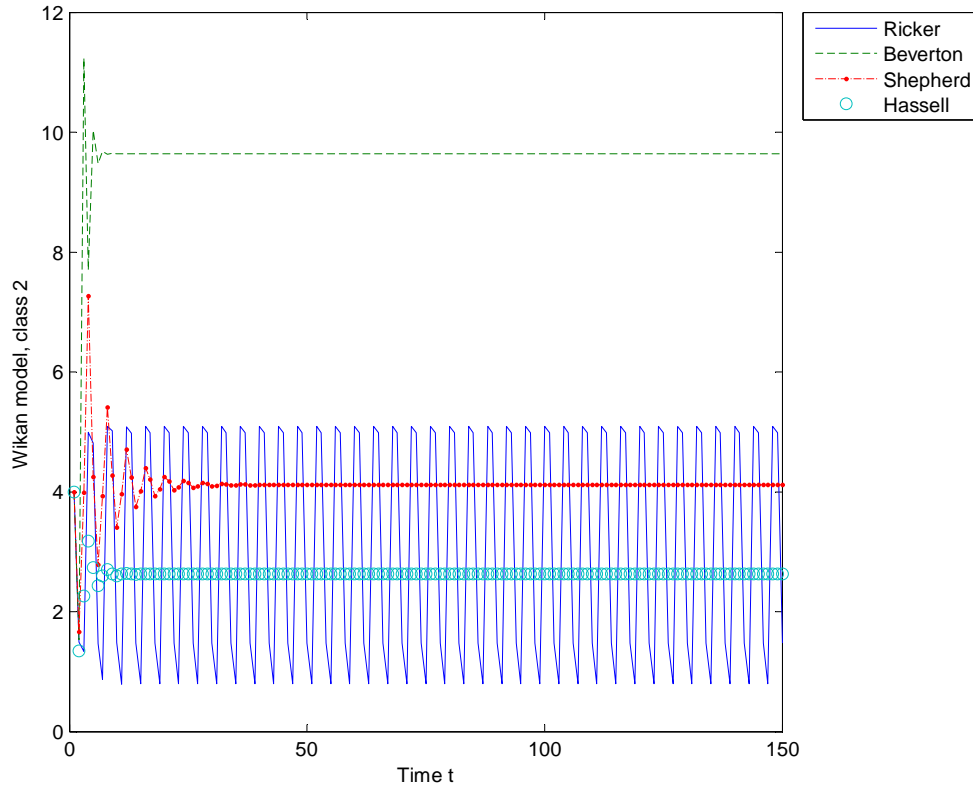
Figure 2.2: These plots are based on cod population model,Wikan model, with too high mortality 0.7769. $n(0) \approx 20$ with $\beta_1 = 1$, $\beta_2 = 0.2$, $\beta_3 = 0.1$. α is 2 for Shepherd and Hassell recruitment functions.

Remark 2.1.7. In order to understand the appearance of periodic or chaotic solutions I have also analyzed the Levin-Goodyear data [40], and model of Wikan and Eide[71], for various recruitment functions. Periodic solutions appeared only for Ricker recruitment and for large $n(0)$ with high mortality. The explanation is as follows. A high population density leads to a small $r(E)$ and thus a small subsequent population. This in turn makes $r(E) \approx 1$ so that the large $n(0)$ makes the follow up population large. Thus the Ricker recruitment function has been the darling of dynamical system theorists because of its simplicity and because it easily leads to

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periodic or even chaotic solutions. This also shows that attempts to explain seemingly periodic behavior of fish catches are weakly founded, because data for large population densities are difficult to come by and are rather insecure so that our knowledge of the recruitment function is at best sketchy. In particular it is unclear how wide the dome in the recruitment function is. The next plots indicate that the model(1.9) for cod or herring with small mortality always leads to stable equilibria even for $n(0)$ as large as 20.
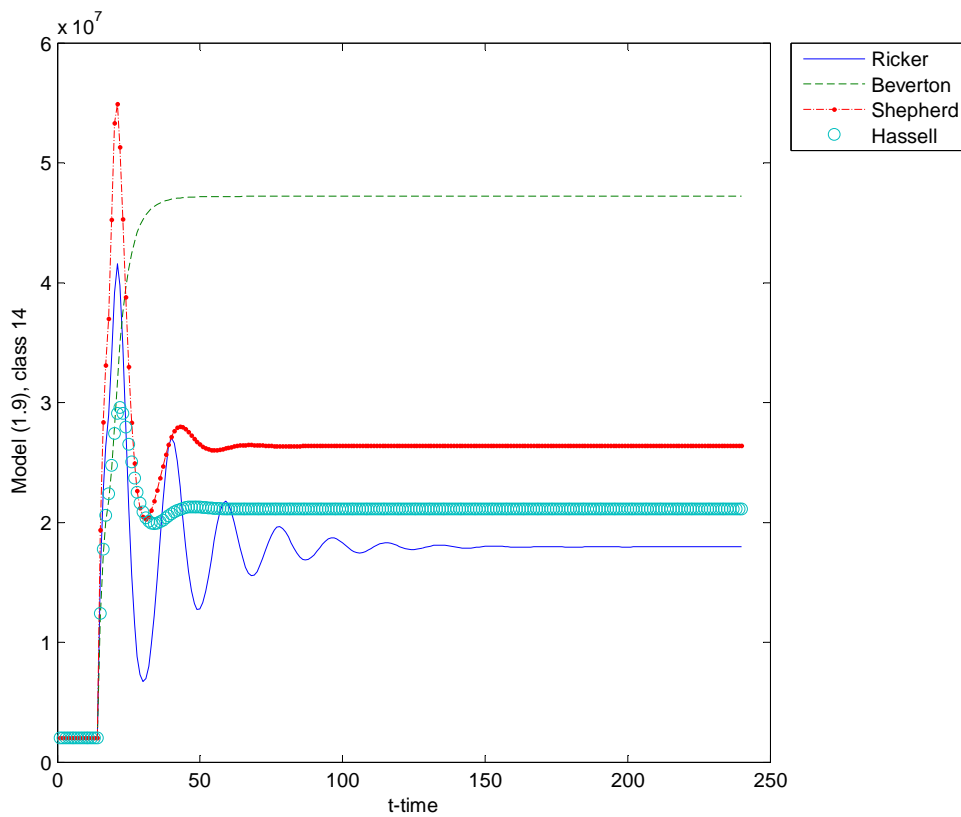
Figure 2.3: Stability for cod population model based on model (1.9). $n(0) = 21$. with $E_0 = 1.64 \times 10^{14}$ for Beverton-Holt and 4.1×10^{14} for others, and α is 2 for Shepherd and Hassell recruitment functions. These plots are based on cod population model with natural mortality 0.2224. Note that Ricker recruitment leads to a more oscillatory behavior and a lower equilibrium level. This plot also indicates that Beverton-Holt recruitment leads to more rapid convergence with larger equilibria.
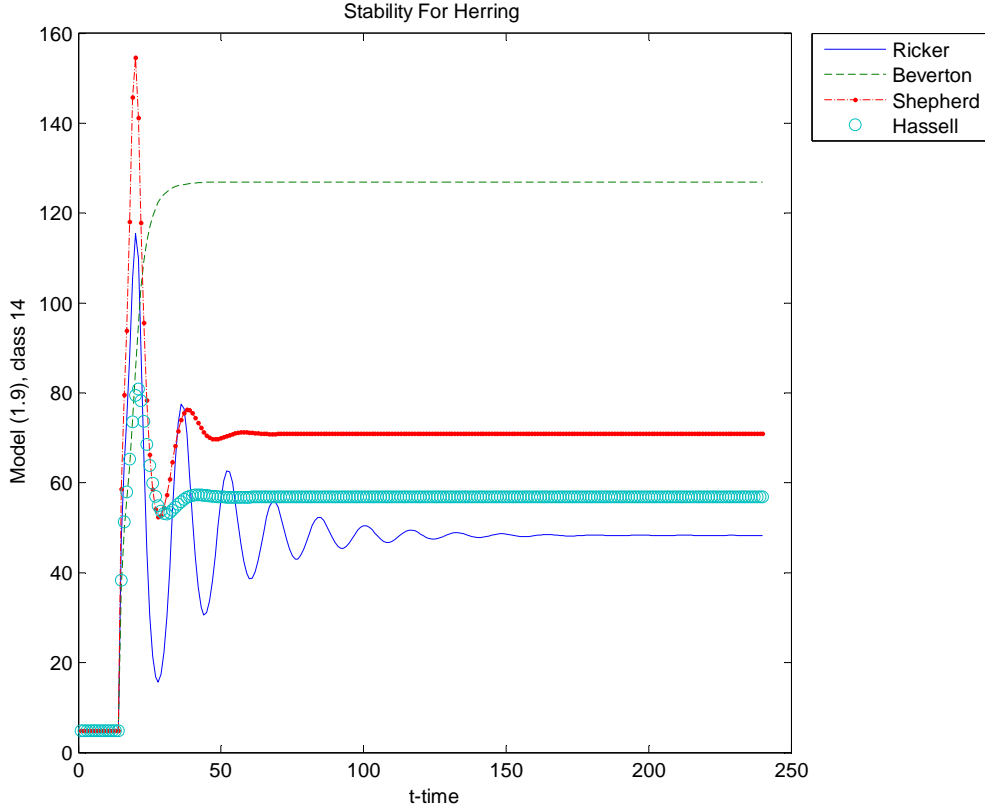
Figure 2.4: The parameter c has been adjusted to give $n(0) = 21$. with $E_0 = 4 \times 10^7$ for Beverton-Holt and 10^8 for others. α is 2 for Shepherd and Hassell recruitment functions. These plots are based on a herring population model with natural mortality 0.2224.

The following lemma can be found in[40].

Lemma 2.1.8. *For the Ricker recruitment function the nontrivial equilibrium value is locally stable if* $0 < ln(n(0)) < 2$

Proof. Let $ln(n(0)) < 2$, then

$$0 < \frac{\bar{E}}{E_0} < 2$$

and $-2 < -\frac{\bar{E}}{E_0} < 0$, so $-1 < 1 - \frac{\bar{E}}{E_0} < 1$,
therefore $-r(\bar{E}) < (1 - \frac{\bar{E}}{E_0})r(\bar{E}) < r(\bar{E})$,
thus
$\left| r(\bar{E}) + \bar{E}\acute{r}(\bar{E}) \right| < r(\bar{E})$ hence $\sigma_1 + \sum_{i=2}^{k} |V_i| < \sigma_1 + \sum_{i=2}^{k} cF_iL_ir(\bar{E}) = 1$, and the result follows. $\square$

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2.2 The Fine Model

One might argue that it is an unjustified simplification to lump the juveniles into one class. The model with $T_1 = T_2 = \ldots\ldots = T_k = 1$, will be called the fine model. Basically nothing changes. Now,however $r_2(x) = \ldots\ldots = r_{T_1-1}(x) = 0$ and of course $\sigma_1 = \ldots\ldots = \sigma_k = 0$. Now the model becomes

$$x(t+1) = Ax(t) \tag{2.16}$$

with the basic matrix

$$
A(x) = \begin{bmatrix}
0 & \cdots & 0 & r_{T_1}(x) & \cdots & r_k(x) \\
\tau_1 & 0 & 0 & \cdots & \cdots & 0 \\
0 & \tau_2 & \ddots & 0 & 0 & 0 \\
\vdots & 0 & \ddots & \ddots & \ddots & \vdots \\
\vdots & 0 & 0 & \ddots & \ddots & 0 \\
0 & \cdots & 0 & 0 & \tau_{k-1} & 0
\end{bmatrix}.
$$

Here, $\tau_i = e^{-\mu_i}$, $r_i = cF_i r(E)$ and $E(t) = \sum_{i=T_1}^{k} F_i x_i(t)$. In fact we can use all previous results if we put $F_2 = \ldots\ldots = F_{T_1-1} = 0$ and $\sigma_1 = 0$. Thus $n(0) = \sum_{i=T_1}^{k} r_i L_i = \sum_{i=T_1}^{k} cF_i r(E)L_i$ where $L_i = e^{-\mu_1}\ldots\ldots e^{-\mu_{i-1}}$. Of course the simpler version with $\tilde{A} = C^{-1}AC$ is easier to work with. It is likewise clear that the Lemmata 2.1.3 to 2.1.8 remain valid. Now (2.9) becomes

$$B_1 = \sum_{i=T_1}^{k} cF_i L_i, \qquad B_1^{-1} = r(\bar{E}) \quad \text{and } c\bar{E} = B_1 \bar{y}_1$$

The local stability of the equilibrium $\bar{y}$ is again determined by the roots of the characteristic polynomial

$$P(\lambda) = -\lambda^k + V_{T_1}\lambda^{k-T_1} + \ldots\ldots + V_k$$

In a numerical study no serious changes were found in the dynamics. It should be noted, however, that the characteristic polynomials for both models are distinct, so there must be small differences in the dynamics. So far the lumping of age classes in models has not been studied systematically. This would be desirable, because

aggregation into age classes is a standard technique for modelers, see eg.[15, 16, 54, 67, 68]. In particular it is not clear if a two age class model-juvenile and adults, leads to realistic predictions. The next plots are based on a cod population, model(2.16) with natural mortality 0.2224 and n(0)=15 as the same above $E_0 = 1.64 \times 10^{14}$ for Beverton-Holt and 4.1×10^{14} for others, and α is 2 for Shepherd and Hassell recruitment functions.
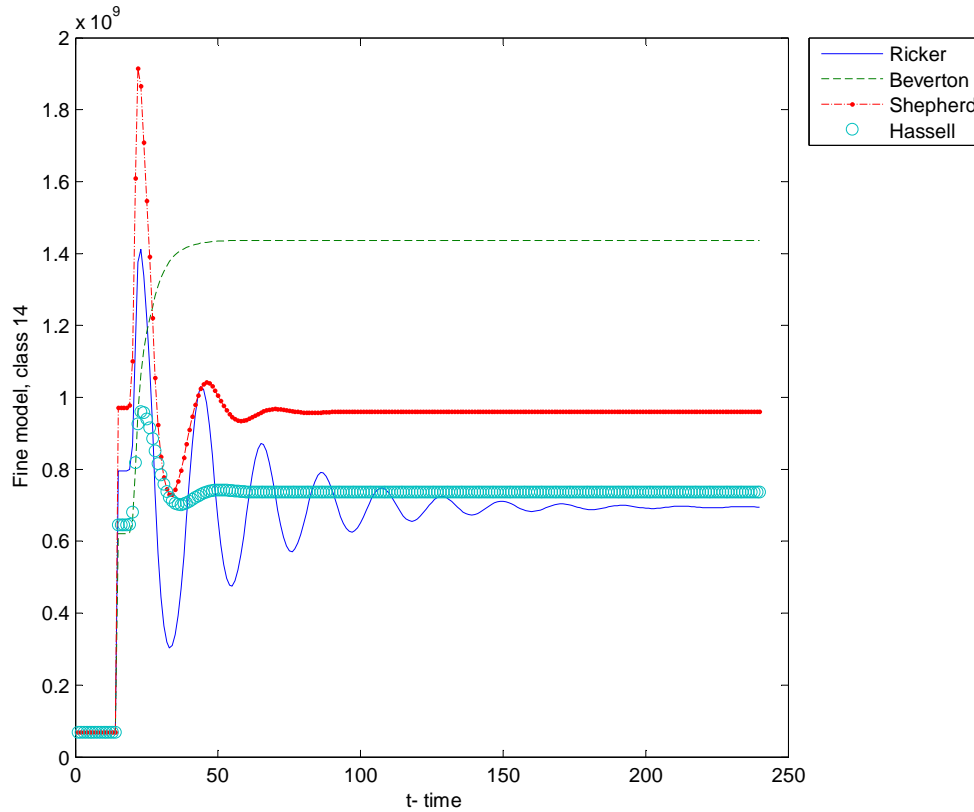
Figure 2.5: Stability for cod population based on the Fine model

2.3 The road to equilibrium

Since nowadays most fish stocks a severely depleted one should also consider how this model describes the return to equilibrium from low levels e.g. 10% of its equilibrium density. It ranges from about 20 to 60 years for different values of $n(0)$. Values of $n(0) \geq 10$, however, are connected with an increasing oscillatory convergence to the equilibrium. The explanation for this is the same as in the Remark 2.1.7. It also shows that the largest eigenvalue has a large imaginary part.

From this point of view it is not surprising that the cod fishing in Newfoundland is not recovered yet [4, 52, 57].
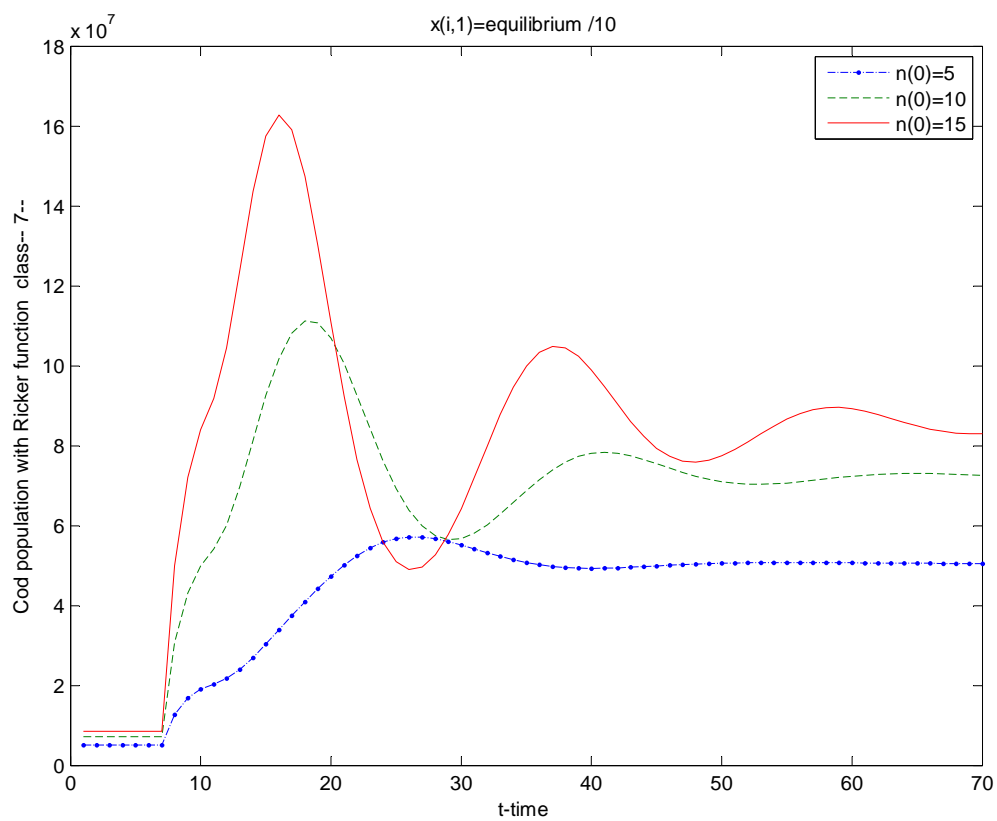
Figure 2.6: The approach to equilibrium is more oscillatory for large $n(0)$
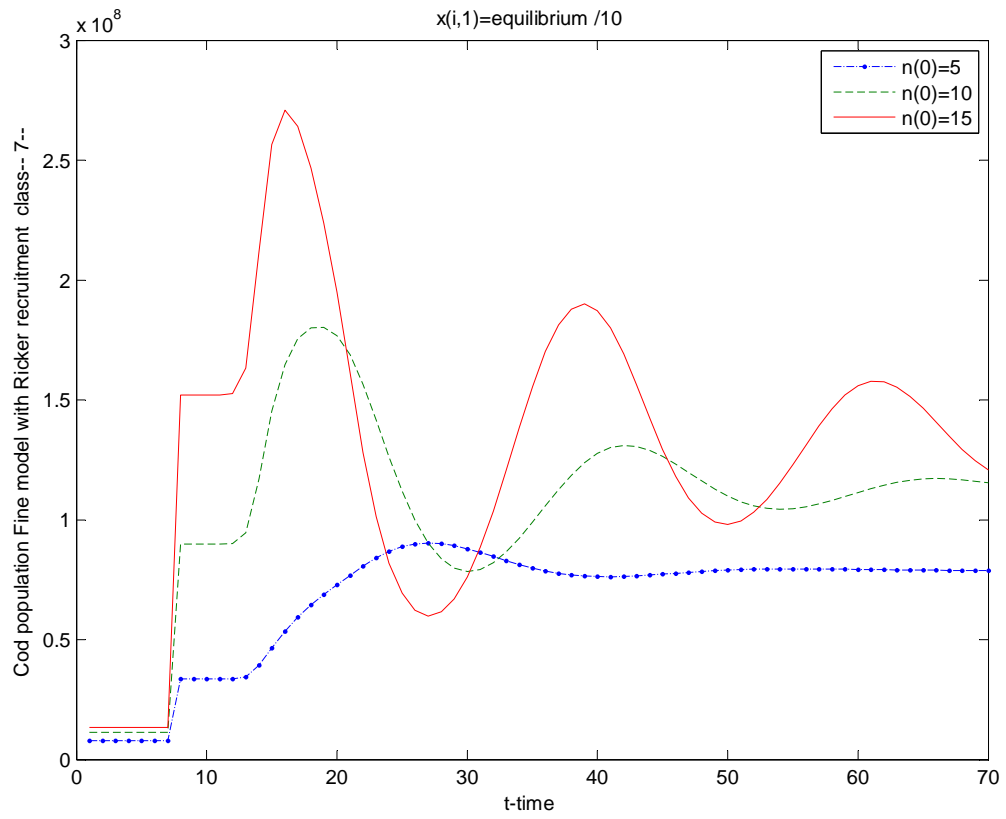
Figure 2.7: The approach to equilibrium in the fine model has the same behaviors as above

Chapter 3

Harvesting

3.1 The effects of harvesting

This section is devoted to the description of harvesting. Basically harvesting acts as a modification of mortality. The ultimate goal is to determine a stable optimal equilibrium solution. This means we are looking for an equilibrium with a maximal yield. Since small external perturbation are expected for any natural population, we are even looking for a stable equilibria. These equilibria depend on the harvesting intensity h and mesh parameter s. In a final step the total harvest H_1 as a function of the harvesting intensity h and mesh parameter s will have to be maximized. This results in a sustainable equilibrium with maximal yield.

The simplest method describes harvesting as an impulse harvesting. Fishing starts with the second class and acts in all classes separately at the end of the season. If h_i is the harvesting intensity of class i, then $H = diag(0, h_2, ..., h_k)$ describes the harvesting matrix and $(1 - H)A$ the escape matrix. The system then becomes

$$x(t+1) = (1 - H)A(t)x(t) \qquad (3.1)$$

with

$$(I - H)A(t) = \begin{bmatrix} \sigma_1 & r_2 & r_3 & \cdots & r_k \\ \tilde{\tau}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{\tau}_2 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{\tau}_{k-1} & 0 \end{bmatrix}$$

where $\tilde{\tau}_{i-1} = (1-h_i).\tau_{i-1}, i = 2, ..., k,\ r_i = cF_i r(E(t))$ and $E(t) = \sum_{i=2}^{k} F_i x_i(t)$. The matrix $(I - H)A$ has the same structure as A. Thus the components of the non-trivial equilibrium vector are

$$\bar{x}_i = \frac{cL_i(h_j)r^{-1}(\frac{1}{n(h_j)})}{n(h_j)(1 - \sigma_1)}$$

where $L_i(h_j) = \tau_1(1 - h_2)\tau_2(1 - h_3)...\tau_{i-1}(1 - h_j)$, and

$n(h_j) = (1 - \sigma_1)^{-1} \sum_{i=2}^{i=k} cF_i L_i(h_j),\ i = 2, 3, ..., k,\ j = 2, 3, 4, ..., k.$

The aim is to determine an optimal sustainable stable fishing strategy, that means an equilibrium $\bar{x}$ of (3.1) with a maximal harvest. Thus we are looking for an $\bar{x} = (1 - H)A\bar{x}$ with $< HA\bar{x}, W >\rightarrow Max$, where $W = (W_1, W_2, ..., W_k)$ is the weight vector and $<,>$ is the inner product. Such an approach is unrealistic for several reasons. It neglects the interactions between natural mortality and fishing mortality. In fact this form describes pulse harvesting at the end of the year. Secondly $h_i = 1$ would imply that all fish of class i are caught. This is unrealistic because fishing becomes more difficult as the sea is depleted[9].

There are good arguments that a two class model will be much more prone to periodic solutions than a model with more than 20 classes. A similar argument might be made for pulse fishing, when it is derived from a two class model. This form of fishing mortality $(1 - h_i)$ will require all sorts of constraints on the h_i. For optimal control models a linear control usually leads to bang-bang solutions, which are difficult to realize in practice.

Thirdly, it is hardly possible to relate fishing effort, i.e number of boats, form and size of gear,..., with h_i. For these reasons we will employ a method which was introduced by Getz [28], which we call the semi-continuous description and the idea that mortality equals fishing mortality plus natural mortality. It is based on a no harvest season of duration t_c just after oviposition. In general t_c will be 1/4 or 1/3 year. If $x(t)$ denotes the number of fish in a population class and if h is the harvesting intensity, one has

$$\frac{dx(t)}{dt} = \begin{cases} -\mu x & : 0 \leq t \leq t_c \\ -(\mu + h)x & : t_c < t \leq 1 \end{cases}$$

From this one easily gets

$$x(t + 1) = x(t)e^{-\mu - h(1 - t_c)} \tag{3.2}$$

This expression also shows, why it is advantageous to express survival in an exponential form. Let h_i denote an fishing effort in class i. Then the model becomes

$$x(t+1) = A(t)x(t) \tag{3.3}$$

with

$$A(t) = \begin{bmatrix} \sigma_1 & r_2 & r_3 & \cdots & r_k \\ \tilde{\tau}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{\tau}_2 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \tilde{\tau}_{k-1} & 0 \end{bmatrix}.$$

$\tilde{\tau}_i = e^{-\mu_i - (1-t_c)h_{i+1}}, i = 1, 2, ..., k-1, \ r_i = cF_i r(E(t))$ and $E(t) = \sum_{i=2}^{k} F_i x_i(t)$.

