

STOCHASTIC MODELS
FOR
BIOLOGICAL SYSTEMS

Doctoral Dissertation

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Synoptic view of the study (Abstract)

The aim of this thesis is to define and study stochastic models of repairable systems and the application of these models to biological systems, especially for cell survival after irradiation with ionizing radiation. The study is divided into four chapters:

The first chapter is considered as an introductory one. In it, basics of renewal processes and reliability theory are defined, for example, renewal process (renewal sequence, renewal time, renewal function and renewal equation), lifetime distributions and failure rates, availability, failure frequency of Markov processes and the supplementary variable technique, Markov renewal process, Markov renewal function, Markov renewal kernel, semi-Markov process, and jump time process.

The second chapter consists of six sections. The first section provides information on stochastic models on repairable systems.

Then in the next section **2.2**, we will study two-unit systems with exponential distribution function of failure and repair time. This section consists of three subsections: in the first subsection, assumptions and states of the system are defined; in subsection **2.2.2**, the system is studied; and in subsection **2.2.3**, availability and special cases of the system are calculated.

In section **2.3**, two-unit systems with general distribution function of repair time and

exponential distribution function of failure time are studied. This section also consists of three subsections: in the first subsection, assumptions and states of the system are defined; in subsection **2.3.2**, the system is studied; and finally in subsection **2.3.3**, special cases and numerical examples of the system are given.

In section **2.4**, the availability characteristics of a two-unit repairable system (series and parallel) with two types of failures are investigated using the supplementary variables method. We study the series and parallel system including an availability analysis.

In section **2.5**, a three-unit model with two repair facilities is investigated; we show how to obtain the undetermined functions, when the supplementary variables method is used. This model is one of the important ones we often encounter in reliability applications. It's difficult to analyze if there some of the random variables have a general distribution. For the model considered here, some of the system equations involve two hazard functions.

In the first subsection the assumptions of the general system are defined, in subsection **2.5.2** the states and equations of the system are presented, in subsection **2.5.3** the solution of the equations of the system is calculated, in subsection **2.5.4** some special cases of the system are investigated, and in subsection **2.5.5** the availability of the system is studied.

In section **2.6**, an exponential limit theorem for two alternating renewal processes is proved. For a two-unit parallel system with alternating operating and repair intervals we consider the first time both units are under repair and show that it is systematically exponential.

The following chapter is concerned with the general structure and functions of cells and the radiation effect. We explain the general structure of the cell.

The radiation effect on the cell is described; this section consists of three subsections. In subsection **3.3.1**, radiation sources are illustrated, the first source is ionizing radiation

(*alpha-*, *beta-*, and *gamma-* radiation). Ionizing radiation exposure can occur from a radiation source outside of the body (external radiation) or as a result of taking radioactive material into the body (internal radiation). The second source is optical radiation, and the third source is non-ionizing radiation. In subsection **3.3.2**, the two theories which prevail in the field of radiobiology (target theory and absorption of radiation) are explained. In subsection **3.3.3**, the radiation effects on cell constituents are considered.

Finally, the last chapter consists of four sections. The first section serves as an introduction and presents the literature on stochastic models for cell survival after irradiation with ionizing radiation.

In section **4.2**, a stochastic two compartment model for cell survival after irradiation is studied. The core of the model is the assumption that once a compartment of the cell is repaired, it will behave like a normal compartment when irradiated further. The probability of survival or damage of an irradiated cell at an arbitrary time instant has been obtained assuming general repair time distributions, which may differ for the two compartments. The availability of the model is also computed. The special cases of exponential repair times and some numerical examples are presented.

In section **4.3**, a n compartments model for cell survival after ionizing irradiation is studied. In this model the cell consists of n regions with different sensitivities; we take into account recovery phenomena with general repair-time and damage distributions, which differ for each region. The probability that the cell is damaged and the mean life-time of the cell are obtained. The special cases are given.

In the final section, a semi-Markovian model for the behavior of a living cell exposed to radiations is studied. We obtain various characteristics of interest pertaining to the cell behavior, for example the probabilities of the cell being in different states and the expected time spent in each state.

Chapter 1

Introduction

In this chapter we review some important background material required in the later chapters.

1.1 Basics of renewal processes and reliability theory

Definition 1.1.1. (Renewal process) Karlin and Taylor [27]:-

Let $(X_n, n \geq 1)$ be a sequence of non-negative random variables defined on a probability space $(\Omega, \mathfrak{F}, \mathbb{P})$. We interpret X_n is the time between the $(n - 1)th$ and the nth event. If $X_n = 0$ then the $(n - 1)th$ and the nth event occur simultaneously.

We assume that $(X_n, n \geq 1)$ is a sequence of independent and identically distributed non-negative random variables with the common distribution function $F(x)$.

Let S_0, S_1, S_2, \dots be the random variables defined by

$$\begin{aligned} S_0 &= X_0 = 0, \\ S_{n+1} &= S_n + X_{n+1}; \quad n \geq 0, \end{aligned} \tag{1.1.1}$$

the sequence $S = (S_n; n \in \mathbb{N})$ is called a **renewal sequence**. The times S_n are called **renewal times**.

Let N_t be the counting process of the **renewal process**, i.e.

$$N_t(\omega) = \sup\{n; S_n \leq t\}; \quad \omega \in \Omega. \tag{1.1.2}$$

$F(x)$ is called the **renewal distribution**. Let

$$M(t) = E[N_t] = \sum_{n=1}^{\infty} F_n(t); \quad t \geq 0, \quad (1.1.3)$$

where

$$F_n(x) = \int_0^x F_{n-1}(x-y)dF(y), \quad (1.1.4)$$

and

$$F_1(x) = F(x).$$

The function $M(t)$ is called the **renewal function**. There are a number of other random variables of interest. Three of these are: the **excess life** (also called the excess random variable), the **current life** (also called the age random variable) and the **total life**, defined, respectively, by

$$\begin{aligned} \gamma_t &= S_{N_t+1} - t && \text{(excess or residual lifetime),} \\ \delta_t &= t - S_{N_t} && \text{(current life or age random variable),} \\ \beta_t &= \gamma_t + \delta_t && \text{(total life),} \end{aligned}$$

a pictorial description of these random variables is given in Figure 1.1.

The equation

$$M(t) = F(t) + \int_0^t M(t-s)dF(s); \quad s \leq t, \quad (1.1.5)$$

is called the **renewal equation**.

Definition 1.1.2. (Lifetime distributions and failure rates) Osaki [40]:-

Let X denote the **lifetime** of a system or a unit subject to random failure, which is, of course, a random variable. The distribution of the lifetime until failure is given by:

$$F(t) = Pr\{X \leq t\}; \quad (t \geq 0).$$

The **survival probability** of X is given by

$$\bar{F}(t) = 1 - F(t) = Pr\{X > t\}; \quad (t \geq 0), \quad (1.1.6)$$

which is the probability that the unit survives up to time t . The probability density of X is assumed to exist and given by

$$f(t) = \frac{dF(t)}{dt}; \quad (t \geq 0), \quad (1.1.7)$$

the **failure rate** or **hazard rate** is defined by

$$r(t) = \frac{f(t)}{\bar{F}(t)}; \quad (t \geq 0). \quad (1.1.8)$$

Definition 1.1.3. (Availability $A(t)$) Osaki [40]:-

The **availability** at time t is defined by

$$A(t) = Pr\{\text{a unit is operating at a specified time } t\}. \quad (1.1.9)$$

This availability is referred to as *pointwise* availability or *instantaneous* availability.

Definition 1.1.4. (Failure frequency, Markov process case) Lam [32]:-

Assume that $\{X(t), t \geq 0\}$ is a continuous-time homogeneous Markov chain with state space $S = \{0, 1, 2, \dots\}$. Denote the infinitesimal matrix for the process by $Q = [q_{ij}]$. Then

$$\begin{aligned} p_{ij}(\Delta t) &= Pr\{X(t + \Delta t) = j \mid X(t) = i\} \\ &= \begin{cases} q_{ij}\Delta t + o(\Delta t); & j \neq i, \\ 1 - q_i\Delta t + o(\Delta t); & j = i, \end{cases} \end{aligned} \quad (1.1.10)$$

where $q_i = -q_{ii}$ and $q_{ij} \geq 0$. Let there further be two kinds of states, up states and down states, where W and F represent, respectively, the sets of the up states and down states. If $S = W \cup F$ and $p_i(t) = Pr\{X(t) = i\}$, then the rate of occurrence of failures (ROCOF) at time t is given by

$$m_f(t) = \sum_{i \in W, j \in F} p_i(t)q_{ij}. \quad (1.1.11)$$

Definition 1.1.5. (Failure frequency, supplementary variable case) Lam [32]:-

If a process $\{X(t), t \geq 0\}$ is not Markovian, one may sometimes introduce some supplementary processes $X_1(t), \dots, X_m(t)$, such that $\{(X(t), X_1(t), X_2(t), \dots, X_m(t)), t \geq 0\}$ is a higher-dimensional Markov process.

Assume again that the process $\{(X(t), X_1(t), X_2(t), \dots, X_m(t)), t \geq 0\}$ has two kinds of states only, namely up states and down states, let W and F represent, respectively, the sets of the up states and down states, so that $S = W \cup F$. Let the transition probability given by

$$\begin{aligned}
& p(i, x_1, x_2, \dots, x_m; j, x'_1, x'_2, \dots, x'_m; \Delta t) \\
&= Pr \left\{ (X(t + \Delta t), X_1(t + \Delta t), X_2(t + \Delta t), \dots, X_m(t + \Delta t)) = (j, x'_1, x'_2, \dots, x'_m) \right. \\
&\quad \left. | (X(t), X_1(t), X_2(t), \dots, X_m(t)) = (i, x_1, x_2, \dots, x_m) \right\} \\
&= \begin{cases} q_{ij}(x_1, x_2, \dots, x_m)\Delta t + o(\Delta t); & j \neq i, \\ 1 - q_i(x_1, x_2, \dots, x_m)\Delta t + o(\Delta t); & j = i, \end{cases} \quad (1.1.12)
\end{aligned}$$

where $q_i(x_1, x_2, \dots, x_m) = -q_{ii}(x_1, x_2, \dots, x_m) \geq 0$ and $q_{ij}(x_1, x_2, \dots, x_m) \geq 0$, for $j \neq i$.

The matrix $Q(x_1, x_2, \dots, x_m) = [q_{ij}(x_1, x_2, \dots, x_m)]$ is called the infinitesimal matrix for the Markov process $\{(X(t), X_1(t), X_2(t), \dots, X_m(t)), t \geq 0\}$.

Let $p_i(t, x_1, x_2, \dots, x_m) = Pr \left\{ (X(t), X_1(t), X_2(t), \dots, X_m(t)) = (i, x_1, x_2, \dots, x_m) \right\}$, then the ROCOF at time t is given by

$$m_f(t) = \sum_{i \in W, j \in F} \int_0^\infty \dots \int_0^\infty p_i(t, x_1, x_2, \dots, x_m) q_{ij}(x_1, x_2, \dots, x_m) dx_1 dx_2 \dots dx_m. \quad (1.1.13)$$

Definition 1.1.6. (Markov renewal process) Çinlar [9] and Kohlas [30]:-

Let, for each $n \in \mathbb{N}$, the random variable X_n take values in a countable set E and the random variable T_n taking values in $\mathfrak{R}_+ = [0, +\infty)$ such that $0 = T_0 \leq T_1 \leq T_2 \leq \dots$

The stochastic process $(X, T) = \{X_n, T_n; n \in \mathbb{N}\}$ is said to be a **Markov renewal process** with state space E if

$$\begin{aligned} Pr\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_0, X_1, \dots, X_n; T_0, T_1, \dots, T_n\} \\ = Pr\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n\}, \end{aligned} \quad (1.1.14)$$

for all $n \in \mathbb{N}$, $j \in E$, and $t \in \mathfrak{R}_+$.

We will always assume that (X, T) is time-homogeneous, that is, for any $i, j \in E, t \in \mathfrak{R}_+$,

$$Pr\{X_{n+1} = j, T_{n+1} - T_n \leq t \mid X_n = i\} = Q_{ij}(t), \quad (1.1.15)$$

is independent of n . The family of probabilities

$$Q(t) = \{Q_{ij}(t); i, j \in E, t \in \mathfrak{R}_+\}, \quad (1.1.16)$$

is called a **semi-Markov kernel** over E .

Definition 1.1.7. (Markov renewal function) Çınlar [9]:-

Let $(X, T) = \{X_n, T_n; n \in \mathbb{N}\}$ be a Markov renewal process with a semi-Markov kernel $Q(t)$ over a countable state space E , we will write $P_i\{A\}$ for the conditional probability $Pr\{A \mid X_0 = i\}$ and, similarly, E_i for the conditional expectations given $\{X_0 = i\}$. Define

$$Q_{ij}^n(t) = P_i\{X_n = j, T_n \leq t\}; \quad i, j \in E, t \in \mathfrak{R}_+, \quad (1.1.17)$$

for all $n \in \mathbb{N}$ and

$$Q_{ij}^0(t) = \begin{cases} 1; & \text{if } i = j, \\ 0; & \text{if } i \neq j, \end{cases} \quad (1.1.18)$$

for all $t \geq 0$.

The expected number of renewals in any finite interval is finite, and

$$R_{ij}(t) = \sum_{n=0}^{\infty} P_i\{X_n = j, T_n \leq t\} = \sum_{n=0}^{\infty} Q_{ij}^n(t), \quad (1.1.19)$$

is finite for any $i, j \in E$ and $t < \infty$. The functions $t \rightarrow R_{ij}(t)$ are called **Markov renewal functions** and the collection $R = \{R_{ij}(\cdot); i, j \in E\}$ of these functions is called **Markov renewal kernel**.

Definition 1.1.8. (Semi-Markov processes) Çinlar [9]:-

Let (X, T) be a Markov renewal process with state space E and semi-Markov kernel $Q(t)$. Define

$$L = \sup_n \{T_n\};$$

then L is the lifetime of (X, T) .

Let the process $Y = \{Y_t; t \geq 0\}$ defined by

$$Y_t = \begin{cases} X_n; & \text{if } T_n \leq t < T_{n+1}, \\ \Delta; & \text{if } t \geq L, \end{cases} \quad (1.1.20)$$

where Δ is a point not in E . This continuous-time parameter process is called the **minimal semi-Markov process** associated with (X, T) .

Definition 1.1.9. (Jump time processes):-

Let $t_0 < s_1 < t_1 < s_2 < \dots$ be random variables. Let

$$X(t) = \begin{cases} 1, & t \in [0, s_1); \\ 0, & t \in [s_1, t_1); \\ 1, & t \in [t_1, s_2); \\ 0, & t \in [s_2, t_2); \\ \dots, & \dots \end{cases} \quad (1.1.21)$$

We define the jump time process $R(t)$ by

$$R(t) = \begin{cases} 0; & t \in \bigcup_{i=0}^{\infty} [t_i, s_i), \\ t - s_n; & s_n \leq t < t_n \text{ for some } n, \end{cases} \quad (1.1.22)$$

and describe by Figure 1.2

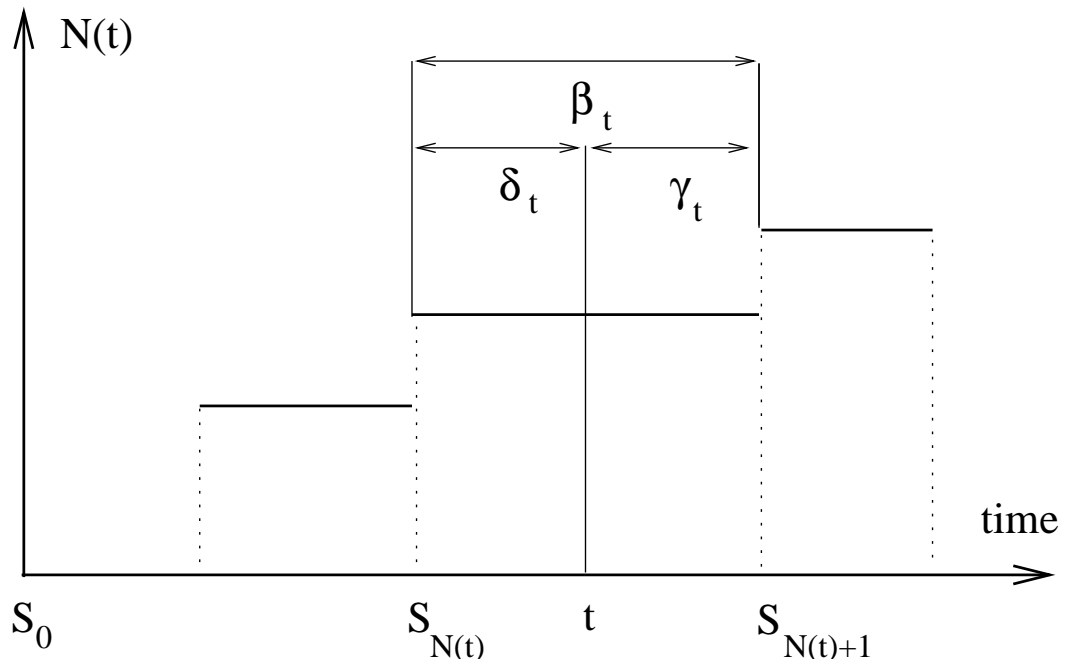


Figure 1.1: The excess life γ_t , the current life δ_t and the total life β_t .

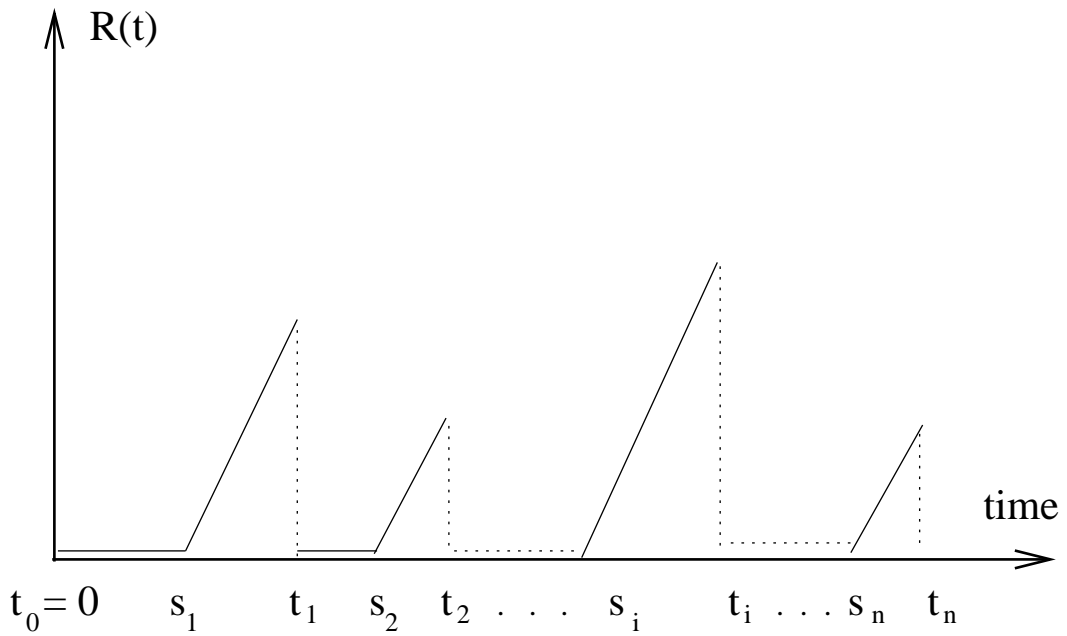


Figure 1.2: The jump time processes $R(t)$.

Chapter 2

Stochastic models of repairable systems

2.1 Introduction and literature

Ever since world war II reliability engineering principles have been applied in many fields, and the study of repairable systems is an important topic in reliability.

There is an extensive literature on availability characteristics of repairable systems with two or three units under varying assumptions on the failures and repairs. Several researchers, including Gaver [14], Srinivasan [45], Osaki and Asakura [41], Osaki and Nakagawa [42], Srinivasan and Gopalan [46], Gopalan and D'Souza ([18] and [17]), Gopalan [16], Li, Alfa and Zhao [35], Dhillon and Rayapati [12], Lam ([31] and [32]), Lam and Zhang ([33] and [34]), Goel, Jaiswal and Gupta [15], Stadje and Zuckerman ([47], [48], [49], [50] and [51]) and the book of Ascher and Feingold [4], have studied stochastic models of repairable systems with several units and repair facilities.

In particular, Li, Alfa and Zhao [35] have studied repair systems with three units and two repair facilities, and (Lam ([31] and [32]) has studied also the two-unit system (series and parallel), defined the rate of occurrence of failures (ROCOF), and obtained the ROCOF in special cases. Osaki and Nakagawa [42] have studied a two-unit standby redundant system

with standby failure.

In most of these papers, exponential distributions are assumed for some system variables and only one type of failure is considered. The methods used in some sections of the existing chapter dealing with non-Markov systems involving many general random variables include *Regenerative point technique* ([23] and [24]) and the *Supplementary variables Method* ([11], [31], [32], [33], [35], [38] and [44]).

Then in the next section **2.2**, we will study two-unit systems with exponential distribution function of failure and repair time. This section consists of three subsections: in the first subsection, assumptions and states of the system are defined; in subsection **2.2.2**, the system is studied; and in subsection **2.2.3**, availability and special cases of the system are calculated.

In section **2.3**, two-unit systems with general distribution function of repair time and exponential distribution function of failure time are studied. This section also consists of three subsections: in the first subsection, assumptions and states of the system are defined; in subsection **2.3.2**, the system is studied; and finally in subsection **2.3.3**, special cases and numerical of the system are given.

In section **2.4**, the availability characteristics of a two-unit repairable system (series and parallel) with two types of failures are investigated using the supplementary variables method. We study the series and parallel system including an availability analysis.

In section **2.5**, a three-unit model with two repair facilities is investigated; we show how to obtain the undetermined functions, when the supplementary variables method is used. This model is one of the important ones we often encounter in reliability applications. It's difficult to analyze if there some of the random variables have a general distribution. For the model considered here, some of the system equations involve two hazard functions. In

the first subsection the assumptions of the general system are defined, in subsection **2.5.2** the states and equations of the system are presented, in subsection **2.5.3** the solution of the equations of the system is calculated, in subsection **2.5.4** some special cases of the system are investigated, and in subsection **2.5.5** the availability of the system is studied.