The harvested amount in class i is then

$$H_i = \int_{t_c}^{1} x_i(t)h_i W_i dt = \frac{h_i W_i x_i(t)e^{-\mu_i t_c}}{\mu_i + h_i}(1 - e^{-(\mu_i + h_i)(1-t_c)}).$$

This shows that in the model only the mortality has changed .In particular all results of previous chapter on equilibria and stability carry over. For a given harvesting strategy $\overrightarrow{h} = (0, h_2, h_3, ..., h_k)$ one can therefore write $n(h), \bar{x}(h), y(h), \bar{y}_1(h)$ for the corresponding quantities. Fishing with seines has the consequence that all fish of a size less than ℓ_1 (in length) escape, while all fish above the size ℓ_2 are caught. The gear selectivity described in[15, 16] suggests a linear interpolation between ℓ_1 and ℓ_2. Knife edge selectivity means $\ell_1 = \ell_2$. Relating the forms of selectivity to age classes leads to harvesting intensity pattern.

$$h_i = \begin{cases} 0 & : i < s \\ \gamma_1 h & : i = s \\ \gamma_2 h & : i = s+1 \\ \gamma_3 h & : i = s+2 \\ h & : s+3 \leq i \leq k \end{cases} \tag{3.4}$$

with $\gamma_1 < \gamma_2 < \gamma_3$. I will mostly only use the knife edge harvesting with

$$h_i = \begin{cases} 0 & : i < s \\ h & : i \geq s \end{cases} \tag{3.5}$$

because numerical results (Table 3.1) have shown that there are no substantially differences between knife edge and gear selectivity. We shall speak of an (h, s) fishing strategy in this case. The total sustainable harvest becomes

$$H_1(h, s) = \sum_{i=s}^{k} (\mu_i + \gamma_i h)^{-1} W_i \gamma_i h x_i(t) e^{-\mu_i t_c} (1 - e^{-(\mu_i + \gamma_i h)(1 - t_c)}) \qquad (3.6)$$

where, $\gamma_{s+i} = 1$ for $i \geq 4$. For an equilibrium value $\bar{x}$ respectively $\bar{y} = \bar{y}(h, s)$ this means

$$H_1(h, s) = \sum_{i=s}^{k} (\mu_i + \gamma_i h)^{-1} W_i \gamma_i h L_i(h, s) \bar{y} e^{-\mu_i t_c} (1 - e^{-(\mu_i + \gamma_i h)(1 - t_c)}) \qquad (3.7)$$

If the price of one unit fish in class i is q_i, one will have to replace W_i above by $q_i W_i$. It is obvious that older and larger fish give a higher price because they have considerably more meat. Unfortunately it is not so easy to get the price per kilogram of an ungutted fish. The following data for cod population have been extracted from [15, 51]. Regression gives in relative units that

$$q_i = \begin{cases} 0.18 + 0.09i & : 2 \leq i \leq 9 \\ 1.08 & : i \geq 10 \end{cases} \qquad (3.8)$$

A higher price of large fish will clearly move the optimal mesh size s up. So the results of [15], " save the young fish " are basically the result of the price list. Note, however, that [54] does not really "save the young ones". For herring population one can use $q_i = 1$ for all i. For $\bar{y}, ...$, one can write now $\bar{y}_1(h, s)$, $B_1 = B_1(h, s) = \sum_{i=2}^{k} cF_i L_i(h, s)$, and $L_i(h, s) = \tilde{\tau}_1(h, s)......\tilde{\tau}_{i-1}(h, s)$. With Ricker recruitment this gives

$$\bar{E} = E_0 ln(\frac{B_1(h, s)}{1 - \sigma_1}) \quad and \quad \bar{y}_1(h, s) = \frac{cE_0}{B_1(h, s)} ln(\frac{B_1(h, s)}{1 - \sigma_1}) \qquad (3.9)$$

and for Beverton-Holt gives

$$\bar{E} = E_0(\frac{B_1(h, s)}{1 - \sigma_1} - 1) \quad and \quad \bar{y}_1(h, s) = \frac{cE_0}{B_1(h, s)}(\frac{B_1(h, s)}{1 - \sigma_1} - 1) \qquad (3.10)$$

In addition we have

$$n(h,s) = (1-\sigma_1)^{-1}\sum_{i=2}^{k} cF_iL_i(h,s) \tag{3.11}$$

There is a nontrivial equilibrium value with nonzero harvest only if $n(h,s) > 1$. This will be locally stable with Ricker recruitment if $ln(n(h,s)) < 2$. In general one would expect that harvesting increases stability. For this one will have to study the characteristic polynomial of $C\mathcal{J}C^{-1}$. This is given by

$$-\lambda^k + \sigma_1\lambda^{k-1} + V_2\lambda^{k-2} + ... + V_k = 0 \tag{3.12}$$

with $V_i = (r + E\frac{dr(E)}{dE})cF_iL_i(h,s), \quad i = 2,3,......,k$. Even though the V_i become smaller with increasing h, standard tests on the characteristic polynomial do not readily yield that the maximal eigenvalue decreases with increasing h. In most cases the largest eigenvalue λ was found to satisfy $\mid \lambda \mid < 0.96$. Since $n(h,s) = (1-\sigma_1)^{-1}\sum_{i=2}^{k} cF_iL_i(h,s)$, the natural reproductive number n, will decrease with increasing mortality which includes fishing mortality. So once $n(h,s) < e^2$ a further increase of h will at least preserve the local stability. We have seen above, Proposition(2.1.6), that the recruitment function has a profound effect on the dynamics. It has also a serious effect on the harvesting. To see this consider (2.9) in the form $r(\bar{E}) = \frac{(1-\sigma)}{B_1}$, and $\bar{y}_1 = \frac{c\bar{E}}{B_1}$. Thus depensation functions with a faster decay will lead to a smaller equilibrium solution and thus to a lower harvest. This indicates a dilemma with the recruitment process. Ultimately R respectively r can only be determined by actual data. These, however, allow at most a proper determination of $n(0)$, the slope at the origin. The highly scattered higher density can be fitted easily and equally well by Beverton-Holt or Ricker functions. These, however, will give quite distinct optimal harvest results.

It remains to discuss various other quantities as functions of the harvesting intensity h and mesh parameter s. It is clear that the transition probabilities τ_i are decreasing with increasing h. Thus the conditional probabilities $L_i(h,s)$ will decrease strongly for larger age indices i. Thus also $n(h,s)$ is a decreasing function of h. Arguing similarly one can see that $n(h,s)$ increases with increasing s, because a large s implies that the first $s-1$ age classes are spared. The same applies to $\bar{E}$. Since E_0 is a scaling parameter populations in an environment with a large supply of food will also lead to a large $\bar{E}$. The function $x^{-1}ln(x)$ on $(0,\infty)$ increases monotonically for

$x < e$, and has a maximum at $x = e$. Beyond $x = e$ it decreases monotonically to 0. Thus $\bar{y}_1(h, s)$ will have a single maximum at $\frac{B_1(h,s)}{1-\sigma_1} = n(h, s) = e$. Thus for Ricker function $\bar{y}_1(h, s)$ will initially increase as a function of h. For large h, however, $n(h, s)$ tends to a value less or equal 1. In this case one would get $\bar{y}_1 = 0$. A larger net parameter s implies a larger value of $n(s, h)$. For larger s the decrease of $n(s, h)$ with h is slower. This holds for other values of $n(0)$ likewise. An increase of μ or a monotonically increasing mortality will obviously decrease the L_i and n. Likewise it will decrease the optimal harvest. Thus there is no need to study sensitivity with respect to μ any further. $n(s, h)$ is clearly a decreasing function of h. But the decrease is less pronounced for larger s.

In general for Ricker recruitment $\bar{y}_1(h, s)$ will increase at first and attain its maximum at $B_1 = (1 - \sigma_1)e$. Then it will decrease rapidly - more rapidly for small s - to 0. For large s, B_1 will never reach $(1 - \sigma_1)e$, so $\bar{y}$ will be increasing monotonically.
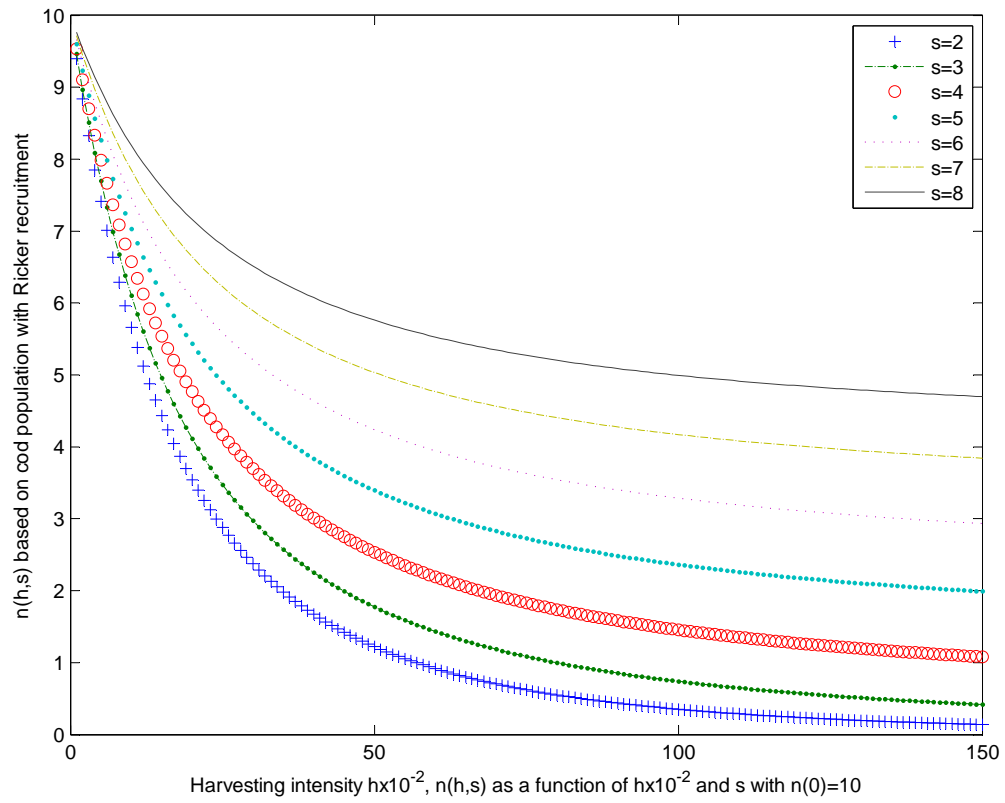
Figure 3.1: $n(h, s)$ is a decreasing function for h. For small s a few fertile classes will remain, the function $n(h, s)$ as a function of h will decrease stronger than for large s. The same holds for other values of $n(0)$
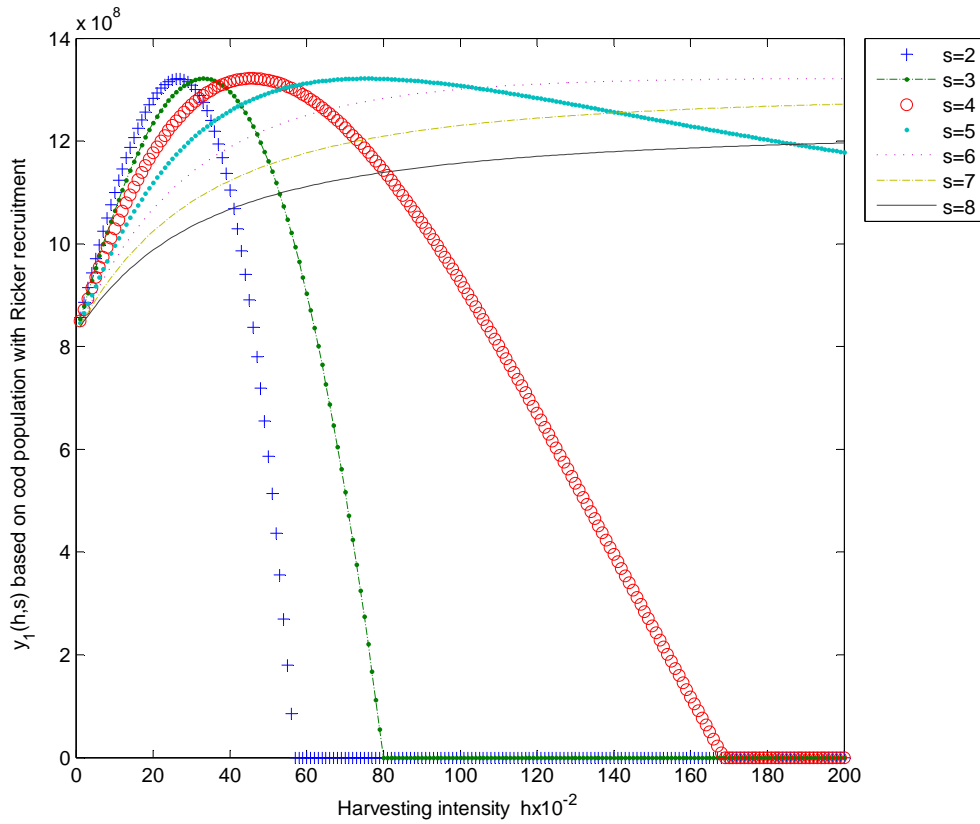
Figure 3.2: $y_1(h, s)$ as a function of $h \times 10^{-2}$ and s with $n(0) = 10$, the function becomes zero fast for small s with higher h. For larger s the influence of h is less pronounced. Since $x^{-1}ln(x)$ has maximum at e, $\bar{y}_1(h, s)$ will initially increase as h increases until $n(h)$ reaches a value beyond e then it will decrease and hit 0 when $n(h)$ becomes less than one for further increasing of h. This decrease is steeper for small s. The same thing holds for other values of $n(0)$.

The expression $H_1(h, s)$ is a sum of terms $\bar{y}_1 L_i(h, s)$ which decrease with h at least for $n(h, s) \geq e$ and terms which increase with h. This makes it likely that the optimal harvesting level h will be such $\bar{y}_1$ has a level slightly below its maximal value. The function $H_1(h, s)$ will in general have the form

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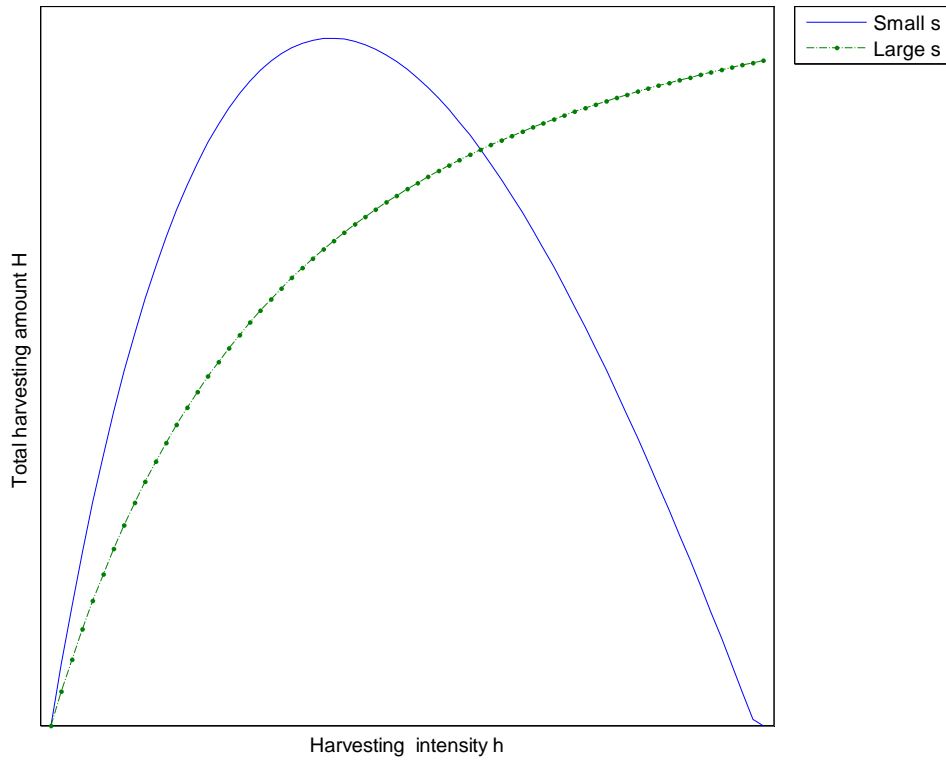
Figure 3.3: The general form of $H_1(h, s)$ for different s.

because for $h = 0$ we have $H_1 = 0$. Then the initial increase of H is checked by the rather rapid decrease of $L_i(h, s)$ in particular those for large i. The fact that even for small harvesting rates h the higher age classes are rather thinly occupied implies also that the optimal s will not be too large. The fact that harvesting decreases the chance of finding large price fish has been observed with many other species.

For Beverton-Holt recruitment $y_1(h, s)$ (3.10) will be a decreasing function with h as well as $n(h, s)$. But $H_1(h, s)$ has the same behavior as Ricker recruitment. The same thing holds for herring populations. It is clear that a small parameter $n(0)$ will require a large value s, because otherwise too many fertile fish are harvested.
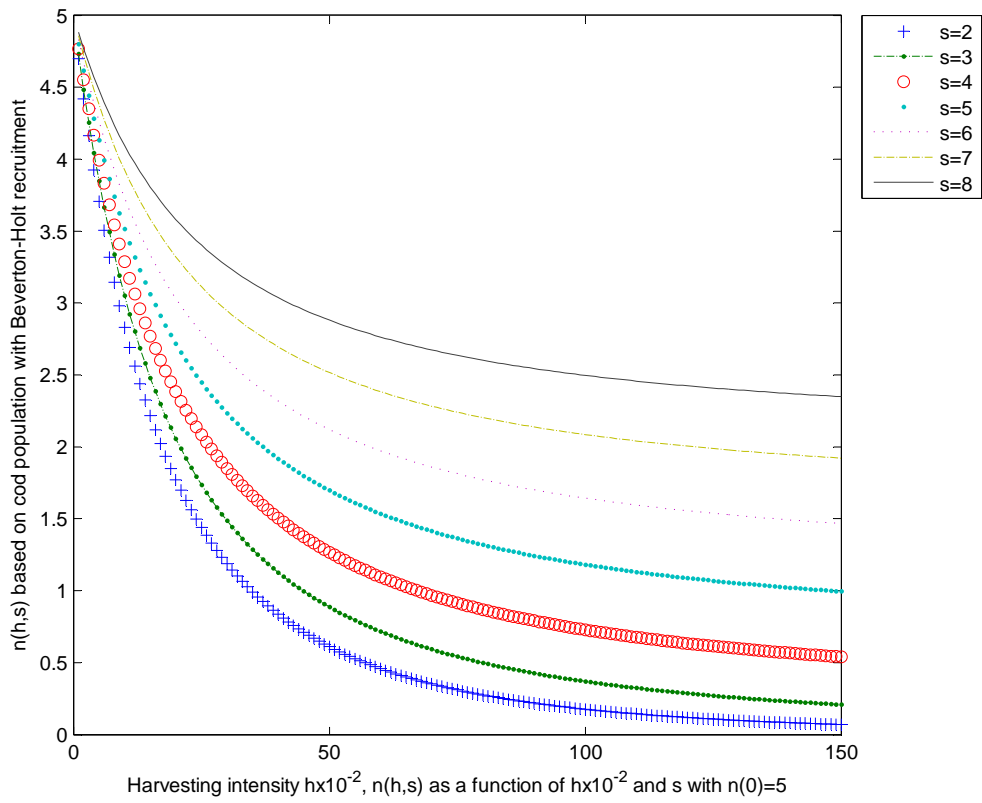
Figure 3.4: The function $n(h, s)$ for cod population with Beverton-Holt recruitment. It has the same behavior as the Ricker recruitment. The same holds for other values of $n(0)$.
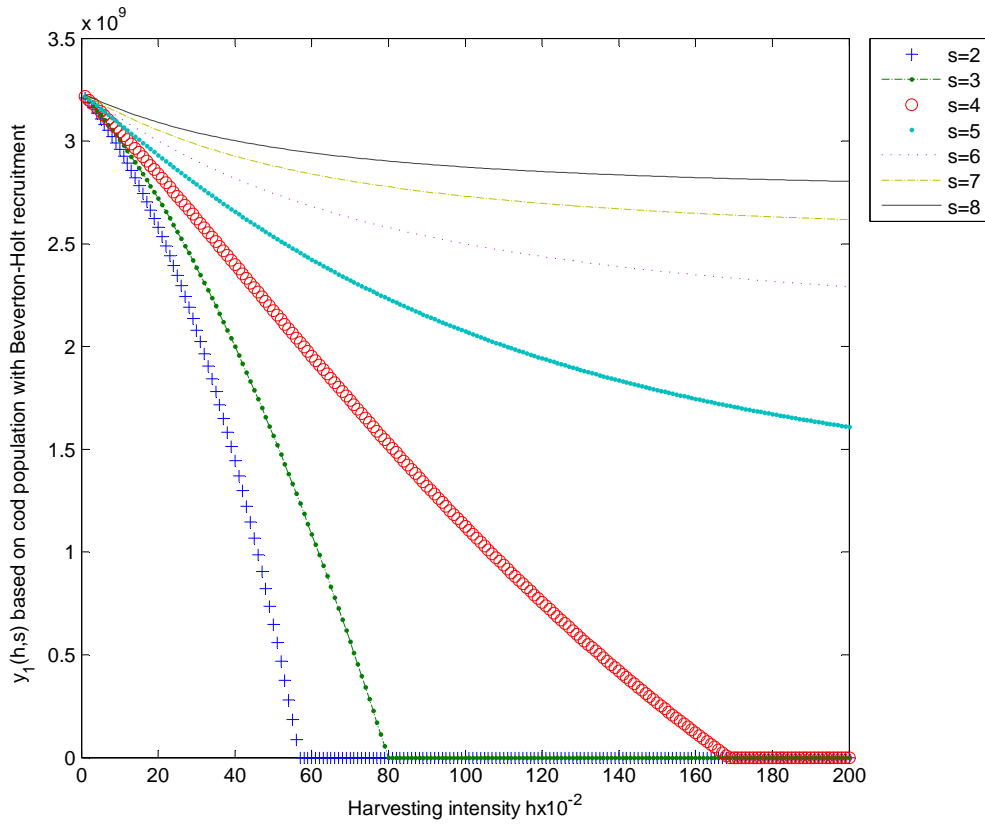
Figure 3.5: $y_1(h, s)$ as a function of $h \times 10^{-2}$ and s with $n(0) = 10$ for Beverton-Holt function has no maximum.

3.2 Stability of the equilibrium value with harvesting:

For an analysis of the local stability with present harvesting one has to compute the roots of the characteristics polynomial of the Jacobian matrix of the system

$$\mathcal{J} = \begin{bmatrix} \sigma_1 & a_2 & a_3 & \cdots & a_k \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad (3.13)$$

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with $a_i = cF_iL_i(h,s)r(E) + \sum_{i=2}^{k} cF_iL_i(h,s)\frac{dr(E)}{dE}\frac{dE}{dy_i} = (r(E) + E\frac{dr(E)}{dE})cF_iL_i(h,s)$, and $E(t) = \sum_{i=2}^{k} F_iL_i(h,s)y_1(h,s)$. Then the characteristics polynomial is, $\sigma_1\lambda^{k-1} + \sum_{i=2}^{k} a_i\lambda^{k-i} - \lambda^k = 0$. By using Ricker function this gives $\sigma_1\lambda^{k-1} + (\frac{1-ln(n(h))}{n(h)})\sum_{i=2}^{k} cF_iL_i(h,s)\lambda^{k-i} - \lambda^k = 0$. Now consider $f(\lambda) = -\lambda^k$, it is clear that all roots of $f(.)$ are located inside the unit circle. So for $|\lambda| = 1$ we have $|g(\lambda)| = \left|\sigma_1\lambda^{k-1} + (\frac{1-ln(n(h))}{n(h)})\sum_{i=2}^{k} cF_iL_i(h,s)\lambda^{k-i}\right| \leq \sigma_1 + \left|(\frac{1-ln(n(h))}{n(h)})\sum_{i=2}^{k} cF_iL_i(h,s)\right|$ $< \sigma_1 + |1 - \sigma_1| = 1 = |f(\lambda)|$, whenever $|ln(n(h)) - 1| < 1$. By Rouche's theorem [61], the equations $f(\lambda) + g(\lambda) = 0$ and $f(\lambda) = 0$ have the same number of zeros located inside unit circle whenever $0 < ln(n(h)) < 2$. It is clear that $ln(n(h)) < ln(n(0))$. So that in general one can say harvesting improves stability of the positive equilibrium value.

3.3 Numerical examples with harvesting and the influence of parameters

In this section the optimal harvesting is studied for h and s. Here h stands for the harvesting intensity or effort, for example the number of boats, people employed,... . The net parameter s measures the width of the meshes of the fishing nets. As argued above data with knife edge harvesting will mainly be studied because extensive computations with gear selective harvesting have only resulted in small changes. The aim is to determine the optimal mesh width s and the maximum sustainable yield (MSY). This will be studied for cod first and later for other species. In addition the local stability is studied of the model in case of harvesting for cod first and later for other species by computing the eigenvalues of Jacobian matrix for various values of h, s, and $n(0)$.

1-Cod

As before according to [41], all parameters for cod population are chosen as above, namely maximum age ,fecundity F_i, weight W_i, mortality μ and scaling parameter E_0. As stated above a modified mortality has a little effect on the harvesting. Set $t_c = 1/4$ in all computations. So the values for H, are relative values. In the actual computations it turns out that for small s , $s = 2$ or 3, the equilibrium value

$\bar{y}(h) = 0$ is reached quickly for rather moderate harvesting intensities. So $s = 2, 3$ will never be optimal and will therefore not be considered. This holds for the fine model too. If one uses prices increasing with weight in (3.8) this will be even more pronounced so that actually only $s > 3$ will be relevant. Similarly net parameters s larger than 11 are inefficient, because they spare too many fish of weight larger than 9 or 10 kg. In the table below a comparison between gear selective harvesting and knife- edge harvesting is given for various net reproductive rates $n(0)$. Gear selective harvesting is based on $\gamma_1 = 0.40, \gamma_2 = 0.60$ and $\gamma_3 = 0.80$. The corresponding knife edge harvesting is based on s.