In section **2.6**, an exponential limit theorem for two alternating renewal processes is proved. For a two-unit parallel system with alternating operating and repair intervals, we consider the first time both units are under repair and show that it is systematically exponential.

2.2 Two-unit system with exponential distributions of failure and repair times

2.2.1 Assumptions of the system

We make the following assumptions:

1. The system consists of two units, at the beginning new units are used, and both are operating. Whenever a unit fails it is repaired.
2. Let $X(t)$ be the state of unit 1 at time t such that:

$$X(t) = \begin{cases} 1; & \text{if unit 1 is operating at time } t, \\ 0; & \text{if unit 1 is under repair at time } t, \end{cases} \quad (2.2.1)$$

with exponential distribution functions with parameters λ_1 and μ_1 for operating and the repair time respectively.

3. Let $Y(t)$ be the state of unit 2 at time t such that:

$$Y(t) = \begin{cases} 1; & \text{if unit 2 is operating at time } t, \\ 0; & \text{if unit 2 is under repair at time } t, \end{cases} \quad (2.2.2)$$

with exponential distribution functions with parameters λ_2 and μ_2 for operating and the repair time respectively.

4. Let $M(t) = (X(t), Y(t))$ be the state of the system at time t , we have the four states of the system $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$.

2.2.2 Study of the system

Under these assumptions, the process $M(t)$ is a finite-state, continuous-time Markov chain and the transition probabilities satisfy:

$$\begin{aligned}
p_{01}(h) &= Pr\{M(t+h) = (0, 1) | M(t) = (0, 0)\} \\
&= \frac{\mu_2}{(\mu_1 + \mu_2)}h + o(h),
\end{aligned} \tag{2.2.3}$$

$$\begin{aligned}
p_{10}(h) &= Pr\{M(t+h) = (0, 0) | M(t) = (0, 1)\} \\
&= \frac{\lambda_2}{(\mu_1 + \lambda_2)}h + o(h),
\end{aligned} \tag{2.2.4}$$

$$\begin{aligned}
p_{02}(h) &= Pr\{M(t+h) = (1, 0) | M(t) = (0, 0)\} \\
&= \frac{\mu_1}{(\mu_1 + \mu_2)}h + o(h),
\end{aligned} \tag{2.2.5}$$

$$\begin{aligned}
p_{20}(h) &= Pr\{M(t+h) = (0, 0) | M(t) = (1, 0)\} \\
&= \frac{\lambda_1}{(\lambda_1 + \mu_2)}h + o(h),
\end{aligned} \tag{2.2.6}$$

$$\begin{aligned}
p_{31}(h) &= Pr\{M(t+h) = (0, 1) | M(t) = (1, 1)\} \\
&= \frac{\lambda_1}{(\lambda_1 + \lambda_2)}h + o(h),
\end{aligned} \tag{2.2.7}$$

$$\begin{aligned}
p_{23}(h) &= Pr\{M(t+h) = (1, 1) | M(t) = (1, 0)\} \\
&= \frac{\mu_2}{(\lambda_1 + \mu_2)}h + o(h),
\end{aligned} \tag{2.2.8}$$

$$\begin{aligned}
p_{32}(h) &= Pr\{M(t+h) = (1, 0) | M(t) = (1, 1)\} \\
&= \frac{\lambda_2}{(\lambda_1 + \lambda_2)}h + o(h),
\end{aligned} \tag{2.2.9}$$

otherwise

$$p_{ij}(h) = o(h); \quad \forall i \neq j, \tag{2.2.10}$$

and

$$\begin{aligned}
 p_{00}(h) &= Pr\{M(t+h) = (0, 0) | M(t) = (0, 0)\} \\
 &= 1 - \left[\frac{\mu_2}{(\mu_1 + \mu_2)} + \frac{\mu_1}{(\mu_1 + \mu_2)} \right] h + o(h) \\
 &= 1 - h + o(h),
 \end{aligned} \tag{2.2.11}$$

also

$$p_{ii}(h) = 1 - h + o(h); \quad i = 0, 1, 2, 3, \tag{2.2.12}$$

such that as $h \rightarrow 0$, we have $p_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta function.

Now we calculate the infinitesimal matrix Q of the Markov chain:

$$q_{ii} = \lim_{h \rightarrow 0^+} \frac{1 - p_{ii}(h)}{h}; \quad i = 0, 1, 2, 3, \tag{2.2.13}$$

and

$$q_{ij} = \lim_{h \rightarrow 0^+} \frac{p_{ij}(h)}{h}; \quad i \neq j. \tag{2.2.14}$$

We have

$$q_{ii} = 1; \quad i = 0, 1, 2, 3,$$

and

$$q_{01} = \frac{\mu_2}{(\mu_1 + \mu_2)};$$

$$q_{10} = \frac{\lambda_2}{(\mu_1 + \lambda_2)};$$

$$q_{02} = \frac{\mu_1}{(\mu_1 + \mu_2)};$$

$$q_{20} = \frac{\lambda_1}{(\lambda_1 + \mu_2)};$$

$$q_{13} = \frac{\mu_1}{(\mu_1 + \lambda_2)};$$

$$q_{31} = \frac{\lambda_1}{(\lambda_1 + \lambda_2)};$$

$$q_{23} = \frac{\mu_2}{(\lambda_1 + \mu_2)};$$

$$q_{32} = \frac{\lambda_2}{(\lambda_1 + \lambda_2)};$$

otherwise

$$q_{ij} = 0; \quad \forall i \neq j.$$

We obtain

$$Q = \begin{pmatrix} -q_{00} & q_{01} & q_{02} & q_{03} \\ q_{10} & -q_{11} & q_{12} & q_{13} \\ q_{20} & q_{21} & -q_{22} & q_{23} \\ q_{30} & q_{31} & q_{32} & -q_{33} \end{pmatrix} = \begin{pmatrix} -1 & \frac{\mu_2}{(\mu_1 + \mu_2)} & \frac{\mu_1}{(\mu_1 + \mu_2)} & 0 \\ \frac{\lambda_2}{(\mu_1 + \lambda_2)} & -1 & 0 & \frac{\mu_1}{(\mu_1 + \lambda_2)} \\ \frac{\lambda_1}{(\lambda_1 + \mu_2)} & 0 & -1 & \frac{\mu_2}{(\lambda_1 + \mu_2)} \\ 0 & \frac{\lambda_1}{(\lambda_1 + \lambda_2)} & \frac{\lambda_2}{(\lambda_1 + \lambda_2)} & -1 \end{pmatrix}. \quad (2.2.15)$$

Let $\pi = (\pi_0, \pi_1, \pi_2, \pi_3)$ be the stationary distribution of the system. Then

$$\sum_{i=0}^3 \pi_i = 1, \quad (2.2.16)$$

and

$$\pi Q = 0. \quad (2.2.17)$$

By equations (2.2.15), (2.2.16) and (2.2.17), we obtain the system of equations

$$\pi_0 = \frac{\lambda_2}{(\lambda_2 + \mu_1)}\pi_1 + \frac{\lambda_1}{(\lambda_1 + \mu_2)}\pi_2, \quad (2.2.18)$$

$$\pi_1 = \frac{\mu_2}{(\mu_1 + \mu_2)}\pi_0 + \frac{\lambda_1}{(\lambda_1 + \lambda_2)}\pi_3, \quad (2.2.19)$$

$$\pi_2 = \frac{\mu_1}{(\mu_1 + \mu_2)}\pi_0 + \frac{\lambda_2}{(\lambda_1 + \lambda_2)}\pi_3, \quad (2.2.20)$$

$$\pi_3 = \frac{\mu_1}{(\lambda_2 + \mu_1)}\pi_1 + \frac{\mu_2}{(\lambda_1 + \mu_2)}\pi_2, \quad (2.2.21)$$

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3. \quad (2.2.22)$$

Solving the linear system (2.2.18)-(2.2.22), we have

$$\pi_0 = \frac{\lambda_1\lambda_2(\mu_1 + \mu_2)}{2\mathbb{R}}, \quad (2.2.23)$$

$$\pi_1 = \frac{\lambda_1\mu_2(\lambda_2 + \mu_1)}{2\mathbb{R}}, \quad (2.2.24)$$

$$\pi_2 = \frac{\lambda_2\mu_1(\lambda_1 + \mu_2)}{2\mathbb{R}}, \quad (2.2.25)$$

$$\pi_3 = \frac{\mu_1\mu_2(\lambda_1 + \lambda_2)}{2\mathbb{R}}, \quad (2.2.26)$$

where

$$\mathbb{R} = \lambda_1\lambda_2\mu_1 + \lambda_1\lambda_2\mu_2 + \lambda_1\mu_1\mu_2 + \lambda_2\mu_1\mu_2.$$

The steady-state probability of two units operating together is given by π_3 , the steady-state probability that both units are under repair is π_0 .

2.2.3 Availability and special cases of the system

The availability of the system, denoted by $A(t)$, is the probability that the system is operating at time t , see definition 1.1.3. Then at least one unit is operating and the steady-state availability of the system, denoted by A , is given by

$$A = \sum_{i=1}^3 \pi_i = \frac{2\mu_1\mu_2(\lambda_1 + \lambda_2) + \lambda_1\lambda_2(\mu_1 + \mu_2)}{2\mathbb{R}}. \quad (2.2.27)$$

Figure 2.1 displays A as a function of λ_1 for a few fixed values of λ_2 .

Let $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$, in equations (2.2.23)- (2.2.26), we have

$$\pi_1 = \pi_2 = \frac{1}{4}, \quad (2.2.28)$$

$$\pi_0 = \frac{\lambda}{2(\lambda + \mu)}, \quad (2.2.29)$$

$$\pi_3 = \frac{\mu}{2(\lambda + \mu)}. \quad (2.2.30)$$

As $t \rightarrow \infty$, the probability of two units operating together at time t converges to π_3 , and the probability of both units being under repair tends to π_0 . The steady-state availability of the system is given by

$$A = \sum_{i=1}^3 \pi_i = \frac{\lambda + 2\mu}{2(\lambda + \mu)}. \quad (2.2.31)$$

2.3 Two-unit system with general distribution function of repair time and exponential distribution function of failure time

2.3.1 Assumptions of the system

We make the following assumptions:

1. A system consists of two units, at the beginning new units are used, and both operating. Whenever a unit fails it can be repaired .
2. Let $X(t)$ describe the state of unit 1 at time t :

$$X(t) = \begin{cases} 1; & \text{if unit 1 is operating at time } t, \\ 0; & \text{if unit 1 is under repair at time } t. \end{cases} \quad (2.3.1)$$

The operating times of unit 1 are exponential with parameter λ , the repair times have an arbitrary distribution function $F_1(t)$ with density $f_1(t)$ and hazard rate $\rho_1(t)$.

3. We define $R_1(t)$ a is the elapsed repair time of unit 1 under repair at time t (the current lifetime of unit 1); $R_1(t)$ is a jump time process (see definition 1.1.9).
4. Let $Y(t)$ describe the state of unit 2 at time t :

$$Y(t) = \begin{cases} 1; & \text{if unit 2 is operating at time } t, \\ 0; & \text{if unit 2 is under repair at time } t. \end{cases} \quad (2.3.2)$$

The operating times of unit 2 are exponential with parameter μ , the repair times have an arbitrary distribution function $F_2(t)$ with density $f_2(t)$ and hazard rate $\rho_2(t)$.

5. We define $R_2(t)$ is the elapsed repair time of unit 2 under repair at time t ; $R_2(t)$ is a jump time process (see definition 1.1.9).

6. Let $M(t) = (X(t), Y(t))$ be the state of the system at time t ; there are four states: $\{(1, 1), (1, 0), (0, 1), (0, 0)\}$.

7. Let $\omega = \{(1, 0), (0, 1)\}$ and define

$$\tau = \inf\{t \geq 0 | M(t) = (0, 0)\}. \quad (2.3.3)$$

2.3.2 Study of the system

Under these assumptions, the probability of the two units operating together at time t without both having simultaneously failed before is

$$p(t) = Pr\{M(t) = (1, 1), M(s) \in \omega \cup \{(1, 1)\} \forall s \leq t\}. \quad (2.3.4)$$

Let

$$Q_1(t, r_1) = \frac{d}{dr_1}\{q_1(t, r_1)\}, \quad (2.3.5)$$

$$Q_2(t, r_2) = \frac{d}{dr_2}\{q_2(t, r_2)\}, \quad (2.3.6)$$

where

$$q_1(t, r_1) = Pr\{M(t) = (1, 0), M(s) \in \omega \cup \{(1, 1)\} \forall s \leq t, R_1(t) \leq r_1\},$$

$$q_2(t, r_2) = Pr\{M(t) = (1, 0), M(s) \in \omega \cup \{(1, 1)\} \forall s \leq t, R_2(t) \leq r_2\};$$

we define

$$Q(t, r_1, r_2) = Q_1(t, r_1) + Q_2(t, r_2). \quad (2.3.7)$$

Then

$$Pr(\tau \in dt) = \left(\int_0^\infty Q_1(t, r_1) \lambda dr_1 + \int_0^\infty Q_2(t, r_2) \mu dr_2 \right) dt, \quad (2.3.8)$$

so that τ has the density function $G(t)$ given by

$$G(t) = \lambda \int_0^\infty Q_1(t, r_1) dr_1 + \mu \int_0^\infty Q_2(t, r_2) dr_2. \quad (2.3.9)$$

These conditions lead to a system of differential equations under the different states of the units. Consider two continuous intervals $(0, t]$ and $(t, t + \epsilon]$ where ϵ is very small. The forward equations for the process may be written as

$$\begin{aligned} p(t + \epsilon) = p(t) & - (\lambda + \mu)p(t)\epsilon + \epsilon \int_0^\infty Q_1(t, r_1)\rho_1(r_1)dr_1 \\ & + \epsilon \int_0^\infty Q_2(t, r_2)\rho_2(r_2)dr_2 + o(\epsilon), \end{aligned} \quad (2.3.10)$$

$$\begin{aligned} Q(t + \epsilon, r_1 + \epsilon, r_2 + \epsilon) & = Q_1(t + \epsilon, r_1 + \epsilon) + Q_2(t + \epsilon, r_2 + \epsilon) \\ & = Q_1(t, r_1) + Q_2(t, r_2) - \epsilon(\lambda + \rho_1(r_1))Q_1(t, r_1) \\ & \quad - \epsilon(\mu + \rho_2(r_2))Q_2(t, r_2) + o(\epsilon). \end{aligned} \quad (2.3.11)$$

As $\epsilon \rightarrow 0$, from equations (2.3.10) and (2.3.11) we conclude that

$$p'(t) = -(\lambda + \mu)p(t) + \int_0^\infty Q_1(t, r_1)\rho_1(r_1)dr_1 + \int_0^\infty Q_2(t, r_2)\rho_2(r_2)dr_2, \quad (2.3.12)$$

$$\begin{aligned} \frac{\partial}{\partial r_1}Q_1(t, r_1) + \frac{\partial}{\partial t}Q_1(t, r_1) + \frac{\partial}{\partial r_2}Q_2(t, r_2) + \frac{\partial}{\partial t}Q_2(t, r_2) \\ = -(\lambda + \rho_1(r_1))Q_1(t, r_1) - (\mu + \rho_2(r_2))Q_2(t, r_2). \end{aligned} \quad (2.3.13)$$

Equation (2.3.12) and (2.3.13) are to be solved subject to boundary conditions.

The first is

$$Q_1(t, 0) = \lambda p(t), \quad (2.3.14)$$

$$Q_2(t, 0) = \mu p(t). \quad (2.3.15)$$

This specifies that as soon as the units of the system enter the states $\{(0, 1), (1, 0)\}$ it goes into the repair process. The second is the initial condition

$$p(0) = 1. \quad (2.3.16)$$

The solution of equations (2.3.12) and (2.3.13) are made easier if they are recast in terms of Laplace transforms.

We have then for equations (2.3.12)-(2.3.15) the transformed equations

$$\begin{aligned} sp^*(s) - p(0) = & -(\lambda + \mu)p^*(s) + \int_0^\infty Q_1^*(s, r_1)\rho_1(r_1)dr_1 \\ & + \int_0^\infty Q_2^*(s, r_2)\rho_2(r_2)dr_2, \end{aligned} \quad (2.3.17)$$

$$\begin{aligned} \frac{\partial}{\partial r_1}Q_1^*(s, r_1) + \frac{\partial}{\partial r_2}Q_2^*(s, r_2) - Q_1(0, r_1) - Q_2(0, r_2) = \\ - (s + \lambda + \rho_1(r_1))Q_1^*(s, r_1) - (s + \mu + \rho_2(r_2))Q_2^*(s, r_2), \end{aligned} \quad (2.3.18)$$

and

$$Q^*(s, 0, 0) = (\lambda + \mu)p^*(s) = Q_1^*(s, 0) + Q_2^*(s, 0), \quad (2.3.19)$$

where

$$Q_1^*(s, 0) = \lambda p^*(s), \quad (2.3.20)$$

$$Q_2^*(s, 0) = \mu p^*(s). \quad (2.3.21)$$

We have $p(0) = 1$ so that $Q_1(0, r_1) = 0$ and $Q_2(0, r_2) = 0$.

$$\begin{aligned} \frac{\partial}{\partial r_1}Q_1^*(s, r_1) + \frac{\partial}{\partial r_2}Q_2^*(s, r_2) = & - (s + \lambda + \rho_1(r_1))Q_1^*(s, r_1) \\ & - (s + \mu + \rho_2(r_2))Q_2^*(s, r_2), \end{aligned} \quad (2.3.22)$$

$$\begin{aligned} (s + \lambda + \mu)p^*(s) = 1 + & \int_0^\infty Q_1^*(s, r_1)\rho_1(r_1)dr_1 \\ & + \int_0^\infty Q_2^*(s, r_2)\rho_2(r_2)dr_2. \end{aligned} \quad (2.3.23)$$

It follows that

$$\frac{\partial}{\partial r_1} Q_1^*(s, r_1) + (s + \lambda + \rho_1(r_1)) Q_1^*(s, r_1) \equiv 0, \quad (2.3.24)$$

$$\frac{\partial}{\partial r_2} Q_2^*(s, r_2) + (s + \mu + \rho_2(r_2)) Q_2^*(s, r_2) \equiv 0. \quad (2.3.25)$$

By integration and simplification, equations (2.3.24) and (2.3.25) become

$$\begin{aligned} Q_1^*(s, r_1) &= (1 - F_1(r_1)) Q_1^*(s, 0) \exp\{-(\lambda + s)r_1\} \\ &= \lambda(1 - F_1(r_1)) P^*(s) \exp\{-(\lambda + s)r_1\}, \end{aligned} \quad (2.3.26)$$

$$\begin{aligned} Q_2^*(s, r_2) &= (1 - F_2(r_2)) Q_2^*(s, 0) \exp\{-(\mu + s)r_2\} \\ &= \mu(1 - F_2(r_2)) P^*(s) \exp\{-(\mu + s)r_2\}, \end{aligned} \quad (2.3.27)$$

and

$$p^*(s) = \frac{1}{[s + \lambda(1 - f_1^*(s + \lambda)) + \mu(1 - f_2^*(s + \mu))]} \quad (2.3.28)$$

By Laplace inversion transform of equations (2.3.26) and (2.3.27), we obtain

$$Q_1(t, r_1) = \lambda(1 - F_1(r_1)) u(t - r_1) p(t - r_1) \exp\{-\lambda r_1\}, \quad (2.3.29)$$

$$Q_2(t, r_1) = \mu(1 - F_2(r_2)) u(t - r_2) p(t - r_2) \exp\{-\mu r_2\}, \quad (2.3.30)$$

where

$$u(t - r) = \begin{cases} 0; & \text{if } t < r, \\ 1; & \text{if } t \geq r, \end{cases} \quad (2.3.31)$$

such that for $t \geq r_1$ and $t \geq r_2$, we have

$$Q_1(t, r_1) = \lambda(1 - F_1(r_1)) p(t - r_1) \exp\{-\lambda r_1\}, \quad (2.3.32)$$

$$Q_2(t, r_1) = \mu(1 - F_2(r_2))p(t - r_2) \exp\{-\mu r_2\}. \quad (2.3.33)$$

In equation (2.3.7), we get

$$\begin{aligned} Q(t, r_1, r_2) &= \lambda(1 - F_1(r_1))p(t - r_1) \exp\{-\lambda r_1\} \\ &\quad + \mu(1 - F_2(r_2))p(t - r_2) \exp\{-\mu r_2\}, \end{aligned} \quad (2.3.34)$$

by equations (2.3.32), (2.3.33) and (2.3.9), we find that

$$\begin{aligned} G(t) &= \lambda^2 \int_0^\infty (1 - F_1(r_1))e^{-\lambda r_1} p(t - r_1) dr_1 \\ &\quad + \mu^2 \int_0^\infty (1 - F_2(r_2))e^{-\mu r_2} p(t - r_2) dr_2. \end{aligned} \quad (2.3.35)$$

The expected value $E(\tau)$ is obtained by differentiating $E[e^{-s\tau}]$. By (2.3.8),

$$\begin{aligned} E[e^{-s\tau}] &= \int_0^\infty e^{-st} \left\{ \int_0^\infty Q_1(t, r_1) \lambda dr_1 + \int_0^\infty Q_2(t, r_2) \mu dr_2 \right\} dt \\ &= \lambda^2 p^*(s) \left\{ \frac{1}{(\lambda + s)} - F_1^*(\lambda + s) \right\} \\ &\quad + \mu^2 p^*(s) \left\{ \frac{1}{(\mu + s)} - F_2^*(\mu + s) \right\}. \end{aligned} \quad (2.3.36)$$

It follows that

$$\begin{aligned} E[\tau] &= \lim_{s \rightarrow 0} \frac{-d}{ds} E[e^{-s\tau}] = \lim_{s \rightarrow 0} E[\tau e^{-s\tau}] \\ &= \lim_{s \rightarrow 0} \frac{-d}{ds} \left\{ \left[\lambda^2 \left(\frac{1}{(\lambda + s)} - F_1^*(\lambda + s) \right) \right. \right. \\ &\quad \left. \left. + \mu^2 \left(\frac{1}{(\mu + s)} - F_2^*(\mu + s) \right) \right] p^*(s) \right\}. \end{aligned} \quad (2.3.37)$$

2.3.3 Special cases and numerical calculations

Let the repair time distribution functions $F_i(t)$, $i = 1, 2$, be exponential with parameters α_i , $i = 1, 2$, respectively. Then

$$f_1^*(\lambda + s) = \frac{\alpha_1}{(\lambda + \alpha_1 + s)}, \quad (2.3.38)$$

$$f_2^*(\mu + s) = \frac{\alpha_2}{(\mu + \alpha_2 + s)}, \quad (2.3.39)$$

$$F_1^*(\lambda + s) = \frac{\alpha_1}{(\lambda + \alpha_1 + s)(\lambda + s)}, \quad (2.3.40)$$

and

$$F_2^*(\mu + s) = \frac{\alpha_2}{(\mu + \alpha_2 + s)(\mu + s)}. \quad (2.3.41)$$

In equations (2.3.28), (2.3.36) and (2.3.37) we obtain

$$p^*(s) = \frac{\theta_1}{\theta}, \quad (2.3.42)$$

$$E[e^{-s\tau}] = \frac{\theta_2}{\theta}, \quad (2.3.43)$$

where

$$\theta = s(\lambda + \alpha_1 + s)(\mu + \alpha_2 + s) + \lambda(\lambda + s)(\mu + \alpha_2 + s) + \mu(\mu + s)(\lambda + \alpha_1 + s),$$

$$\theta_1 = (\lambda + \alpha_1 + s)(\mu + \alpha_2 + s),$$

$$\theta_2 = \mu^2(\lambda + \alpha_1 + s) + \lambda^2(\mu + \alpha_2 + s),$$

and we have

$$E[\tau] = \frac{\mu(\lambda + \alpha_1) + \lambda(\mu + \alpha_2) + (\lambda + \alpha_1)(\mu + \alpha_2)}{\mu^2(\lambda + \alpha_1) + \lambda^2(\mu + \alpha_2)}. \quad (2.3.44)$$

Let $\mu = \lambda$ and $\alpha_1 = \alpha_2 = \alpha$. In this case

$$p^*(s) = \frac{(\lambda + \alpha + s)}{s(\lambda + \alpha + s) + 2\lambda(\lambda + s)}, \quad (2.3.45)$$

$$E[e^{-s\tau}] = \frac{2\lambda^2}{s(\lambda + \alpha + s) + 2\lambda(\lambda + s)}, \quad (2.3.46)$$

and

$$E[\tau] = \frac{3\lambda + \alpha}{2\lambda^2}. \quad (2.3.47)$$

We consider two numerical examples from [31]. The parameter set in Example 1 is $\lambda = 1.0$, $\mu = 2.0$, $\alpha_1 = 4.0$ and $\alpha_2 = 3.0$, and in Example 2 $\lambda = 2.0$, $\mu = 1.0$, $\alpha_1 = 5.0$ and $\alpha_2 = 3.0$. By (2.3.44) the expected first time at which both units are under repair is $E[\tau] = 1.6$ and $E[\tau] = 1.86957$, respectively.

The probability of the two units operating together at time t is given by (2.3.42) and depicted in Figure 2.2.

Now consider (2.3.45)- (2.3.47) for $\lambda = 1.0$ and $\alpha = 4.0$, and $\lambda = 2.0$ and $\alpha = 5.0$. The expected first time at which both units are not operating is $E[\tau] = 3.5$ or $E[\tau] = 1.375$, respectively.

The probability of the two units operating together at time t is given by Figure 2.3.

2.4 Stochastic analysis of a repairable two-unit system with two types of failures

2.4.1 Assumptions of the system

We consider the following assumptions:

1. A system consists of two units and one repairman. At the beginning new units are used so that both are operating. Whenever a unit fails either due to type I failures or due to type II failures it can be repaired by the repairman.
2. A repaired unit is as good as a new one.
3. The probability that a repair of unit i is due to type I (II) failure is p_i (q_i), where $p_i + q_i = 1$.
4. Let X_{in} be the operating time of unit i after its $(n - 1)$ th repair (due to type I or type II failures), $i = 1, 2$. Then the sequences $\{X_{in}, n = 1, 2, \dots\}$ are independent and identically distributed random variables; each X_{in} has an exponential distribution function $F_i(x)$, $i = 1, 2$, with density function

$$f_i(x) = \begin{cases} \lambda_i \exp\{-\lambda_i x\}; & \text{if } x > 0, \\ 0; & \text{otherwise.} \end{cases} \quad (2.4.1)$$

5. Let Y_{in} be the repair times due to type I failures of unit i , $i = 1, 2$. The sequence $\{Y_{in}, n = 1, 2, \dots\}$ is i.i.d., Y_{in} has distribution function $G_i(y)$ with density function $g_i(y)$ and

$$\sigma_i = \int_0^{\infty} yg_i(y) dy < \infty. \quad (2.4.2)$$

6. Let Z_{in} be the repair times due to type II failures of unit i , $i = 1, 2$. The sequence $\{Z_{in}, n = 1, 2, \dots\}$ is i.i.d., each Z_{in} has distribution function $K_i(z)$ with density function $k_i(z)$ and

$$\delta_i = \int_0^{\infty} z k_i(z) dz < \infty. \quad (2.4.3)$$

7. The sequences $\{X_{1n}\}$, $\{X_{2n}\}$, $\{Y_{1n}\}$, $\{Y_{2n}\}$, $\{Z_{1n}\}$ and $\{Z_{2n}\}$ are all independent.

2.4.2 The series system

Now, we discuss the models for a series system. In this case, the system is in an up state if and only if both units are operating. Whenever a unit fails it is either repaired due to type I or type II failures, and the system as a whole fails.

2.4.2.1 States and equations of the system

Let $X(t)$ be the state of the system at time t . Then, there are five states:

State 0 means the two units are both operating.

State 1 means unit 1 is under repair due to type I failure.

State 2 means unit 2 is under repair due to type I failure.

State 3 means unit 1 is under repair due to type II failure.

State 4 means unit 2 is under repair due to type II failure.

Transitions among the states are shown in Figure 2.4.

Now, let $W = \{0\}$ (the up state) and $F = \{1, 2, 3, 4\}$ (the down state), then $S = W \cup F$ is the state space. Furthermore, it is clear from Assumptions 3 and 4 that $\{X(t), t \geq 0\}$ is not a Markov chain. However, it can be extended to a two-dimensional Markov process by introducing two supplementary variables ([11], [31], [32], [33], [35], [38] and [44]). Suppose

that unit i is under repair due to type I or type II failures, and let the repair time spent since the beginning of repair due to type I and type II failures be $Y_i(t)$ and $Z_i(t)$, respectively.

Then define

$$U(t) = \begin{cases} X(t); & X(t)=0, \\ (X(t), Y_1(t)); & X(t)=1, \\ (X(t), Y_2(t)); & X(t)=2, \\ (X(t), Z_1(t)); & X(t)=3, \\ (X(t), Z_2(t)); & X(t)=4, \end{cases} \quad (2.4.4)$$

$\{U(t), t \geq 0\}$ is a Markov process.

Furthermore, let

$$P_0(t) = Pr\{X(t) = 0\}, \quad (2.4.5)$$

$$P_i(t, x)dx = Pr\{X(t) = i, x < Y_i(t) \leq x + dx\}; \quad i = 1, 2, \quad (2.4.6)$$

and

$$P_i(t, x)dx = Pr\{X(t) = i, x < Z_{i-2}(t) \leq x + dx\}; \quad i = 3, 4. \quad (2.4.7)$$

$P_i(t, x)$, $i = 1, 2, 3, 4$, is defined only for $0 \leq x < t$; we set $P_i(t, x) = 0$, for $x \geq t$, $i = 1, 2, 3, 4$.

We need the hazard functions of $G_i(y)$ and $K_i(z)$:

$$\begin{aligned} r_i(y)\Delta t &= Pr\{y < Y_i(t) \leq y + \Delta t | Y_i(t) = y\} \\ &= \frac{g_i(y)\Delta t}{1 - G_i(y)}; \quad i = 1, 2, \end{aligned} \quad (2.4.8)$$

$$\begin{aligned} r_i(z)\Delta t &= Pr\{z < Z_{i-2}(t) \leq z + \Delta t | Z_{i-2}(t) = z\} \\ &= \frac{k_{i-2}(z)\Delta t}{1 - K_{i-2}(z)}; \quad i = 3, 4. \end{aligned} \quad (2.4.9)$$

Then the infinitesimal matrix of the process is given by

$$\begin{aligned}
Q(x) &= [q_{ij}(x)]_{\{i,j=0,\dots,4\}} \\
&= \begin{pmatrix}
-(\lambda_1 + \lambda_2) & p_1\lambda_1 & p_2\lambda_2 & q_1\lambda_1 & q_2\lambda_2 \\
r_1(x) & -r_1(x) & 0 & 0 & 0 \\
r_2(x) & 0 & -r_2(x) & 0 & 0 \\
r_3(x) & 0 & 0 & -r_3(x) & 0 \\
r_4(x) & 0 & 0 & 0 & -r_4(x)
\end{pmatrix}. \tag{2.4.10}
\end{aligned}$$

Consider the two real intervals $(0, t]$ and $(t, t + h]$ where h is very small. The forward equations for the model can be written as

$$\begin{aligned}
P_0(t+h) &= P_0(t) \{ (1 - p_1\lambda_1 h)(1 - q_1\lambda_1 h)(1 - p_2\lambda_2 h)(1 - q_2\lambda_2 h) \} \\
&+ \int_0^\infty [P_1(t, x)r_1(x) + P_2(t, x)r_2(x) \\
&+ P_3(t, x)r_3(x) + P_4(t, x)r_4(x)] h dx + o(h), \tag{2.4.11}
\end{aligned}$$

$$P_i(x+h, t+h) = P_i(x, t)(1 - r_i(x)h) + o(h); \quad x > 0, i = 1, 2, 3, 4, \tag{2.4.12}$$

As $h \rightarrow 0$ in equations (2.4.11) and (2.4.12), we arrive at the following differential equations:

$$\begin{aligned}
\frac{dP_0(t)}{dt} &= -(\lambda_1 + \lambda_2)P_0(t) + \int_0^\infty [P_1(t, x)r_1(x) + P_2(t, x)r_2(x) \\
&+ P_3(t, x)r_3(x) + P_4(t, x)r_4(x)] dx, \tag{2.4.13}
\end{aligned}$$

$$\frac{\partial P_i(t, x)}{\partial t} + \frac{\partial P_i(t, x)}{\partial x} = -r_i(x)P_i(t, x); \quad x > 0, i = 1, 2, 3, 4. \tag{2.4.14}$$

The boundary conditions are

$$P_i(t, 0) = p_i\lambda_i P_0(t); \quad i = 1, 2, \tag{2.4.15}$$

$$P_i(t, 0) = q_{i-2}\lambda_{i-2} P_0(t); \quad i = 3, 4, \tag{2.4.16}$$

and the initial conditions:

$$P_0(0) = 1, \tag{2.4.17}$$

$$P_i(0, x) = 0; \quad i = 1, 2, 3, 4. \quad (2.4.18)$$

Taking the Laplace transforms of the equations (2.4.13)-(2.4.16), it follows that:

$$(s + \lambda_1 + \lambda_2)P_0^*(s) = 1 + \int_0^\infty [P_1^*(s, x)r_1(x) + P_2^*(s, x)r_2(x) + P_3^*(s, x)r_3(x) + P_4^*(s, x)r_4(x)]dx, \quad (2.4.19)$$

$$\frac{\partial P_i^*(s, x)}{\partial x} + (s + r_i(x))P_i^*(s, x) = 0; \quad x > 0, i = 1, 2, 3, 4, \quad (2.4.20)$$

and

$$P_i^*(s, 0) = p_i \lambda_i P_0^*(s); \quad i = 1, 2, \quad (2.4.21)$$

$$P_i^*(s, 0) = q_{i-2} \lambda_{i-2} P_0^*(s); \quad i = 3, 4. \quad (2.4.22)$$

The solutions to equations (2.4.20)-(2.4.22) are given by

$$P_i^*(s, x) = p_i \lambda_i P_0^*(s)(1 - G_i(x))e^{-sx}; \quad x > 0, i = 1, 2, \quad (2.4.23)$$

and

$$P_i^*(s, x) = q_{i-2} \lambda_{i-2} P_0^*(s)(1 - K_{i-2}(x))e^{-sx}; \quad x > 0, i = 3, 4. \quad (2.4.24)$$

Furthermore, with equations (2.4.23) and (2.4.24), it follows from (2.4.19) that

$$P_0^*(s) = \frac{1}{s + \lambda_1[1 - p_1 g_1^*(s) - q_1 k_1^*(s)] + \lambda_2[1 - p_2 g_2^*(s) - q_2 k_2^*(s)]}. \quad (2.4.25)$$

2.4.2.2 Availability analysis of the system

According to the analysis of the series system given above, we can obtain the transient and equilibrium availability characteristics of this system as follows.

(1) Availability of the series system

The availability $A(t)$ of the series system (the probability that the system is operating at time t) is clearly given by

$$A(t) = P_0(t), \quad (2.4.26)$$

and from equation (2.4.25) we have

$$A^*(s) = \frac{1}{s + \lambda_1[1 - p_1g_1^*(s) - q_1k_1^*(s)] + \lambda_2[1 - p_2g_2^*(s) - q_2k_2^*(s)]}, \quad (2.4.27)$$

and the steady-state availability of the series system is given by

$$\begin{aligned} A(\infty) &= \lim_{t \rightarrow \infty} A(t) \\ &= \lim_{s \rightarrow 0} sA^*(s) \\ &= \frac{1}{1 + \lambda_1[p_1\sigma_1 + q_1\delta_1] + \lambda_2[p_2\sigma_2 + q_2\delta_2]}. \end{aligned} \quad (2.4.28)$$

(2) Failure frequency of the series system

By using definition 1.1.4, we have the rate of occurrence of failures (ROCOF) by

$$\begin{aligned} m_f(t) &= \sum_{j=1}^4 P_0(t)q_{0j}(x) \\ &= (p_1\lambda_1 + p_2\lambda_2 + q_1\lambda_1 + q_2\lambda_2)P_0(t) \\ &= (\lambda_1 + \lambda_2)P_0(t). \end{aligned} \quad (2.4.29)$$

For the Laplace transform we find that

$$m_f^*(s) = (\lambda_1 + \lambda_2)P_0^*(s), \quad (2.4.30)$$

and from equation (2.4.25) we have the ROCOF given by

$$m_f^*(s) = \frac{\lambda_1 + \lambda_2}{s + \lambda_1[1 - p_1g_1^*(s) - q_1k_1^*(s)] + \lambda_2[1 - p_2g_2^*(s) - q_2k_2^*(s)]}. \quad (2.4.31)$$

Therefore, we can calculate the ROCOF $m_f(t)$ by inverting its Laplace transform (2.4.31).

By using a Tauberian theorem, the limit of ROCOF (which is the steady-state RO-COF) is given by

$$\begin{aligned} m_f(\infty) &= \lim_{t \rightarrow \infty} m_f(t) \\ &= \lim_{s \rightarrow 0} sm_f^*(s) \\ &= \frac{\lambda_1 + \lambda_2}{1 + \lambda_1[p_1\sigma_1 + q_1\delta_1] + \lambda_2[p_2\sigma_2 + q_2\delta_2]}. \end{aligned} \quad (2.4.32)$$

(3) Renewal frequency of the series system

Let $m_r(t)$ denote the renewal frequency, i.e., the derivative of the expected number of renewals of the system having occurred up to time t . A renewal of the series system means return to the initial state. By using definition 1.1.5, we have

$$\begin{aligned} m_r(t) &= \int_0^\infty \{P_1(t, x)r_1(x) + P_2(t, x)r_2(x) \\ &\quad + P_3(t, x)r_3(x) + P_4(t, x)r_4(x)\} dx. \end{aligned} \quad (2.4.33)$$

Its Laplace transform is

$$\begin{aligned} m_r^*(s) &= \int_0^\infty \{P_1^*(s, x)r_1(x) + P_2^*(s, x)r_2(x) \\ &\quad + P_3^*(s, x)r_3(x) + P_4^*(s, x)r_4(x)\} dx, \end{aligned} \quad (2.4.34)$$

and from equations (2.4.23)-(2.4.25), it follows that

$$m_r^*(s) = \frac{\lambda_1[p_1g_1^*(s) + q_1k_1^*(s)] + \lambda_2[p_2g_2^*(s) + q_2k_2^*(s)]}{s + \lambda_1[1 - p_1g_1^*(s) - q_1k_1^*(s)] + \lambda_2[1 - p_2g_2^*(s) - q_2k_2^*(s)]}. \quad (2.4.35)$$

The steady-state renewal frequency of the series system is given by

$$\begin{aligned} m_r(\infty) &= \lim_{t \rightarrow \infty} m_r(t) \\ &= \lim_{s \rightarrow 0} sm_r^*(s) \\ &= \frac{\lambda_1 + \lambda_2}{1 + \lambda_1[p_1\sigma_1 + q_1\delta_1] + \lambda_2[p_2\sigma_2 + q_2\delta_2]}. \end{aligned} \quad (2.4.36)$$

(4) **Special case**

When the system has only type I failures, i.e., $p_1 = p_2 = 1$, $q_1 = q_2 = 0$, we see that

$$P_0^*(s) = \frac{1}{s + \lambda_1[1 - g_1^*(s)] + \lambda_2[1 - g_2^*(s)]}, \quad (2.4.37)$$

$$A^*(s) = \frac{1}{s + \lambda_1[1 - g_1^*(s)] + \lambda_2[1 - g_2^*(s)]}, \quad (2.4.38)$$

$$m_f^*(s) = \frac{\lambda_1 + \lambda_2}{s + \lambda_1[1 - g_1^*(s)] + \lambda_2[1 - g_2^*(s)]}. \quad (2.4.39)$$

(see also Lam [32]). The renewal frequency is given by

$$m_r^*(s) = \frac{\lambda_1 g_1^*(s) + \lambda_2 g_2^*(s)}{s + \lambda_1[1 - g_1^*(s)] + \lambda_2[1 - g_2^*(s)]}. \quad (2.4.40)$$

2.4.2.3 Example

Now, suppose that the repair times Y_{in} and Z_{in} due to type I and type II failures have exponential distribution functions with parameters μ_i and α_i respectively, $i = 1, 2$. Then

$$\begin{aligned} g_i^*(s) &= \frac{\mu_i}{s + \mu_i}; & i = 1, 2, \\ k_i^*(s) &= \frac{\alpha_i}{s + \alpha_i}; & i = 1, 2, \end{aligned}$$

and we obtain

$$P_0^*(s) = \frac{\bar{h}_1}{\bar{h}}, \quad (2.4.41)$$

$$A^*(s) = \frac{\bar{h}_1}{\bar{h}}, \quad (2.4.42)$$

$$m_f^*(s) = \frac{(\lambda_1 + \lambda_2)\bar{h}_1}{\bar{h}}, \quad (2.4.43)$$

$$m_r^*(s) = \frac{\bar{h}_2}{\bar{h}}, \quad (2.4.44)$$

where

$$\begin{aligned}\bar{h} &= s(\bar{h}_1 + \lambda_1(s + \mu_2)(s + \alpha_2)[s + p_1\alpha_1 + q_1\mu_1] \\ &\quad + \lambda_2(s + \mu_1)(s + \alpha_1)[s + p_2\alpha_2 + q_2\mu_2]), \\ \bar{h}_1 &= (s + \mu_1)(s + \mu_2)(s + \alpha_1)(s + \alpha_2), \\ \bar{h}_2 &= \lambda_1(s + \mu_2)(s + \alpha_2)[\mu_1\alpha_1 + s(q_1\alpha_1 + p_1\mu_1)] \\ &\quad + \lambda_2(s + \mu_1)(s + \alpha_1)[\mu_2\alpha_2 + s(q_2\alpha_2 + p_2\mu_2)].\end{aligned}$$

Plots for the above equations are shown in Figures 2.5, 2.6 and 2.7. These results indicate that the availability $A(t)$ and the failure frequency $m_f(t)$ are decreasing functions quickly approaching their limits. The renewal frequency $m_r(t)$ is unimodal, first quickly increasing and then decreasing to its limit.

These findings are in agreement with those of Lam and Zhang [34].

As $t \rightarrow \infty$, we obtain the limits

$$A(\infty) = \frac{\mu_1\mu_2\alpha_1\alpha_2}{\mu_1\mu_2\alpha_1\alpha_2 + \lambda_1\mu_2\alpha_2[p_1\alpha_1 + q_1\mu_1] + \lambda_2\mu_1\alpha_1[p_2\alpha_2 + q_2\mu_2]}, \quad (2.4.45)$$

$$m_f(\infty) = \frac{\mu_1\mu_2\alpha_1\alpha_2(\lambda_1 + \lambda_2)}{\mu_1\mu_2\alpha_1\alpha_2 + \lambda_1\mu_2\alpha_2[p_1\alpha_1 + q_1\mu_1] + \lambda_2\mu_1\alpha_1[p_2\alpha_2 + q_2\mu_2]}, \quad (2.4.46)$$

$$m_r(\infty) = \frac{\mu_1\mu_2\alpha_1\alpha_2(\lambda_1 + \lambda_2)}{\mu_1\mu_2\alpha_1\alpha_2 + \lambda_1\mu_2\alpha_2[p_1\alpha_1 + q_1\mu_1] + \lambda_2\mu_1\alpha_1[p_2\alpha_2 + q_2\mu_2]}. \quad (2.4.47)$$

When there are only failures of type I ,i.e., $p_1 = p_2 = 1$ or $q_1 = q_2 = 0$, we get

$$A^*(s) = \frac{(s + \mu_1)(s + \mu_2)}{s((s + \mu_1)(s + \mu_2) + \lambda_1(s + \mu_2) + \lambda_2(s + \mu_1))}, \quad (2.4.48)$$

$$m_f^*(s) = \frac{(s + \mu_1)(s + \mu_2)(\lambda_1 + \lambda_2)}{s((s + \mu_1)(s + \mu_2) + \lambda_1(s + \mu_2) + \lambda_2(s + \mu_1))}, \quad (2.4.49)$$

$$m_r^*(s) = \frac{\lambda_1(s + \mu_2)\mu_1 + \lambda_2(s + \mu_1)\mu_2}{s((s + \mu_1)(s + \mu_2) + \lambda_1(s + \mu_2) + \lambda_2(s + \mu_1))}. \quad (2.4.50)$$

2.4.3 The parallel system

Now we discuss the models for a parallel system, in which the system is in an up state if and only if at least one unit is operating. If a unit fails while the other one is repaired (due to either type I or type II failures) the system fails.