$n(0)$	s_m	$n(h_{max}, s)$ gear sel.	$n(h_{max}, s)$ knife edge	$H(h, s)$ gear sel.	$H_1(h, s)$ knife edge
3	10	1.79	1.79	1.33×10^8	1.33×10^8
5	9	2.76	2.62	2.13×10^8	2.19×10^8
7	8	3.87	3.08	2.94×10^8	3.01×10^8
10	7	3.89	3.53	3.94×10^8	4.01×10^8
15	6	4.48	3.90	5.26×10^8	5.36×10^8
22	5	4.50	3.63	6.80×10^8	6.89×10^8

Table 3.1: Comparing the optimal harvesting for gear selective harvesting $\gamma_1 = 0.40, \gamma_2 = 0.60$ and $\gamma_3 = 0.80$ for cod population with knife edge harvesting. Maximum effort is 3

We put a limit on the maximal harvesting intensity, because the graphs for $H_1(h, s)$ for $s \geq 6$ are monotonically increasing. But for $h = 3$ values close to the maximal value are attained. Needless to say that $h \geq 1$ is rather unrealistic.
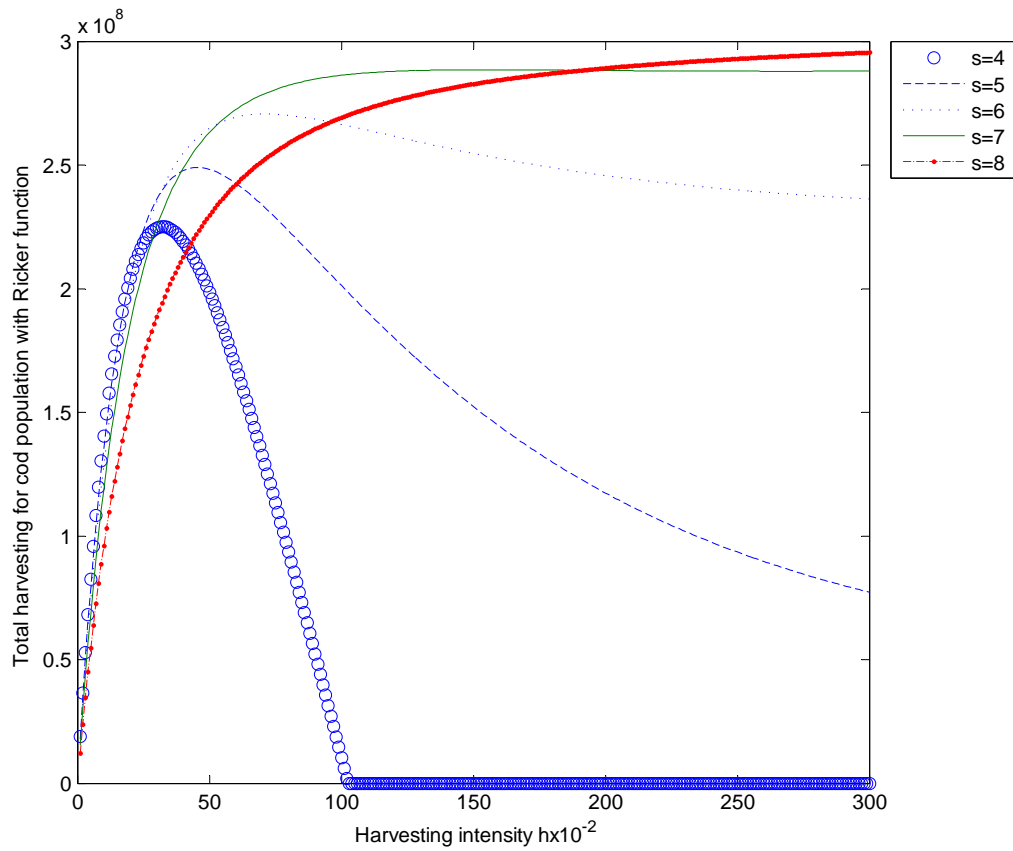
Figure 3.6: A typical plot of total harvested amount $H_1(h,s)$ as a function of h, s with $n(0) = 7$. For small s an increase in h will have strong effect on $n(h,s), \bar{y}_1(h)$, and thus the total harvesting. But for large s the total harvested amount may stay at a high level different from 0, because it is preserved by the remaining small age classes. The same holds for others values of $n(0)$.
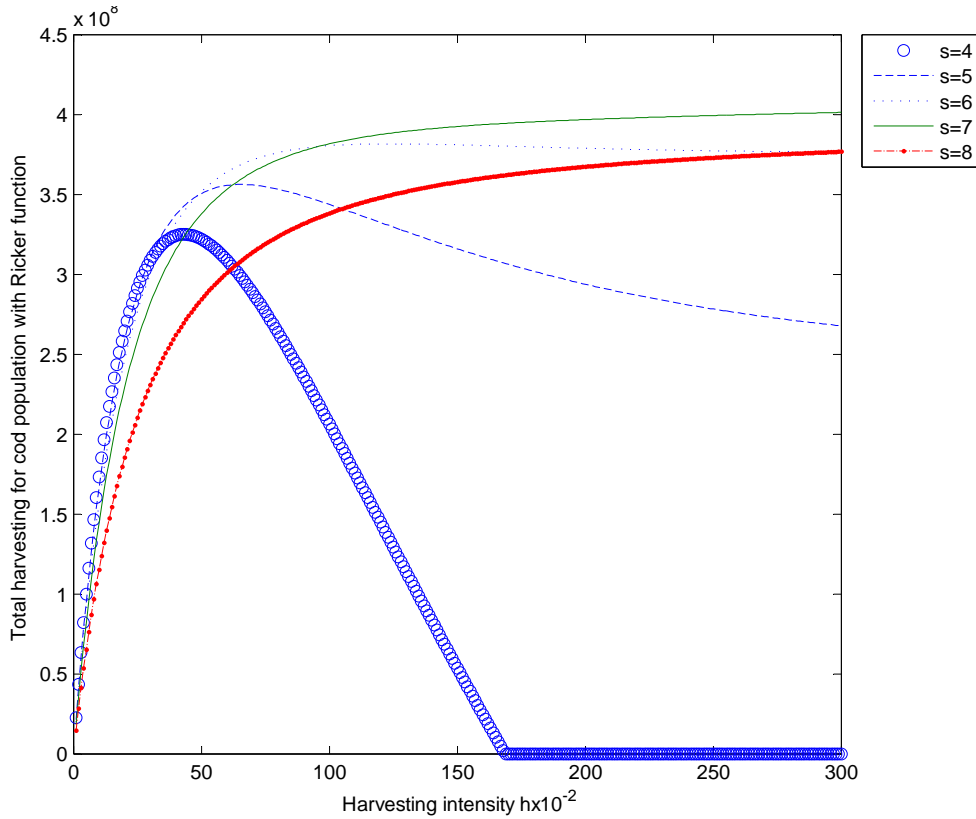
Figure 3.7: The total harvest as a function of $h \times 10^{-2}$ with $n(0) = 10$

For small s an increase in h will have strong effect on $n(h, s), \bar{y}_1(h)$, and then the total harvesting. The harvested amount will initially increase rapidly, until $\bar{y}_1(h)$ decreases. As $\bar{y}_1(h)$ hits 0 the total harvesting becomes 0 likewise. For large s the remaining small age classes preserve $\bar{y}_1(h)$ and it may never tend 0. Thus the total harvest may stay at a nonnegative level. As expected the price plays an important role in harvesting. It is obvious that larger fish are relatively more valuable, because they give more meat. Likewise it is clear that a higher price for older fish will increase the optimal s and the higher the price more it is in advantageous to use nets with larger meshes. From a conservation point of view it is also advantageous to save the young fish.

As expected the maximal harvest increases with $n(0)$, while s decreases with $n(0)$. According to the table 3.1,there is no important difference between the knife-edge strategy and gear selective strategy. Thus throughout the remainder only knife-edge selectivity will be used. Extensive calculations with the fine model also showed that there is no substantial difference between the fine model and the model above.

In the following tables some numerical values for various quantities are presented. These are the largest eigenvalue of the the Jacobian i.e the largest root in (2.8), the net reproductive rate $n(h)$ and the equilibrium value as a function of h and s.

The mesh width s=3										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.902	0.742	0.926	0.967	0.999	0.946	0.900	0.811	0.751	0.680
the reproductive rate n(h)	7	2.882	1.578	1.241	1.004	0.699	0.518	0.293	0.196	0.118
the equilibrium point $\times 10^8$:	6.992	9.238	7.272	4.379	0.09660	0	0	0	0	0
The mesh width s=4										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.902	0.784	0.851	0.895	0.925	0.967	0.997	0.952	0.921	0.886
the reproductive rate n(h)	7	3.337	2.106	1.774	1.534	1.217	1.019	0.756	0.632	0.524
the equilibrium point $\times 10^8$:	6.992	9.083	8.895	8.127	7.018	4.053	0.4626	0	0	0
The mesh width s=5										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.902	0.808	0.673	0.791	0.831	0.875	0.901	0.937	0.960	0.973
the reproductive rate n(h)	7	3.804	2.685	2.377	2.151	1.847	1.655	1.394	1.268	1.156
the equilibrium point $\times 10^8$	6.992	8.834	9.252	9.167	8.956	8.356	7.657	5.993	4.708	3.149
The mesh width s=6										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.902	0.833	0.747	0.703	0.611	0.746	0.789	0.834	0.853	0.8697
the reproductive rate n(h)	7	4.245	3.248	2.968	2.761	2.480	2.301	2.053	1.932	1.824
the equilibrium point $\times 10^8$	6.992	8.566	9.123	9.219	9.252	9.212	9.109	8.813	8.574	8.287
The mesh width s=7										
Harvesting intensity h :	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue :	0.902	0.855	0.815	0.795	0.773	0.729	0.680	0.606	0.693	0.733
the reproductive rate n(h) :	7	4.650	3.773	3.526	3.3338	3.084	2.919	2.691	2.578	2.476
the equilibrium point $\times 10^8$	6.992	8.313	8.852	8.991	9.083	9.185	9.231	9.251	9.240	9.210

Table 3.2: The net reproductive rate n(0)=7. One can see for $s = 3$ or 4 The net reproductive rate becomes less than one for an increase in h, but for $s \geq 5$ the $n(0)$ will never reach one and the equilibrium will take a positive value.The equilibrium point as a function of h will increase first until $n(0)$ becomes less than e then it will decreasing.

The mesh width s=3									
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.9152	0.8192	0.824	0.894	0.938	0.999	0.952	0.859	0.794	0.717
the reproductive rate n(h)	10	4.117	2.255	1.773	1.434	0.9991	0.741	0.419	0.280	0.169
the equilibrium point $\times 10^8$	8.274	12.35	12.96	11.61	9.033	0	0	0	0	0
The mesh width s=4									
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue:	0.915	0.845	0.708	0.741	0.819	0.888	0.928	0.986	0.981	0.946
the reproductive rate n(h)	10	4.767	3.008	2.534	2.192	1.738	1.456	1.080	0.903	0.748
the equilibrium point $\times 10^8$	8.274	11.77	13.155	13.18	12.87	11.43	9.267	2.564	0	0
The mesh width s=5									
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.915	0.866	0.799	0.758	0.705	0.661	0.762	0.835	0.865	0.889
the reproductive rate n(h)	10	5.434	3.836	3.395	3.073	2.639	2.364	1.991	1.811	1.651
the equilibrium point $\times 10^8$	8.274	11.19	12.59	12.94	13.13	13.21	13.08	12.43	11.78	10.91
The mesh width s=6									
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.915	0.885	0.850	0.832	814	0.777	0.741	0.649	0.523	0.657
the reproductive rate n(h)	10	6.065	4.640	4.240	3.945	3.544	4.049	2.933	2.760	2.605
the equilibrium point $\times 10^8$	8.274	10.68	11.88	12.248	12.50	12.83	13.09	13.18	1.322	13.21
The mesh width s=7									
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.915	0.901	0.883	0.875	0.867	0.852	0.839	0.815	0.799	0.781
the reproductive rate n(h)	10	6.643	5.391	5.034	4.769	4.405	4.171	3.844	3.683	3.537
the equilibrium point $\times 10^8$	8.274	10.24	11.23	11.54	11.77	12.10	12.30	12.57	12.72	12.83

Table 3.3: The same vital parameters are used but the net reproductive rate n(0)=10

	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The mesh width s=3										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.933	0.879	0.758	0.676	0.826	0.924	0.982	0.920	0.850	0.765
the reproductive rate n(h)	15	6.175	3.382	2.660	2.151	1.499	1.111	0.628	0.419	0.253
the equilibrium point $\times 10^8$	9.735	15.89	19.42	19.82	19.19	14.55	5.100	0	0	0
The mesh width s=4										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.933	0.896	0.829	0.785	0.729	0.675	0.798	0.894	0.937	0.977
the reproductive rate n(h)	15	7.150	4.513	3.801	3.288	2.607	2.183	1.620	1.354	1.123
the equilibrium point $\times 10^8$	9.730	14.83	18.00	18.93	19.51	19.81	19.28	16.05	12.07	5.554
The mesh width s=5										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.933	0.912	0.876	0.857	0.838	0.797	0.756	0.639	0.537	0.699
the reproductive rate n(h) :	15	8.152	5.754	5.093	4.609	3.958	3.546	2.987	2.717	2.477
the equilibrium point $\times 10^8$	9.730	13.87	16.39	17.23	17.87	18.73	19.24	19.75	19.83	19.74
The mesh width s=6										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.933	0.926	0.910	0.902	0.924	0.879	0.865	0.838	0.820	0.799
the reproductive rate n(h)	15	9.097	6.960	6.360	5.917	5.315	4.930	4.400	4.141	3.908
the equilibrium point $\times 10^8$	9.730	13.08	15.02	15.68	16.19	16.94	1.744	18.15	18.50	18.80

Table 3.4: The net reproductive rate n(0)=15

2-Herring

In chapter 1 all vital parameters were fixed namely fecundity F_i, weight W_i, mortality μ, maximum age, scaling parameter E_0. c was also computed for various values of $n(0)$. As expected the results are not much different than those for cod and as before the key variable for a fish stock are $n(0)$, the modification of prices, and the choice of the recruitment function. It is quite clear that this model can be adapted easily to other species of fish with similar results. Thus the fluctuations observed with fish landings will most likely be caused by climatic and other external influences.

This model also shows that such harvesting systems considered in isolation are expected to be sustainable and stable. But stability may be endangered through other species (bycatch),environmental deterioration and subsidies of the fishing industry.The data for Beverton-Holt recruitment look quite similar, though the MSY for $n(0) = 10$ is considerably larger, as expected. Multi decade variation of the equilibrium and $n(0)$ will lead to a slow variation of the equilibrium. High harvesting intensities and bad climate conditions might then even lead to a collapse of fisheries, at least in the low $n(0)$ regime.It will be one of the challenges to buffer natural fish populations against climatic variations and excessive harvesting.

In the table below a comparison between gear selective harvesting and knife-edge harvesting is given for various net reproductive rates $n(0)$. Gear selective harvesting is also based on $\gamma_1 = 0.40, \gamma_2 = 0.60$ and $\gamma_3 = 0.80$. The corresponding knife edge harvesting is based on s. All computations are based on herring population

$n(0)$	s_m	$n(h_{max}, s)$ gear sel.	$n(h_{max}, s)$ knife edge	$H(h, s)$ gear sel.	$H(h, s)$ knife edge
3	8	1.68	1.62	46.0172	46.9441
5	7	2.29	2.18	104.0443	108.0685
7	6	2.40	2.21	163.7329	168.1948
10	5	2.33	2.29	245.2344	244.7242
15	5	3.45	3.08	364.4324	382.4892
22	4	2.94	2.47	528.5488	532.8835

Table 3.5: Comparing the optimal harvesting for gear selective harvesting $\gamma_1 = 0.40, \gamma_2 = 0.60$ and $\gamma_3 = 0.80$ for herring population with knife edge harvesting. Maximum effort is 5

Here one has also to put a limit on the maximal harvesting intensity, because the graphs for $H_1(h, s)$ for $s \geq 6$ are also monotonically increasing.
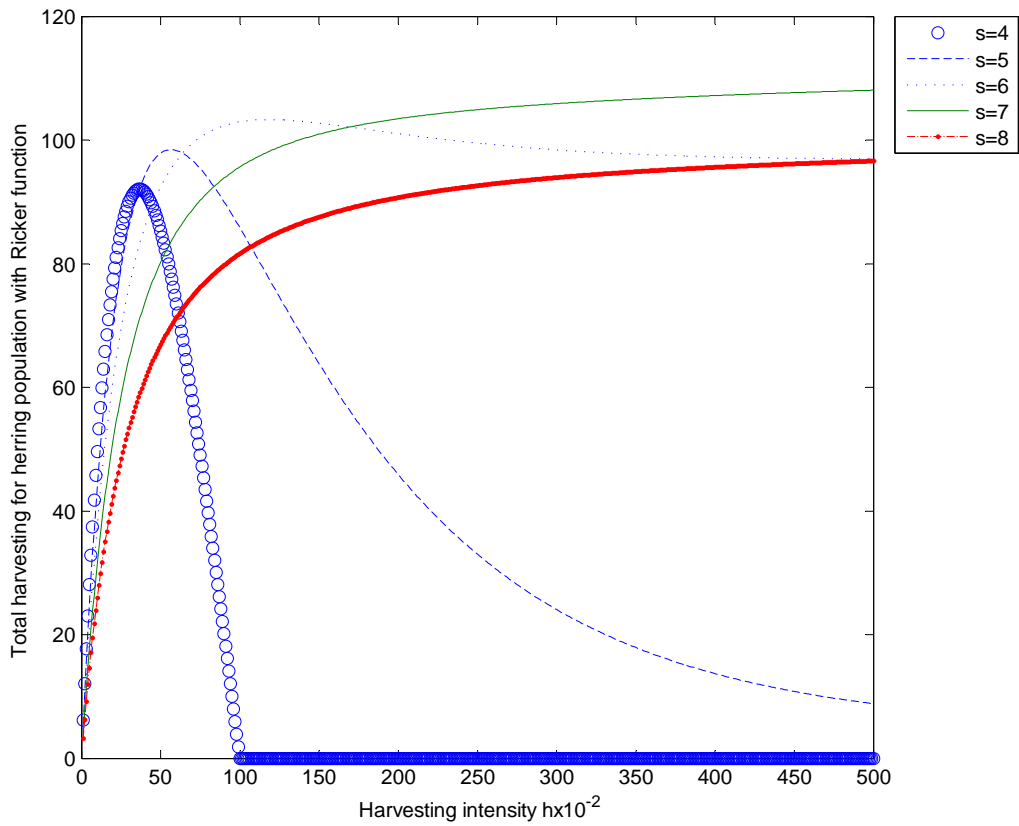
Figure 3.8: Total harvest $H_1(h, s)$ as a function of h, s with $n(0) = 5$ for a herring population with Ricker recruitment function. Initially increased harvesting increases the harvest. But when $n(0)$ declines to much the total harvest decreases. It reaches 0 when $n(0) = 1$.
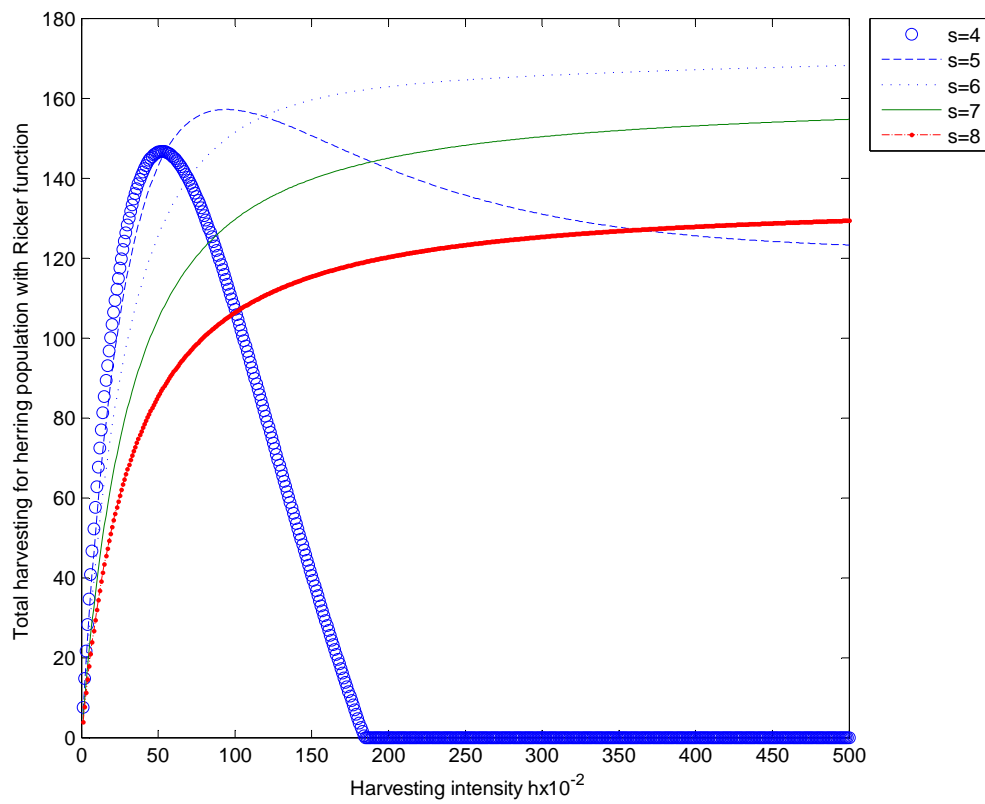
Figure 3.9: The total harvest $H_1(h, s)$ of a herring population with Ricker recruitment as a function of $h \times 10^{-2}$ with $n(0) = 7$

In the following tables we present some numerical values of various quantities, the largest eigenvalue, the net reproductive rate and the the equilibrium value which are depend on the harvesting intensity h and the mesh width s, for different values of net reproductive rate at zero density $n(0)$.

The mesh width s=4										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.863	0.639	0.875	0.909	0.934	0.972	0.999	0.952	0.923	0.893
the reproductive rate n(h)	5	2.735	1.852	1.598	1.411	1.159	0.999	0.789	0.690	0.607
the equilibrium point $\times 10^3$	1.076	1.23	1.112	0.980	0.815	0. 424	0	0	0	0
The mesh width s=5										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.863	0.739	0.795	0.833	0.859	0.893	0.915	0.949	0.967	0.984
the reproductive rate n(h)	5	3.088	2.306	2.074	1.900	1.658	1.500	1.281	1.174	1.079
the equilibrium point $\times 10^3$	1.076	1.220	1.211	1.175	1.129	1.019	0.90	0.645	0.456	0.235
The mesh width s=6										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.863	0.767	0.615	0.713	0.758	0.801	0.826	0.858	0.874	0.889
the reproductive rate n(h) :	5	3.424	2.758	2.557	2.404	2.187	2.044	1.837	1.733	1.637
the equilibrium point $\times 10^3$	1.076	1.201	1.229	1.227	1.219	1.196	1.169	1.106	1.060	1.006

Table 3.6: The $n(h, s)$ and the equilibrium point as a function of h and s have the same behavior as before. All results applied on herring population with Ricker function. The net reproductive rate n(0)=5

The mesh width s=4										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.891	0.834	0.726	0.626	0.693	0.804	0.853	0.916	0.948	0.979
the reproductive rate n(h)	9	4.923	3.332	2.877	2.540	2.085	1.799	1.419	1.243	1.093
the equilibrium point $\times 10^3$	1.468	1.947	2.172	2.209	2.207	2.120	1.963	1.48	1.051	0.488
The mesh width s=5										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.896	0.859	0.805	0.773	0.738	0.648	0.521	0.736	0.777	0.811
the reproductive rate n(h)	9	5.559	4.151	3.734	3.420	2.984	2.700	2.305	2.112	1.942
the equilibrium point $\times 10^3$	1.468	1.856	2.062	2.122	2.163	2.204	2.213	2.18	2.129	2.055
The mesh width s=6										
Harvesting intensity h	0	0.2	0.4	0.5	0.6	0.8	1	1.5	2	3
The largest eigenvalue	0.896	0.880	0.854	0.840	0.827	0.801	0.777	0.725	0.682	0.623
the reproductive rate n(h)	9	6.164	4.965	4.603	4.327	3.937	3.678	3.307	3.119	2.945
the equilibrium point $\times 10^3$:	1.468	1.775	1.941	1.995	2.036	2.094	2.130	2.175	2.194	2.206

Table 3.7: The net reproductive rate n(0)=9

3.4   Effect of cost in semi-continuous harvesting :

Harvesting incurs a considerable amount of cost. These are fixed cost in terms of boats and processing factories. The cost proportional to the harvesting intensity arise from fuel, wages and depreciation of boats and gear. In addition there are costs arising from processing the fish. These are proportional to the harvested amount and can be taken care of through the price. Thus we write

$$C(h) = a + bh \tag{3.14}$$

for the cost. Here a describes the fixed cost and b is the cost parameter of the harvesting effort h. In order to determine the parameters a and b, we assume that the cost amount to a fixed percentage of the maximal yield. In general we would expect the cost to be to about 60% to 80% of the total yield. It will be very difficult to determine the real cost because of subsidies, taxation and other economic factors. This is even more complicated because in reality fishing is not of the sole owner type as the model suggests. It is obvious that for the optimization only the term bh is

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relevant. Since $H_1(h,s)$ increases only little for larger h, the effect of harvesting cost amounts to a decrease of the optimal h. We will always assume that the non fixed cost are proportional to the total yield and one can write $bh_{max} = k_1 H_{1max}$, with $k_1 = 0.3$ or 0.4. For simplicity we set $a = bh_{max}$. This restricts k_1 to less than 0.5. If bh amounts to a fixed percentage of $H_1(h,s)$. b is defined only implicitly and has to be determined by a fixed point algorithm. For the moment this seems the most reasonable way to fix the cost parameters. The main effect of the cost is to reduce the optimal h. The net gain is then the difference between the yield $H_1(h,s)$ and the total cost $C(h)$ i.e.

$$NH(h) = H_1(h,s) - C(h) \tag{3.15}$$

This also shows that subsidies for fuel or wages, which decrease b, will lead to a larger optimal harvesting intensity. This has already been observed with single class models [8]. This effect is stronger if the cost term h is larger. The importance for this is that it keeps the harvested populations more stable. Subsidies for fuel and wages will decrease b and thus increase the optimal fishing mortality. Thus all in all subsidies have a detrimental effect on the stock in particular on a severely depleted stock. In particular subsidies may lead to harvesting intensities beyond the break even intensity.