2.4.3.1 States and equations of the system

Again, let $X(t)$ be the state of the system at time t . Then there are thirteen states:

State 0 means two units are operating.

State 1 means unit 1 is under repair due to type I failure and unit 2 is operating.

State 2 means unit 2 is under repair due to type I failure and unit 1 is operating.

State 3 means unit 1 is under repair due to type I failure and unit 2 is waiting for repair due to type II failure.

State 4 means unit 2 is under repair due to type I failure and unit 1 is waiting for repair due to type II failure.

State 5 means unit 1 is under repair due to type I failure and unit 2 is waiting for repair due to type I failure.

State 6 means unit 2 is under repair due to type I failure and unit 1 is waiting for repair due to type I failure.

State 7 means unit 1 is under repair due to type II failure and unit 2 is operating.

State 8 means unit 2 is under repair due to type II failure and unit 1 is operating.

State 9 means unit 1 is under repair due to type II failure and unit 2 is waiting for repair due to type I failure.

State 10 means unit 2 is under repair due to type II failure and unit 1 is waiting for repair due to type I failure.

State 11 means unit 1 is under repair due to type II failure and unit 2 is waiting for

repair due to type II failure.

State 12 means unit 2 is under repair due to type II failure and unit 1 is waiting for repair due to type II failure.

Transitions among the states are shown in Figure 2.8.

Now we have $W = \{0, 1, 2, 7, 8\}$, $F = \{3, 4, 5, 6, 9, 10, 11, 12\}$ and $S = W \cup F$. Clearly $\{X(t), t \geq 0\}$ is not a Markov chain.

As in the case of a series system, we introduce supplementary variables ([11], [31], [32], [33], [35], [38] and [44]). Using $Y_i(t)$ and $Z_i(t), i = 1, 2$ as defined above (see subsection 2.4.2), we define

$$U(t) = \begin{cases} X(t); & X(t)=0, \\ (X(t), Y_1(t)); & X(t)=1,3,5, \\ (X(t), Y_2(t)); & X(t)=2,4,6, \\ (X(t), Z_1(t)); & X(t)=7,9,11, \\ (X(t), Z_2(t)); & X(t)=8,10,12. \end{cases} \quad (2.4.51)$$

Then $\{U(t), t \geq 0\}$ is a Markov process. Furthermore, let

$$P_0(t) = Pr\{X(t) = 0\}, \quad (2.4.52)$$

$$P_i(t, x)dx = Pr\{X(t) = i, x < Y(t) \leq x + dx\}; \quad i = 1, 2, 3, 4, 5, 6, \quad (2.4.53)$$

$$Y(t) = \begin{cases} Y_1(t); & i=1,3,5, \\ Y_2(t); & i=2,4,6, \end{cases} \quad (2.4.54)$$

and

$$P_i(t, x)dx = Pr\{X(t) = i, x < Z(t) \leq x + dx\}; \quad i = 7, 8, 9, 10, 11, 12, \quad (2.4.55)$$

$$Z(t) = \begin{cases} Z_1(t); & i=7,9,11, \\ Z_2(t); & i=8,10,12. \end{cases} \quad (2.4.56)$$

We set

$$P_i(t, x) = 0; \quad x \geq t, i = 1, \dots, 12. \quad (2.4.57)$$

Then, the infinitesimal matrix of the process is given by

$$Q(x) = [q_{ij}(x)]_{\{i,j=0,\dots,12\}}$$

$$= \begin{pmatrix} -a & a_1 & a_2 & 0 & 0 & 0 & 0 & a_3 & a_4 & 0 & 0 & 0 & 0 \\ B & -B_1 & 0 & a_4 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ D & 0 & -D_1 & 0 & a_3 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B & 0 & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -D & 0 & 0 & D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B & 0 & 0 & -B & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & D & 0 & 0 & 0 & 0 & -D & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & -b_1 & 0 & a_2 & 0 & a_4 & 0 \\ d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d_1 & 0 & a_1 & 0 & a_3 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 & 0 & -b & 0 & 0 & 0 \\ 0 & d & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -d & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & -b & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & d & 0 & 0 & 0 & 0 & -d \end{pmatrix} \quad (2.4.58)$$

where,

$$\begin{aligned} a &= \lambda_1 + \lambda_2, & a_1 &= p_1 \lambda_1, & a_2 &= p_2 \lambda_2, \\ B_1 &= r_1(x) + \lambda_2, & a_3 &= q_1 \lambda_1, & a_4 &= q_2 \lambda_2, \\ D_1 &= r_2(x) + \lambda_1, & B &= r_1(x), & D &= r_2(x), \\ b_1 &= r_3(x) + \lambda_2, & b &= r_3(x), & d &= r_4(x), \\ d_1 &= r_4(x) + \lambda_1. \end{aligned}$$

Similar to the model for a series system above, consider the two real intervals $(0, t]$ and

$(t, t + h]$ where h is very small. The forward equations for the model can be written as

$$\begin{aligned} P_0(t+h) &= P_0(t) \{ (1 - p_1 \lambda_1 h)(1 - q_1 \lambda_1 h)(1 - p_2 \lambda_2 h)(1 - q_2 \lambda_2 h) \} \\ &+ \int_0^\infty [P_1(t, x)r_1(x) + P_2(t, x)r_2(x) \\ &+ P_7(t, x)r_3(x) + P_8(t, x)r_4(x)] h dx + o(h), \end{aligned} \quad (2.4.59)$$

$$\begin{aligned} P_1(x+h, t+h) &= P_1(x, t) \{ (1 - r_1(x)h)(1 - q_2 \lambda_2 h)(1 - p_2 \lambda_2 h) \} \\ &+ o(h), \end{aligned} \quad (2.4.60)$$

$$\begin{aligned} P_2(x+h, t+h) &= P_2(x, t) \{ (1 - r_2(x)h)(1 - q_1 \lambda_1 h)(1 - p_1 \lambda_1 h) \} \\ &+ o(h), \end{aligned} \quad (2.4.61)$$

$$\begin{aligned} P_7(x+h, t+h) &= P_7(x, t) \{ (1 - r_3(x)h)(1 - q_2 \lambda_2 h)(1 - p_2 \lambda_2 h) \} \\ &+ o(h), \end{aligned} \quad (2.4.62)$$

$$\begin{aligned} P_8(x+h, t+h) &= P_8(x, t) \{ (1 - r_4(x)h)(1 - q_1 \lambda_1 h)(1 - p_1 \lambda_1 h) \} \\ &+ o(h), \end{aligned} \quad (2.4.63)$$

$$P_3(x+h, t+h) = P_3(x, t)(1 - r_1(x)h) + q_2 \lambda_2 h P_1(x, t) + o(h), \quad (2.4.64)$$

$$P_4(x+h, t+h) = P_4(x, t)(1 - r_2(x)h) + q_1 \lambda_1 h P_2(x, t) + o(h), \quad (2.4.65)$$

$$P_5(x+h, t+h) = P_5(x, t)(1 - r_1(x)h) + p_2 \lambda_2 h P_1(x, t) + o(h), \quad (2.4.66)$$

$$P_6(x+h, t+h) = P_6(x, t)(1 - r_2(x)h) + p_1 \lambda_1 h P_2(x, t) + o(h), \quad (2.4.67)$$

$$P_9(x+h, t+h) = P_9(x, t)(1 - r_3(x)h) + p_2 \lambda_2 h P_7(x, t) + o(h), \quad (2.4.68)$$

$$P_{10}(x+h, t+h) = P_{10}(x, t)(1 - r_4(x)h) + p_1 \lambda_1 h P_8(x, t) + o(h), \quad (2.4.69)$$

$$P_{11}(x+h, t+h) = P_{11}(x, t)(1 - r_3(x)h) + q_2 \lambda_2 h P_7(x, t) + o(h), \quad (2.4.70)$$

$$P_{12}(x+h, t+h) = P_{12}(x, t)(1 - r_4(x)h) + q_1 \lambda_1 h P_8(x, t) + o(h). \quad (2.4.71)$$

As $h \rightarrow 0$ in equations (2.4.59)-(2.4.71), we arrive at the following differential equations:

$$\begin{aligned} \frac{dP_0(t)}{dt} = & -(\lambda_1 + \lambda_2)P_0(t) + \int_0^\infty [P_1(t, x)r_1(x) + P_2(t, x)r_2(x) \\ & + P_7(t, x)r_3(x) + P_8(t, x)r_4(x)] dx, \end{aligned} \quad (2.4.72)$$

$$\frac{\partial P_1(t, x)}{\partial t} + \frac{\partial P_1(t, x)}{\partial x} = -(\lambda_2 + r_1(x))P_1(t, x), \quad (2.4.73)$$

$$\frac{\partial P_2(t, x)}{\partial t} + \frac{\partial P_2(t, x)}{\partial x} = -(\lambda_1 + r_2(x))P_2(t, x), \quad (2.4.74)$$

$$\frac{\partial P_7(t, x)}{\partial t} + \frac{\partial P_7(t, x)}{\partial x} = -(\lambda_2 + r_3(x))P_7(t, x), \quad (2.4.75)$$

$$\frac{\partial P_8(t, x)}{\partial t} + \frac{\partial P_8(t, x)}{\partial x} = -(\lambda_1 + r_4(x))P_8(t, x), \quad (2.4.76)$$

$$\frac{\partial P_3(t, x)}{\partial t} + \frac{\partial P_3(t, x)}{\partial x} = q_2\lambda_2P_1(t, x) - r_1(x)P_3(t, x), \quad (2.4.77)$$

$$\frac{\partial P_4(t, x)}{\partial t} + \frac{\partial P_4(t, x)}{\partial x} = q_1\lambda_1P_2(t, x) - r_2(x)P_4(t, x), \quad (2.4.78)$$

$$\frac{\partial P_5(t, x)}{\partial t} + \frac{\partial P_5(t, x)}{\partial x} = p_2\lambda_2P_1(t, x) - r_1(x)P_5(t, x), \quad (2.4.79)$$

$$\frac{\partial P_6(t, x)}{\partial t} + \frac{\partial P_6(t, x)}{\partial x} = p_1\lambda_1P_2(t, x) - r_2(x)P_6(t, x), \quad (2.4.80)$$

$$\frac{\partial P_9(t, x)}{\partial t} + \frac{\partial P_9(t, x)}{\partial x} = p_2\lambda_2P_7(t, x) - r_3(x)P_9(t, x), \quad (2.4.81)$$

$$\frac{\partial P_{10}(t, x)}{\partial t} + \frac{\partial P_{10}(t, x)}{\partial x} = p_1\lambda_1P_8(t, x) - r_4(x)P_{10}(t, x), \quad (2.4.82)$$

$$\frac{\partial P_{11}(t, x)}{\partial t} + \frac{\partial P_{11}(t, x)}{\partial x} = q_2 \lambda_2 P_7(t, x) - r_3(x) P_{11}(t, x), \quad (2.4.83)$$

$$\frac{\partial P_{12}(t, x)}{\partial t} + \frac{\partial P_{12}(t, x)}{\partial x} = q_1 \lambda_1 P_8(t, x) - r_4(x) P_{12}(t, x), \quad (2.4.84)$$

where $x > 0$, with the boundary conditions

$$P_1(t, 0) = p_1 \lambda_1 P_0(t) + \int_0^\infty [P_6(t, x) r_2(x) + P_{10}(t, x) r_4(x)] dx, \quad (2.4.85)$$

$$P_2(t, 0) = p_2 \lambda_2 P_0(t) + \int_0^\infty [P_5(t, x) r_1(x) + P_9(t, x) r_3(x)] dx, \quad (2.4.86)$$

$$P_7(t, 0) = q_1 \lambda_1 P_0(t) + \int_0^\infty [P_4(t, x) r_2(x) + P_{12}(t, x) r_4(x)] dx, \quad (2.4.87)$$

$$P_8(t, 0) = q_2 \lambda_2 P_0(t) + \int_0^\infty [P_3(t, x) r_1(x) + P_{11}(t, x) r_3(x)] dx, \quad (2.4.88)$$

$$P_i(t, 0) = 0; \quad i = 3, 4, 5, 6, 9, 10, 11, 12, \quad (2.4.89)$$

and the initial conditions

$$P_0(0) = 1, \quad (2.4.90)$$

$$P_i(0, x) = 0; \quad i = 1, \dots, 12. \quad (2.4.91)$$

Taking Laplace transforms in the equations above, it follows that:

$$(s + \lambda_1 + \lambda_2) P_0^*(s) = 1 + \int_0^\infty [P_1^*(s, x) r_1(x) + P_2^*(s, x) r_2(x) + P_7^*(s, x) r_3(x) + P_8^*(s, x) r_4(x)] dx, \quad (2.4.92)$$

$$\frac{\partial P_1^*(s, x)}{\partial x} + (s + \lambda_2 + r_1(x)) P_1^*(s, x) = 0, \quad (2.4.93)$$

$$\frac{\partial P_2^*(s, x)}{\partial x} + (s + \lambda_1 + r_2(x)) P_2^*(s, x) = 0, \quad (2.4.94)$$

$$\frac{\partial P_7^*(s, x)}{\partial x} + (s + \lambda_2 + r_3(x))P_7^*(s, x) = 0, \quad (2.4.95)$$

$$\frac{\partial P_8^*(s, x)}{\partial x} + (s + \lambda_1 + r_4(x))P_8^*(s, x) = 0, \quad (2.4.96)$$

$$\frac{\partial P_3^*(s, x)}{\partial x} + (s + r_1(x))P_3^*(s, x) = q_2\lambda_2P_1^*(s, x), \quad (2.4.97)$$

$$\frac{\partial P_4^*(s, x)}{\partial x} + (s + r_2(x))P_4^*(s, x) = q_1\lambda_1P_2^*(s, x), \quad (2.4.98)$$

$$\frac{\partial P_5^*(s, x)}{\partial x} + (s + r_1(x))P_5^*(s, x) = p_2\lambda_2P_1^*(s, x), \quad (2.4.99)$$

$$\frac{\partial P_6^*(s, x)}{\partial x} + (s + r_2(x))P_6^*(s, x) = p_1\lambda_1P_2^*(s, x), \quad (2.4.100)$$

$$\frac{\partial P_9^*(s, x)}{\partial x} + (s + r_3(x))P_9^*(s, x) = p_2\lambda_2P_7^*(s, x), \quad (2.4.101)$$

$$\frac{\partial P_{10}^*(s, x)}{\partial x} + (s + r_4(x))P_{10}^*(s, x) = p_1\lambda_1P_8^*(s, x), \quad (2.4.102)$$

$$\frac{\partial P_{11}^*(s, x)}{\partial x} + (s + r_3(x))P_{11}^*(s, x) = q_2\lambda_2P_7^*(s, x), \quad (2.4.103)$$

$$\frac{\partial P_{12}^*(s, x)}{\partial x} + (s + r_4(x))P_{12}^*(s, x) = q_1\lambda_1P_8^*(s, x), \quad (2.4.104)$$

$$P_1^*(s, 0) = p_1\lambda_1P_0^*(s) + \int_0^\infty [P_6^*(s, x)r_2(x) + P_{10}^*(s, x)r_4(x)]dx, \quad (2.4.105)$$

$$P_2^*(s, 0) = p_2\lambda_2P_0^*(s) + \int_0^\infty [P_5^*(s, x)r_1(x) + P_9^*(s, x)r_3(x)]dx, \quad (2.4.106)$$

$$P_7^*(s, 0) = q_1 \lambda_1 P_0^*(s) + \int_0^\infty [P_4^*(s, x) r_2(x) + P_{12}^*(s, x) r_4(x)] dx, \quad (2.4.107)$$

$$P_8^*(s, 0) = q_2 \lambda_2 P_0^*(s) + \int_0^\infty [P_3^*(s, x) r_1(x) + P_{11}^*(s, x) r_3(x)] dx, \quad (2.4.108)$$

As before, the solutions to equations (2.4.93)-(2.4.96) are given by

$$P_1^*(s, x) = P_1^*(s, 0)(1 - G_1(x))e^{-(s+\lambda_2)x}, \quad (2.4.109)$$

$$P_2^*(s, x) = P_2^*(s, 0)(1 - G_2(x))e^{-(s+\lambda_1)x}, \quad (2.4.110)$$

$$P_7^*(s, x) = P_7^*(s, 0)(1 - K_1(x))e^{-(s+\lambda_2)x}, \quad (2.4.111)$$

$$P_8^*(s, x) = P_8^*(s, 0)(1 - K_2(x))e^{-(s+\lambda_1)x}. \quad (2.4.112)$$

Furthermore, with equations (2.4.109)-(2.4.112), it follows from (2.4.92) that

$$P_0^*(s) = \frac{1 + \aleph_1}{(s + \lambda_1 + \lambda_2)}, \quad (2.4.113)$$

where

$$\aleph_1 = P_1^*(s, 0)g_1^*(s + \lambda_2) + P_2^*(s, 0)g_2^*(s + \lambda_1) + P_7^*(s, 0)k_1^*(s + \lambda_2) + P_8^*(s, 0)k_2^*(s + \lambda_1).$$

Thus from equations (2.4.97)-(2.4.104), we have

$$P_3^*(s, x) = q_2 P_1^*(s, 0)(1 - G_1(x))e^{-sx}[1 - e^{-\lambda_2 x}], \quad (2.4.114)$$

$$P_4^*(s, x) = q_1 P_2^*(s, 0)(1 - G_2(x))e^{-sx}[1 - e^{-\lambda_1 x}], \quad (2.4.115)$$

$$P_5^*(s, x) = p_2 P_1^*(s, 0)(1 - G_1(x))e^{-sx}[1 - e^{-\lambda_2 x}], \quad (2.4.116)$$

$$P_6^*(s, x) = p_1 P_2^*(s, 0)(1 - G_2(x))e^{-sx}[1 - e^{-\lambda_1 x}], \quad (2.4.117)$$

$$P_9^*(s, x) = p_2 P_7^*(s, 0)(1 - K_1(x))e^{-sx}[1 - e^{-\lambda_2 x}], \quad (2.4.118)$$

$$P_{10}^*(s, x) = p_1 P_8^*(s, 0)(1 - K_2(x))e^{-sx}[1 - e^{-\lambda_1 x}], \quad (2.4.119)$$

$$P_{11}^*(s, x) = q_2 P_7^*(s, 0)(1 - K_1(x))e^{-sx}[1 - e^{-\lambda_2 x}], \quad (2.4.120)$$

$$P_{12}^*(s, x) = q_1 P_8^*(s, 0)(1 - K_2(x))e^{-sx}[1 - e^{-\lambda_1 x}]. \quad (2.4.121)$$

On the basis of equations (2.4.113)-(2.4.121), we obtain from (2.4.105)-(2.4.108)

$$P_1^*(s, 0) = \frac{p_1 P_0^*(s) \Delta_1}{\Delta} = \frac{p_1 \Delta_1}{\Lambda}, \quad (2.4.122)$$

$$P_2^*(s, 0) = \frac{p_2 P_0^*(s) \Delta_2}{\Delta} = \frac{p_2 \Delta_2}{\Lambda}, \quad (2.4.123)$$

$$P_7^*(s, 0) = \frac{q_1 P_0^*(s) \Delta_1}{\Delta} = \frac{q_1 \Delta_1}{\Lambda}, \quad (2.4.124)$$

$$P_8^*(s, 0) = \frac{q_2 P_0^*(s) \Delta_2}{\Delta} = \frac{q_2 \Delta_2}{\Lambda}, \quad (2.4.125)$$

$$P_0^*(s) = \frac{\Delta}{\Lambda}, \quad (2.4.126)$$

where

$$\begin{aligned} \Delta &= 1 - (p_2[g_2^*(s) - g_2^*(s + \lambda_1)] + q_2[k_2^*(s) - k_2^*(s + \lambda_1)]) \\ &\quad \cdot (p_1[g_1^*(s) - g_1^*(s + \lambda_2)] + q_1[k_1^*(s) - k_1^*(s + \lambda_2)]), \\ \Delta_1 &= \lambda_1 + \lambda_2(p_2[g_2^*(s) - g_2^*(s + \lambda_1)] + q_2[k_2^*(s) - k_2^*(s + \lambda_1)]), \\ \Delta_2 &= \lambda_2 + \lambda_1(p_1[g_1^*(s) - g_1^*(s + \lambda_2)] + q_1[k_1^*(s) - k_1^*(s + \lambda_2)]), \\ \theta_1 &= (\lambda_1 + \lambda_2)\{1 - (p_1 g_1^*(s) + q_1 k_1^*(s))(p_2 g_2^*(s) + q_2 k_2^*(s))\}, \\ \theta_2 &= \lambda_1(p_1 g_1^*(s + \lambda_2) + q_1 k_1^*(s + \lambda_2))[p_2(1 - g_2^*(s)) + q_2(1 - k_2^*(s))], \\ \theta_3 &= \lambda_2(p_2 g_2^*(s + \lambda_1) + q_2 k_2^*(s + \lambda_1))[p_1(1 - g_1^*(s)) + q_1(1 - k_1^*(s))], \\ \Lambda &= s\Delta + \theta_1 - \theta_2 - \theta_3. \end{aligned}$$

2.4.3.2 Availability analysis of the system

From the above probability analysis of this parallel system, we can obtain the transient and equilibrium availability characteristics of the parallel system as follows.

(1) **Availability of the parallel system**

The availability of the parallel system, denoted by $A_p(t)$, is the probability that the system is operating at time t . Then

$$A_p(t) = P_0(t) + \int_0^\infty \{P_1(t, x) + P_2(t, x) + P_7(t, x) + P_8(t, x)\} dx. \quad (2.4.127)$$

For the Laplace transform we have

$$A_p^*(s) = P_0^*(s) + \int_0^\infty \{P_1^*(s, x) + P_2^*(s, x) + P_7^*(s, x) + P_8^*(s, x)\} dx, \quad (2.4.128)$$

and from equations (2.4.109)-(2.4.112) and (2.4.122)-(2.4.126), we have

$$\begin{aligned} A_p^*(s) &= \frac{\Delta}{\Lambda} + \frac{\Delta_1}{\Lambda(s + \lambda_2)} \{p_1(1 - g_1^*(s + \lambda_2)) + q_1(1 - k_1^*(s + \lambda_2))\} \\ &+ \frac{\Delta_2}{\Lambda(s + \lambda_1)} \{p_2(1 - g_2^*(s + \lambda_1)) + q_2(1 - k_2^*(s + \lambda_1))\}, \end{aligned} \quad (2.4.129)$$

and the steady-state availability of the parallel system, is given by

$$\begin{aligned} A_p(\infty) &= \lim_{t \rightarrow \infty} A_p(t) \\ &= \lim_{s \rightarrow 0} sA_p^*(s). \end{aligned} \quad (2.4.130)$$

(2) **Failure frequency of the parallel system**

The rate of occurrence of failures (ROCOF) is

$$m_f(t) = \sum_{j \in F} \int_0^\infty P_0(t) q_{0j}(x) dx + \sum_{i \in W \setminus \{0\}} \sum_{j \in F} \int_0^\infty P_i(t, x) q_{ij}(x) dx. \quad (2.4.131)$$

We have

$$\begin{aligned} m_f(t) &= (a_4 + a_2) \int_0^\infty [P_1(t, x) + P_7(t, x)] dx \\ &+ (a_3 + a_1) \int_0^\infty [P_2(t, x) + P_8(t, x)] dx, \end{aligned} \quad (2.4.132)$$

$$m_f(t) = \lambda_2 \int_0^\infty [P_1(t, x) + P_7(t, x)] dx + \lambda_1 \int_0^\infty [P_2(t, x) + P_8(t, x)] dx. \quad (2.4.133)$$

By Laplace transform we obtain,

$$m_f^*(s) = \lambda_2 \int_0^\infty [P_1^*(s, x) + P_7^*(s, x)] dx + \lambda_1 \int_0^\infty [P_2^*(s, x) + P_8^*(s, x)] dx, \quad (2.4.134)$$

and from equations (2.4.109)-(2.4.112) and (2.4.122)-(2.4.126), we find that

$$\begin{aligned} m_f^*(s) &= \frac{\lambda_2 \Delta_1}{\Lambda(s + \lambda_2)} \{p_1(1 - g_1^*(s + \lambda_2)) + q_1(1 - k_1^*(s + \lambda_2))\} \\ &+ \frac{\lambda_1 \Delta_2}{\Lambda(s + \lambda_1)} \{p_2(1 - g_2^*(s + \lambda_1)) + q_2(1 - k_2^*(s + \lambda_1))\}. \end{aligned} \quad (2.4.135)$$

Therefore, we can calculate the ROCOF $m_f(t)$ by inversion of the Laplace transform from equation (2.4.135) the steady-state ROCOF is given by

$$\begin{aligned} m_f(\infty) &= \lim_{t \rightarrow \infty} m_f(t) \\ &= \lim_{s \rightarrow 0} s m_f^*(s). \end{aligned} \quad (2.4.136)$$

(3) Renewal frequency of the parallel system:

Let $m_r(t)$ denote the derivative of the expected number of renewals of the system having occurred up to time t . It is called the renewal frequency. A renewal of the parallel system means that the state of the system returns to the initial state. Then

$$\begin{aligned} m_r(t) &= \int_0^\infty \{P_1(t, x)r_1(x) + P_2(t, x)r_2(x) \\ &+ P_7(t, x)r_3(x) + P_8(t, x)r_4(x)\} dx, \end{aligned} \quad (2.4.137)$$

and the Laplace transform is

$$\begin{aligned} m_r^*(s) &= \int_0^\infty \{P_1^*(s, x)r_1(x) + P_2^*(s, x)r_2(x) \\ &+ P_7^*(s, x)r_3(x) + P_8^*(s, x)r_4(x)\} dx. \end{aligned} \quad (2.4.138)$$

From equations (2.4.109)-(2.4.112) and (2.4.122)-(2.4.126) we get

$$m_r^*(s) = \left[\Delta_1 \{p_1 g_1^*(s + \lambda_2) + q_1 k_1^*(s + \lambda_2)\} + \Delta_2 \{p_2 g_2^*(s + \lambda_1) + q_2 k_2^*(s + \lambda_1)\} \right] \Lambda^{-1}. \quad (2.4.139)$$

The steady-state renewal frequency of the parallel system is

$$\begin{aligned} m_r(\infty) &= \lim_{t \rightarrow \infty} m_r(t) \\ &= \lim_{s \rightarrow 0} s m_r^*(s). \end{aligned} \quad (2.4.140)$$

(4) Special case

When the system has only type I failures, i.e., $p_1 = p_2 = 1$, $q_1 = q_2 = 0$, we see that

$$P_1^*(s, 0) = \frac{\lambda_1 + \lambda_2([g_2^*(s) - g_2^*(s + \lambda_1)])}{\Lambda^*}, \quad (2.4.141)$$

$$P_2^*(s, 0) = \frac{\lambda_2 + \lambda_1([g_1^*(s) - g_1^*(s + \lambda_2)])}{\Lambda^*}, \quad (2.4.142)$$

$$P_7^*(s, 0) = P_8^*(s, 0) = 0, \quad (2.4.143)$$

$$P_0^*(s) = \frac{1 - [g_2^*(s) - g_2^*(s + \lambda_1)][g_1^*(s) - g_1^*(s + \lambda_2)]}{\Lambda^*}, \quad (2.4.144)$$

where

$$\begin{aligned} \Lambda^* &= s\{1 - [g_2^*(s) - g_2^*(s + \lambda_1)][g_1^*(s) - g_1^*(s + \lambda_2)]\} \\ &\quad + (\lambda_1 + \lambda_2)\{1 - g_1^*(s)g_2^*(s)\} - \lambda_1 g_1^*(s + \lambda_2)(1 - g_2^*(s)) \\ &\quad - \lambda_2 g_2^*(s + \lambda_1)(1 - g_1^*(s)). \end{aligned}$$

The availability is given by

$$\begin{aligned}
A_p^*(s) &= \{(s + \lambda_1)(1 - g_1^*(s + \lambda_2))(\lambda_1 + \lambda_2[g_2^*(s) - g_2^*(s + \lambda_1)]) \\
&\quad + (s + \lambda_2)(1 - g_2^*(s + \lambda_1))(\lambda_2 + \lambda_1[g_1^*(s) - g_1^*(s + \lambda_2)])\} \\
&\quad \{(s + \lambda_1)(s + \lambda_2)\Lambda^*\}^{-1}, \tag{2.4.145}
\end{aligned}$$

and the ROCOF by its Laplace transform

$$\begin{aligned}
m_f^*(s) &= \{\lambda_2(s + \lambda_1)(1 - g_1^*(s + \lambda_2))(\lambda_1 + \lambda_2[g_2^*(s) - g_2^*(s + \lambda_1)]) \\
&\quad + \lambda_1(s + \lambda_2)(1 - g_2^*(s + \lambda_1))(\lambda_2 + \lambda_1[g_1^*(s) - g_1^*(s + \lambda_2)])\} \\
&\quad \{(s + \lambda_1)(s + \lambda_2)\Lambda^*\}^{-1}. \tag{2.4.146}
\end{aligned}$$

Equation (2.4.146) was obtained by Lam [32]. The Laplace transform of the renewal frequency is

$$\begin{aligned}
m_r^*(s) &= \{\Lambda^*\}^{-1}\{g_1^*(s + \lambda_2)(\lambda_1 + \lambda_2[g_2^*(s) - g_2^*(s + \lambda_1)]) \\
&\quad + g_2^*(s + \lambda_1)(\lambda_2 + \lambda_1[g_1^*(s) - g_1^*(s + \lambda_2)])\}. \tag{2.4.147}
\end{aligned}$$

2.4.3.3 Example

As a special case, suppose that the repair times Y_{in} and Z_{in} due to type I and type II failures, respectively, have an exponential distribution functions with parameters μ_i and α_i , respectively, $i = 1, 2$. Then we obtain that $A_p(\infty)$, $m_f(\infty)$ and $m_r(\infty)$ are given by

$$A_p(\infty) = \frac{\Phi}{\Psi}, \tag{2.4.148}$$

$$m_f(\infty) = \frac{\Phi_1}{\Psi}, \tag{2.4.149}$$

$$m_r(\infty) = \frac{\Phi_2}{\Psi}, \tag{2.4.150}$$

where

$$\begin{aligned}
\Phi &= \mu_1\mu_2\alpha_1\alpha_2\{\lambda_1\lambda_2[p_2\alpha_2 + q_2\mu_2 + \lambda_1][p_1\alpha_1 + q_1\mu_1 + \lambda_2] \\
&\quad + \lambda_2(\alpha_1 + \lambda_2)(\mu_1 + \lambda_2)[p_2\alpha_2 + q_2\mu_2 + \lambda_1] \\
&\quad + \lambda_1(\mu_2 + \lambda_1)(\alpha_2 + \lambda_1)[p_1\alpha_1 + q_1\mu_1 + \lambda_2] \\
&\quad + (\alpha_2 + \lambda_1)(\mu_2 + \lambda_1)(\alpha_1 + \lambda_2)(\mu_1 + \lambda_2)\}, \\
\Phi_1 &= \lambda_1\lambda_2\mu_1\mu_2\alpha_1\alpha_2\{(\lambda_1 + \lambda_2)[p_2\alpha_2 + q_2\mu_2 + \lambda_1][p_1\alpha_1 + q_1\mu_1 + \lambda_2] \\
&\quad + (\alpha_2 + \lambda_1)(\mu_2 + \lambda_1)[p_1\alpha_1 + q_1\mu_1 + \lambda_2] \\
&\quad + (\alpha_1 + \lambda_2)(\mu_1 + \lambda_2)[p_2\alpha_2 + q_2\mu_2 + \lambda_1]\}, \\
\Phi_2 &= \mu_1\mu_2\alpha_1\alpha_2\{[\lambda_1(\mu_2 + \lambda_1)(\alpha_2 + \lambda_1) + \lambda_1\lambda_2(p_2\alpha_2 + q_2\mu_2 + \lambda_1)] \\
&\quad \cdot [\alpha_1\mu_1 + \lambda_2(p_1\mu_1 + q_1\alpha_1)] + [\alpha_2\mu_2 + \lambda_1(p_2\mu_2 + q_2\alpha_2)] \\
&\quad \cdot [\lambda_2(\mu_1 + \lambda_2)(\alpha_1 + \lambda_2) + \lambda_1\lambda_2(p_1\alpha_1 + q_1\mu_1 + \lambda_2)]\}, \\
\Psi &= (\alpha_2 + \lambda_1)(\mu_2 + \lambda_1)(\alpha_1 + \lambda_2)(\mu_1 + \lambda_2)\{\mu_1\mu_2\alpha_1\alpha_2 + (\lambda_1 + \lambda_2) \\
&\quad \cdot [\mu_1\mu_2(q_2\alpha_1 + q_1\alpha_2) + \alpha_1\alpha_2(p_2\mu_1 + p_1\mu_2)]\} \\
&\quad - \lambda_1\lambda_2\mu_1\mu_2\alpha_1\alpha_2[p_1\alpha_1 + q_1\mu_1 + \lambda_2][p_2\alpha_2 + q_2\mu_2 + \lambda_1] \\
&\quad - \lambda_1\mu_1\alpha_1(\alpha_2 + \lambda_1)(\mu_2 + \lambda_1)[p_2\alpha_2 + q_2\mu_2][\mu_1\alpha_1 + \lambda_2(q_1\alpha_1 + p_1\mu_1)] \\
&\quad - \lambda_2\mu_2\alpha_2(\alpha_1 + \lambda_2)(\mu_1 + \lambda_2)[p_1\alpha_1 + q_1\mu_1][\mu_2\alpha_2 + \lambda_1(q_2\alpha_2 + p_2\mu_2)].
\end{aligned}$$

Plots corresponding to the above equations are displayed in Figures 2.9, 2.10 and 2.11. They show the dependence of the steady-state availability $A_p(\infty)$ and of the renewal frequencies $m_f(\infty)$ and $m_r(\infty)$ on λ_2 .

These results are in agreement with Lam and Zhang [34].

For the system with only type I failure, we have $p_1 = p_2 = 1$, $q_1 = q_2 = 0$, and

$$A_p(\infty) = \frac{\mu_1\mu_2\{\mu_1\mu_2 + (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \mu_1 + \mu_2)\}}{\hbar_3}, \quad (2.4.151)$$

$$m_f(\infty) = \frac{\mu_1\mu_2\lambda_1\lambda_2(2\lambda_1 + 2\lambda_2 + \mu_1 + \mu_2)}{\bar{h}_3}, \quad (2.4.152)$$

(this result was obtained by Lam [32]), and

$$m_r(\infty) = \frac{\mu_1\mu_2(\lambda_1 + \lambda_2)(\mu_1\mu_2 + \lambda_1\mu_1 + \lambda_2\mu_2)}{\bar{h}_3}, \quad (2.4.153)$$

where

$$\bar{h}_3 = \lambda_1\mu_2(\lambda_2 + \mu_1)(\lambda_1 + \lambda_2 + \mu_2) + \lambda_2\mu_1(\lambda_1 + \mu_2)(\lambda_1 + \lambda_2 + \mu_1) + \mu_1\mu_2(\mu_1\mu_2 + \lambda_1\mu_1 + \lambda_2\mu_2).$$

2.5 Repairable system with three units and two repair facilities

2.5.1 Assumptions

We assume the following:

1. A system consists of three units and two different repair facilities.
2. Repair facility 1 can repair either failed unit 1 or unit 2, and repair facility 2 is responsible to repairing only unit 3 due to type I or type II failures. .
3. A repaired unit is as good as a new one.
4. The cost of repair due to type I failure is smaller than the cost of repair due to type II failure.
5. The probability of unit 3 breaking down due to type I and type II failures is given by p and q , respectively, where $p + q = 1$.
6. Let X_{in} be the operating time of unit i after its $(n - 1)th$ repair, $i = 1, 2, 3$. Then the sequence $\{X_{in}, n = 1, 2, \dots\}$ is independent and identically distributed common exponential with density function:

$$f_i(x) = \begin{cases} \lambda_i \exp\{-\lambda_i x\}; & \text{if } x > 0, \\ 0; & \text{otherwise,} \end{cases}, \quad i = 1, 2, 3. \quad (2.5.1)$$

7. Let Y_{in} be the repair time of unit i after its nth failure, $i = 1, 2$. The sequence $\{Y_{in}, n = 1, 2, \dots\}, i = 1, 2$ is i.i.d. with distribution $G_i(y), i = 1, 2$, and density function $g_i(y), i = 1, 2$. Let $\mu_i(y), i = 1, 2$ be the hazard rate function of $Y_{in}, i = 1, 2$, so that

$$G_i(y) = 1 - \exp\left\{-\int_0^y \mu_i(t) dt\right\}; \quad i = 1, 2. \quad (2.5.2)$$

8. Let Z_{in} be the repair time due to type I failure (for $i = 1$) or due to type II failure (for $i = 2$) of unit 3 after its $(n + 1)$ th failure. The sequence $\{Z_{in}, n = 1, 2, \dots\}, i = 1, 2$ is i.i.d. with distribution $K_i(z), i = 1, 2$, and density function $k_i(z), i = 1, 2$. Let $\alpha_i(z), i = 1, 2$ be the hazard rate function of $Z_{in}, i = 1, 2$, so that

$$K_i(z) = 1 - \exp\left\{-\int_0^z \alpha_i(t) dt\right\}; \quad i = 1, 2. \quad (2.5.3)$$

9. The sequences $\{X_{1n}\}, \{X_{2n}\}, \{Y_{1n}\}, \{Y_{2n}\}, \{Z_{1n}\}$ and $\{Z_{2n}\}$ are all independent.

2.5.2 The states and equations of the system

The system is in an up state if and only if unit 3 operating and at least one unit of (1 or 2) is operating.

The system is in a down up state If unit 3 has failed or one unit of (1 or 2) failed while the other one is being repaired.

Let $S(t)$ be the state of the system at time t . Then, there are eleven states:

State 0 means three units are operating.

State 1 means unit 1 is under repair and units 2, 3 are operating.

State 2 means unit 2 is under repair and units 1, 3 are operating.

State 3 means unit 3 is under repair due to type I failure and units 1,2 are operating.

State 4 means unit 3 is under repair due to type II failure and units 1,2 are operating.

State 5 means unit 1 is under repair, unit 2 is operating and unit 3 is under repair due to type I failure.

State 6 means unit 1 is under repair, unit 2 is operating and unit 3 is under repair due to type II failure.

State 7 means unit 1 is under repair, unit 2 is waiting for repair and unit 3 is operating.

State 8 means unit 2 is under repair, unit 1 is operating and unit 3 is under repair due to type I failure.

State 9 means unit 2 is under repair, unit 1 is operating and unit 3 is under repair due to type II failure.

State 10 means unit 2 is under repair, unit 1 is waiting for repair and unit 3 is operating. Transitions among the states are shown in Figure 2.12.

Now, let $W = \{0, 1, 2\}$ the set of the up states and F be the set of down states, i.e., $F = \{3, 4, 5, 6, 7, 8, 9, 10\}$; then $S = W \cup F$.

Furthermore, it is clear from Assumptions 7 and 8 that $\{S(t), t \geq 0\}$ is not a Markov process. The elapsed repair time $(Y_1(t), Y_2(t))$ for the units 1, 2 and repair times due to type I and type II failures $(Z_1(t), Z_2(t))$ for unit 3 at time t need to be introduced. Following a standard probabilistic argument (for example, see [10] and [11]), we can show that the process $U(t) = \{S(t), Y_1(t), Y_2(t), Z_1(t), Z_2(t)\}$ defined by

$$U(t) = \begin{cases} S(t); & S(t)=0, \\ (S(t), Y_1(t)); & S(t)=1,7, \\ (S(t), Y_2(t)); & S(t)=2,10, \\ (S(t), Z_1(t)); & S(t)=3, \\ (S(t), Z_2(t)); & S(t)=4, \\ (S(t), Y_1(t), Z_1(t)); & S(t)=5, \\ (S(t), Y_1(t), Z_2(t)); & S(t)=6, \\ (S(t), Y_2(t), Z_1(t)); & S(t)=8, \\ (S(t), Y_2(t), Z_2(t)); & S(t)=9, \end{cases}$$

forms a Markov process. Now let

$$\begin{aligned} P_0(t) &= Pr\{S(t) = 0\}, \\ P_i(t, x)dx &= Pr\{S(t) = i, x < Y_1(t) \leq x + dx\}; \quad i = 1, 7, \\ P_i(t, y)dy &= Pr\{S(t) = i, y < Y_2(t) \leq y + dy\}; \quad i = 2, 10, \\ P_i(t, z)dz &= Pr\{S(t) = i, z < Z(t) \leq z + dz\}; \quad i = 3, 4, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \mu_1(x) + \lambda_2 + \lambda_3, & a_2 &= \mu_1(x) + \alpha_1(z), & a_3 &= \mu_2(y) + \alpha_1(z), \\ b_1 &= \mu_2(y) + \lambda_1 + \lambda_3, & b_2 &= \mu_1(y) + \alpha_2(z), & b_3 &= \mu_2(y) + \alpha_2(z), \\ a &= \lambda_1 + \lambda_2 + \lambda_3. \end{aligned}$$

Similar to the models in Section 2.4 above, we can derive the following differential equations:

$$\begin{aligned} \frac{dP_0(t)}{dt} &= -(\lambda_1 + \lambda_2 + \lambda_3)P_0(t) + \int_0^\infty \left\{ P_1(t, x)\mu_1(x)dx + P_2(t, y)\mu_2(y)dy \right. \\ &\quad \left. + [P_3(t, z)\alpha_1(z) + P_4(t, z)\alpha_2(z)]dz \right\}, \end{aligned} \quad (2.5.4)$$

$$\begin{aligned} \frac{\partial P_1(t, x)}{\partial t} + \frac{\partial P_1(t, x)}{\partial x} &= -(\lambda_2 + \lambda_3 + \mu_1(x))P_1(t, x) \\ &\quad + \int_0^x \left\{ P_5(t, x, z)\alpha_1(z) + P_6(t, x, z)\alpha_2(z) \right\} dz, \end{aligned} \quad (2.5.5)$$

$$\begin{aligned} \frac{\partial P_2(t, y)}{\partial t} + \frac{\partial P_2(t, y)}{\partial y} &= -(\lambda_1 + \lambda_3 + \mu_2(y))P_2(t, y) \\ &\quad + \int_0^y \left\{ P_8(t, y, z)\alpha_1(z) + P_9(t, y, z)\alpha_2(z) \right\} dz, \end{aligned} \quad (2.5.6)$$

$$\begin{aligned} \frac{\partial P_3(t, z)}{\partial t} + \frac{\partial P_3(t, z)}{\partial z} &= -\alpha_1(z)P_3(t, z) + \int_z^\infty \left\{ P_5(t, x, z)\mu_1(x)dx \right. \\ &\quad \left. + P_8(t, y, z)\mu_2(z)dy \right\}, \end{aligned} \quad (2.5.7)$$

$$\begin{aligned} \frac{\partial P_4(t, z)}{\partial t} + \frac{\partial P_4(t, z)}{\partial z} &= -\alpha_2(z)P_4(t, z) + \int_z^\infty \left\{ P_6(t, x, z)\mu_1(x)dx \right. \\ &\quad \left. + P_9(t, y, z)\mu_2(y)dy \right\}, \end{aligned} \quad (2.5.8)$$

where $y, x > 0$, and

$$P_1(t, 0) = \lambda_1 P_0(t) + \int_0^\infty P_{10}(t, y)\mu_2(y)dy, \quad (2.5.9)$$

$$P_2(t, 0) = \lambda_2 P_0(t) + \int_0^\infty P_7(t, x)\mu_1(x)dx, \quad (2.5.10)$$

$$P_3(t, 0) = p\lambda_3 P_0(t), \quad (2.5.11)$$

$$P_4(t, 0) = q\lambda_3 P_0(t), \quad (2.5.12)$$

$$\begin{aligned} \frac{\partial P_5(t, x, z)}{\partial t} + \frac{\partial P_5(t, x, z)}{\partial x} \\ + \frac{\partial P_5(t, x, z)}{\partial z} = -[\mu_1(x) + \alpha_1(z)]P_5(t, x, z), \end{aligned} \quad (2.5.13)$$

$$\begin{aligned} \frac{\partial P_6(t, x, z)}{\partial t} + \frac{\partial P_6(t, x, z)}{\partial x} \\ + \frac{\partial P_6(t, x, z)}{\partial z} = -[\mu_1(x) + \alpha_2(z)]P_6(t, x, z), \end{aligned} \quad (2.5.14)$$