$n(0)$	s_0	I			II		
		h_{opt}	b1	$NH(h_{opt})$	h_{opt}	b2	$NH(h_{opt})$
3	9	0.323	8.43×10^7	3.63×10^7	0.244	1.35×10^8	1.65×10^7
5	8 or 7	0.391	1.37×10^8	7.13×10^7	0.252	3.56×10^8	3.22×10^7
7	7	0.395	2.48×10^8	9.91×10^7	0.299	3.01×10^8	4.49×10^7
10	6	0.41	2.49×10^8	1.33×10^8	0.307	3.94×10^8	6.05×10^7
15	5	0.421	3.18×10^8	1.78×10^8	0.326	5.00×10^8	8.16×10^7

Table 3.8: The optimal harvesting intensity for I: $h_{opt}b_1 = 0.3H_1(h,s)$ and II: $h_{opt}b_2 = 0.4H_1(h,s)$ with $NH(h,s) = H_1(h,s) - a - bh$. s is likewise optimized. All these results based on data for cod population with Ricker recruitment

$n(0)$	s_0	I			II		
		h_{opt}	b1	$NH(h_{opt})$	h_{opt}	b2	$NH(h_{opt})$
3	6	0.265	45.0558	15.920	0.220	67.7935	7.4574
5	5	0.36	76.4054	36.6749	0.298	115.1861	17.1629
7	5	0.499	85.9247	57.1690	0.389	132.6979	26.4071
10	5 or 4	0.678	94.1195	85.0845	0.424	188.0900	39.8753
13	4	0.63	134.061	112.6115	0.516	203.471	52.4959

Table 3.9: The optimal harvesting intensity for I: $h_{opt}b_1 = 0.3H_1(h, s)$ and II: $h_{opt}b_2 = 0.4H_1(h, s)$ with $NH(h, s) = H_1(h, s) - a - bh$. s is likewise optimized. All these results based on data for herring population with Ricker recruitment
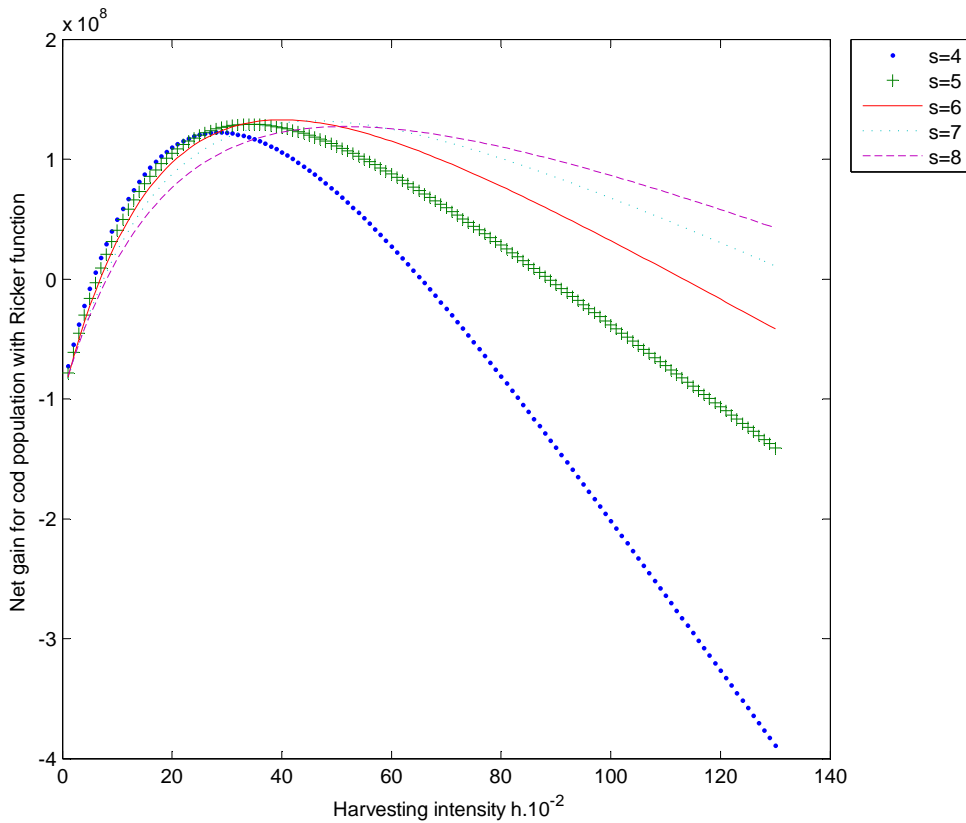
Figure 3.10: The net gain as a function of $h \times 10^2$. The cost parameters are $a = 0.3H_1$ and $bh_{op} = 0.3H_1$ with $n(0) = 10$.

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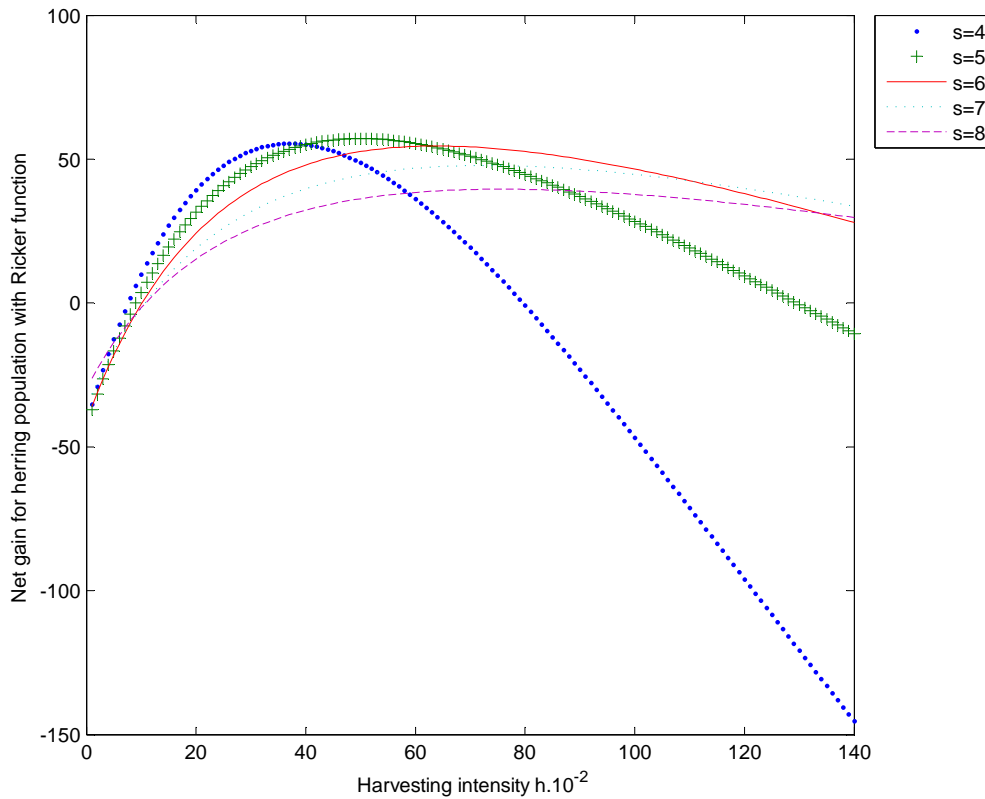
Figure 3.11: The net gain as a function of $h \times 10^2$. The cost parameters are $a = 0.3H_1$ and $bh_{op} = 0.3H_1$ with $n(0) = 7$.

A larger cost parameter b will lead to a lower harvesting intensity. Typical optimal fishing mortality then will be about 0.4 to 0.6. Remember that the effective fishing mortality is only $(1 - t_c)h$, i.e $0.75 \times h$ in most cases.

It is obvious that a higher price for larger fish will shift the net parameter to higher values. A higher price will of course also lead to higher revenues. The above graphs, however, are not altered substantially.

3.5 The influence of the recruitment function

So far the choice of the recruitment functions has played little role in the literatures, even though their impact on the dynamics can be dramatic as we have seen with proposition 2.1.6 and the results of Levin Goodyear (Figure 2.1). Thus it

remains to analyze the influence of the recruitment function on the total harvest. A little thought shows that the Beverton-Holt function should result in a higher yield, because it describes a lower competition at a higher densities. Nevertheless the choice of recruitment function has played no role up to now in the literature even though there is a strong pressure for more precise and accurate models. The problem, however, is even more serious because we do not even know what the correct Ricker counterpart to a given Beverton-Holt function is. One invariant is clearly $n(0)$ i.e the slope of R at zero density. If one chooses the same E_0, the parameter which describes the onset of depensation we have from (2.9) with $B_1 = \sum_{i=2}^{k} cF_iL_i$, $r(\bar{E}) = \frac{(1-\sigma_1)}{B_1}$ and $\bar{y}_1 = \frac{c\bar{E}}{B_1}$. Since σ_1, c, F_i and L_i are constants the equilibrium value is essentially determined by the equation $r(\bar{E}) =$constant. Since r is monotonically decreasing the equilibrium will be smaller for those r , which decrease faster. Thus with the same E_0 the Ricker equilibrium will be smaller than the Beverton-Holt equilibrium, because $e^{-x} < (1+x)^{-1}$ for $x > 0$. Since Beverton-Holt recruitment has a larger impact for large x, the corresponding mesh parameter will be larger. Beyond this it is clear that all statements made for Ricker recruitment function will extend to the Beverton-Holt realm too. Thus the optimal harvesting intensity is reduced when cost are taken into account, or that a higher price for larger fish tends to increase s. Beyond this the recruitment function will have an impact on s and the total yield. Since the Beverton-Holt recruitment has lesser effect for higher densities, we expect the optimal s to increase for Beverton-Holt recruitment (table 3.10). Thus more care should be taken with its choice. The following results for optimal harvesting show the importance of the recruitment function. Thus the choice of it has more influence than gear selectivity, the weight distribution or even the price.

$n(0)$	Ricker function			Beverton-Holt function		
	h_{opt}	s_o	$NH(h_{opt})$	h_{opt}	s_o	$NH(h_{opt})$
5	0.391	8	$7.13{\times}10^7$	0.314	9	$1.21{\times}10^8$
7	0.395	7	9.91×10^7	0.345	9	1.95×10^8
10	0.401	6	1.33×10^8	0.309	8	$3.065{\times}10^8$
15	0.421	5	1.78×10^8	0.326	8	$4.96{\times}10^8$

Table 3.10: The results of optimal harvesting are based on cod population for different recruitment functions with the same other parameters

Another more realistic way of comparing the recruitment is to use $n(0)$, the slope of the recruitment function at zero density, and $\bar{E}$ as invariants. Both constants have a direct biological meaning. From (2.9),(2.10) we obtain $r(E)^{-1} = n(0)$ so that $E_{0R}ln(n(0)) = \bar{E}_R$ and $\bar{E}_{BH} = (n(0) - 1)E_{0BH}$. Here R respectively BH stands for Ricker respectively Beverton-Holt. If $n(0)$ and $\bar{E}$ are used as invariants one can get $(n(0) - 1)E_{0BH} = E_{0R}ln(n(0))$ and $E_{0BH} = \frac{E_{0R}ln(n(0))}{(n(0)-1)}$.

The following table we use the same parameters as in table 3.10. But $E_{0BH} = \frac{E_{0R}ln(n(0))}{(n(0)-1)}$

$n(0)$	Ricker function			Beverton-Holt function		
	h_{opt}	s_o	$NH(h_{opt})$	h_{opt}	s_o	$NH(h_{opt})$
7	0.395	7	9.91×10^7	0.345	9	$6.3194{\times}10^7$
10	0.401	6	1.33×10^8	0.309	8	$7.8413{\times}10^8$

Table 3.11: The results of optimal harvesting are based on cod population for different recruitment functions. All other parameters are the as before but $E_{0BH} = \frac{E_{0R}ln(n(0))}{(n(0)-1)}$

The discrepancy of these results shows that also this method does not yield acceptable results. So the question of a proper choice of recruitment function remains open. For this either a model based derivation or precise measurements at high densities are needed. Because of a lack of high density data this question will probably be open for long time.

Chapter 4

Optimal harvesting of fish populations with other strategies

4.1 The optimal control harvesting model

In the previous chapters of this dissertation the essential vital parameters of the model have been determined and the harvesting parameters s and h, where determined for the optimized equilibrium model. The aim in the following chapter is to treat s and h as dynamic variables and to compute an optimal control solution. The key parameters are $n(0), E_0, \mu$, and W_i. E_0 as a scaling parameter played no dominant role. From these the optimal harvesting role h_{opt} and the optimal net parameters s_o as well as the cost parameters a and b will be derived. Another important factor in optimal control is the discount factor δ. From actual data $\delta = (1 + \frac{p_o}{100})^{-1}$, $p_o = 4$ or 5 seems to be reasonable. They will be used as inputs for the optimal control model. The larger δ, the larger the impact of the future. The extension to such an optimal control model is necessary, because by now many fish species have reached a catastrophically low level so that a good policy is needed to reach an optimal sustainable solution.In addition $\delta > 0$ may be needed for convergence.

Mathematically this is a well defined problem of optimal control, which only depends on the time horizon $\mathcal{T}$ considered. Beyond that, however, other solutions will be considered, which are close to optimal, but politically and economically more acceptable. To put it bluntly : A bang-bang solution with $h = 0$ for a time interval

of about 10 years or so, is not acceptable politically or economically. Finally the results will have to be interpreted with respect to feasibility. An important aspect in optimal control is the shadow price interpretation of the covariables, which has been used successfully in understanding economic or epidemic problems.

In this section, we will study the discrete time optimal control problem in order to maximize the total yield (profit) of fishing in the time interval $[0, \mathcal{T}]$. In principle we have two parameters to control (s, h). The class parameter s describes the width of mesh of fishing nets in as much as all fish of age more or equal to s, are caught. Note that selective gear fishing does not yield substantially different results than knife-edge fishing. $h(t)$ denotes the harvesting intensity. In practical situations s would arise in the form of restrictions of fishing gear. While the bounds for $h(t)$ would amount to a fishing quota. As noted above $h(t)$ can be understood as limited by the fishing industry, i.e the number of boats, their size, people employed,..etc. The static equilibrium model was mainly analyzed to fix the range of parameters and to study their sensitivity. In addition we expect the static equilibrium model to appear as the limit or singular solution in the general optimal control model.

Our strategy of optimal control is as therefore as follows: Fix the key vital parameters $n(0), \mu, F_i, W_i$, the mesh parameter s and the recruitment function. Determine the optimal equilibrium solution for these parameters and from these derive the cost parameters a and b. It is the aim of the optimal control theory to determine a harvesting strategy $h = (h_1, h_2, \ldots, h_{\mathcal{T}})$ on $[0, \mathcal{T}]$ which gives rise to a maximal revenue $J(h)$. It is known that for optimal control problems which are linear in the control the solutions are quite often of bang-bang type. For more details in this particular field see [8],[14],[37]. In our case bang-bang solutions are not expected, because the equations are not linear in $h(t)$.

Let $h = (h_1, h_2, \ldots, h_{\mathcal{T}})$ be a given harvesting strategy, where $\mathcal{T}$ is the total number of years of fishing considered. Then the evolution of the population is uniquely determined. Compute $x_i(t)$ for $i = 1, 2, \ldots, k$ and t=1,....,$\mathcal{T}$

$$x_1(t+1) = \sigma_1 x_1(t) + R(E) \tag{4.1}$$

$$x_i(t+1) = \tau_{i-1} x_{i-1}(t) \qquad\qquad 2 \le i \le s$$

$$x_i(t+1) = \tilde{\tau}_{i-1}x_{i-1}(t) \qquad\qquad s < i \leq k$$

where $\tau_1 = \frac{e^{-\mu_1}}{3}, \tau_i = e^{-\mu_i}$, for $i = 2, 3, .., s, \tilde{\tau}_i = e^{-\mu_i-(1-t_c)h_i(t)}$ for $i = s+1, s+2, .., k$, $R(E) = cEr(E) = \sum_{i=2}^{k} cF_i x_i(t)r(E)$. The Ricker recruitment function will mostly be used ,but Beverton-Holt recruitment functions will be employed too in order to determine the influence of the recruitment function. The Beverton-Holt recruitment function was used by Tahvonen [67] and Diekert et al [15] solely without qualms.

In optimal control theory the vector $\overrightarrow{x}(t) = (x_1(t), x_2(t), ..., x_k(t))$ is called the state vector and the individual components x_i are called the state variables. The vector $\overrightarrow{h} = (h_1, h_2, ..., h_{\mathcal{T}-1})$ is called the control vector and the $(h_j)_{j=1}^{\mathcal{T}-1}$ are called the control variables.

Throughout we assume that $\mu_1 = \mu_2 = ... = \mu_{k-1} = \mu$, and $h_s(t) = h_{s+1}(t) = ... = h_k(t) = h(t)$, i.e the mortality and the harvesting amount are fixed for all classes. The reason for this is that it makes no sense to change the nets or gear from one year to the next. Then it follows that $\tau_1 = \frac{e^{-\mu}}{3}, \tau_i = e^{-\mu}$, and $\tilde{\tau}_i = e^{-\mu-(1-t_c)h(t)}$ for $i = 2, 3, .., s$ and $i = s+1, s+2, .., k$ respectively. The net return in the year t is given by

$$NH(t) = \sum_{i=s}^{k}(\mu + h(t))^{-1}(q_i(t)h(t)W_i(t)x_i(t)e^{-\mu t_c}(1 - e^{-(\mu+h(t))(1-t_c)})) - a - bh(t).$$

The total yield from fishing with discounting factor δ in the time interval $[0, \mathcal{T}]$, is

$$J(h) = \sum_{t=1}^{\mathcal{T}} NH(t)\delta^t, \quad \delta = (1 + \frac{p_o}{100})^{-1} \tag{4.2}$$

Treating fishery as a sole owner problem, the optimal control problem is to choose a harvesting strategy h which maximizes J. For general differential equations optimal control problem the existence proof is rather more difficult which involves a measurable choice of the control variable see [22]. Here the problem is rather simple. Assume the harvesting intensity is restricted to some interval $[0, k_3]$. Then the function

$$[0, k_3]^{\mathcal{T}} \ni (h(t)) \Longrightarrow J(h) \tag{4.3}$$

depends continuously and differentially on h. Since $[0, k_3]^{\mathcal{T}}$ is compact, this function has at least one maximum h^*. There are good reasons that in this case h^* is unique. In that case the Lagrange method of a constrained extremum comes into play. Thus there exist covariables $(\lambda_i(t))_{i=1}^k$, the Lagrange multipliers , so that the corresponding Lagrange function can be used to determine h^* and the corresponding state $x_i^*(t))_{i=1}^k$. Since all this describes a discrete dynamical system a more elegant formulation is given through the Pontryagin Maximum Principle (PMP). For details and derivation of the result, the reader is referred to the excellent book of S.Lenhart and J.T. Workman[37]. For more details refer to ([18],[30],[32],[37],[43],[53],[63],).

In this theory one defines the Hamiltonian as follows

$$\mathcal{H}(t) = \sum_{i=s}^k [q_i h(t) W_i x_i(t) e^{-\mu t_c} (\mu + h(t))^{-1} (1 - e^{-(\mu+h(t))(1-t_c)}) - a - bh(t)] \delta^t$$

$$+ \lambda_1(t+1)(\sigma_1 x_1(t) + R(E)) + \sum_{i=2}^s \lambda_i(t+1) e^{-\mu} x_{i-1}(t)$$

$$+ \sum_{i=s+1}^k \lambda_i(t+1) e^{-\mu-(1-t_c)h(t)} x_{i-1}(t), \qquad for \qquad t \leq \mathcal{T} - 1$$

The Hamiltonian is thus made up of the yield term $NH(t)$ and the dynamic equations with the corresponding covariables. This can be rewritten as follows

$$\mathcal{H}(t) = \sum_{i=3}^s [\lambda_i(t+1) e^{-\mu} x_{i-1}(t) + \lambda_1(t+1) c F_i x_i(t) r(E)] + \lambda_1(t+1) c F_2 x_2(t) r(E)$$

$$+ \lambda_1(t+1)\sigma_1 x_1(t) + \lambda_2(t+1)\frac{e^{-\mu}}{3} x_1(t) + \sum_{i=s+1}^k [\lambda_i(t+1) e^{-\mu-(1-t_c)h(t)} x_{i-1}(t) +$$

$$\delta^t x_i(t) q_i W_i h(t) e^{-\mu tc} (1 - e^{-(\mu+h(t))(1-tc)}) (\mu + h(t))^{-1} + \lambda_1(t+1) c F_i x_i(t) r(E)]$$

$$+ \delta^t x_s(t) q_s W_s h(t) e^{-\mu tc} (1 - e^{-(\mu+h(t))(1-tc)}) (\mu + h(t))^{-1} - a\delta^t - b\delta^t h(t).$$

Here the Ricker function is used. The covariables λ_i, satisfy

$$\lambda_i(t) = \frac{\partial \mathcal{H}(t)}{\partial x_i(t)} \tag{4.4}$$

The equations of the covariables λ_i, are thus given by

$$\lambda_1(t) = \frac{\partial \mathcal{H}(t)}{\partial x_1(t)} = \lambda_1(t+1)\sigma_1 + \lambda_2(t+1)\tau_1$$

$$\lambda_2(t) \quad = \frac{\partial \mathcal{H}(t)}{\partial x_2(t)} \quad = \lambda_1(t+1)cF_2 e^{-\frac{E(t)}{E_0}}(1-\frac{E(t)}{E_0}) + \lambda_3(t+1)\tau_2.$$

$$\lambda_{s-1}(t) = \frac{\partial \mathcal{H}(t)}{\partial x_{s-1}(t)} = \lambda_1(t+1)cF_{s-1} e^{-\frac{E(t)}{E_0}}(1-\frac{E(t)}{E_0}) + \lambda_s(t+1)\tau_s.$$

$$\lambda_s(t) = \frac{\partial \mathcal{H}(t)}{\partial x_s(t)} \quad = (\mu + h_s(t))^{-1}(\delta^t h_s(t) q_s W_s e^{-\mu t_c})(1 - e^{-(\mu+h_s(t))(1-t_c)})$$

$$+ \lambda_1(t+1)cF_s e^{-\frac{E(t)}{E_0}}(1-\frac{E(t)}{E_0}) + \lambda_{s+1}(t+1)\tilde{\tau}_{s+1}.$$

$$\lambda_{k-1}(t) = \frac{\partial \mathcal{H}(t)}{\partial x_{k-1}(t)} = \frac{\delta^t h_s(t) q_{k-1} W_{k-1} e^{-\mu t_c}}{\mu + h_s(t)}(1 - e^{-(\mu+h_s(t))(1-t_c)})$$

$$+ \lambda_1(t+1)cF_{k-1} e^{-\frac{E(t)}{E_0}}(1-\frac{E(t)}{E_0}) + \lambda_k(t+1)\tilde{\tau}_k.$$

$$\lambda_k(t) = \frac{\partial \mathcal{H}(t)}{\partial x_k(t)} = (\delta^t h_s(t) q_k W_k e^{-\mu t_c})(\mu_k + h_k(t))^{-1}(1 - e^{-(\mu+h_k(t))(1-t_c)})$$

$$+ \lambda_1(t+1)cF_k e^{-\frac{E(t)}{E_0}}(1-\frac{E(t)}{E_0}).$$

These covariables measure the sensitivity of the value functional J with respect to the state variables x_i. In the literature they are commonly called the shadow prices[8, 37]. The PMP now states

$$\mathcal{H}(t, x^*, \lambda, h^*) \geq \mathcal{H}(t, x, \lambda, h)$$

where (x, h) is any other solution system for the harvesting strategy h. The application of the PMP is complicated by the fact that the state variables are uniquely determined by the initial conditions at 1 and the dynamics (4.1), while the covariables have no initial conditions and have fixed final time conditions $\lambda_i(\mathcal{T}) = 0$. In addition they depend on the $(x_i(t))_{i=1}^k$ and the dynamics (4.4). So the solution of both systems will have to be determined by an iterative gradient type algorithm. This is described in [37] and will be outlined below through the following steps :

1. Define all parameters namely $\mathcal{T}, k, s, W, \mu, E_0$, and the initial conditions for the states $(x_i(1))_i^k$. Compute the equilibrium solution as well as a and b.

2. Guess an initial harvesting strategy $h_{in} = (h_{in}(t))$. Mostly we take $h_{in}(t) = 0.02$ for $t < 11$ and $h_{in}(t) = h_{opt}$ for $11 \leq t \leq \mathcal{T} - 1$.

3. Solve the state equations (4.1) forward with h_{in}, which gives $\vec{x}_{in}(t)$ for all $t = 1, 2, ..., \mathcal{T}$.

4. Solve the adjoint equations (4.4) backward with h_{in}, and $\vec{x}_{in}(t)$. This gives $\vec{\lambda}_{in}(t)$ for all $t = \mathcal{T} - 1, ..., 1$.

5. Form the Hamiltonian function $\mathcal{H}(t, \vec{x}_{in}, \vec{\lambda}_{in}, u)$ and maximize it with respect to $u(t)$ subject to the constraints for the harvesting strategy u. This gives $u(t)$ for all $t = 1, 2, ..., \mathcal{T} - 1$.

6. If $h_{in}(t) - u(t)$ is sufficiently small for all $t = 1, 2, ..., \mathcal{T} - 1$ stop. Then $h_{in}(t)$ is the desired solution and go to Step 7. Otherwise use a convex combination between $h_{in}(t)$ and $u(t)$ as the new input in step 3 and repeat this process. This averaging is introduced to stabilize this recursion procedure. In many cases this recursive determination of $h(t)$ will converge to a desired solution.

7. Use the optimal harvesting with corresponding solution to compute the optimal total yield and other results.

4.2 Optimal harvesting strategies

Many one class models of optimal harvesting, in epidemiology or even in marketing lead to solution of the fastest approach type [8] . In this case the optimal control problem consists of three phases. The first phase leads to the optimal equilibrium solution, which for differential equation models agrees with the singular solution. In the middle phase the solution is constant, while in the last phase the control is extremal.

In epidemiology the last phase is usually the no care no vaccination phase, while in the Nerlove- Arrow advertising model the last phase means no advertisement. For multi-classes models with a nonlinear control a fastest approach has no direct meaning. Nevertheless all solutions encountered with this model are of this three phase type, see Fig 4.1,4.3. Phase 1, the recovery phase is a time interval with no harvesting in which the population recovers from a precariously low level. This is followed by a constant harvesting rate $h = h_1$ until in the last phase, unrestricted harvesting sets in.

The first and the last phase are typically of the order $0.2 \times k_2$. The duration and form of the first phase depends of course on the initial value. That $h_1 \approx h_{opt}$ is plausible, but in all cases we found $h_1 > h_{opt}$. However, in the middle phase they are the same if $\delta = 1$. From the results it is obvious that $h_1 - h_{opt}$ is a decreasing function of δ. It is clear that the optimal control solution leads to a lower net parameter s_o, because the constants a and b are based on the optimized equilibrium model. If one makes s_o lower that means one permits to catch more fish at lower cost. But for $n(0) \geq 10$ or greater one can not reduce s_o more than 1 since the fertile classes will not be large enough to produce more eggs in order to get a better results. The same argument holds for making s_o higher. This means there is not enough fish to balance the higher cost, particularly for $n(0) = 3$ or less. That h_1 is larger than h_{opt} follows from the fact that with a smaller δ the future of the ecosystem is of lesser importance. The extreme case $\delta = 0$ for example results in h, to be maximal.

Unexpected, however, is a dip in the harvesting intensity to low levels before maximal harvesting sets in. It could possibly be explained by an additional population growth before all out harvesting sets in. As expected, the last phase harvesting profiles are mostly independent of $\mathcal{T}$, if $\mathcal{T}$ is large enough. The first phase, however, depends on the initial conditions and will be longer if the initial population is smaller. Thus the solutions are almost of the fastest approach type. The difference to differential equation models is of course also a consequence of the built in time delay. It is important to note that there is no indication for periodic or pulse fishing solutions. Pulse fishing solutions are excluded, because a reduction of a stock to almost zero require h to be extremely large and corresponding cost to be very large. For periodic solution the argument is similar, because a reduction of the population to low levels requires a large h. This is not easy compensated by the low h phases, because the marginal effect of h decreases with in increasing h. This can easily be seen from the $H_1(h, s)$ curves.

The following tables give the main results for J as a function of $n(0) = 5, 7, 10, 15$. and s_o the optimal mesh parameter with $k_1 = 0.3$. This means a profit of 40%. Where h_{optc} is the solution from optimal control with $h_{in} = 0.02$ for $t \leq 10$ $h_{in} = h_{opt}$ for $t \geq 11$, $\delta = 0.96$ and initial condition of the x_i equal to 25 percent of the equilibrium solution. We denote this by $x_i(1) = \frac{eq}{4}$. For illustration purpose one

can also use other initial condition namely x_i equal to 10 percent of the equilibrium or random but small initial conditions. The idea here is, that even with excessive harvesting the population has attained a form of equilibrium.

$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
5	8	0.391	7.4487×10^8	6.7798×10^8	5.1866×10^8	2.8179×10^8
7	7	0.395	1.1768×10^9	1.1370×10^9	9.6051×10^8	6.6894×10^8
10	6	0.401	1.7136×10^9	1.7407×10^9	1.5712×10^9	1.2342×10^9
15	5	0.421	2.4105×10^8	2.5986×10^9	2.4615×10^9	2.0778×10^9

Table 4.1: All vital parameters are applied for cod population with Ricker function which are fixed in the previous chapter. As one expects that J is increasing function of n but s_0 is decreasing function of n. $\mathcal{T} = 3k_2$ and $\delta = 0.96$.

The following table gives the results for $n(0) = 5, 7, 10, 15$ by using all parameters as above but $\mathcal{T} = 4k_2$.

$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
5	8	0.391	7.7578×10^8	7.0964×10^8	5.4799×10^8	3.0597×10^8
7	7	0.395	1.2174×10^9	1.1796×10^9	1.0010×10^9	7.0370×10^8
10	6	0.401	1.7631×10^9	1.7955×10^9	1.6250×10^9	1.2818×10^9
15	5	0.421	2.4690×10^9	2.6655×10^9	2.5318×10^9	2.1418×10^9

Table 4.2: The same parameters as before but $\mathcal{T} = 4k_2$. J has the same behavior as before and it is independent of $\mathcal{T}$.

The following table gives the results for $n(0) = 5, 7, 10, 15$ by using all parameters as above but $x_i(1) = \frac{eq}{10}$. It is obvious that in this case the total yield is smaller.

$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
5	8	0.391	2.9795×10^8	2.4375×10^8	1.1780×10^8	-6.8056×10^7
7	7	0.395	6.6349×10^8	6.2904×10^8	4.8129×10^8	2.3890×10^8
10	6	0.401	1.1483×10^9	1.1703×10^9	1.0227×10^9	7.3043×10^8
15	5	0.421	1.7976×10^9	1.9626×10^9	1.8395×10^9	1.4958×10^9

Table 4.3: Results of total yield for cod population with $x_i(1) = \frac{eq}{10}$, $k_1 = 0.3$ and $\mathcal{T} = 3k_2$

The following table gives the results for $n(0) = 5, 7, 10, 15$ by using all parameters as above but the initial state conditions are taken as $x_i(1) = \frac{eq}{10} + k_0\frac{eq}{10}$ where k_0 is a random number between 0 and 1.

$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
5	8	0.391	4.8351×10^8	4.0329×10^8	2.5057×10^8	6.5176×10^7
7	7	0.395	8.6586×10^8	8.6649×10^8	6.8770×10^8	4.2020×10^8
10	6	0.401	1.3749×10^9	1.4236×10^9	1.2711×10^9	9.6465×10^8
15	5	0.421	2.0303×10^9	2.2918×10^9	2.0168×10^9	1.7455×10^9

Table 4.4: Results of total yield for cod population with $x_i(1) = \frac{eq}{10} + k_0\frac{eq}{10}$, $k_1 = 0.3$ and, $\mathcal{T} = 3k_2$

The following table gives the results for $n(0) = 7, 10$ by using all parameters as above but $\delta = 0.98$ and $x_i(t) = \frac{eq}{4}$. Obviously the total harvest depends monotonically on the discount factor.

$\mathcal{T}$	$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
$3k_2$	7	7	0.395	2.9279×10^9	2.8774×10^9	2.5593×10^9	2.0037×10^9
$4k_2$	7	7	0.395	3.2834×10^9	3.2477×10^9	2.9077×10^9	2.3016×10^9
$3k_2$	10	6	0.401	4.0242×10^9	4.0997×10^9	3.7968×10^9	3.1646×10^9
$4k_2$	10	6	0.401	4.4715×10^9	4.5772×10^9	4.2606×10^9	3.5743×10^9

Table 4.5: Results of total yield for cod population with $x_i(1) = \frac{eq}{4}$, $k_1 = 0.3$ and $\delta = 0.98$

The following table gives the results for $n(0) = 7, 10$ by using all parameters as above but $\delta = 0.98$ and $x_i(1) = \frac{eq}{10}$.

$\mathcal{T}$	$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
$3k_2$	7	7	0.395	2.2823×10^9	2.2334×10^9	1.9461×10^9	1.4475×10^9
$4k_2$	7	7	0.395	2.6402×10^9	2.6035×10^9	2.2941×10^9	1.7446×10^9
$3k_2$	10	6	0.401	3.3399×10^9	3.4026×10^9	3.1242×10^9	2.5424×10^9
$4k_2$	10	6	0.401	3.7872×10^9	3.8819×10^9	3.5878×10^9	2.9515×10^9

Table 4.6: Results of total yield for cod population with $x_i(1) = \frac{eq}{10}$, $k_1 = 0.3$ and $\delta = 0.98$

These results indicate that the optimal control solution leads to a lower net parameter as was noted before. In addition if one reduces s_o by more than 2 units, the total harvest decreases. The results will be presented as the next table for $n(0) = 7$, but the same situation prevails for other values of $n(0)$ and δ.

$\mathcal{T}$	s_o	h_{opt}	$J(s_o - 3, h_{optc})$	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
$3k_2$	7	0.395	2.7143×10^9	2.9279×10^9	2.8774×10^9	2.5593×10^9	2.0037×10^9
$4k_2$	7	0.395	3.0306×10^9	3.2834×10^9	3.2477×10^9	2.9077×10^9	2.3016×10^9

Table 4.7: Results of total yield for cod population with $x_i(1) = \frac{eq}{4}$, $k_1 = 0.3$ and $\delta = 0.98$.

The following plots give typical harvesting profiles. As we noted before the first phase is a time for recovering to population from low level,then harvesting at rate $\approx h_{opt}$ and the final phase,the unrestricted harvesting sets in.

Figure 4.1: Optimal harvesting for cod populations and Ricker function with $x_i(1) = \frac{eq}{4}$, $bh_{opt} = 0.3H, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 10$.
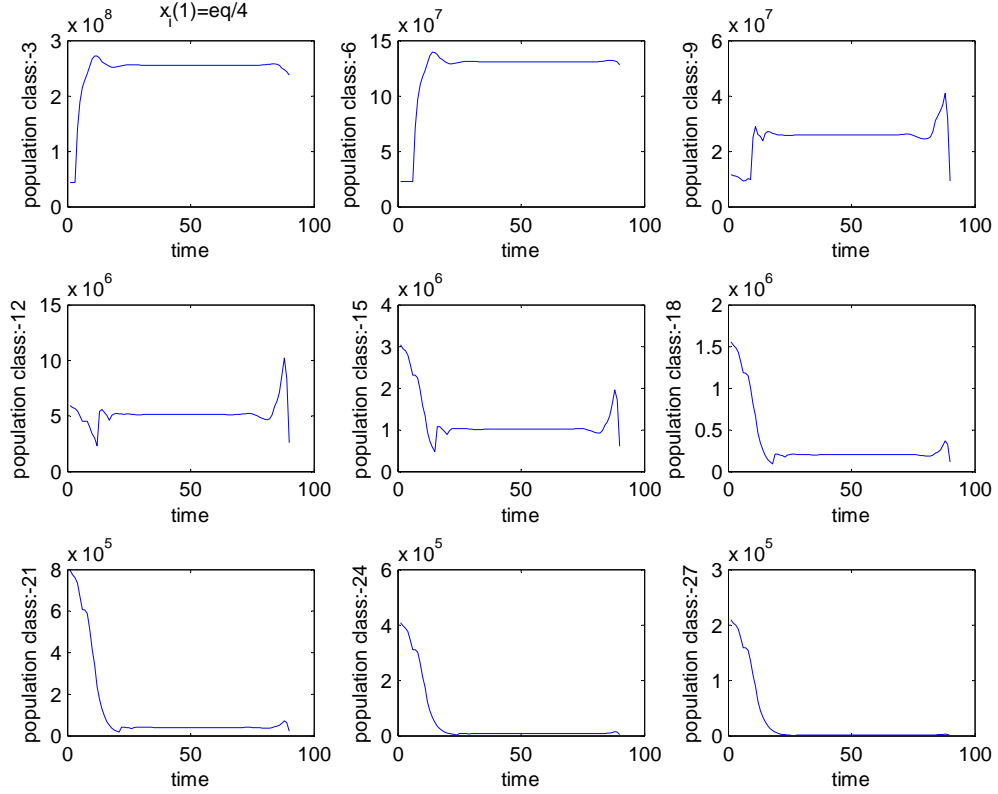
Figure 4.2: Cod population for several age classes after harvesting, Ricker function with $x_i(1) = \frac{eq}{4}$, $bh_{opt} = 0.3H_1, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 10$.

Note that even for moderate fishing intensities the higher classes $i > 18$ are very thinly occupied. Remember that harvesting essentially set in for $i > \frac{k_2}{3}$. Thus with optimal harvesting there will be much fewer large fish. The results for the various classes indicate that it seems to be advantageous to build up the middle classes $i = 9$ to 16 for higher egg production first in all out harvesting. It would be interesting if this phenomenon also arises in the recovery phase.
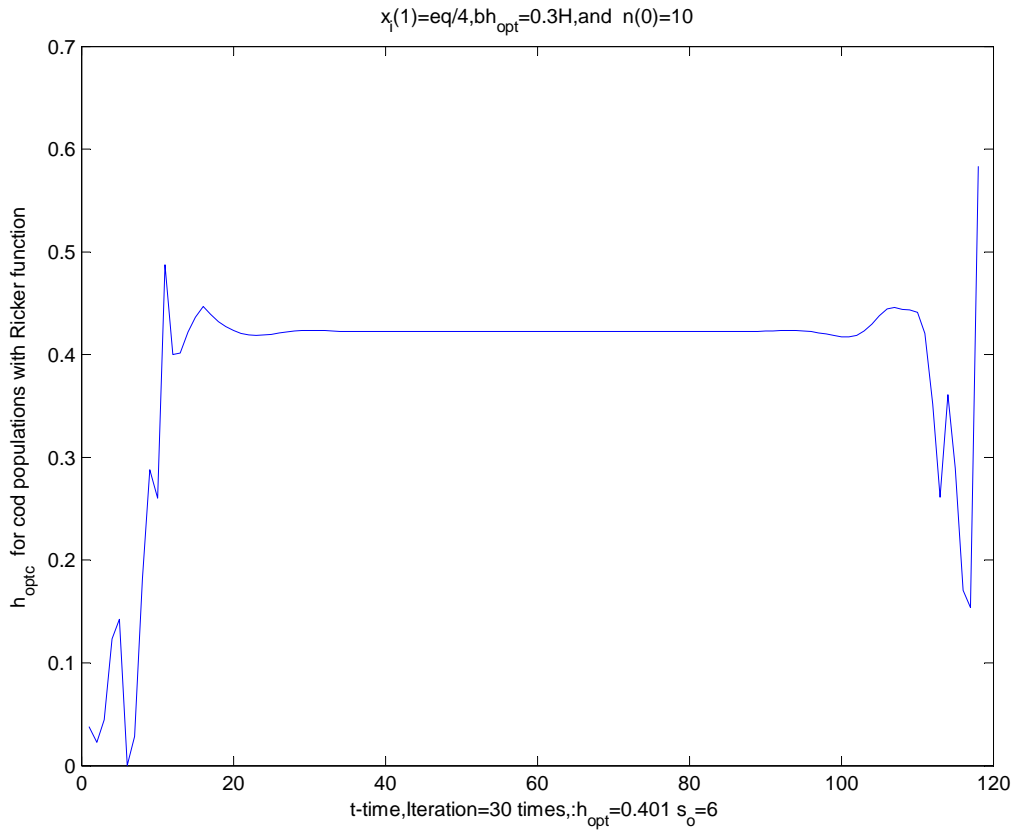
Figure 4.3: The optimal harvesting is independent of $\mathcal{T}$. Here $\mathcal{T} = 4k_2$. The last 11 years give the same profile as $\mathcal{T} = 3k_2$.

We have noted above that the optimal model with discount factor $0 < \delta < 1$ will lead to a higher h_1, the equilibrium of the optimal control model. Numerically $h_1(\delta)$ is monotone decreasing. Likewise one expects this effect to be more dominant for larger $n(0)$.
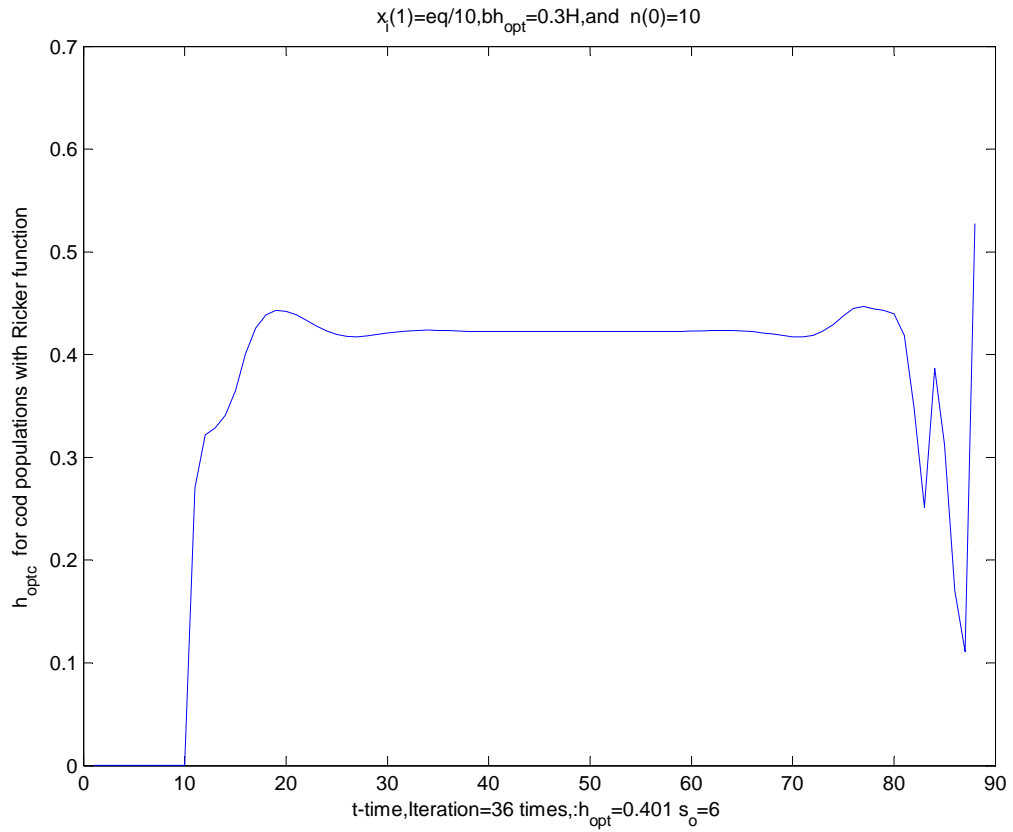
Figure 4.4: The first phase depends on the initial value of population. Optimal harvesting for cod populations and Ricker function with the same parameters but $x_i(1) = \frac{eq}{10}$
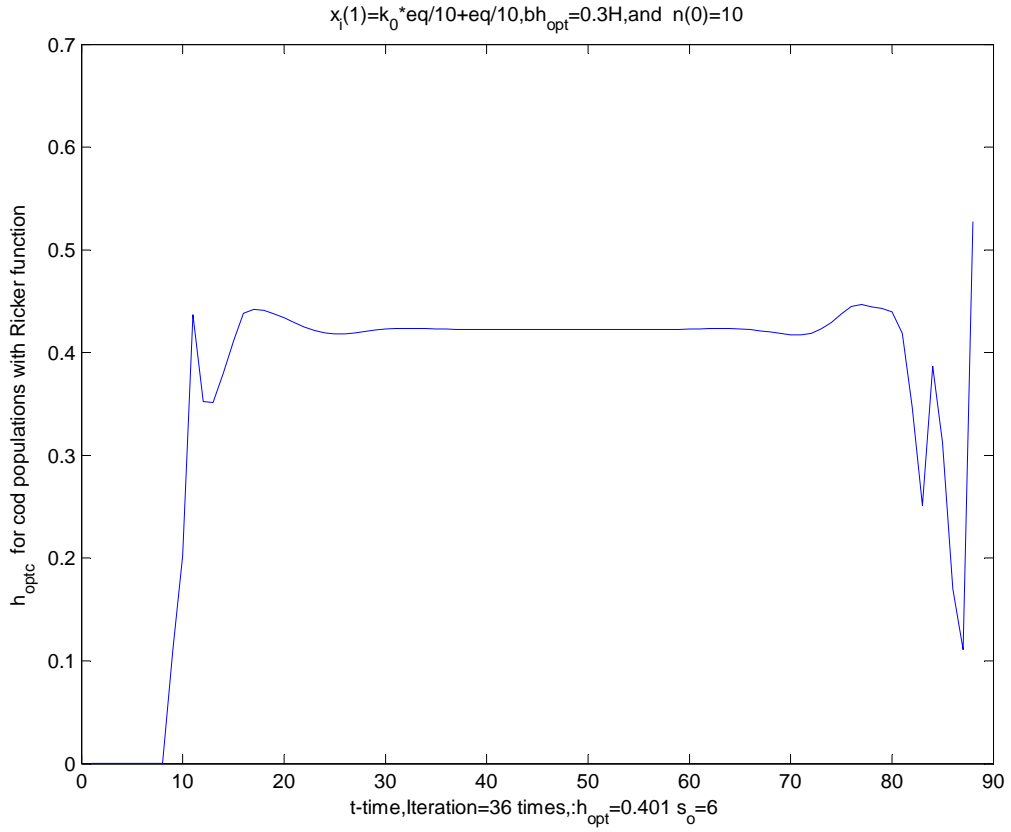
Figure 4.5: Optimal harvesting for cod populations and Ricker function with $x_i(1) = \frac{eq}{10} + k_0 \frac{eq}{10}$, $bh_{opt} = 0.3H_1, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 10$.
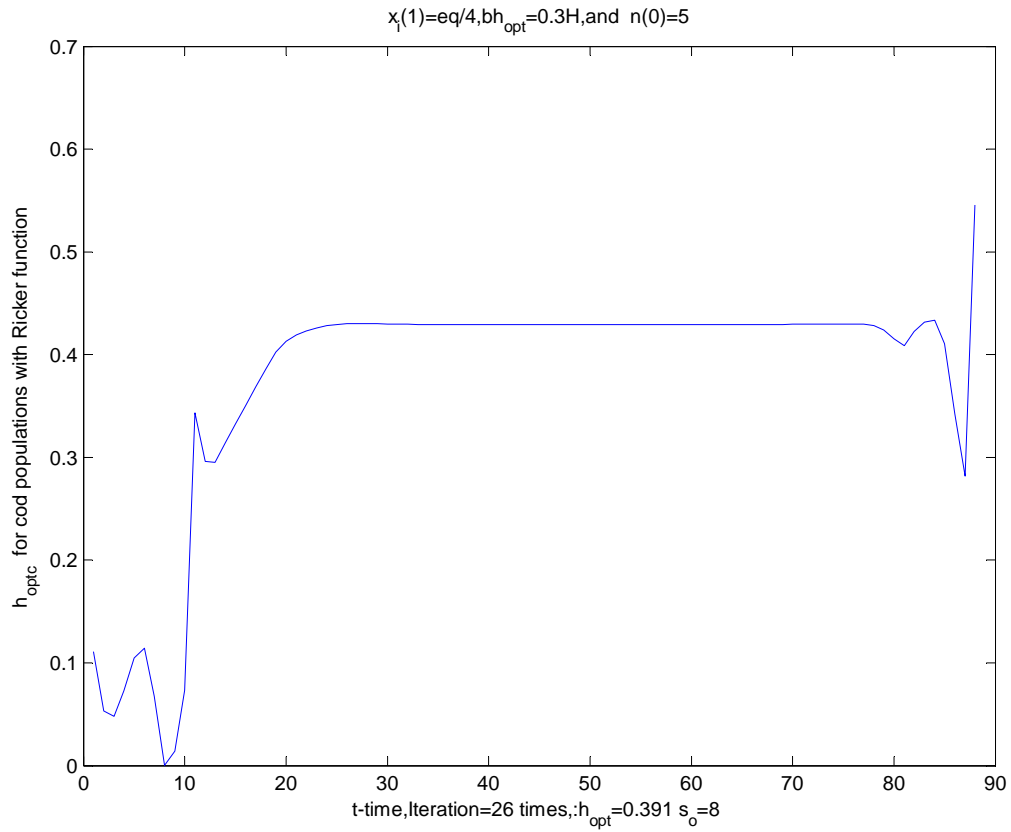
Figure 4.6: Optimal harvesting for cod populations and Ricker function with $x_i(1) = \frac{eq}{4}$, $bh_{opt} = 0.3H_1, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 5$. Note that the dip in the last phase is smaller if $n(0)$ is smaller.
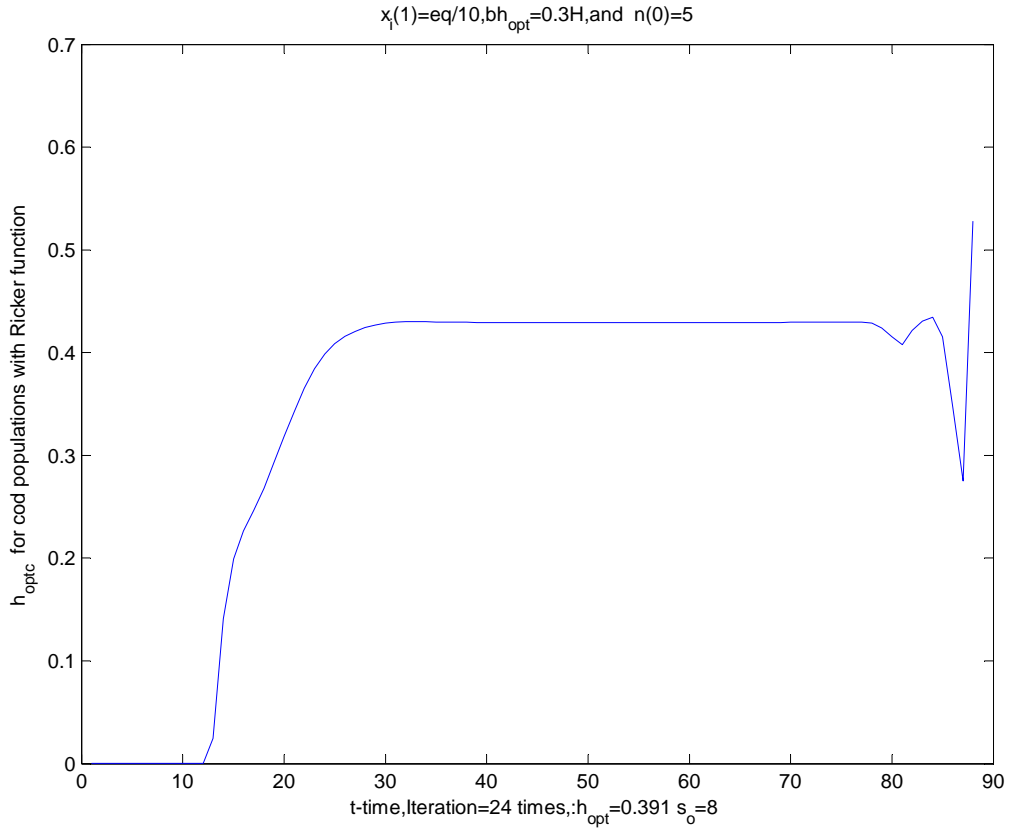
Figure 4.7: Optimal harvesting for cod populations and Ricker function with the same parameters but $x_i(1) = \frac{eq}{10}$ and $\mathcal{T} = 3k_2$.