$$\begin{aligned} \frac{\partial P_8(t, y, z)}{\partial t} + \frac{\partial P_8(t, y, z)}{\partial y} \\ + \frac{\partial P_8(t, y, z)}{\partial z} = -[\mu_2(y) + \alpha_1(z)]P_8(t, y, z), \end{aligned} \quad (2.5.15)$$

$$\begin{aligned} \frac{\partial P_9(t, y, z)}{\partial t} + \frac{\partial P_9(t, y, z)}{\partial y} \\ + \frac{\partial P_9(t, y, z)}{\partial z} = -[\mu_2(y) + \alpha_2(z)]P_8(t, y, z), \end{aligned} \quad (2.5.16)$$

$$\frac{\partial P_7(t, x)}{\partial t} + \frac{\partial P_7(t, x)}{\partial x} = \lambda_2 P_1(t, x) - \mu_1(x)P_7(t, x), \quad (2.5.17)$$

$$\frac{\partial P_{10}(t, y)}{\partial t} + \frac{\partial P_{10}(t, y)}{\partial y} = \lambda_1 P_2(t, y) - \mu_2(y)P_{10}(t, y), \quad (2.5.18)$$

with the initial conditions

$$P_0(0) = 1, \quad (2.5.19)$$

and

$$P_i(0, x) = 0; \quad i = 1, 7, \quad (2.5.20)$$

$$P_i(0, y) = 0; \quad i = 2, 10, \quad (2.5.21)$$

$$P_i(0, z) = 0; \quad i = 3, 4, \quad (2.5.22)$$

$$P_i(0, x, z) = 0; \quad i = 5, 6, \quad (2.5.23)$$

$$P_i(0, y, z) = 0; \quad i = 8, 9, \quad (2.5.24)$$

and the boundary conditions

$$P_5(t, x, 0) = p\lambda_3 P_1(t, x), \quad (2.5.25)$$

$$P_6(t, x, 0) = q\lambda_3 P_1(t, x), \quad (2.5.26)$$

$$P_8(t, y, 0) = p\lambda_3 P_2(t, y), \quad (2.5.27)$$

$$P_9(t, y, 0) = q\lambda_3 P_2(t, y), \quad (2.5.28)$$

$$P_i(t, 0) = 0, \quad i = 7, 10. \quad (2.5.29)$$

2.5.3 Solution of the equations

The complements of the distribution functions $G(\cdot)$ and $K(\cdot)$ are denoted by

$$\bar{G}(\cdot) = 1 - G(\cdot),$$

$$\bar{K}(\cdot) = 1 - K(\cdot),$$

and the inverse Laplace transforms of a function $D(u)$ is denoted by

$$L^{-1}[D^*(\eta)] = D(u).$$

Taking Laplace transforms in the equations (2.5.4) -(2.5.18) and (2.5.25)-(2.5.28), it follows that:

$$\begin{aligned} (s + \lambda_1 + \lambda_2 + \lambda_3) P_0^*(s) &= 1 + \int_0^\infty \left\{ P_1^*(t, s) \mu_1(x) dx + P_2^*(s, y) \mu_2(y) dy \right. \\ &\quad \left. + [P_3^*(s, z) \alpha_1(z) + P_4^*(s, z) \alpha_2(z)] dz \right\}, \quad (2.5.30) \end{aligned}$$

$$\begin{aligned}\frac{\partial P_1^*(s, x)}{\partial x} &= -(s + \lambda_2 + \lambda_3 + \mu_1(x))P_1^*(s, x) \\ &+ \int_0^x \left\{ P_5^*(s, x, z)\alpha_1(z) + P_6^*(s, x, z)\alpha_2(z) \right\} dz,\end{aligned}\quad (2.5.31)$$

$$\begin{aligned}\frac{\partial P_2^*(s, y)}{\partial y} &= -(s + \lambda_1 + \lambda_3 + \mu_2(y))P_2^*(s, y) \\ &+ \int_0^y \left\{ P_8^*(s, y, z)\alpha_1(z) + P_9^*(s, y, z)\alpha_2(z) \right\} dz,\end{aligned}\quad (2.5.32)$$

$$\begin{aligned}\frac{\partial P_3^*(s, z)}{\partial z} &= -(s + \alpha_1(z))P_3^*(s, z) \\ &+ \int_z^\infty \left\{ P_5^*(s, x, z)\mu_1(x)dx + P_8^*(s, y, z)\mu_2(z)dy \right\},\end{aligned}\quad (2.5.33)$$

$$\begin{aligned}\frac{\partial P_4^*(s, z)}{\partial z} &= -(s + \alpha_2(z))P_4^*(s, z) \\ &+ \int_z^\infty \left\{ P_6^*(s, x, z)\mu_1(x)dx + P_9^*(s, y, z)\mu_2(y)dy \right\},\end{aligned}\quad (2.5.34)$$

$$P_1^*(s, 0) = \lambda_1 P_0^*(s) + \int_0^\infty P_{10}^*(s, y)\mu_2(y)dy,\quad (2.5.35)$$

$$P_2^*(s, 0) = \lambda_2 P_0^*(s) + \int_0^\infty P_7^*(s, x)\mu_1(x)dx,\quad (2.5.36)$$

$$P_3^*(s, 0) = p\lambda_3 P_0^*(s),\quad (2.5.37)$$

$$P_4^*(s, 0) = q\lambda_2 P_0^*(s),\quad (2.5.38)$$

$$\frac{\partial P_5^*(s, x, z)}{\partial x} + \frac{\partial P_5^*(s, x, z)}{\partial z} + [s + \mu_1(x) + \alpha_1(z)]P_5^*(s, x, z) = 0,\quad (2.5.39)$$

$$\frac{\partial P_6^*(s, x, z)}{\partial x} + \frac{\partial P_6^*(s, x, z)}{\partial z} + [s + \mu_1(x) + \alpha_2(z)]P_6^*(s, x, z) = 0,\quad (2.5.40)$$

$$\frac{\partial P_8^*(s, y, z)}{\partial y} + \frac{\partial P_8^*(s, y, z)}{\partial z} + [s + \mu_2(y) + \alpha_1(z)]P_8^*(s, y, z) = 0,\quad (2.5.41)$$

$$\frac{\partial P_9^*(s, y, z)}{\partial y} + \frac{\partial P_9^*(s, y, z)}{\partial z} + [s + \mu_2(y) + \alpha_2(z)]P_9^*(s, y, z) = 0,\quad (2.5.42)$$

$$\frac{\partial P_7^*(s, x)}{\partial x} + (s + \mu_1(x))P_7^*(s, x) = \lambda_2 P_1^*(s, x), \quad (2.5.43)$$

$$\frac{\partial P_{10}^*(s, y)}{\partial y} + (s + \mu_2(y))P_{10}^*(s, y) = \lambda_1 P_2^*(s, y), \quad (2.5.44)$$

$$P_5^*(s, x, 0) = p\lambda_3 P_1^*(s, x), \quad (2.5.45)$$

$$P_6^*(s, x, 0) = q\lambda_3 P_1^*(s, x), \quad (2.5.46)$$

$$P_8^*(s, y, 0) = p\lambda_3 P_2^*(s, y), \quad (2.5.47)$$

$$P_9^*(s, y, 0) = q\lambda_3 P_2^*(s, y). \quad (2.5.48)$$

As before, the solutions to equations (2.5.39)- (2.5.42) are given by

$$P_5^*(s, x, z) = \bar{G}_1(x)\bar{K}_1(z)H_1(s, x - z)e^{-sx}, \quad (2.5.49)$$

$$P_6^*(s, x, z) = \bar{G}_1(x)\bar{K}_2(z)H_2(s, x - z)e^{-sx}, \quad (2.5.50)$$

$$P_8^*(s, y, z) = \bar{G}_2(y)\bar{K}_1(z)H_3(s, y - z)e^{-sy}, \quad (2.5.51)$$

$$P_9^*(s, y, z) = \bar{G}_2(y)\bar{K}_2(z)H_4(s, y - z)e^{-sy}, \quad (2.5.52)$$

where $H_i(s, u)$ ($i = 1, 2, 3, 4$) are functions to be determined. Substituting equations (2.5.49)-(2.5.52) in equations (2.5.31)-(2.5.34) we obtain

$$P_1^*(s, x) = e^{-(s+\lambda_2+\lambda_3)x}\bar{G}_1(x)C_1(s, x), \quad (2.5.53)$$

$$P_2^*(s, y) = e^{-(s+\lambda_1+\lambda_3)y}\bar{G}_2(y)C_2(s, y), \quad (2.5.54)$$

$$P_3^*(s, z) = e^{-sz}\bar{K}_1(z)C_3(s, z), \quad (2.5.55)$$

$$P_4^*(s, z) = e^{-sz}\bar{K}_2(z)C_4(s, z), \quad (2.5.56)$$

where

$$C_1(s, x) = \int_0^x e^{(\lambda_2+\lambda_3)u} \{k_1(u) * H_1(s, u) + k_2(u) * H_2(s, u)\} du + C_1(s),$$

$$C_2(s, y) = \int_0^y e^{(\lambda_1+\lambda_3)u} \{k_1(u) * H_3(s, u) + k_2(u) * H_4(s, u)\} du + C_2(s),$$

$$C_3(s, z) = \int_0^\infty e^{-su} H_1(s, u) [\overline{G}_1(u) - \overline{G}_1(u+z)] du \\ + \int_0^\infty e^{-su} H_3(s, u) [\overline{G}_2(u) - \overline{G}_2(u+z)] du + C_3(s),$$

$$C_4(s, z) = \int_0^\infty e^{-su} H_2(s, u) [\overline{G}_1(u) - \overline{G}_1(u+z)] du \\ + \int_0^\infty e^{-su} H_4(s, u) [\overline{G}_2(u) - \overline{G}_2(u+z)] du + C_4(s),$$

and the functions $C_i(s)$ ($i = 1, 2, 3, 4$) are still to be determined. Combining (2.5.53) and (2.5.54) with (2.5.43) and (2.5.44), we see that

$$P_7^*(s, x) = \lambda_2 e^{-sx} \overline{G}_1(x) \int_0^x e^{-(\lambda_2 + \lambda_3)u} C_1(s, u) du, \quad (2.5.57)$$

$$P_{10}^*(s, y) = \lambda_1 e^{-sy} \overline{G}_2(y) \int_0^y e^{-(\lambda_1 + \lambda_3)u} C_2(s, u) du. \quad (2.5.58)$$

Using equations (2.5.53)-(2.5.56), and noticing (2.5.30), we find that

$$P_0^*(s) = \frac{1}{(s + \lambda_1 + \lambda_2 + \lambda_3)} \left\{ 1 \right. \\ \left. + \int_0^\infty e^{-(s + \lambda_2 + \lambda_3)x} g_1(x) C_1(s, x) dx + \int_0^\infty e^{-(s + \lambda_1 + \lambda_3)y} g_2(y) C_2(s, y) dy \right. \\ \left. + \int_0^\infty e^{-sz} k_1(z) C_3(s, z) dz + \int_0^\infty e^{-sz} k_2(z) C_4(s, z) dz \right\}. \quad (2.5.59)$$

Therefore, to obtain the explicit solution for equations (2.5.49)-(2.5.52) to (2.5.59), we have to determine $H_i(s, u)$ and $C_i(s)$ ($i = 1, 2, 3, 4$).

Lemma 2.5.1. *The functions $H_i(s, u)$, ($i = 1, 2, 3, 4$) can be expressed as products of $C_j(s)$ ($j = 1, 2$) and functions of u . We have*

$$H_1(s, u) = p\lambda_3 C_1(s) D_1(u), \quad (2.5.60)$$

$$H_2(s, u) = q\lambda_3 C_1(s) D_1(u), \quad (2.5.61)$$

$$H_3(s, u) = p\lambda_3 C_2(s) D_2(u), \quad (2.5.62)$$

$$H_4(s, u) = q\lambda_3 C_2(s) D_2(u), \quad (2.5.63)$$

where

$$D_1(u) = L^{-1} \left[\frac{1}{\lambda_2 + \eta \{1 + \lambda_3 [p\overline{K}_1^*(\eta) + q\overline{K}_2^*(\eta)]\}} \right],$$

$$D_2(u) = L^{-1} \left[\frac{1}{\lambda_1 + \eta \{1 + \lambda_3 [p\overline{K}_1^*(\eta) + q\overline{K}_2^*(\eta)]\}} \right],$$

Proof. Substitute (2.5.49)-(2.5.52) in the equations (2.5.45) - (2.5.48). This yields

$$H_1(s, x)e^{(\lambda_2+\lambda_3)x} = p\lambda_3 \left\{ C_1(s) + \int_0^x e^{(\lambda_2+\lambda_3)u} \{k_1(u) * H_1(s, u) + k_2(u) * H_2(s, u)\} du \right\},$$

$$H_2(s, x)e^{(\lambda_2+\lambda_3)x} = q\lambda_3 \left\{ C_1(s) + \int_0^x e^{(\lambda_2+\lambda_3)u} \{k_1(u) * H_1(s, u) + k_2(u) * H_2(s, u)\} du \right\},$$

$$H_3(s, y)e^{(\lambda_1+\lambda_3)y} = p\lambda_3 \left\{ C_2(s) + \int_0^y e^{(\lambda_1+\lambda_3)u} \{k_1(u) * H_3(s, u) + k_2(u) * H_4(s, u)\} du \right\},$$

$$H_4(s, y)e^{(\lambda_1+\lambda_3)y} = q\lambda_3 \left\{ C_2(s) + \int_0^y e^{(\lambda_1+\lambda_3)u} \{k_1(u) * H_3(s, u) + k_2(u) * H_4(s, u)\} du \right\}.$$

Thus,

$$H_1(s, 0) = p\lambda_3 C_1(s), \quad H_2(s, 0) = q\lambda_3 C_1(s),$$

$$H_3(s, 0) = p\lambda_3 C_2(s), \quad H_4(s, 0) = q\lambda_3 C_2(s),$$

and taking derivatives we obtain

$$H_1(s, x)(\lambda_2 + \lambda_3) + \frac{\partial}{\partial x} H_1(s, x) = p\lambda_3 \{k_1(x) * H_1(s, x) + k_2(x) * H_2(s, x)\},$$

$$H_2(s, x)(\lambda_2 + \lambda_3) + \frac{\partial}{\partial x} H_2(s, x) = q\lambda_3 \{k_1(x) * H_1(s, x) + k_2(x) * H_2(s, x)\},$$

$$H_3(s, y)(\lambda_1 + \lambda_3)y + \frac{\partial}{\partial y} H_3(s, y) = p\lambda_3 \{k_1(y) * H_3(s, y) + k_2(y) * H_4(s, y)\},$$

$$H_4(s, y)(\lambda_1 + \lambda_3)y + \frac{\partial}{\partial y} H_4(s, y) = q\lambda_3 \{k_1(y) * H_3(s, y) + k_2(y) * H_4(s, y)\}.$$

By taking Laplace transforms in the above equations with respect to x and y , we get

$$(\lambda_2 + \lambda_3 + \eta)H_1^*(s, \eta) - H_1(s, 0) = p\lambda_3 \{k_1^*(\eta) * H_1^*(s, \eta) + k_2^*(\eta) * H_2^*(s, \eta)\},$$

$$(\lambda_2 + \lambda_3 + \eta)H_2^*(s, \eta) - H_2(s, 0) = q\lambda_3 \{k_1^*(\eta) * H_1^*(s, \eta) + k_2^*(\eta) * H_2^*(s, \eta)\},$$

$$(\lambda_1 + \lambda_3 + \eta)H_3^*(s, \eta) - H_3(s, 0) = p\lambda_3 \{k_1^*(\eta) * H_3^*(s, \eta) + k_2^*(\eta) * H_4^*(s, \eta)\},$$

$$(\lambda_1 + \lambda_3 + \eta)H_4^*(s, \eta) - H_4(s, 0) = q\lambda_3 \{k_1^*(\eta) * H_3^*(s, \eta) + k_2^*(\eta) * H_4^*(s, \eta)\}.$$

The solutions are

$$H_1^*(s, \eta) = \frac{p\lambda_3 C_1(s)}{\lambda_2 + \eta[1 + \lambda_3\{pk_1^*(\eta) + qk_2^*(\eta)\}]},$$

$$H_2^*(s, \eta) = \frac{q\lambda_3 C_1(s)}{\lambda_2 + \eta[1 + \lambda_3\{pk_1^*(\eta) + qk_2^*(\eta)\}]},$$

$$H_3^*(s, \eta) = \frac{p\lambda_3 C_2(s)}{\lambda_1 + \eta[1 + \lambda_3\{pk_1^*(\eta) + qk_2^*(\eta)\}]},$$

$$H_4^*(s, \eta) = \frac{q\lambda_3 C_2(s)}{\lambda_1 + \eta[1 + \lambda_3\{pk_1^*(\eta) + qk_2^*(\eta)\}]},$$

which is equivalent to equations (2.5.60)-(2.5.63) by taking inverse Laplace transforms. \square

Lemma 2.5.2. *The functions $C_i(s)$ ($i = 1, 2, 3, 4$) are determined by*

$$\begin{aligned} C_1(s) &= \frac{1}{L_3(s)}, \\ C_2(s) &= \frac{L_1(s)}{L_3(s)}, \\ C_3(s) &= p\lambda_3 C(s), \\ C_4(s) &= q\lambda_3 C(s), \end{aligned}$$

where

$$\begin{aligned} L_1(s) &= \frac{\lambda_2 + \lambda_2\lambda_1 \int_0^\infty e^{-sx} g_1(x) \left(\int_0^x e^{-(\lambda_2+\lambda_3)u} C_{01}(u) du \right) dx}{\lambda_1 + \lambda_2\lambda_1 \int_0^\infty e^{-sy} g_2(y) \left(\int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u) du \right) dy}, \\ L_2(s) &= \frac{1}{\lambda_1} \left[1 - \lambda_1 L_1(s) \int_0^\infty e^{-sy} g_2(y) \left(\int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u) du \right) dy \right], \\ L_3(s) &= [s + \lambda_1 + \lambda_2 + \lambda_3 - \lambda_3(pk_1^*(s) + qk_2^*(s))] L_2(s) \\ &\quad - \left\{ \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x) dx \right. \\ &\quad \left. + \lambda_3 \int_0^\infty e^{-sz} [pk_1(z) + qk_2(z)] C_{03}(z) dz \right\} \\ &\quad - L_1(s) \left\{ \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y) dy \right. \\ &\quad \left. + \lambda_3 \int_0^\infty e^{-sz} [pk_1(z) + qk_2(z)] C_{04}(z) dz \right\}, \\ C_{01}(u) &= 1 + \lambda_3 \int_0^u e^{-(\lambda_2+\lambda_3)r} [pk_1(r) * D_1(r) + qk_2(r) * D_1(r)] dr, \\ C_{02}(u) &= 1 + \lambda_3 \int_0^u e^{-(\lambda_1+\lambda_3)r} [pk_1(r) * D_2(r) + qk_2(r) * D_2(r)] dr, \\ C_{03}(u) &= \int_0^\infty e^{-sr} [\overline{G}_1(r) - \overline{G}_1(r+u)] D_1(r) dr, \\ C_{04}(u) &= \int_0^\infty e^{-sr} [\overline{G}_2(r) - \overline{G}_2(r+u)] D_2(r) dr, \end{aligned}$$

$$C(s) = \frac{L_2(s)}{L_3(s)}.$$

Proof. Substituting equations (2.5.53) - (2.5.56) in (2.5.59) and using Lemma 2.5.1, we have

$$\begin{aligned} P_0^*(s) = & \frac{1}{(s + \lambda_1 + \lambda_2 + \lambda_3)} \left(1 + C_1(s) \left\{ \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x) dx \right. \right. \\ & + p\lambda_3 \int_0^\infty e^{-sz} k_1(z) \left(\int_0^\infty e^{-su} D_1(u) [\overline{G}_1(u) - \overline{G}_1(u+z)] du \right) dz \\ & + q\lambda_3 \int_0^\infty e^{-sz} k_2(z) \left(\int_0^\infty e^{-su} D_1(u) [\overline{G}_1(u) - \overline{G}_1(u+z)] du \right) dz \left. \right\} \\ & + C_3(s) k_1^*(s) + C_4(s) k_2^*(s) + C_2(s) \left\{ \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y) dy \right. \\ & + p\lambda_3 \int_0^\infty e^{-sz} k_1(z) \left(\int_0^\infty e^{-su} D_2(u) [\overline{G}_2(u) - \overline{G}_2(u+z)] du \right) dz \\ & \left. \left. + q\lambda_3 \int_0^\infty e^{-sz} k_2(z) \left(\int_0^\infty e^{-su} D_2(u) [\overline{G}_2(u) - \overline{G}_2(u+z)] du \right) dz \right\} \right). \end{aligned}$$

By equations (2.5.35)-(2.5.38), as well as Lemma 2.5.1, we have

$$C_1(s) = \lambda_1 P_0^*(s) + \lambda_1 C_2(s) \int_0^\infty e^{-sy} g_2(y) \left(\int_0^y e^{-(s+\lambda_1+\lambda_3)u} C_{02}(u) du \right) dy,$$

$$C_2(s) = \lambda_2 P_0^*(s) + \lambda_2 C_1(s) \int_0^\infty e^{-sx} g_1(x) \left(\int_0^x e^{-(s+\lambda_2+\lambda_3)u} C_{01}(u) du \right) dx,$$

$$C_3(s) = p\lambda_3 P_0^*(s),$$

$$C_4(s) = q\lambda_3 P_0^*(s).$$

Now, the result is straightforward. \square

From the above equations, the Laplace transforms of the explicit solutions of the system are given by

$$P_5^*(s, x, z) = p\lambda_3 e^{-sx} \overline{G}_1(x) \overline{K}_1(z) D_1(x-z) C_1(s), \quad (2.5.64)$$

$$P_6^*(s, x, z) = q\lambda_3 e^{-sx} \overline{G}_1(x) \overline{K}_2(z) D_1(x-z) C_1(s), \quad (2.5.65)$$

$$P_8^*(s, y, z) = p\lambda_3 e^{-sy} \overline{G}_2(y) \overline{K}_1(z) D_2(y-z) C_2(s), \quad (2.5.66)$$

$$P_9^*(s, y, z) = q\lambda_3 e^{-sy} \overline{G}_2(y) \overline{K}_2(z) D_2(y-z) C_2(s), \quad (2.5.67)$$

$$P_1^*(s, x) = e^{-(s+\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) C_1(s), \quad (2.5.68)$$

$$P_2^*(s, y) = e^{-(s+\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) C_2(s), \quad (2.5.69)$$

$$P_3^*(s, z) = p\lambda_3 e^{-sz} \overline{K}_1(z) \left(C_1(s) C_{03}(z) + C_2(s) C_{04}(z) + C(s) \right), \quad (2.5.70)$$

$$P_4^*(s, z) = q\lambda_3 e^{-sz} \overline{K}_2(z) \left(C_1(s) C_{03}(z) + C_2(s) C_{04}(z) + C(s) \right), \quad (2.5.71)$$

$$P_7^*(s, x) = \lambda_2 e^{-sx} \overline{G}_1(x) C_1(s) \int_0^x e^{-(\lambda_2+\lambda_3)u} C_{01}(u) du, \quad (2.5.72)$$

$$P_{10}^*(s, y) = \lambda_1 e^{-sy} \overline{G}_2(y) C_2(s) \int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u) du, \quad (2.5.73)$$

$$\begin{aligned} P_0^*(s) = & \frac{1}{(s + \lambda_1 + \lambda_2 + \lambda_3)} \left(1 + C_1(s) \left\{ \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x) dx \right. \right. \\ & + p\lambda_3 \int_0^\infty e^{-sz} k_1(z) C_{03}(z) dz + q\lambda_3 \int_0^\infty e^{-sz} k_2(z) C_{03}(z) dz \left. \right\} \\ & + \lambda_3 C(s) \left[pk_1^*(s) + qk_2^*(s) \right] + C_2(s) \left\{ \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y) dy \right. \\ & \left. \left. + p\lambda_3 \int_0^\infty e^{-sz} k_1(z) C_{04}(z) dz + q\lambda_3 \int_0^\infty e^{-sz} k_2(z) C_{04}(z) dz \right\} \right), \quad (2.5.74) \end{aligned}$$

where $D_i(u)$, $C_{0j}(u)$ and $C_j(s)$ ($i = 1, 2, j = 1, 2, 3, 4$) are defined above.