For completeness sake we also compare Ricker and Beverton-Holt optimal control harvesting for $n(0) = 5, 7, 10, 15, \mathcal{T} = 3k_2, \; \delta = 0.96, s_0 - 1$ and s_0. All computation based on cod population.

$n(0)$	Ricker function				Beverton-Holt function			
	h_{opt}	s_o	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	h_{opt}	s_o	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$
5	0.391	8	6.7798×10^8	5.1866×10^8	0.314	9	9.5334×10^8	6.2094×10^8
7	0.395	7	1.1370×10^9	9.6051×10^8	0.345	9	1.8269×10^9	1.2420×10^9
10	0.401	6	1.7407×10^9	1.5712×10^9	0.309	8	3.1095×10^9	2.4506×10^9
15	0.421	5	2.5986×10^9	2.4615×10^9	0.326	8	5.5114×10^9	4.3647×10^9

Table 4.8: The results of optimal harvesting for different recruitment functions with the same other parameters .As we noted before the Beverton-Holt recruitment gives a higher results than Ricker function.

For using $E_{0BH} = \frac{E_{0R} ln(n(0))}{(n(0)-1)}$ the results for Ricker and Beverton-Holt optimal control harvesting are also given for $n(0) = 7, 10, \mathcal{T} = 3k_2, \delta = 0.96, s_0 - 1$ and s_0.

$n(0)$	Ricker function				Beverton-Holt function			
	h_{opt}	s_o	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	h_{opt}	s_o	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$
7	0.395	7	1.137×10^9	9.6051×10^8	0.345	9	5.9249×10^8	4.0278×10^8
10	0.401	6	1.7407×10^9	1.5712×10^9	0.309	8	7.9550×10^8	6.2699×10^8

Table 4.9: The results of optimal harvesting for different recruitment functions with the same other parameters,but $E_{0BH} = \frac{E_{0R} ln(n(0))}{(n(0)-1)}$

This result also shows that the discrepancy for adjusted recruitment functions prevails.
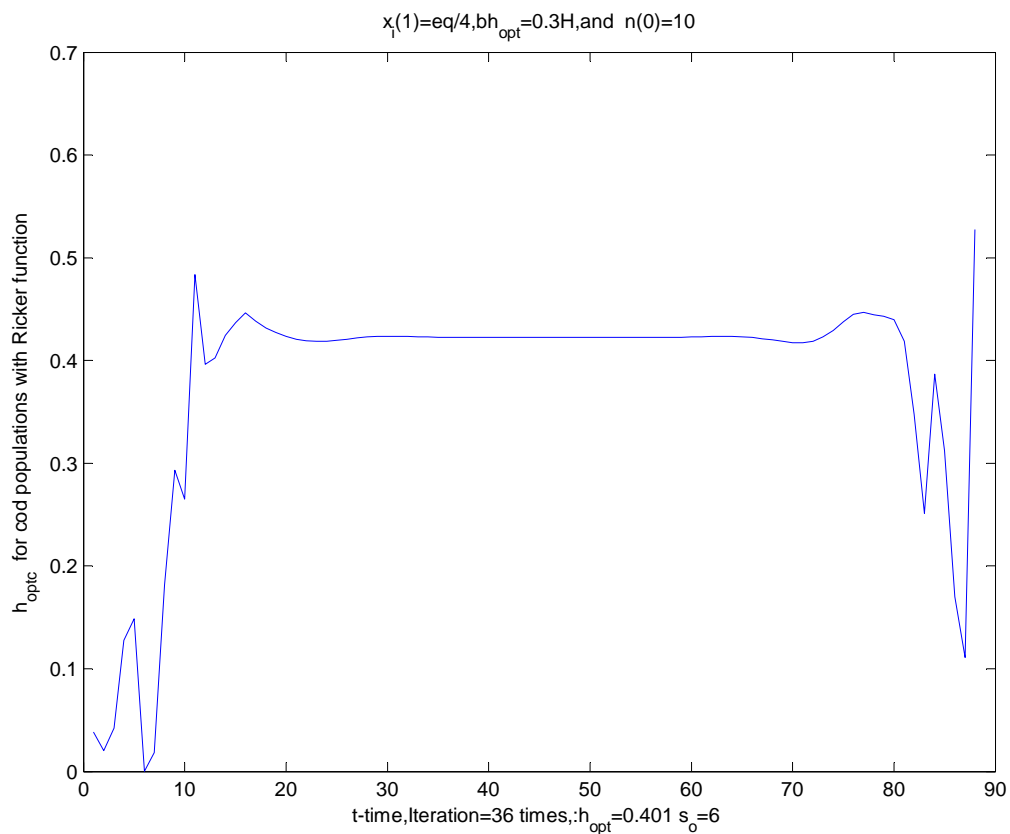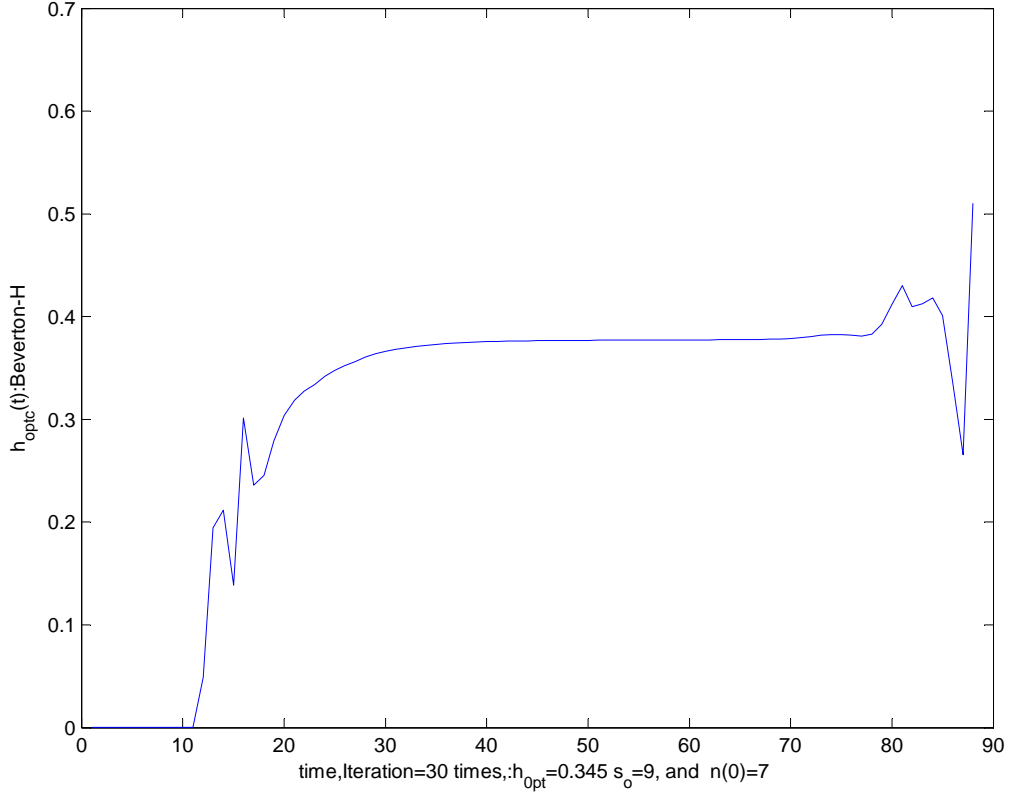
Figure 4.8: Optimal harvesting for cod populations and Beverton-Holt function with $x_i(1) = \frac{eq}{10}$, $bh_{opt} = 0.3H_1, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 10$. The same as before there are three phases of optimal harvesting the first one is a short time to recover of population from low density, then the harvesting with rate h_1 and the last phase is the unrestricted harvesting. Note that the constant middle phase is lower than for Ricker recruitment.

4.3 Near optimal strategies

The general optimal harvesting strategy is approximately of the following form: For some initial time interval $[0, n_1] = I$ the harvesting intensity is nearly zero. On an interval $I_2 = [n_1 + 1, n_2]$, $n_1 \approx \frac{k_2}{3}, n_2 \approx \mathcal{T} - \frac{k_2}{3}$, the harvesting rate is almost constant with $h_1 \approx h_{opt}$, while it is maximal on the final interval $[n_2, \mathcal{T}]$. In the interval I_1 the population is built up to a sufficiently high level. In I_2 this level is

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kept, while is it optimal to exploit the stock as much as possible in the last interval, because no one cares what happens after $\mathcal{T}$. The solution is thus of the almost fastest approach type. It is quite clear that a considerable no harvest interval I_1 is economically and politically not feasible. for that reason we propose the following near optimal strategies :

Step 1:

Fix all parameters as above and determine an optimal harvesting intensity h_{opt} based on the equilibrium model .

Step 2:

For the time horizon $\mathcal{T}$ define the harvesting strategy as

$$
h_t = \begin{cases} \alpha_1 h_{opt} & : 1 \le t \le n_1 \quad \alpha_1 = 0.2 \text{ or } 0.3 \\ h_{opt} & : n_1 + 1 \le t < \mathcal{T} \end{cases} \tag{4.5}
$$

$\mathcal{T}$ should be large, at least $3k_2$. We will compute the total yield h_t and compare it with the optimal control solution.

A few remarks can be made to begin with. n_1 should be approximately $\frac{k_2}{3}$ to $\frac{k_2}{2}$. Once the second stage is reached the solution should approach the equilibrium optimal harvest rate rather rapidly because most of the eigenvalues λ of the equilibrium Jacobian are ≤ 0.96.

The following tables give the main results of intensive computations for $n(0) = 3, 5, 7, 10, 15$, and s_o the optimal mesh parameter with $k_1 = 0.3$, $\delta = 0.96$. The initial conditions are x_i equal to 25 percent of the optimal solution and $n_1 = 10$. The idea here is, that even with excessive harvesting the population has attained some form of an equilibrium.

$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3	9	0.323	1.0236×10^8	8.4559×10^7	8.0175×10^7
5	8	0.391	5.1866×10^8	5.0577×10^8	4.9802×10^8
7	7	0.395	9.6051×10^8	9.4889×10	9.4207×10^8
10	6	0.401	1.5712×10^9	1.5402×10^9	1.5416×10^9
15	5	0.421	2.4615×10^9	2.3588×10^9	2.3828×10^9

Table 4.10: Results of total yield for cod population and Ricker function with $k_1 = 0.3, \mathcal{T} = 3k_2$

From the results in the table(4.10), one can see that for small $n(0)$ choosing $\alpha_1 = 0.2$ strategy is better than $\alpha_1 = 0.3$ because it gives the population time to recover. But for large $n(0)$ i.e $n(0) \geq 10$ the $\alpha_1 = 0.3$ is much better.
The following table gives the results for $n(0) = 3, 5, 7, 10, 15$ by using all parameters as above but $\mathcal{T} = 4k_2$.

$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3	9	0.323	1.1693×10^8	1.0082×10^8	9.6437×10^7
5	8	0.391	5.4799×10^8	5.3769×10^8	5.2994×10^8
7	7	0.395	1.0010×10^9	9.9326×10^8	9.8645×10^8
10	6	0.401	1.6250×10^9	1.5997×10^9	1.6011×10^9
15	5	0.421	2.5318×10^9	2.4387×10^9	2.4627×10^9

Table 4.11: Results of total yield for cod population and Ricker function with $k_1 = 0.3, T = 4k$

The following table gives the results for $n(0) = 5, 7, 10$ with $\mathcal{T} = 3k_2$ and $\mathcal{T} = 4k_2$ respectively by using $\delta = 0.98$.

T	$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3k	5	8	0.391	1.6072×10^9	1.5640×10^9	1.5478×10^9
4k	5	8	0.391	1.8588×10^9	1.8269×10^9	1.8107×10^9
3k	7	7	0.395	2.5593×10^9	2.5131×10^9	2.4982×10^9
4k	7	7	0.395	2.9077×10^9	2.8785×10^9	2.8636×10^9
3k	10	6	0.401	3.7968×10^9	3.7151×10^9	3.7108×10^9
4k	10	6	0.401	4.2606×10^9	4.2053×10^9	4.2009×10^9

Table 4.12: The same parameters are used for cod population and Ricker function, but $\delta = 0.98$

The following table gives the results for cod population for $n(0) = 5, 7, 10$, $\delta = 0.96$ with $\mathcal{T} = 3k_2$ and $\mathcal{T} = 4k_2$ respectively but Beverton-Holt function is used

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T	$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3k	5	9	0.314	6.2094×10^8	5.6660×10^8	5.7073×10^8
4k	5	9	0.314	6.6937×10^8	6.2093×10^8	6.2506×10^8
3k	7	9	0.345	1.2420×10^9	1.1143×10^9	1.1311×10^9
4k	7	9	0.345	1.3207×10^9	1.2012×10^9	1.2179×10^9
3k	10	8	0.309	2.4506×10^9	2.3175×10^9	2.3417×10^9
4k	10	8	0.309	2.5727×10^9	2.4548×10^9	2.4790×10^9

Table 4.13: The same parameters are used for cod population and Beverton-Holt function, but $\delta = 0.96$

These results indicate that such a policy would give the fishing industry sufficient time to adapt and restructure. This should involve result in lower subsidies, adequate fishing gear, reduction of the fleet and reduction of the work force.

The following tables give results of another strategy which is more politically acceptable and more realistic because most of fish stocks are a very low level far from equilibrium, so all results are given here with the same vital parameters and the same harvesting intensity but $n_1 = 15$ with α_1 in (4.5) will be 0.3 or 0.4 and the initial conditions are $x_i \approx 17\%$ of the equilibrium solution. The idea is also here

to search and choose a better strategy which is more applicable politically and economically ,because one can not say that a stop of fishing and close fishery for n_1 years.

$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.3$	$J(h_t, \alpha_1 = 0.4)$
5	8	0.391	3.2352×10^8	2.6672×10^8	2.4023×10^8
7	7	0.395	7.3364×10^8	6.4035×10	6.1728×10^8
10	6	0.401	1.3179×10^9	1.1305×10^9	1.1284×10^9
15	5	0.421	2.1825×10^9	1.7889×10^9	1.8387×10^9

Table 4.14: All results applied on cod population with $k_1 = 0.3, \mathcal{T} = 3k_2$

The following table gives the results for a cod population with $n(0) = 5, 7, 10, 15$ by using all parameters as above but $\mathcal{T} = 4k_2$.

$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.3)$	$J(h_t, \alpha_1 = 0.4)$
5	8	0.391	3.5276×10^8	2.9865×10^8	2.7215×10^8
7	7	0.395	7.7408×10^8	6.8472×10^8	6.6165×10^8
10	6	0.401	1.3714×10^9	1.1900×10^9	1.1879×10^9
15	5	0.421	2.2525×10^9	1.8688×10^9	1.9186×10^9

Table 4.15: The same parameters as above but, $\mathcal{T} = 4k_2$

The following table gives the results for cod population for $n(0) = 7, 10, \delta = 0.96$ with $\mathcal{T} = 3k_2$ and $\mathcal{T} = 4k_2$ respectively but Beverton-Holt function is used.

T	$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.3)$	$J(h_t, \alpha_1 = 0.4)$
3k	7	9	0.345	6.6975×10^8	5.7630×10^8	5.3548×10^8
4k	7	9	0.345	7.4883×10^8	6.6359×10^8	6.2276×10^8
3k	10	8	0.309	1.6517×10^9	1.4693×10^9	1.4295×10^9
4k	10	8	0.309	1.7742×10^9	1.6066×10^9	1.5667×10^9

Table 4.16: The same parameters are used for cod population with Beverton-Holt function.

4.4 Some numerical results for herring

The main results of intensive computations are presented for herring populations for $n(0) = 3, 5, 7$, and 10 as well as all vital parameters are chosen as before with $k_1 = 0.3$. Here we also denote that h_{optc} is the solution from optimal control with initial input $h_{in} = 0.02$ for $t \leq 10$, $h_{in} = h_{opt}$ for $t \geq 11$, $\delta = 0.96$ and initial condition of the $x_i(1)$ equal to 25 percent of the equilibrium solution. Other initial conditions will be used namely $x_i(1)$ equal to 10 percent of the equilibrium. One can also see that even with excessive harvesting the population has attained a form of equilibrium.

$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
3	6	0.265	158.9959	142.1745	95.7644	25.5892
5	5	0.36	440.4588	458.9407	377.5458	206.1034
7	5	0.499	824.9086	853.5809	683.8780	354.9264
10	5	0.678	1.4628×10^3	1.4865×10^3	1.1348×10^3	569.0673

Table 4.17: Results of total yield based on herring population with Ricker function and $k_1 = 0.3, \mathcal{T} = 3k_2$

The following table gives the results for $n(0) = 5$ by using all parameters as above but $x_i(1) = \frac{eq}{10}$ $\mathcal{T} = 3k_2$, and $\mathcal{T} = 4k_2$. It is also in this case the total yield is smaller than before.

T	$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
3k	5	5	0.36	231.0874	244.6137	178.1270	39.7865
4k	5	5	0.36	253.3328	269.9631	202.7933	58.8149

Table 4.18: The same parameters with $k_1 = 0.3, \mathcal{T} = 3k_2$, and $\mathcal{T} = 4k_2$ for $n(0) = 5$, but $x_i(1) = \frac{eq}{10}$
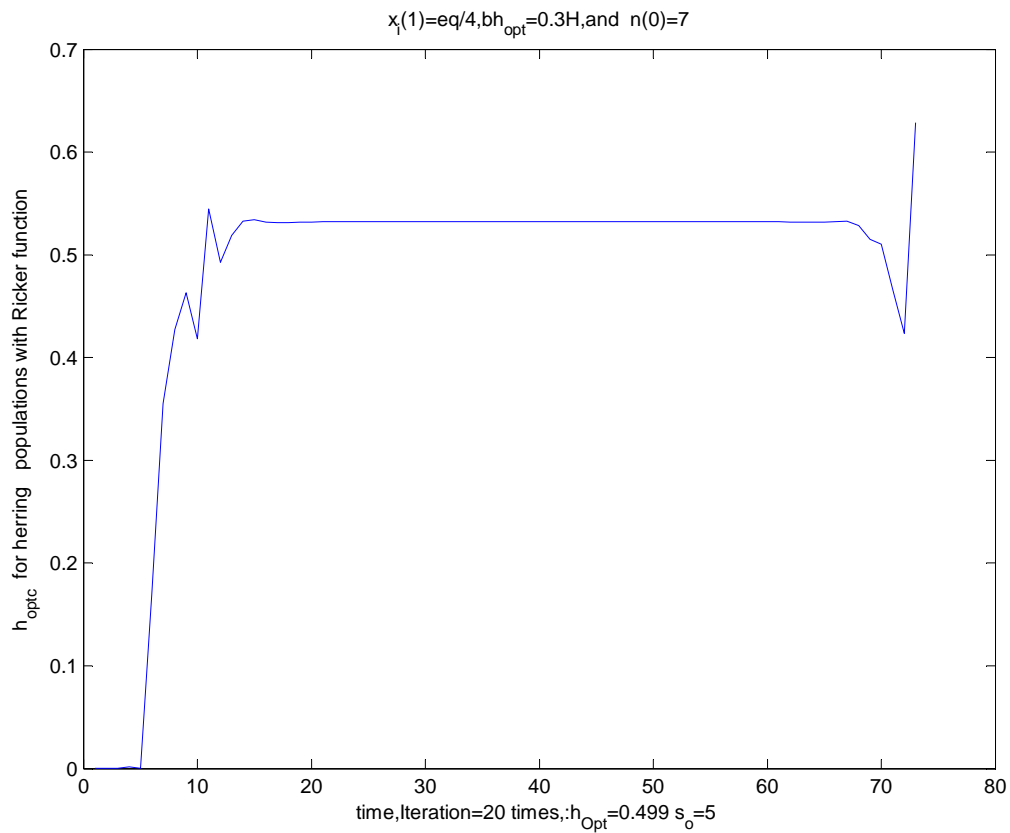
Figure 4.9: The optimal harvesting profile for herring population has the same behavior as for cod population. Ricker function is used with $x_i(1) = \frac{eq}{4}$, $bh_{opt} = 0.3H_1, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 7$.

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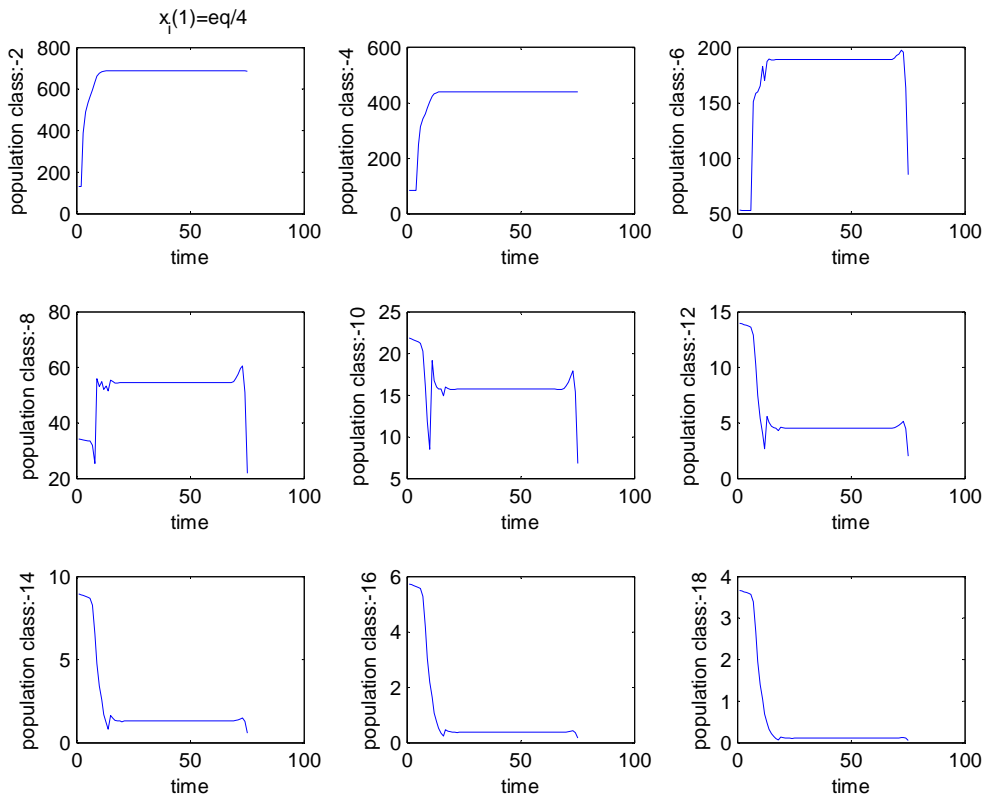
Figure 4.10: Herring population for several age classes after harvesting, Ricker function with $x_i(1) = \frac{eq}{4}$, $bh_{opt} = 0.3H_1, \delta = 0.96, \mathcal{T} = 3k_2$ and $n(0) = 7$.
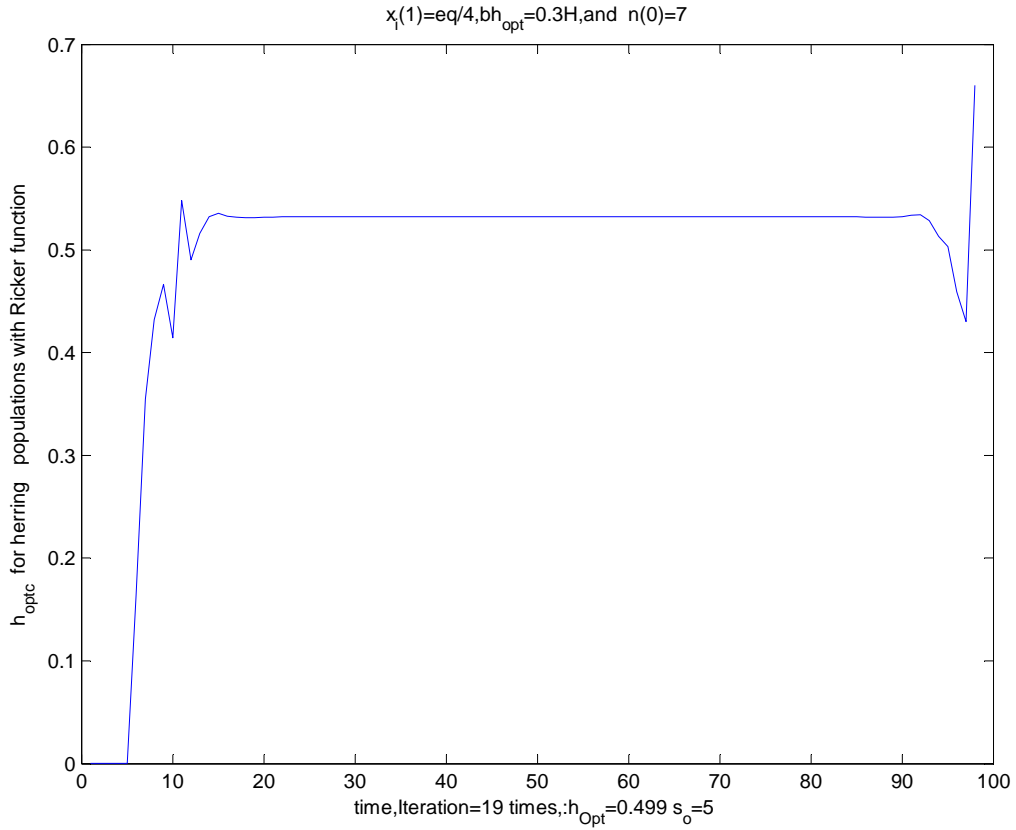
Figure 4.11: As before the optimal harvesting is independent on $\mathcal{T}$. $\mathcal{T} = 4k_2$. The first and the last 11 years give the same profile as $\mathcal{T} = 3k_2$

Some results will be presented for near optimal strategies which based on herring population. The following tables give results for $n(0) = 3, 5, 7, 10$ and s_0 the optimal mesh parameter with $k_1 = 0.3$, $\delta = 0.96, \delta = 0.98, \mathcal{T} = 3k_2$, and $\mathcal{T} = 4k_2$. The initial conditions are $x_i(1)$ equal to 25 percent of the optimal solution.