Let $P_\gamma(t) = Pr\{S(t) = \gamma\}$ for $\gamma \in S = F \cup W$. The Laplace transforms of the explicit state probabilities of the system are given by

$$P_5^*(s) = p\lambda_3 \int_0^\infty \int_0^x e^{-sx} \overline{G}_1(x) \overline{K}_1(z) D_1(x-z) dx dz C_1(s),$$

$$P_6^*(s) = q\lambda_3 \int_0^\infty \int_0^x e^{-sx} \overline{G}_1(x) \overline{K}_2(z) D_1(x-z) dx dz C_1(s),$$

$$P_8^*(s) = p\lambda_3 \int_0^\infty \int_0^y e^{-sy} \overline{G}_2(y) \overline{K}_1(z) D_2(y-z) dy dz C_2(s),$$

$$P_9^*(s) = q\lambda_3 \int_0^\infty \int_0^y e^{-sy} \overline{G}_2(y) \overline{K}_2(z) D_2(y-z) dy dz C_2(s),$$

$$P_1^*(s) = \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) dx C_1(s),$$

$$P_2^*(s) = \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) dy C_2(s),$$

$$P_3^*(s) = p\lambda_3 \int_0^\infty e^{-sz} \overline{K}_1(z) \left(C_1(s) C_{03}(z) + C_2(s) C_{04}(z) + C(s) \right) dz,$$

$$P_4^*(s) = q\lambda_3 \int_0^\infty e^{-sz} \overline{K}_2(z) \left(C_1(s) C_{03}(z) + C_2(s) C_{04}(z) + C(s) \right) dz,$$

$$P_7^*(s) = \lambda_2 \int_0^\infty e^{-sx} \overline{G}_1(x) \left(\int_0^x e^{-(\lambda_2+\lambda_3)u} C_{01}(u) du \right) dx C_1(s),$$

$$P_{10}^*(s) = \lambda_1 \int_0^\infty e^{-sy} \overline{G}_2(y) \left(\int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u) du \right) dy C_2(s),$$

$$\begin{aligned} P_0^*(s) = & \frac{1}{(s + \lambda_1 + \lambda_2 + \lambda_3)} \left(1 + C_1(s) \left\{ \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x) dx \right. \right. \\ & + p\lambda_3 \int_0^\infty e^{-sz} k_1(z) C_{03}(z) dz + q\lambda_3 \int_0^\infty e^{-sz} k_2(z) C_{03}(z) dz \left. \right\} \\ & + \lambda_3 C(s) \left[pk_1^*(s) + qk_2^*(s) \right] + C_2(s) \left\{ \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y) dy \right. \\ & \left. \left. + p\lambda_3 \int_0^\infty e^{-sz} k_1(z) C_{04}(z) dz + q\lambda_3 \int_0^\infty e^{-sz} k_2(z) C_{04}(z) dz \right\} \right). \end{aligned}$$

2.5.4 Special cases

When unit 3 is only subject to type I failures, we have $p = 1$ and from equations (2.5.64)-(2.5.74) we obtain

$$P_1^*(s, x) = e^{-(s+\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) C_1(s),$$

$$P_2^*(s, y) = e^{-(s+\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) C_2(s),$$

$$P_3^*(s, z) = \lambda_3 e^{-sz} \overline{K}_1(z) \left(C_1(s) C_{03}(z) + C_2(s) C_{04}(z) + C(s) \right),$$

$$P_5^*(s, x, z) = \lambda_3 e^{-sx} \overline{G}_1(x) \overline{K}_1(z) D_1(x-z) C_1(s),$$

$$P_7^*(s, x) = \lambda_2 e^{-sx} \overline{G}_1(x) C_1(s) \int_0^x e^{-(\lambda_2+\lambda_3)u} C_{01}(u) du,$$

$$P_8^*(s, y, z) = \lambda_3 e^{-sy} \overline{G}_2(y) \overline{K}_1(z) D_2(y-z) C_2(s),$$

$$P_{10}^*(s, y) = \lambda_1 e^{-sy} \overline{G}_2(y) C_2(s) \int_0^y e^{-(\lambda_1+\lambda_3)u} C_{02}(u) du,$$

$$\begin{aligned} P_0^*(s) &= \frac{1}{(s + \lambda_1 + \lambda_2 + \lambda_3)} \left(1 + \lambda_3 k_1^*(s) C(s) \right. \\ &\quad \left. + C_1(s) \left\{ \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x) dx + \lambda_3 \int_0^\infty e^{-sz} k_1(z) C_{03}(z) dz \right\} \right. \\ &\quad \left. + C_2(s) \left\{ \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y) dy + \lambda_3 \int_0^\infty e^{-sz} k_1(z) C_{04}(z) dz \right\} \right), \end{aligned}$$

and

$$P_4^*(s, z) = P_6^*(s, x, z) = P_9^*(s, y, z) = 0,$$

where

$$\begin{aligned}
L_3(s) &= \left[s + \lambda_1 + \lambda_2 + \lambda_3 - \lambda_3 k_1^*(s) \right] L_2(s), \\
&\quad - \left\{ \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} g_1(x) C_{01}(x) dx + \lambda_3 \int_0^\infty e^{-sz} C_{03}(z) k_1(z) dz \right\}, \\
&\quad - L_1(s) \left\{ \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} g_2(y) C_{02}(y) dy + \lambda_3 \int_0^\infty e^{-sz} C_{04}(z) k_1(z) dz \right\},
\end{aligned}$$

$$C_{01}(u) = 1 + \lambda_3 \int_0^u e^{-(\lambda_2+\lambda_3)r} k_1(r) * D_1(r) dr,$$

$$C_{02}(u) = 1 + \lambda_3 \int_0^u e^{-(\lambda_1+\lambda_3)r} k_1(r) * D_2(r) dr,$$

$$C_{03}(u) = \int_0^\infty e^{-sr} D_1(r) \left[\bar{G}_1(r) - \bar{G}_1(r+u) \right] dr,$$

$$C_{04}(u) = \int_0^\infty e^{-sr} D_2(r) \left[\bar{G}_2(r) - \bar{G}_2(r+u) \right] dr,$$

$$D_1(u) = L^{-1} \left[\frac{1}{\lambda_2 + \eta \{1 + \lambda_3 \bar{K}_1^*(\eta)\}} \right],$$

$$D_2(u) = L^{-1} \left[\frac{1}{\lambda_1 + \eta \{1 + \lambda_3 \bar{K}_1^*(\eta)\}} \right].$$

These result were obtained by Li et.al. [35]. However, their formulas for $C_3(s)$ and $P_3(s, z)$ were not correct.

2.5.5 Availability

According to the analysis of the system, we can obtain the transient and equilibrium availability characteristics of the system as follows. First,

$$A(t) = P_0(t) + \int_0^\infty P_1(t, x) dx + \int_0^\infty P_2(t, y) dy,$$

so that its Laplace transform is given by

$$A^*(s) = P_0^*(s) + \int_0^\infty P_1^*(s, x) dx + \int_0^\infty P_2^*(s, y) dy.$$

From equations (2.5.68) and (2.5.69), we have

$$\begin{aligned} A^*(s) = P_0^*(s) &+ C_1(s) \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) dx \\ &+ C_2(s) \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) dy, \end{aligned} \quad (2.5.75)$$

and the steady-state availability of the system is

$$\begin{aligned} A &= \lim_{t \rightarrow \infty} A(t) \\ &= \lim_{s \rightarrow 0} sA^*(s) \\ &= P_0 + C_1 \int_0^\infty e^{-(\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) dx \\ &\quad + C_2 \int_0^\infty e^{-(\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) dy, \end{aligned}$$

where

$$C_i = \lim_{s \rightarrow 0} sC_i(s), \quad i = 1, 2,$$

and

$$P_0 = \lim_{s \rightarrow 0} sP_0^*(s).$$

Second, the failure frequency $m_f(t)$, is

$$m_f(t) = \sum_{j \in F} \int_0^\infty P_0(t) q_{0j}(x) dx + \sum_{i \in W \setminus \{0\}} \sum_{j \in F} \int_0^\infty P_i(t, x) q_{ij}(x) dx.$$

Hence,

$$\begin{aligned}
m_f(t) &= [p\lambda_3 + q\lambda_3]P_0(t) + [p\lambda_3 + q\lambda_3 + \lambda_2] \int_0^\infty P_1(t, x)dx \\
&\quad + [p\lambda_3 + q\lambda_3 + \lambda_1] \int_0^\infty P_2(t, y)dy, \\
&= \lambda_3 \left\{ P_0(t) + \int_0^\infty P_1(t, x)dx + \int_0^\infty P_2(t, y)dy \right\} \\
&\quad + \lambda_2 \int_0^\infty P_1(t, x)dx + \lambda_1 \int_0^\infty P_2(t, y)dy \\
&= \lambda_3 A(t) + \lambda_2 \int_0^\infty P_1(t, x)dx + \lambda_1 \int_0^\infty P_2(t, y)dy,
\end{aligned}$$

and the Laplace transform is given by

$$\begin{aligned}
m_f^*(s) &= \lambda_3 A^*(s) + \lambda_2 \int_0^\infty P_1^*(s, x)dx + \lambda_1 \int_0^\infty P_2^*(s, y)dy \\
&= \lambda_3 A^*(s) + \lambda_2 C_1(s) \int_0^\infty e^{-(s+\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) dx \\
&\quad + \lambda_1 C_2(s) \int_0^\infty e^{-(s+\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) dy.
\end{aligned} \tag{2.5.76}$$

We obtain the steady-state failure frequency:

$$\begin{aligned}
m_f(\infty) &= \lim_{t \rightarrow \infty} m_f(t) \\
&= \lim_{s \rightarrow 0} s m_f^*(s) \\
&= \lambda_3 A + \lambda_2 C_1 \int_0^\infty e^{-(\lambda_2+\lambda_3)x} \overline{G}_1(x) C_{01}(x) dx \\
&\quad + \lambda_1 C_2 \int_0^\infty e^{-(\lambda_1+\lambda_3)y} \overline{G}_2(y) C_{02}(y) dy.
\end{aligned}$$

Third, let $m_r(t)$ denote the derivative of the expected number of renewals of the system, renewal of the system means that the state of the system returns to the initial state. The renewal frequency is

$$\begin{aligned}
m_r(t) &= \int_0^\infty P_1(t, x) \mu_1(x) dx + \int_0^\infty P_2(t, y) \mu_2(y) dy \\
&\quad + \int_0^\infty P_3(t, z) \alpha_1(z) dz + \int_0^\infty P_4(t, z) \alpha_2(z) dz,
\end{aligned} \tag{2.5.77}$$

and its Laplace transform is thus

$$\begin{aligned} m_r^*(s) &= \int_0^\infty P_1^*(s, x)\mu_1(x)dx + \int_0^\infty P_2^*(s, y)\mu_2(y)dy \\ &\quad + \int_0^\infty P_3^*(s, z)\alpha_1(z)dz + \int_0^\infty P_4^*(s, z)\alpha_2(z)dz. \end{aligned} \quad (2.5.78)$$

From equation (2.5.30), we get

$$m_r^*(s) = (s + \lambda_1 + \lambda_2 + \lambda_3)P_0^*(s) - 1. \quad (2.5.79)$$

The steady-state renewal frequency of the system is given by

$$\begin{aligned} m_r &= \lim_{t \rightarrow \infty} m_r(t) \\ &= \lim_{s \rightarrow 0} sm_r^*(s) \\ &= (\lambda_1 + \lambda_2 + \lambda_3)P_0. \end{aligned}$$

2.6 An exponential limit theorem for a two-unit system

We consider the following two-unit system. Both units work independently until failure, are repaired, work again until the next failure, and so on. For both units the operating times are i.i.d. random variables with a common distribution function F and the repair times are i.i.d. random variables with a common distribution function $G_\alpha(y) = G(y/\alpha)$, where $\alpha > 0$ is a parameter and G is a fixed distribution function. We consider the time instants $t_1^{(\alpha)}, t_2^{(\alpha)}, \dots$ at which one unit is under repair and the other unit fails. As $\alpha \rightarrow 0$, the repair times become smaller and tend stochastically to zero. We will show that under certain conditions the first time one unit fails while the other is under repair is asymptotically exponentially distributed after a normalization.

We introduce the following quantities:

- (1) F, G are distribution functions on $(0, \infty)$ with continuous densities f and g , respectively;
- (2) $\{X_k\}, \{\tilde{X}_k\}, \{Y_k^{(\alpha)}\}, \{\tilde{Y}_k^{(\alpha)}\}, k \in \mathbb{N}$ are independent sequences ($\alpha > 0$) having distribution functions $F(x)$ and $G_\alpha(y) = G(y/\alpha)$, where the X_k and \tilde{X}_k denote the lengths of the operating intervals and the $Y_k^{(\alpha)}$ and $\tilde{Y}_k^{(\alpha)}$ denote the lengths of the repair intervals;
- (3) λ is the expected value of F , the expected value of G is 1;
- (4) $t_1^{(\alpha)} < t_2^{(\alpha)} < t_3^{(\alpha)} < \dots$ are the times at which one system is already under repair and the other one breaks down.

The proof applies the Fourier analysis of renewal theory (Alsmeyer [3] and Breiman [8]).

Let F_α be the convolution of F and G_α . Let

$$U_\alpha = \sum_{n=1}^{\infty} F_\alpha^{*n}$$

be the renewal measure of F_α . Let

$$\varphi(z) = \int_0^\infty e^{-zt} dF(t),$$

$$\psi_\alpha(z) = \int_0^\infty e^{-zt} dG_\alpha(t),$$

be the Laplace transforms of F and G_α .

We make the following assumptions:

(a) The characteristic function $\tilde{\varphi}(t) = \varphi(-it)$, $t \in \mathbb{R}$, is integrable on \mathbb{R} .

(b) There is a distribution function $H(x)$ on $(0, \infty)$ with a finite expected value such that

$$\frac{1 - G(u+x)}{1 - G(u)} \leq 1 - H(x), \quad x, u \geq 0. \quad (2.6.1)$$

We will show

Theorem 2.6.1. *As $\alpha \searrow 0$, the distribution of $\alpha t_1^{(\alpha)}$ converges to the exponential distribution with parameter $2\lambda^{-2}$.*

Lemma 2.6.2. *Let $W^{(\alpha)}(t)$ be the number of points $\alpha t_1^{(\alpha)}, \alpha t_2^{(\alpha)}, \dots$ in $[0, t]$. Then*

$$\sup_{0 < \alpha \leq 1} E(W^{(\alpha)}(t)) < \infty \text{ for all } t \geq 0 \quad (2.6.2)$$

and

$$\lim_{\alpha \rightarrow 0} E(W^{(\alpha)}(t)) = 2\lambda^{-2}t \text{ for all } t \geq 0. \quad (2.6.3)$$

Proof. Let

$$I^{(\alpha)}(t) = \begin{cases} 1; & \text{first unit under repair at time } t, \\ 0; & \text{otherwise,} \end{cases}$$

$$\tilde{I}^{(\alpha)}(t) = \begin{cases} 1; & \text{first unit under repair at time } t, \\ 0; & \text{otherwise,} \end{cases}$$

$W_i^{(\alpha)}(t)$ = number of times of total failure when unit i was already broken and the other unit breaks down.

$$B^{(\alpha)} = \{t \geq 0 \mid \text{unit 2 under repair at time } t\},$$

$$B_t^{(\alpha)} = B^{(\alpha)} \cap [0, t],$$

$$T^{(\alpha)}(B) = \text{number of repair times of unit 1 beginning in } B, B \subset [0, \infty),$$

$$\tau^{(\alpha)}(B) = E[T^{(\alpha)}(B)], \tau^{(\alpha)}(t) = E[T^{(\alpha)}[0, t]]$$

The following basic relations hold:

$$W^{(\alpha)}(t) = W_1^{(\alpha)}\left(\frac{t}{\alpha}\right) + W_2^{(\alpha)}\left(\frac{t}{\alpha}\right) \quad (2.6.4)$$

$$E[W^{(\alpha)}(t)] = 2E\left[W_1^{(\alpha)}\left(\frac{t}{\alpha}\right)\right] \quad (2.6.5)$$

$$\begin{aligned} E[W^{(\alpha)}(t)] &= E[T^{(\alpha)}(B_t^{(\alpha)})] \\ &= E[E[T^{(\alpha)}(B_t^{(\alpha)}) \mid B_t^{(\alpha)}]] \\ &= E[\tau^{(\alpha)}(B_t^{(\alpha)})] \\ &= E\left[\int_{[0,t]} \tilde{I}_s^{(\alpha)} d\tau^{(\alpha)}(s)\right] \\ &= \int_{[0,t]} E[\tilde{I}_s^{(\alpha)}] d\tau^{(\alpha)}(s). \end{aligned} \quad (2.6.6)$$

Let $w_1^{(\alpha)}(t) = E[W_1^{(\alpha)}(t)]$. One main step of the proof is to decompose $\tau^{(\alpha)}(t)$ follows:

$$\tau^{(\alpha)}(t) = \tau_1^{(\alpha)}(t) + \tau_2^{(\alpha)}(t) \quad (2.6.7)$$

where $(\tau_1^{(\alpha)})_{\alpha \in (0,1]}$ is tight, i.e.

$$\lim_{K \rightarrow \infty} \sup_{\alpha \in [0,1]} \tau_1^{(\alpha)}([K, \infty)) = 0, \quad (2.6.8)$$

and $\tau_2^{(\alpha)}$ has a density $|p_2^{(\alpha)}| \leq M < \infty$ for some constant M , satisfying

$$\lim_{\substack{K \rightarrow \infty \\ \beta \rightarrow 0}} \sup_{\alpha \in (0,1]} \sup_{t \geq K} |p_2^{(\alpha)}(t) - \frac{1}{\lambda}| = 0. \quad (2.6.9)$$

This will be proved later.

We have to show that $w_1^{(\alpha)}(t/\alpha) \rightarrow \lambda^{-2}t$, as $\alpha \rightarrow 0$. For any $K \in (0, t/\alpha)$,

$$\begin{aligned} |w_1^{(\alpha)}(t/\alpha) - \lambda^{-2}t| &= |E[\int_{B_{t/\alpha}^{(\alpha)}} d\tau^{(\alpha)}(s)] - \lambda^{-2}t| \\ &\leq E[\int_{B_K^{(\alpha)}} d\tau^{(\alpha)}(s)] + \tau_1^{(\alpha)}([K, \infty)) \\ &\quad + \lambda^{-1}E[l(B_K^{(\alpha)})] + \lambda^{-1} |E[l(B_{t/\alpha}^{(\alpha)})] - \lambda^{-1}t| \\ &\quad + E[\int_{B_{t/\alpha}^{(\alpha)} \setminus B_K^{(\alpha)}} |p_2^{(\alpha)}(s) - \lambda^{-1}| ds], \end{aligned} \quad (2.6.10)$$

where l is the Lebesgue measure. Clearly, $E[l(B_K^{(\alpha)})] \rightarrow 0$, as $\alpha \rightarrow 0$. By tightness, for every $\varepsilon > 0$, there is a $K_\varepsilon > 0$ such that $\tau_1^{(\alpha)}([K, \infty)) < \varepsilon$ for all $\alpha \in (0, 1]$ if $K \geq K_\varepsilon$. For large K , the integrand in the last term is less than ε , so that the term itself is bounded by $\varepsilon E[l(B_{t/\alpha}^{(\alpha)})]$. Thus, the proof of Lemma 1 will be complete if we can show that

$$\lim_{\alpha \rightarrow 0} E[\int_{B_K^{(\alpha)}} d\tau^{(\alpha)}(s)] = 0 \quad \text{for all } K > 0 \quad (2.6.11)$$

and

$$\lim_{\alpha \rightarrow 0} E[l(B_{t/\alpha}^{(\alpha)})] = \lambda^{-1}t \quad \text{for all } t > 0. \quad (2.6.12)$$

We have

$$\begin{aligned} E[\int_{B_K^{(\alpha)}} d\tau^{(\alpha)}(s)] &= \int_{[0, K]} E[\tilde{I}_s^{(\alpha)}] d\tau^{(\alpha)}(s) \\ &\leq e^{Kz} \int_0^\infty e^{-zs} E[\tilde{I}_s^{(\alpha)}] d\tau^{(\alpha)}(s) \end{aligned} \quad (2.6.13)$$

for any $z \in (0, \infty)$. Therefore, (2.6.11) follows if we can prove that for $q_\alpha(s) = E[\tilde{I}_s^{(\alpha)}]$ we have

$$\lim_{\alpha \rightarrow 0} \int_0^\infty e^{-zs} q_\alpha(s) d\tau^{(\alpha)}(s) = 0 \text{ for some } z \in (0, \infty). \quad (2.6.14)$$

The integral in (2.6.14) is equal to

$$\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \hat{q}_\alpha(u) \hat{\tau}^{(\alpha)}(z-u) du$$

for any $a \in (0, z)$ where \hat{q}_α and $\hat{\tau}^{(\alpha)}$ are the corresponding Laplace transforms of q_α and $\tau^{(\alpha)}$.