$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3	6	0.265	95.7644	85.5454	82.5975
5	5	0.36	377.5458	355.4343	352.2861
7	5	0.499	683.8780	639.2437	642.6193
10	5	0.678	1.1348×10^3	1.0331×10^3	1.0546×10^3

Table 4.19: The total yield for near optimal strategy with $k_1 = 0.3, \mathcal{T} = 3k_2, \delta = 0.96$

The following table gives the results for $n(0) = 3, 5, 7, 10$ by using all parameters as above but $\mathcal{T} = 4k_2$.

$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3	6	0.265	105.8719	97.4584	94.5099
5	5	0.36	402.3794	382.8855	379.7373
7	5	0.499	723.9822	682.0348	685.4104
10	5	0.678	1.1956×10^3	1.0968×10^3	1.1182×10^3

Table 4.20: The same all parameters as above but $\mathcal{T} = 4k_2$

The following table gives the results for $n(0) = 3, 5, 7$ with $\mathcal{T} = 3k_2$ and $\mathcal{T} = 4k_2$ respectively by using $\delta = 0.98$.

T	$n(0)$	s_o	h_{opt}	$J(s_o, h_{optc})$	$J(h_t, \alpha_1 = 0.2)$	$J(h_t, \alpha_1 = 0.3)$
3k	3	6	0.265	298.5738	271.2311	264.8985
4k	3	6	0.265	362.4683	340.5807	334.2454
3k	5	5	0.36	893.5978	854.3392	846.9929
4k	5	5	0.36	1.0456×10^3	1.0141×10^3	1.0068×10^3
3k	7	5	0.499	1.5061×10^3	1.4407×10^3	1.4417×10^3
4k	7	5	0.499	1.7475×10^3	1.6898×10^3	1.6908×10^3

Table 4.21: The same parameters aa above but $\delta = 0.98$ is used.

The following tables give results for $n(0) = 3, 5, 7$ by using Beverton-Holt function and s_0 the optimal mesh parameter with $k_1 = 0.3$, $\delta = 0.96$, $\mathcal{T} = 3k_2$, and $\mathcal{T} = 4k_2$. The initial conditions are $x_i(1)$ equal to 25 percent of the equilibrium solution.

$\mathcal{T}$	$n(0)$	s_o	h_{opt}	$J(s_o - 2, h_{optc})$	$J(s_o - 1, h_{optc})$	$J(s_o, h_{optc})$	$J(s_o + 1, h_{optc})$
$3k_2$	3	7	0.256	246.5031	190.9195	103.5260	-1.7514
$4k_2$	3	6	0.265	259.9937	205.1952	116.7521	8.4515
$3k_2$	5	6	0.2870	776.3458	685.7707	480.0812	199.6436
$4k_2$	5	6	0.287	813.0615	724.6400	516.0599	227.7294
$3k_2$	7	6	0.339	1.4929×10^3	1.3136×10^3	924.3678	413.9924
$4k_2$	7	6	0.339	1.5611×10^3	1.3837×10^3	986.7503	460.7569

Table 4.22: The results by using Beverton-Holt function. The negative sign indicates the harvesting is uneconomic

Chapter 5

Life near extinction

5.1   Life at the limit

The fact that most commercial marine fish stocks are near collapse in meanwhile common knowledge. The sharply reduced landings are clear evidence for this. Despite the fact that fishery scientists propose moderate fishing quotas or different gear, these suggestions are consistently ignored by politicians acting on the advice of a well entrenched lobby [23]. The threat of ten thousand or more unemployed fishermen alone generally suffices to lay the ground for irrational behaviour. On the other hand it has to be admitted that scientists are most often too cautious in their statements and the complex world of fishery science teems with contradictory statements to what extent overfishing, environment or climate are responsible for the decline in landings. Froese at al [24] argue vehemently that the decline is largely due to overfishing and in a model calculation they show that even a rather moderate change in fishing policy might yield higher returns within a 5 year interval. These calculations, however, are based on a bulk model and will not stand up to calculations based on this model. Fernandes and Cook [21] argue that the status of fish stock is not as bad as anticipated and is even rebounding. Some authors attribute the population variability mostly to environmental effects, while Hilborn et al [35] see the parent stock as the main factor. Thus there are a large number of models to explain various aspects of the marine fish ecosystem. On top of this one has a very insecure data basis, based on landings and a few trawler surveys. As an example for this state of affairs consider the rather rudimentary understanding of the recruit-

ment process which is rather important for the life cycle of marine fish. Another important aspect in this analysis is depensation at low densities (Allee effect). So far there are good indications that there is no depensation at low densities. Nonetheless there are rather large insecurities in setting up a good causal predictive model. The only reasonable predictive models so far are multivariate autoregressive statistical models[39], which are hardly based on the biology of marine fish. Nonetheless we will study here the effect of various factors on marine fish populations, which are described by the above model in order to get some idea on the size of various factors. First these factors will be studied separately in order to determine their impact and their sensitivity. Then various combinations will be analyzed and then a further aspect recovery scenarios will be studied. The fine model based on year classes will be used in all computation throughout this chapter.

5.2   Fishing:

Most people see overfishing as the main danger, and even when quotas are set, these are generally too high. In addition illegal fishing and piracy take their toll. An important aspect in the bycatch which may threaten the young fish. Froese [23] estimated the profit margin in the EU as 3 to 6%. Together with the piracy and substantial subsidies it is therefore plausible to set the actual harvesting intensity near the break even intensity. The break even intensity is defined as the fishing mortality at which cost equals profit. This seems to be justified in view of the subsidies for fuel, unemployment benefit, and the fact that piracy, illegal fishing and the bycatch etc. amount to 5-10% of the harvest. Thus it seems plausible that most of the European cod fishery is run near the break even intensity.
Another aspect is highgrading. This occurs because mature fish are more valuable than younger ones. Thus fishermen will discard the part of a catch consisting mainly of younger fish, in order to stay within the allowed quotas. Highgrading occurs because the harvest of young fish is too high and because the mesh width is too small. Overfishing thus has two aspects, the quotas are too high and the meshes of the fishing nets are too fine. As expected fishermen will opt for smaller mesh width. For this reason we will also have to consider the use fishing nets with finer meshes. We model this as $s_0 \longrightarrow s_0 - 1, \; s_0 - 2$.

5.3   Environmental factors

That the environment plays an important role for marine fish can be seen from the large variation in the net reproductive number. Environmental factors act in various phases of the life cycle

1-Oviposition:

Oviposition is influenced by the temperature of the water, its salinity, presence of predators and status of the environment. The general increase in temperature in recent years and the damage inflicted by aggressive fishing gear (ground nets) will certainly have a negative effect. We will model these negative effects by

$$c \longrightarrow ce^{-P_0} \tag{5.1}$$

Problematic is of course to estimate P_0 properly. In view of the large variation of $n(0)$, we can expect

$$0 \le P_0 \le 0.2 \tag{5.2}$$

and model calculations will be made for $P_0 = 0, 0.1, 0.2$. Note that the oviposition effect reduces $n(0)$ to approximately $n(0)exp(-P_0)$. Thus the sensitivity of P_0 will be essentially proportional of $n(0)$. The effect of P_0 is only to reduce the optimal intensity h_{opt} as well as the net gain. In the remainder we will speak of a perturbed model, if the perturbation $c \longrightarrow ce^{-P_0}$ is applied. This applies similarly to other perturbations introduced below.

2-The larval and juvenile phase:

The larval stage is the most important phase in the life of marine fish. Since the larvae have a high demand for food. Competition and lack of food and their fate as bycatch determine their development. In addition the larvae and young fish are preyed upon and sickness and cannibalism take their toll. Thus in bad years their progression can be severely curtailed. We model this by

$$\tau_i = e^{-\mu} \longrightarrow e^{-(\mu+P_1)} \qquad \text{with} \quad 0 \le P_1 \le 0.2 \tag{5.3}$$

for $i = 1, 2, 3, 4$. It is obvious that this rather global factor influences the development of the population more than P_0, because it acts over several years.

3-The adult phase

Lack of food and being preyed upon will be the dominant factor influencing marine fish in the adult phase. Again, this can be caused by environmental effects and the competition with other species or increased harvesting through piracy illegal harvesting. Misreporting will also influence our picture. We express it in general as

$$\tau_i = e^{-\mu} \longrightarrow e^{-(\mu+P_2)} \qquad \text{with} \quad 0 \leq P_2 \leq 0.2 \tag{5.4}$$

Note that the fishing mortality is about 0.3 to 0.4.

5.4   Harvesting near the break even point

Unrestricted competitive harvesting of fish will continue until is not profitable any more. With subsidies fishing will even go beyond that. Currently harvesting of fish in the EU is close to that and the profit margin is 3-6% [23], though even that is uncertain because of the many forms of subsidies handed out by the European Union. So in this section we will take a 4% profit margin as our baseline scenario. That means the cost parameters a and b are equal to 0.48 from the total harvested amount H i.e $k_1 = 0.48$, because the fishermen have agreed to harvest as much as it is economically feasible. On the other hand we have additional perturbations coming from the environmental factors. Likewise we will determine the sensitivity with respect these parameters. This will be studied for the environmental factors $P_0, P_1 \ and \ P_2$ separately for values between 0 and 0.2. In addition we will take a reduction in the width of fishing nets into account. But generally these factors may operate together. Thus if the fish stocks continue to suffer from a high harvesting intensity, their state may be out of safe biologically as well as economically limits [24]. The environmental factors will be considered via oviposition, juvenile or larval phase and adult phase as we noted above. The basic questions are then the influence on the profit (N_H) and survival of species $n(0)$. The negative effects listed above will in general lead to a negative net gain. To be sure, this will be made up by subsidies. This, however, is not part of our model. Thus we will rather concentrate on the population and n as a function of h, P_0, P_1, and P_2. It is obvious that

$$n(h, P_0.P_1, P_2) \approx e^{-P_0} n(h, 0.P_1, P_2)$$

This, however, holds only approximately, because P_0 will shift the optimal h. Since P_1 acts for 4 or 5 years its effect will be stronger than that of P_0. The greatest effect,however, is due to P_2, which acts throughout the whole adult life.

It is also clear that this model can not predict a total collapse of a population. Even moderate $n \geq 2$. will lead to an eventual recovery of the population. The main reason is that we treat the population as a continuous variable. At very low densities we might in fact have an Allee effect, which then would lead to extinction. This, however, is no reason to rejoice, because a longer period of low density will have a severe effect on the food web. This applies in particular to cod, which is the top predator. So the effect of low density might in fact result in an extinction because cod might be replaced by another predator, leaving little room for recovery. This, however, goes beyond this model.

The computations will be made for cod only and $\mu \approx 0.2$ as well as the weight distribution W_i will be fixed. So the only factor that distinguishes a stock is $n(0)$ the natural reproduction rate. It clear that a stock with $n(0)$ larger than 6 or 7 will recover more rapidly from disaster and will not go extinct than a stock with small $n(0)$. Thus we will also limit ourselves to $2 \leq n(0) \leq 5$.

The results in the following table based on cod population with Ricker recruitment. Here the cost terms a and b will be defined by a 4% profit. Otherwise we use the same vital parameters namely μ, F_i, and W_i, which have been used in the previous chapters.

$n(0)$	Unperturbed model with optimal s_0					Perturbed model $P_0 = 0.1$			
	s_o	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	10	0.1092	1.6149	3.113×10^8	9.469×10^6	0.0923	1.5014	2.684×10^8	8.309×10^6
3	10	0.1765	2.2080	4.809×10^8	1.357×10^7	0.1603	2.0386	4.395×10^8	1.264×10^7
4	9	0.1930	2.6836	5.884×10^8	1.446×10^7	0.1785	2.4801	5.492×10^8	1.386×10^7
5	9	0.2239	3.2164	6.751×10^8	1.566×10^7	0.2103	2.9631	6.363×10^8	1.514×10^7

$n(0)$	Perturbed model $P_0 = 0.2$			
	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	0.0757	1.3979	2.2491×10^8	7.0811×10^6
3	0.1438	1.8855	3.9789×10^8	1.1657×10^7
4	0.1638	2.2954	5.0974×10^8	1.3209×10^7
5	0.1963	2.7336	5.9741×10^8	1.4592×10^7

Table 5.1: All quantities here are a decreasing function of parameter P_0. All results based on cod populations with Ricker function, $k_1 = 0.48$ from total harvested amount $P_0 = 0, 0.1$ and 0.2 with $P_1 = P_2 = 0$.

The following table gives results based on cod populations with Ricker function with the same vital parameters as above but values $P_1 = 0.1$, and 02 are used.

$n(0)$	Unperturbed model with optimal s_0					Perturbed model $P_1 = 0.1$			
	s_o	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	10	0.1092	1.6149	3.113×10^8	9.469×10^6	0.0436	1.2168	1.390×10^8	4.417×10^6
3	10	0.1765	2.2080	4.809×10^8	1.357×10^7	0.1101	1.6215	3.206×10^8	9.531×10^6
4	9	0.1930	2.6836	5.889×10^8	1.446×10^7	0.1341	1.9748	4.398×10^8	1.174×10^7
5	9	0.2239	3.2164	6.751×10^8	1.566×10^7	0.1673	2.3368	5.311×10^8	1.337×10^7

$n(0)$	Perturbed model $P_1 = 0.2$			
	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	0.00001	0.8987	0×10^8	0×10^6
3	0.0444	1.2214	1.4483×10^8	4.4918×10^6
4	0.0758	1.4837	2.7655×10^8	7.9538×10^6
5	0.108	1.7364	3.7485×10^8	1.0218×10^7

Table 5.2: This table shows that the larval and juvenile factor has more severe effect on results than the oviposition factor because it operates for several years. All results based on cod populations with Ricker function, a and b are 0.4% profit, $P_1 = 0, 0.1$ and 0.2 with $P_0 = P_2 = 0$.

Note that for large values of P_1 will make the $n(0)$ less than one, especially for $n(0) = 2$, then the population will die out and this will effect on other quantities likewise.

The following table gives result based on cod populations with Ricker function with the same vital parameters but $P_0 = 0$, $P_1 = 0$, and $P_2 = 0.1, 0.2$. In case $n(0) = 2$, or 3 and for large values of $P_2 > 0.1$ the $n(h_{opt})$ becomes less or equal 1, then the population will die out, on the other hand as soon as the $n(h_{opt})$ is less than 1.02 or 1.03 , the population is principle in danger to die out.

$n(0)$	Unperturbed model with optimal s_0					Perturbed model $P_2 = 0.1$			
	s_o	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	10	0.1092	1.6149	3.113×10^8	9.469×10^6	0.0163	1.0399	4.119×10^7	8.640×10^5
3	10	0.1765	2.2080	4.809×10^8	1.357×10^7	0.1390	1.3870	3.171×10^8	5.680×10^6
4	9	0.1930	2.6836	5.884×10^8	1.446×10^7	0.1785	1.7056	5.030×10^8	7.584×10^6
5	9	0.2239	3.2164	6.751×10^8	1.566×10^7	0.2307	2.0407	6.457×10^8	8.746×10^6

$n(0)$	Perturbed model $P_2 = 0.2$			
	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	0.00001	0.8987	0×10^8	0×10^6
3	0.00001	0.6587	0×10^8	0×10^6
4	0.1079	1.2083	2.6165×10^8	2.6040×10^6
5	0.1989	1.430	4.6874×10^8	3.8413×10^6

Table 5.3: All results based on cod populations with Ricker function, $k_1 = 0.48, P_0 = 0, 0.1$ and 0.2 with $P_1 = P_2 = 0$. One can see the effect of P_2 on results is more pronounced than others because it acts on the whole life .

In the following table, one can study the sensitivity of h_{opt}, and $n(h_{opt})$, with respect to $P_0, P_1,$ and P_2.

$n(0)$	$\frac{\triangle h_{opt}}{\triangle P_0}$	$\frac{\triangle h_{opt}}{\triangle P_1}$	$\frac{\triangle h_{opt}}{\triangle P_2}$	$\frac{\triangle n(h_{opt})}{\triangle P_0}$	$\frac{\triangle n(h_{opt})}{\triangle P_1}$	$\frac{\triangle n(h_{opt})}{\triangle P_2}$
2	-0.169	-0.656	-0.929	-1.135	-3.981	-5.75
3	-0.162	-0.664	-0.375	-1.694	-5.865	-8.21
4	-0.145	-0.589	-0.145	-2.035	-7.088	-9.78
5	-0.136	-0.566	0.068	-2.533	-8.796	-11.757

Table 5.4: As expected P_2 acts most severely on h and n. All results based on cod populations with Ricker function, $k_1 = 0.48$, and $\triangle P_0 = 0 - 0.1, \triangle P_1 = 0 - 0.1$ and $\triangle P_2 = 0 - 0.1$

Next figure exhibits the effect of the environmental factors on the total harvesting amount for $n(0) = 3$. As expected P_2 acts most severely on n, and h followed by P_1 and finally by P_0. For other values of $n(0)$, it has the same behavior.
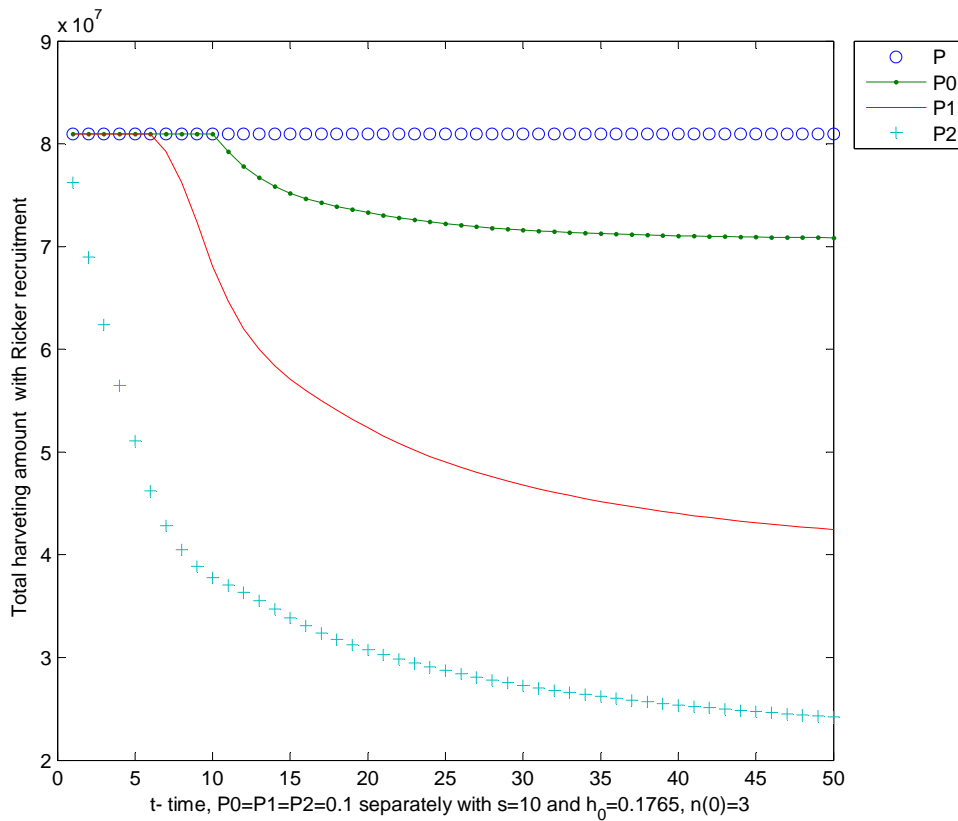
Figure 5.1: The total harvesting H is based on cod population with Ricker function with $n(0) = 3, s_o = 10,$ and $k_1 = 4\%$ profits. Here P represents the case when $P_0 = P_1 = P_2 = 0$. The behavior of total harvesting amount which is decreasing function of P_0, P_1 and P_2. One can also see that the adult environmental factor has more effect on total harvesting amount than oviposition as well as juvenile environmental factor. All environmental factors are done separately

To complete our investigation on the results when one reduces the optimal mesh width s_0 to $s_0 - 1$ and $s_0 - 2$ respectively. The following tables give results for n(0)=2, 3,4,5. All standard parameters are the same as before and the environmental factors are done separately.

$n(0)$	$s_o - 1$	Unperturbed model				Perturbed model $P_0 = 0.1$			
		h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	9	0.0910	1.5985	3.093×10^8	9.088×10^6	0.0767	1.4906	2.670×10^8	8.029×10^6
3	9	0.1508	2.1474	4.749×10^8	1.260×10^7	0.1360	1.9927	4.348×10^8	1.184×10^7
4	8	0.1653	2.5875	5.781×10^8	1.359×10^7	0.1519	2.4049	5.406×10^8	1.313×10^7
5	8	0.1951	3.0588	6.603×10^8	1.443×10^7	0.1818	2.8359	6.237×10^8	1.409×10^7

$n(0)$	Perturbed model $P_1 = 0.1$				Perturbed model $P_2 = 0.1$			
	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	0.0361	1.2149	1.387×10^8	4.344×10^6	0.0125	1.0398	4.113×10^7	8.598×10^5
3	0.0918	1.6048	3.186×10^8	9.144×10^6	0.1073	1.3748	3.158×10^8	5.496×10^6
4	0.1125	1.9421	4.353×10^8	1.133×10^7	0.1400	1.6746	4.988×10^8	7.447×10^6
5	0.1417	2.2750	5.237×10^8	1.273×10^7	0.1852	1.9762	6.378×10^8	8.432×10^6

Table 5.5: All results based on cod fish populations with Ricker function. The same parameters are used but $P_0 = 0.1, P_1 = 0.1$ and $P_2 = 0.1$ with the optimal mesh width $s_0 - 1$.

$n(0)$	$s_o - 2$	Unperturbed model				Perturbed model $P_0 = 0.1$			
		h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	8	0.0759	1.5869	3.073×10^8	8.883×10^6	0.0639	1.4830	2.656×10^8	7.877×10^6
3	8	0.1271	2.1018	4.691×10^8	1.208×10^7	0.1141	1.9589	4.302×10^8	1.142×10^7
4	7	0.1400	2.5144	5.676×10^8	1.320×10^7	0.1282	2.3485	5.319×10^8	1.280×10^7
5	7	0.1671	2.9340	6.452×10^8	1.385×10^7	0.1548	2.7370	6.108×10^8	1.360×10^7

$n(0)$	Perturbed model $P_1 = 0.1$				Perturbed model $P_2 = 0.1$			
	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
2	0.0301	1.2136	1.383×10^8	4.303×10^6	0.0097	1.0397	4.106×10^7	8.577×10^5
3	0.0765	1.5929	3.166×10^8	8.935×10^6	0.0829	1.3662	3.144×10^8	5.455×10^6
4	0.0943	1.9186	4.306×10^8	1.115×10^7	0.1096	1.6522	4.943×10^8	7.514×10^6
5	0.1193	2.2293	5.162×10^8	1.245×10^7	0.1466	1.9275	6.296×10^8	8.481×10^6

Table 5.6: The same as before but $P_0 = 0.1, P_1 = 0.1$ and $P_2 = 0.1$ with the optimal mesh width $s_0 - 2$.

By the same way in the next tables the sensitivity is studied for $h_{opt}, n(h_{opt})$ and N_{opt} with respect to P_0, P_1 and P_2, but for $s_0 - 1$.

$n(0)$	$s_0 - 1$	$\frac{\triangle h_{opt}}{\triangle P_0}$	$\frac{\triangle h_{opt}}{\triangle P_1}$	$\frac{\triangle h_{opt}}{\triangle P_2}$	$\frac{\triangle n(h_{opt})}{\triangle P_0}$	$\frac{\triangle n(h_{opt})}{\triangle P_1}$	$\frac{\triangle n(h_{opt})}{\triangle P_2}$
2	9	-0.143	-0.549	-0.785	-1.079	-3.836	-5.587
3	9	-0.148	-0.590	-0.435	-1.547	-5.426	-7.726
4	8	-0.134	-0.528	-0.253	-1.826	-6.454	-9.129
5	8	-0.133	-0.534	-0.099	-2.229	-7.838	-10.826

Table 5.7: All results based on cod populations with Ricker function, $k_1 = 0.48, \triangle P_0 = 0 - 0.1 \triangle P_1 = 0 - 0.1$ and $\triangle P_0 = 0 - 0.1$, but $s_0 - 1$

$n(0)$	$s_0 - 2$	$\frac{\triangle h_{opt}}{\triangle P_0}$	$\frac{\triangle h_{opt}}{\triangle P_1}$	$\frac{\triangle h_{opt}}{\triangle P_2}$	$\frac{\triangle n(h_{opt})}{\triangle P_0}$	$\frac{\triangle n(h_{opt})}{\triangle P_1}$	$\frac{\triangle n(h_{opt})}{\triangle P_2}$
2	8	-0.120	-0.458	-0.662	-1.039	-3.733	-5.542
3	8	-0.130	-0.506	-0.442	-1.429	-5.089	-7.356
4	7	-0.118	-0.457	-0.304	-1.659	-5.958	-8.622
5	7	-0.123	-0.478	-0.205	-1.97	-7.047	-10.065

Table 5.8: All results based on cod populations with Ricker function, $k_1 = 0.48, \triangle P_0 = 0 - 0.1 \triangle P_1 = 0 - 0.1$ and $\triangle P_0 = 0 - 0.1$, but $s_0 - 2$

To conclude our study of the effect of environmental factors, we will combine all these factors together. That means we will introduce perturbation $P_0 = P_1 = P_2 = 0.1$. We will also combine $P_0 = P_1 = 0.1$ and $P_2 = 0.2$. The following tables give the result of a cod population with Ricker function. All other vital parameters are the same as before but $n(0) = 4$. The net gain is based on 4% profits. The optimal mesh width s_0 is also modified to $s_0 - 1$ and $s_0 - 2$.

$n(0) = 4$	Unperturbed model				Perturbed model $P_0 = P_1 = P_2 = 0.1$			
	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$	h_{opt}	$n(h_{opt})$	$\sum_{i=4}^{30} \bar{x}_i$	$\bar{x}_{12}$
s_0	0.1930	2.684	5.884×10^8	1.446×10^7	0.0560	1.184	1.787×10^8	3.347×10^6
$s_0 - 1$	0.1653	2.588	5.781×10^8	1.359×10^7	0.0433	1.183	1.783×10^8	3.333×10^6
$s_0 - 2$	0.14	2.514	5.676×10^8	1.320×10^7	0.0339	1.181	1.779×10^8	3.342×10^6

Table 5.9: All results based on cod fish populations with Ricker function. Here the perturbations are $P_0 = P_1 = P_2 = 0.1$ with $n(0) = 4$.

Note that we found in all cases that the net reproductive rate $n(h_{opt})$ becomes less than one when we used $P_0 = P_1 = 0.1$ and $P_2 = 0.2$. Thus the equilibrium value will become 0.

we also computed the time development of a population with a harvesting rate of h_{opt} at 4% profit. We used $\bar{x}_i$, the equilibrium value of the static 4% profit margin model as the initial value. This describes how a population evolves which has been harvested for long time at rate h_{opt} at the 4% profit level. The following table gives the evolution of such a population with $P_0 = P_1 = P_2 = 0.1$. So we will assume that the stock will suffer from these factors for short time e.g. 5 years with harvesting rate h_{opt} at 4% profits. The initial value is the equilibrium value with $n(0) = 3, 4$ respectively. All other vital parameters are the same as before. The optimal mesh width s_0 is also modified to $s_0 - 1$ and $s_0 - 2$. All results based on cod population with Ricker function.

s	h_{opt}	Unperturbed model		Perturbed model $P_2 = P_1 = P_0 = 0.1$	
		$\sum_{i=4}^{30} x_i(5)$	$x_{12}(5)$	$\sum_{i=4}^{30} x_i(5)$	$x_{12}(5)$
s_0	0.1765	4.7912×10^8	1.3571×10^7	3.2117×10^8	9.0971×10^6
$s_0 - 1$	0.1508	4.7259×10^8	1.2596×10^7	3.1678×10^8	8.4436×10^6
$s_0 - 2$	0.1271	4.6625×10^8	1.2078×10^7	3.125×10^8	8.0964×10^6

Table 5.10: All other vital parameters are the same but $P_0 = P_1 = P_2 = 0.1$ and $n(0) = 3$.

s	h_{opt}	Unperturbed model		Perturbed model $P_2 = P_1 = P_0 = 0.1$	
		$\sum_{i=4}^{30} x_i(5)$	$x_{12}(5)$	$\sum_{i=4}^{30} x_i(5)$	$x_{12}(5)$
s_0	0.193	5.9614×10^8	1.446×10^7	3.996×10^8	9.691×10^6
$s_0 - 1$	0.1653	5.854×10^8	1.3594×10^7	3.968×10^8	9.112×10^6
$s_0 - 2$	0.1400	5.7448×10^8	1.3204×10^7	3.851×10^8	8.851×10^6

Table 5.11: All other vital parameters are the same but $P_0 = P_1 = P_2 = 0.1$ and $n(0) = 4$.

As a final example we study a crash scenario evolution of a population with $s_0 - 2, P_0 = P_1 = 0.1$ and $P_2 = 0.2$ lasting for 5 or 10 years for various $n(0) = 2, 3, 4, 5$. The initial value is the equilibrium value at 4% profit. The time evolution total population $\sum_{i=4}^{30} x_i$ is computed with harvesting rate at the 4% profit level. Note that in this scenario the net reproductive rate $n(0)$ is less than one in the static model. The crash scenario results are given in the following tables .

$n(0)$	h_{opt}	Unperturbed model			Perturbed model $P_0 = P_1 = 0.1, P_2 = 0.2$		
		$\sum_{i=4}^{30} x_i(5)$	$x_5(5)$	$x_{12}(5)$	$\sum_{i=4}^{30} x_i(5)$	$x_5(5)$	$x_{12}(5)$
2	0.0759	3.0173×10^8	5.3126×10^7	8.8815×10^6	1.7211×10^8	3.5612×10^7	3.9907×10^6
3	0.1271	4.6625×10^8	8.4244×10^7	1.2078×10^6	2.688×10^8	5.6471×10^7	5.427×10^6
4	0.1400	5.7448×10^8	1.0633×10^8	1.3204×10^7	3.350×10^8	7.1274×10^7	5.933×10^6
5	0.1671	6.6688×10^8	1.2346×10^8	1.3850×10^7	3.919×10^8	8.2756×10^7	6.223×10^6

Table 5.12: All other vital parameters are the same but $P_0 = P_1 = 0.1$ and $P_2 = 0.2$. and $s_o - 2$

$n(0)$	h_{opt}	Unperturbed model			Perturbed model $P_0 = P_1 = 0.1, P_2 = 0.2$		
		II	$x_5(10)$	$x_{12}(10)$	II	$x_5(10)$	$x_{12}(10)$
2	0.0759	$2.868{\times}10^8$	$4.865{\times}10^7$	$8.882{\times}10^6$	$9.167{\times}10^7$	$1.884{\times}10^7$	$1.793{\times}10^6$
3	0.127	$4.588{\times}10^8$	$8.200{\times}10^7$	$1.208{\times}10^7$	$1.638{\times}10^8$	$3.580{\times}10^7$	$2.439{\times}10^6$
4	0.1400	$5.923{\times}10^8$	$1.118{\times}10^8$	$1.320{\times}10^7$	$2.255{\times}10^8$	$5.166{\times}10^7$	$2.666{\times}10^6$
5	0.1671	$7.226{\times}10^8$	$1.408{\times}10^8$	$1.385{\times}10^7$	$2.884{\times}10^8$	$6.774{\times}10^7$	$2.796{\times}10^6$

$n(0)$	h_{opt}	Perturbed model $P_0 = P_1 = 0, P_2 = 0.2$		
		II	$x_5(10)$	$x_{12}(10)$
2	0.0759	$1.487{\times}10^8$	$3.287{\times}10^7$	$2.190{\times}10^6$
3	0.1271	$2.678{\times}10^8$	$6.224{\times}10^7$	$2.979{\times}10^6$
4	0.1400	$3.706{\times}10^8$	$8.974{\times}10^7$	$3.256{\times}10^6$
5	0.1671	$4.756{\times}10^8$	$1.175{\times}10^8$	$3.415{\times}10^6$

Table 5.13: All other vital parameters are the same but $P_2 = 0.2$. and $s_o - 2$. II$=\sum_{i=4}^{30} x_i(10)$

The results indicate that illegal fishing is more dangerous than environmental factors. It is clear that with our model any effect needs a long time to be seen. The next figure indicates that the effect of environmental factors.
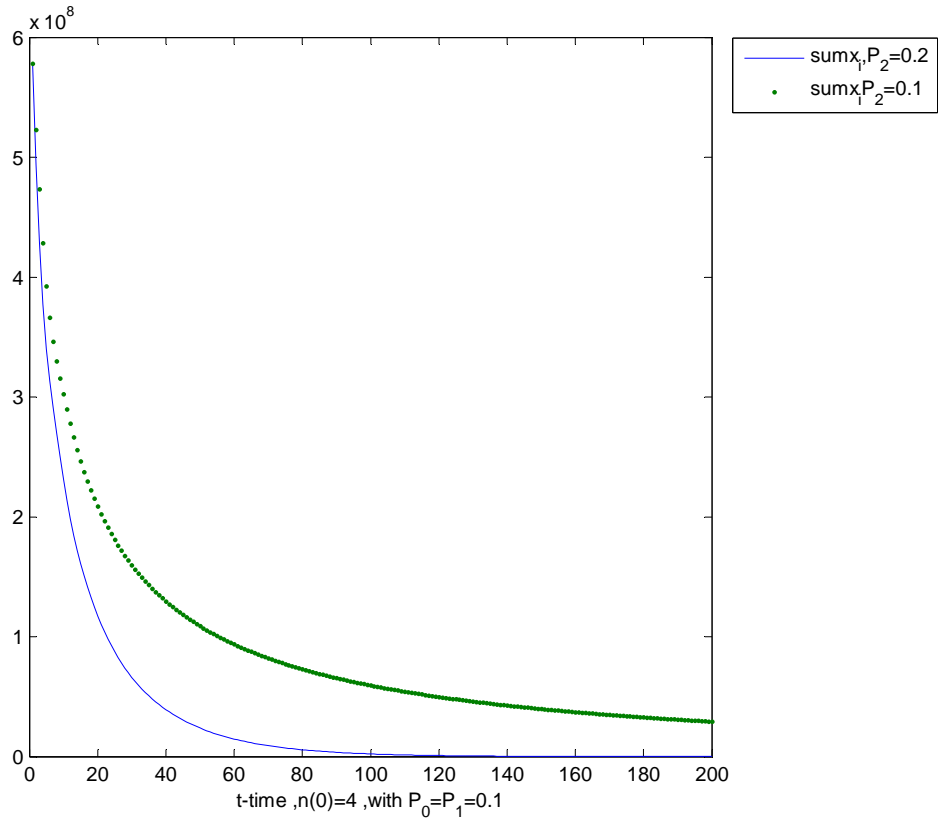
Figure 5.2: The total cod population with P_0, P_1 and P_2 for $n(0) = 4$. with Ricker function The net profit is 4% from total harvesting amount.

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Appendices

Appendix A

Compute the parameter c from $n(0)$.

1- Parameters and Initialization

$k = 28; n(0) = 10; \mu = 0.223$;q=0;Y=0; $\sigma_1 = e^{-\mu} * \frac{2}{3}$;(The number of age classes, the net reproductive rate and mortality)

p(2)=0.01;p(3)=0.13;p(4)=0.63;p(5)=0.96;p(6:k)=1;(The growth of the gonads in s_1 years)

2- Compute the weight,survival rate,and fecundity in each class

W(1)=0;$t(1) = e^{-\mu}$; l(1)=1;

for i=2:1:k; $t(i) = exp(-\mu)$; l(i)=t(i-1)*l(i-1); $W(i) = 17 * (1 - e^{-0.12*(i+2)})^3$;

d(i)=p(i)*W(i)*250000; q=q+d(i); T(i)=p(i)*W(i)*l(i)*250000; Y=Y+T(i); end

3- $c = \frac{n(0)*(1-\sigma_1)}{Y}$

Appendix B

Convergent model program

1- Parameters and Initialization

k=28; n(0)=10; $E_0 = 4.1 * 10^{14}, \mu = 0.223$; The number of age classes, Scaling parameter and Mortality

q=0;Y=0; $\sigma_1 = e^{-\mu} * \frac{2}{3}$; $t(1) = e^{-\mu}$; $W(1) = 0; p(2) = 0.01; p(3) = 0.13; p(4) = 0.63; p(5) = 0.96; p(6:k) = 1$; (The growth of the gonads in s_1 years)

$x(1:k,1) = \frac{equilibrium\ value}{4}$; l(1)=1; Initial value

2- Compute the weight,survival rate and fecundity in each class

for i=2:1:k-1; $t(i) = exp(-\mu)$; end for i=2:1:k; l(i)=t(i-1)*l(i-1); $W(i) = 17 * (1 - e^{-0.12*(i+2)})^3$;d(i)=p(i)*W(i)*250000; q=q+d(i); T(i)=p(i)*W(i)*l(i)*250000; Y=Y+T(i); end

3- for s=1:1:280; r(s)=0; Tem=0; for i=2:1:k; Tem=Tem+d(i)*x(i,s); end E(s)=Tem; Number of eggs

$g(s) = \frac{E(s)}{E_0}$; c=$\frac{n(0)*(1-\sigma_1)}{Y}$; $r(s) = exp(-g(s))$; Ricker function

$$A = \begin{bmatrix} \sigma_1 & c*d(2)*r(s) & c*d(3)*r(s) & \cdots & c*d(k)*r(s) \\ t(1) & 0 & 0 & \cdots & 0 \\ 0 & t(2) & \ddots & \vdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t(k-1) & 0 \end{bmatrix}$$

x(:,s+1)=A*x(:,s); end

5- Plots

i=1:1:80; B(i)=x(14,i); plot (i,B(i));

ylabel ('Model with n(0)=10 age class x(14,t)'); xlabel ('t-time ');

Appendix C

Determine the cost parameters a and b.

1- Parameters and Initialization

$k = 28$; tc=0.25; s=6; (The number of age classes, the duration of no harvesting and the optimal mesh width)

$E_0 = 4.1 * (10)^{14}; \mu = 0.223$ (Scaling parameter and Mortality)

p(2)=0.01;p(3)=0.13;p(4)=0.63;p(5)=0.96;p(6:k)=1; (The growth of the gonads in s_1 years)

$\sigma_1 = exp(\mu) * \frac{2}{3}$; $t(1) = exp(-\mu) * \frac{1}{3}$; $c = 0.000004089100217777939$; (The survival rate and the parameter c)

T=0; l(1)=1;d(1)=0;Q(1)=0;BB=0;D=0;F=0; $h = 1$ (Initial guess)

2- check=1; count=1; $while(check > 0)$;

oldh=h; To save the initial guess in oldh

(Compute the weight, fecundity,and price function)

for i=2:1:s-1; t(i)=exp(-mu); end for i=s:1:27; t(i)=exp(-mu-(h*(1-tc))); end for i=2:1:k; if $i < 10$; q(i)=0.18+0.09*i; else; q(i)=1.08;

$l(i) = t(i-1)*l(i-1)$; $W(i) = 17*(1-e^{-0.12(i+2)})^3$; $d(i) = l(i)*p(i)*W(i)*250000$; end; $BB = BB + d(i); Q(i) = q(i)*w(i)*l(i)$;

for i=s:1:k; T=T+Q(i); end $n = \frac{(c*BB)}{G}; if\ n > 1\ y = \frac{(log(n)*E0)}{BB}$; else y=0; end

$NN = (exp(-mu) * 2/3) + c * BB * exp(\frac{-(y*BB)}{E0})$; (Equilibrium condition)

$A = \frac{h}{(mu+h)}$; S=1-exp(-(mu+h)*(1-tc)); H=A*T*y*S*exp(-mu*tc); $a = k_1 * H$;

$b = \frac{(k_1*H)}{h}$; $M(count) = H; M1(count) = a; M2(count) = b$;

3- Compute the net gain

for h1=1:1:2000; T1(h1)=0; l1(1)=1; d1(1,h1)=0; Q1(1,h1)=0; BB1(h1)=0;

$L1(h1) = h1 * 10^{-3}$;D1(h1)=0;F1(h1)=0;S1(h1)=0;

for i=2:1:s-1; t1(i)=exp(-mu); end

for i=s:1:27; $t1(i) = exp(-mu - (L1(h1) * (1 - tc)))$; end

for i=2:1:28; l1(i)=t1(i-1)*l1(i-1); d1(i,h1)=p(i)*w(i)*l1(i)*250000;
BB1(h1)=BB1(h1)+d1(i,h1); Q1(i,h1)=q(i)*w(i)*l1(i); end
for i=s:1:28 T1(h1)=T1(h1)+Q1(i,h1); end
$n1(h1) = \frac{(c*BB1(h1))}{G}$; (the net reproductive rate at h_1)
if $n1(h1) > 1$ $y1(h1) = \frac{(log(n1(h1))*E0)}{BB1(h1)}$; else y1(h1)=0; end
$NN1(h1) = (exp(-mu) * 2/3) + c * BB1(h1) * exp(\frac{-(y1(h1)*BB1(h1))}{E0})$; (Equilibrium
condition)
A1(h1)=L1(h1)/(mu+L1(h1)); S1(h1)=1-exp(-(mu+L1(h1))*(1-tc));
H1(h1)=A1(h1)*T1(h1)*y1(h1)*S1(h1)*exp(-mu*tc); end

4- Compute the net gain

for i=1:1:2000; $Z(i) = H1(i) - a - b * L1(i)$; end $[W, Y] = max(Z)$;
$Y1 = Y * 10^{-3}$; (the rate of harvesting which maximize the net gain)
YY(count)=W; (maximum net gain)
Y2(count)=Y1; test=abs(oldh-Y1)-0.000005;check condition
h=0.5*Y1+0.5*oldh; if $count > 100$ break; end
count=count+1; end
$a = M1(count - 1)$
$b = M2(count - 1)$
$h_{opt} = Y2(count - 1)$
NH=YY(count-1).

Appendix D

Optimal control program and Near optimal strategy

1- Parameters and Initialization

$n(0) = 10$ (The net reproductive rate)

Time-step=90 (The number of time steps)

$k = 28$; tc=0.25; s=6; (The number of age classes, the duration of no harvesting and the optimal mesh width)

a=9.9657e+007; b=2.4852e+008; $\delta = 0.96$, (The cost parameters a and b with discount factor)

$h_{opt} = 0.401; \alpha = 0.3$,(The optimal harvesting rate from static model,alpha is related to the near optimal strategy)

$E_0 = 4.1 * (10)^{14}; \mu = 0.223$ (Scaling parameter and Mortality)

p(2)=0.01;p(3)=0.13;p(4)=0.63;p(5)=0.96;p(6:k)=1; (The growth of the gonads in s_1 years)

$\sigma_1 = exp(\mu) * \frac{2}{3}; c = 4.0891 \times 10^{-6}$; (The survival rate and the parameter c)

x=zeros(k,time-step-1);lambda=zeros(k,time-step); $h(1 : 10) = 0.02$, $h(11$:time-step-1)= h_{opt}; (Initial guess)

$x(1 : k, 1) = \frac{equilibrium\ value}{4}$; $y(1 : k, 1) = \frac{equilibrium\ value}{4}$; lambda(1:k,time-step)=0 (Initial value and tranversality condition)

2- Compute the weight, fecundity,and price function

for i=1:1:k; if $i < 10$; q(i)=0.18+0.09*i;   else; q(i)=1.08;

$W(i) = 17 * (1 - e^{-0.12(i+2)})^3; F(i) = p(i) * W(i) * 250000$; end;

3- check=1; count=1; $while(check > 0)$;

oldh(1:time-step-1)=h(1:time-step-1); To save the initial guess in oldh

4- Compute the state variables forward $x_i(t)$ by using system 4.1 for all t

for t=2:1:time-step-1;

Tem=0; for i=1:1:k; Tem=Tem+F(i)*x(i,t);end;

E(t)=Tem; (The number of eggs at each t)

$r(t) = exp(\frac{-E(t)}{E_0})$; (The Ricker function)

$x(1, t+1) = \sigma_1 * x(1,t) + c * E(t) * r(t)$; $x(2, t+1) = \frac{exp(-\mu)}{3} * x(1,t)$;

for i=3:1:s; $x(i, t+1) = exp(-\mu) * x(i-1,t)$; end;

for i=s+1:1:k; $x(i, t+1) = exp(-\mu - (1-tc) * h(t)) * x(i-1,t)$; end;

end;

5- Compute the covariables $\lambda_i(t)$ backward from 4.4

for L=time-step-1:-1:1; $\lambda(1, L) = \lambda(1, L+1) * \sigma + \lambda(2, L+1) * t(1)$;

for j=2:1:s-1; $\lambda(j, L) = \lambda(1, L+1) * c * F(j) * r(j) * (1 - \frac{E(L)}{E_o}) + \lambda(j+1, L) * exp(-\mu)$;

end; for j=s:1:L-1; $\lambda(J, L) = h(L) * (\mu + h(L))^{-1} * q(j)W(j) * exp(-\mu * tc) * (1 - exp(-(\mu + h(L)) * (1-tc))) * \delta^L + \lambda(1, L+1) * c * F(j) * r(j) * (1 - \frac{E(L)}{E_o}) + \lambda(j+1, L) * exp(-\mu - h(L) * (1-tc))$; end; $\lambda(k, L) = h(L) * q(k) * W(k) * exp((-\mu * tc)) * ((\mu + h(L))^{-}1) * (1 - exp(-(mu + h(L)) * (1-tc))) * \delta^L + \lambda(1, L+1) * c * F(k) * r(L) * (1 - \frac{E(L)}{E_o})$; end;

6- Define the Hamiltonian as a function of h and find the maximizer of it for each t

for t=1:1:time-step-1;

u(t)=0; A1(t)=0; A3(t)=0; R1(t)=0; R2(t)=0; R3(t)=0; $A3(t) = \lambda(1, t+1) * c * F(2) * x(2,t) * r(t) + (\lambda(2, t+1) * x(1,t) * t(1) + \lambda(1, t+1) * \sigma_1 * x(1,t)$;

for i=3:1:s; $A1(t) = A1(t) + \lambda(1, t+1) * c * F(i) * x(i,t) * r(t) + (\lambda(i, t+1) * x(i-1,t) * exp(-\mu)$; end;

$DDD1(t) = x(s,t) * q(s) * W(s) * exp(-\mu * tc)$;

for i=s+1:1:k; $R1(t) = R1(t) + x(i,t) * q(i) * W(i) * exp(-\mu * tc)$; $R2(t) = R2(t) + \lambda(i, t+1) * x(i-1,t)$; $R3(t) = R3(t) + \lambda(1, t+1) * c * F(i) * x(i,t) * r(t)$; end;

$H = @(N) - (A1(t) + ((R1(t) + DDD1(t)) * (\frac{N}{\mu+N}) * (1 - exp(-(\mu + N) * (1-tc)))) * \delta^t + R2(t) * exp((-\mu - (1-tc) * N)) + R3(t) + A3(t) - (a + b*N) * \delta^t)$; u(t)=fminbnd(H,0,5);end;

7- h=u*0.1+oldh*0.9; (A convex combination to update a new vector at each t)

8- Compute the optimal net yield J for each iterate by using the new h

for t=1:1:time-step-1; Tem1=0;

for j=2:1:k; Tem1=Tem1+F(j)*y(j,t); end

EE(t)=Tem1; $rr(t) = exp(-EE(t)/E0)$; $y(1, t+1) = \sigma_1 * y(1,t) + c * EE(t) * rr(t)$; $y(2, t+1) = \frac{exp(-\mu)}{3} * y(1,t)$;

for i=3:1:s; $y(i, t+1) = exp(-\mu) * y(i-1,t)$; end

for i=s+1:1:k; $y(i, t+1) = exp(-\mu - (1 - tc) * h(t)) * y(i-1, t)$; end

G(t)=0; for j=s:1:k;

$G(t) = G(t) + (q(j) * W(j) * h(t) * y(j,t) * exp(-\mu * tc) * (1 - e^{-(\mu+h(t))*(1-tc)}) * \delta^t * (\mu + h(t))^{-1})$; end $uu1(t) = (a + b * h(t)) * \delta^t$; end

$J_{optc}(count) = sum(G - uu1)$;

9- Test=norm(u-oldh)-0.05*norm(u); check=Test;(Convergence test, one can also use 0.01 or less)

if count> 100 break; end count=count+1; (Account new iterate)

end (This end for loop while in step 3)

10- To compute the net yield for the near optimal solutions initialize as above with

$h_t(1 : 10) = \alpha * h_{opt}; h_t(11:\text{time-step-1})= h_{opt}; xn(1 : k, 1) = \frac{equilibrium\ value}{4}$; for

t=1:1:time-step-1; Tem2=0; for j=2:1:k; Tem2= Tem2+F(j)*xn(j,t); end

En(t)=Tem2; $rn(t) = exp(-En(t)/E0); xn(1, t+1) = \sigma * xn(1, t) + c * En(t) * rn(t)$;

$xn(2, t + 1) = \frac{exp(-\mu)}{3} * xn(1, t)$; for i=3:1:s; $xn(i, t+1) = exp(-\mu) * xn(i - 1, t)$;

end for i=s+1:1:k; $xn(i, t + 1) = (exp(-\mu - (1 - tc) * h_t(t))) * (xn(i - 1, t))$; end

Gn(t)=0; for j=s:1:k; $Gn(t) = Gn(t) + (q(j) * W(j) * h_t(t) * xn(j,t) * exp(-\mu * tc) * (1 - exp(-(\mu + h_t(t)) * (1 - tc))) * \delta^t * (\mu + h_t(t))^{-1})$; end $un(t) = (a + b * h_t(t)) * \delta^t$;

end $J_{nearopt} = sum(Gn - un)$

11- Plot the optimal u and several age classes

for i=1:1:9; subplot(3,3,i) plot(t1,x(3*i,:),'-');

xlabel('time'); ylabel(strcat('population class:-',num2str(3*i))); if i==1

title('$x_i(1) = eq/4'$); else end end figure i=1:1:time-step-2;

plot(i,u(i),'-'); xlabel('time,Iteration=36 times,:$h_{Opt} = 0.401 s_o = 6'$);

ylabel('h_{optc} for cod populations with Ricker function');

title('$x_i(1) = eq/4, bh_{opt}$=0.3H,T=3k,and n(0)=10');

Remark1: One can also find the maximizer of the Hamiltonian in step 6 by another way as following, but we used the way in step 6

for t=1:1:time-step-1; u(t)=0; A2(t)=0; A3(t)=0; R1(t)=0; R2(t)=0; R3(t)=0;DDD1(t)=0;

A3(t)=$\lambda_1(t+1) * c * F_2 * x_2(t) * r(t) + (\lambda_2(t+1) * x_1(t) * exp(-\mu) * \frac{1}{3} + \lambda_1(t+1) * \sigma * x_1(t)$;

for i=3:1:s;

$A2(t) = A2(t) + \lambda_1(t + 1) * c * F_i * x_i(t) * r(t) + (\lambda_i(t + 1) * x_{i-1}(t) * exp(-\mu))$; end

$DDD1(t) = x_s(t) * q_s * W_s * exp(-\mu * tc)$; for i=s+1:1:k;

$R1(t) = R1(t) + (x_i(t) * q_i * W_i * exp(-\mu * tc))$;

$R2(t) = R2(t) + \lambda_i(t+1) * x_{i-1}(t);$

$R3(t) = R3(t) + \lambda_1(t+1) * c * F_i * x_i(t) * r(t);$

end

for e=1:1:5 $\times 10^4$; $N(e) = e * 10^{-4}$;

$H(e,t) = A2(t) + (((R1(t) + DDD1(t)) * (\frac{N(e)}{(mu+N(e))})) * (1 - exp(-(\mu + N(e)) * (1 - tc)))) * \delta^t) + R2(t) * exp(-\mu - (1 - tc) * N(e)) + R3(t) + A3(t) - ((aa + bb * N(e)) * \delta^t);$ (Each column represents the Hamiltonian function at each t

end end

$[rrr, uu] = max(H, [], 1);$ To find the maximum value and maximizer for The Hamiltonian at each t

for t=1:time-step-1

u(t)=uu(t)*0.0001; end.