By renewal theory, we find that

$$\tau^{(\alpha)}(t) = F(t) + (F * U_\alpha)(t) \quad (2.6.15)$$

and

$$1 - E[\tilde{I}_t^{(\alpha)}] = 1 - F(t) + ((1 - F) * U_\alpha)(t). \quad (2.6.16)$$

Note that

$$U_\alpha(t) = \sum_{n=1}^{\infty} (F * G_\alpha)^{*n}(t).$$

It follows that

$$\hat{\tau}^{(\alpha)}(u) = \varphi(u) / [1 - \varphi(u)\psi_\alpha(u)] \quad (2.6.17)$$

$$\hat{q}_\alpha(u) = \frac{\varphi(u)(1 - \psi_\alpha(u))}{u[1 - \varphi(u)\psi_\alpha(u)]}. \quad (2.6.18)$$

Hence, by Laplace inversion

$$\int_0^\infty e^{-zs} q_\alpha(s) d\tau^{(\alpha)}(s) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \frac{\varphi(u)[1 - \psi_\alpha(u)]}{u[1 - \varphi(u)\psi_\alpha(u)]} \cdot \frac{\varphi(z-u)}{1 - \varphi(z-u)\psi_\alpha(z-u)} du, \quad (2.6.19)$$

where we may take $z = 2a$, $a > 0$. The integrand on the right-hand side tends to 0, as $\alpha \rightarrow 0$, and is in absolute value bounded by $C|\varphi(u)|^2/|u|$, where the constant C does not depend on u and α . From assumption (a), it follows that

$$\int_{a-i\infty}^{a+i\infty} \frac{|\varphi(u)|^2}{|u|} du < \infty.$$

Thus (2.6.14) follows by Lebesgue's convergence theorem.

Next we have

$$E[l(B_t^{(\alpha)})] = \int_0^t E[\tilde{I}_s^{(\alpha)}] ds,$$

so that

$$\int_0^\infty e^{-zt} E[l(B_{t/\alpha}^{(\alpha)})] dt = \frac{\varphi(\alpha z)[1 - \psi_\alpha(\alpha z)]}{\alpha z^2[1 - \varphi(\alpha z)\psi_\alpha(\alpha z)]}. \quad (2.6.20)$$

Note that $\psi_\alpha(z) = \psi_1(\alpha z)$, $E(Y_\alpha) = \alpha$.

By Taylor expansion we have

$$\begin{aligned} \psi_\alpha(\alpha z) &= 1 - \alpha^2 z + o(\alpha^2) \\ \varphi(\alpha z)\psi_\alpha(\alpha z) &= 1 - \lambda \alpha z + O(\alpha^2), \end{aligned}$$

which yields

$$\lim_{\alpha \rightarrow 0} \int_0^\infty e^{-zt} E[l(B_{t/\alpha}^{(\alpha)})] dt = \frac{1}{\lambda z^2}$$

by (2.6.20). By the Tauberian theorem, it follows that

$$\lim_{\alpha \rightarrow 0} E[l(B_{t/\alpha}^{(\alpha)})] = \frac{t}{\lambda}.$$

The lemma is proved. □

We still have to show the decomposition (2.6.7) with the properties described above. This follows from the following renewal theoretic result. We can decompose U_α as follows:

$U_\alpha = U_\alpha^{(1)} + U_\alpha^{(2)}$, where the measures $(U_\alpha^{(1)})_{\alpha \in (0,1]}$ are tight and the measures $(U_\alpha^{(2)})_{\alpha \in (0,1]}$ have densities u_α which satisfy

$$u_\alpha^{(2)}(x) \leq C < \infty \text{ for all } x \geq 0 \text{ and } \alpha \in (0, 1] \quad (2.6.21)$$

for some constant C and

$$\lim_{\beta \rightarrow 0, K \rightarrow \infty} \sup_{0 < \alpha \leq \beta, x \geq k} |u_\alpha^{(2)}(x) - \lambda^{-1}| = 0. \quad (2.6.22)$$

For the proof we have to follow the arguments in (Stone [52] and Breiman [8]) closely and see if they can be extended in our special case to obtain uniform statements with respect to α .

We write $F = pF_1 + qF_2$, where $p > 0, p + q = 1$ and F_1 has a continuous density f_1 that is zero outside of a compact interval. Let

$$\begin{aligned} U_\alpha^{(1)} &= (F * G_\alpha) * \sum_{n=0}^{\infty} q^n (F_2 * G_\alpha)^{*n} \\ U_\alpha^{(2)} &= pU_\alpha^{(1)} * G_\alpha * F_1 * U_\alpha. \end{aligned}$$

Obviously we have

$$U_\alpha = U_\alpha^{(1)} + U_\alpha^{(2)}.$$

It is clear that the measures $(U_\alpha^{(1)})_{\alpha \in (0,1]}$ are tight. It remains to show the properties of $(U_\alpha^{(2)})_{\alpha \in (0,1]}$. Two Fourier analytic lemmas are needed. Recall that $\tilde{\varphi}(t) = \varphi(-it)$. Let $\tilde{\psi}_\alpha(t) = \psi_\alpha(-it)$.

Lemma 2.6.3. (a) For sufficiently small α and t ,

$$|1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)| \geq \lambda t/2.$$

(b)

$$\lim_{\alpha \rightarrow 0} \int_{-\infty}^{\infty} \left| \Re \left(\frac{1}{1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)} - \frac{1}{1 - \tilde{\varphi}(t)} \right) \right| dt = 0.$$

Proof. (a) Let $F_\alpha = F * G_\alpha$. Clearly we have

$$\begin{aligned} |1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t) - (\lambda + \alpha)it| &\leq \int_0^\infty |e^{itx} - 1 - itx| dF_\alpha(x) \\ &\leq (t^2/2) \int_{0 \leq x \leq 2/|t|} x^2 dF_\alpha(x) + 2|t| \int_{x > 2/|t|} |x| dF_\alpha(x). \end{aligned} \quad (2.6.23)$$

Let $\varepsilon > 0$ be given. For every $\delta \in (0, 2]$ we have

$$\begin{aligned} (t^2/2) \int_{0 \leq x \leq 2/|t|} x^2 dF_\alpha(x) &= (t^2/2) \left[\int_{0 \leq x \leq \delta/|t|} + \int_{\delta/|t| < x \leq 2/|t|} \right] \\ &\leq (\delta|t|/2)(\lambda + \alpha) + |t|(1 - F_\alpha(\delta/|t|)). \end{aligned} \quad (2.6.24)$$

Choose $\delta < \varepsilon/(2(\lambda + \alpha))$. Then if $|t|$ is small enough, the first term on the right-hand side of (2.24) is smaller than $|t|\varepsilon/2$. For small $|t|$ the second term on the right-hand side of (2.6.24) is also smaller than $|t|\varepsilon/2$. It follows that

$$|1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t) - (\lambda + \alpha)it| \leq |t|\varepsilon. \quad (2.6.25)$$

Thus (a) is proved.

(b) Let $\varepsilon > 0$ be given. We have

$$\begin{aligned} I(t) &= \frac{1}{1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)} - \frac{1}{1 - \tilde{\varphi}(t)} \\ &= \frac{\tilde{\varphi}(t)(\tilde{\psi}_\alpha(t) - 1)}{(1 - \tilde{\varphi}(t))(1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t))}. \end{aligned} \quad (2.6.26)$$

$\tilde{\varphi}(t)$ is integrable on \mathbb{R} by assumption (a). We have $\lim_{|t| \rightarrow \infty} \tilde{\varphi}(t) = 0$ by the Riemann-Lebesgue lemma. Since $|\tilde{\psi}_\alpha(t)| \leq 1$, it follows from (2.6.26) that for some $K > 0$

$$\int_{|t| \geq K} |\Re I(t)| dt < \varepsilon. \quad (2.6.27)$$

For every $\delta > 0$ the denominator $1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)$ remains bounded away from zero on $[\delta, K]$ and converges to $1 - \tilde{\varphi}(t)$ as $\alpha \rightarrow 0$. Thus

$$\int_{\delta \leq |t| \leq K} |\Re I(t)| dt < \varepsilon \quad (2.6.28)$$

for sufficiently small α .

It remains to consider the region $|t| < \delta$. By (a) we have

$$\begin{aligned} \left| \Re \left(\frac{1}{1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)} \right) \right| &= \frac{|\Re(1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t))|}{|1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)|^2} \\ &\leq \frac{4}{\lambda^2} t^{-2} |\Re(1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t))| \end{aligned} \quad (2.6.29)$$

for small $|t|$ and α . For the right-hand side we obtain

$$\begin{aligned} \int_{|t|<\delta} t^{-2} |\Re(1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t))| dt &\leq \int_0^\infty \int_{|t|<\delta} t^{-2} |e^{itu} - 1 - i \sin tu| dt dF_\alpha(u) \\ &\leq \int_0^\infty u \int_{|s|<\delta u} s^{-2} |e^{is} - 1 - i \sin s| ds dF_\alpha(u). \end{aligned} \quad (2.6.30)$$

It now follows easily that

$$\int_{|t|<\delta} \left| \Re \left(\frac{1}{1 - \tilde{\varphi}(t)\tilde{\psi}_\alpha(t)} \right) \right| dt < \varepsilon \quad (2.6.31)$$

for sufficiently small α and δ . □

Lemma 2.6.4. *For every $c > 0$ we have*

$$\sup_{0 < \alpha \leq \beta, x \geq K} |U_\alpha([x - c, x + c]) - (2c/\lambda)| \rightarrow 0 \quad (2.6.32)$$

as $\beta \rightarrow 0$ and $K \rightarrow \infty$.

Proof. If $h : [0, \infty) \rightarrow \mathbb{R}$ is an integrable function with an integrable Fourier transform \tilde{h} , then

$$\int_0^\infty h(u) U_\alpha(x + du) = \frac{2\pi \tilde{h}(0)}{\lambda + \alpha} + 2 \int_{\mathbb{R}} e^{-isx} \tilde{h}(s) \Re \left(\frac{1}{1 - \tilde{\varphi}(s)\tilde{\psi}_\alpha(s)} \right) ds \quad (2.6.33)$$

(Breiman [8], p. 221). Now we approximate the interval $[-c, c]$ from above and from below.

Fix $\delta \in (0, c)$. We use the functions $h_\delta^{(1)} = h_{\delta,c}$ and $h_\delta^{(2)} = h_{\delta,c-\delta}$, where $h_{\delta,c}$ is the continuous

function which is 1 on $[-c, c]$, 0 on $(-\infty, c - \delta]$ and $[c + \delta, \infty)$ and has straight line pieces on $(c - \delta, c)$ and $(c, c + \delta)$. Then we have

$$\int_0^{\infty} h_{\delta}^{(1)}(u) U_{\alpha}(x + du) \geq U_{\alpha}([x - c, x + c]) \geq \int_0^{\infty} h_{\delta}^{(2)}(u) U_{\alpha}(x + du). \quad (2.6.34)$$

The Fourier transform of $h_{\delta,c}$ is

$$\tilde{h}_{\delta,c}(t) = \frac{\cos ct - \cos(c + \delta)t}{\pi \delta t^2}.$$

It is integrable and bounded. From (0.33) we obtain

$$\int_0^{\infty} h_{\delta}^{(1)}(u) U_{\alpha}(x + du) = \frac{2c + \delta}{\lambda + \alpha} + 2 \int_{\mathbb{R}} \cos sx \tilde{h}_{\delta}^{(1)}(s) \Re \left(\frac{1}{1 - \tilde{\varphi}(s) \tilde{\psi}_{\alpha}(s)} \right) ds. \quad (2.6.35)$$

A similar relation holds for $h_{\delta}^{(2)}$.

We show that the integral on the right-hand side of (2.6.35) is small when x is large and α is small. Then (2.6.32) follows from (2.6.34) by letting $\delta \rightarrow 0$. Now the integral on the right-hand side of (2.6.35) is equal to the sum

$$\begin{aligned} & \int_0^{\infty} \cos sx \tilde{h}_{\delta}^{(1)}(s) \Re \left(\frac{1}{1 - \tilde{\varphi}(s)} \right) ds \\ & + \int_0^{\infty} \cos sx \tilde{h}_{\delta}^{(1)}(s) \left[\Re \left(\frac{1}{1 - \tilde{\varphi}(s) \psi_{\alpha}(s)} \right) - \Re \left(\frac{1}{1 - \tilde{\varphi}(s)} \right) \right] ds. \end{aligned} \quad (2.6.36)$$

As $x \rightarrow \infty$, the first integral tends to 0 by the Riemann-Lebesgue lemma. The second integral tends to 0 uniformly in x as $\alpha \rightarrow 0$ by Lemma 2(b). \square

Now we can show (2.6.21) and (2.6.22). Let $v_{\alpha}(x)$ be the density of $F_1 * U_{\alpha}$. Then

$$v_{\alpha}(x) = \int_0^x u_{\alpha}(x - y) f_1(y) dy \quad (2.6.37)$$

and f_1 is zero outside a finite interval $[0, c]$. Thus it follows from Lemma 3 that

$$\sup_{0 < \alpha \leq \beta, x \geq K} |v_{\alpha}(x) - (1/\lambda)| \rightarrow 0 \quad (2.6.38)$$

as $\beta \rightarrow 0$ and $K \rightarrow \infty$. It follows from the proof of Lemma 3 that

$$\sup_{0 < \alpha \leq 1, x > 0} U_\alpha([x - c, x + c]) < \infty.$$

Therefore,

$$\sup_{0 < \alpha \leq 1, x > 0} v_\alpha(x) \leq \sup_{0 < \alpha \leq 1, x > 0} U_\alpha([x - c, x]) \sup_{u > 0} f_1(u) < \infty. \quad (2.6.39)$$

By its definition $u_\alpha^{(2)}(x)$ is the convolution of $v_\alpha(x)$ and the density $w_\alpha(x)$ of $pU_\alpha^{(1)} * G_\alpha$. Therefore and by (2.6.39), $u_\alpha^{(2)}(x)$ is uniformly bounded in α and x , i.e., (2.6.21) holds. It is clear that $w_\alpha(x)$ is a probability density and that the family $pU_\alpha^{(1)} * G_\alpha$, $0, \alpha \leq 1$, is tight.

Then for $x > K$

$$\begin{aligned} |u_\alpha^{(2)}(x) - (1/\lambda)| &= \left| \int_0^x w_\alpha(x-y)v_\alpha(y)dy - (1/\lambda) \right| \\ &\leq \int_K^x w_\alpha(x-y)|v_\alpha(y) - (1/\lambda)|dy \\ &\quad + \int_0^K w_\alpha(x-y)|v_\alpha(y) - (1/\lambda)|dy \\ &\quad + (1/\lambda) \int_x^\infty w_\alpha(u)du. \end{aligned} \quad (2.6.40)$$

Now we take $\sup_{0 < \alpha \leq \beta, x \geq K}$ for all terms in (2.6.40). Then as $\beta \rightarrow 0$ and $K \rightarrow \infty$ all three suprema on the right-hand side tend to 0, the first by (2.6.38), the second by the tightness of $pU_\alpha^{(1)} * G_\alpha$, $0, \alpha \leq 1$, and the boundedness of $v_\alpha(y)$, and the third by tightness. (2.6.21) and (2.6.22) are proved.

To get the representation

$$\tau^{(\alpha)}(t) = \tau_1^{(\alpha)}(t) + \tau_2^{(\alpha)}(t),$$

we can now use (2.6.15). Let

$$\tau_1^{(\alpha)}(t) = F(t) + (U_\alpha^{(1)} * F)(t)$$

and

$$\tau_2^{(\alpha)}(t) = (U_\alpha^{(2)} * F)(t).$$

Then the properties of $\tau_1^{(\alpha)}(t)$ and $\tau_2^{(\alpha)}(t)$ follow immediately from those of $U_\alpha^{(1)}(t)$ and $U_\alpha^{(2)}(t)$ and (2.6.15).

Proof of the theorem. Let $w^{(\alpha)} = \alpha t_1^{(\alpha)}$ be the smallest point of $W^{(\alpha)}$. Let $\alpha_n > 0, \alpha_n \rightarrow 0$, such that the distribution of $w^{(\alpha_n)}$ has a limiting distribution μ on $[0, \infty]$ (including the end points 0 and ∞). We have to show that μ is the exponential distribution with parameter $2/\lambda^2$.

Clearly,

$$E[W^{(\alpha_n)}(t)] = \int_0^t E[W^{(\alpha_n)}(t) | w^{(\alpha_n)} = x] P(w_1^{(\alpha_n)} \in dx).$$

The left-hand side converges to $2\lambda^{-2}t$. Now we show that

$$E[W^{(\alpha_n)}(t) | w^{(\alpha)} = x] \rightarrow 1 + 2\lambda^{-2}(t - x) \quad (2.6.41)$$

uniformly in $x \in [0, t]$. Then it follows that

$$2\lambda^{-2}t = \int_0^t [1 + 2\lambda^{-2}(t - x)] d\mu(x).$$

Taking Laplace transforms on both sides yields

$$2\lambda^{-2}s^{-2} = s^{-1}\hat{\mu}(s) + 2\lambda^{-2}s^{-2}\hat{\mu}(s),$$

where $\hat{\mu}(s) = \int_0^\infty e^{-st} d\mu(t)$. Thus

$$\hat{\mu}(s) = 2\lambda^{-2}/(s + 2\lambda^{-2})$$

and the theorem is proved. It remains to show (2.6.41).

Let $s_i^{(\alpha)}$, $i = 1, 2$, be the smallest point of $W_i^{(\alpha)}$. We have by symmetry

$$\begin{aligned} E[W^{(\alpha)}(t) \mid w^{(\alpha)} = x] &= E[W^{(\alpha)}(t) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha] \\ &= E[W_1^{(\alpha)}(t) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha] \\ &\quad + E[W_2^{(\alpha)}(t) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha]. \end{aligned} \quad (2.6.42)$$

Then (2.6.41) follows from (2.6.42) and the relations

$$E[W_1^{(\alpha)}(t) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha] \rightarrow 1 + \lambda^{-2}(t - x) \quad (2.6.43)$$

$$E[W_2^{(\alpha)}(t) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha] \rightarrow \lambda^{-2}(t - x). \quad (2.6.44)$$

We prove (2.6.43). (2.6.44) is proved in the same way.

To prove (2.6.43), we write

$$\begin{aligned} E[W_1^{(\alpha)}(t) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha] &= \\ 1 + E \left[\int_{B^{(\alpha)} \cap [x/\alpha, t/\alpha]} U_\alpha(ds - (x/\alpha)) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha \right]. \end{aligned} \quad (2.6.45)$$

Now we follow the proof in (2.6.10). Therefore it suffices to show that we have, uniformly in $x \in [0, t]$,

$$\lim_{\alpha \rightarrow 0} E \left[\int_{B^{(\alpha)} \cap [x/\alpha, (x/\alpha) + \varepsilon]} U_\alpha(ds - (x/\alpha)) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha \right] = 0 \quad (2.6.46)$$

for every $\varepsilon > 0$ and

$$\lim_{\alpha \rightarrow 0} E[l(B^{(\alpha)} \cap [x/\alpha, t/\alpha]) \mid s_1^{(\alpha)} = x/\alpha, s_2^{(\alpha)} > x/\alpha] = \lambda^{-1}(t - x) \quad (2.6.47)$$

for every $t > 0$.

(2.6.46) is analogous to (2.6.11). However, by the conditioning, at time x/α the first unit fails and the second one is under repair and has some residual random repair time $V_x^{(\alpha)}$, which has the distribution function $R_x^{(\alpha)}(v)$. By assumption (b), $V_x^{(\alpha)}$ converges to 0

in distribution uniformly in x . Let $\rho_x^{(\alpha)}(u)$ be the Laplace transform of the distribution $R_x^{(\alpha)}$. We proceed as in (2.6.13) and (2.6.19). In (2.6.19) we have to use the Laplace transforms

$$\frac{\varphi(u)(1 - \psi_\alpha(u))}{1 - \varphi(u)\psi_\alpha(u)} = \varphi(u)(1 - \psi_\alpha(u)) + \frac{\varphi(u)^2(1 - \psi_\alpha(u)\psi_\alpha(u))}{1 - \varphi(u)\psi_\alpha(u)} \quad (2.6.48)$$

and

$$u^{-1}(1 - \rho_x^{(\alpha)}(u)) + \frac{\rho_x^{(\alpha)}(u)\varphi(u)(1 - \psi_\alpha(u))}{u(1 - \varphi(u)\psi_\alpha(u))} \quad (2.6.49)$$

instead of (2.6.17) and (2.6.18). Integrating the product of (2.6.48) and (2.6.49) as in (2.6.19) we can again use the Lebesgue convergence theorem and prove (2.6.46).

To show (2.6.47), we remark first that the convergence in (2.6.12) is uniform in $t \in [0, T]$ for every $T > 0$, because $E[l(B_{t/\alpha}^{(\alpha)})]$ is nondecreasing in t for every α and the limit $\lambda^{-1}t$ is a continuous function of t . The following inequalities are obvious:

$$l(B_{(t/\alpha)+V_x^{(\alpha)}}^{(\alpha)} \setminus B_{(x/\alpha)+V_x^{(\alpha)}}^{(\alpha)}) \leq l(B_{t/\alpha}^{(\alpha)} \setminus B_{x/\alpha}^{(\alpha)}) \leq l(B_{(t/\alpha)+V_x^{(\alpha)}}^{(\alpha)} \setminus B_{(x/\alpha)+V_x^{(\alpha)}}^{(\alpha)}) + V_x^{(\alpha)}. \quad (2.6.50)$$

The distribution of $l(B_{(t/\alpha)+V_x^{(\alpha)}}^{(\alpha)} \setminus B_{(x/\alpha)+V_x^{(\alpha)}}^{(\alpha)})$ is the same as that of $l(B_{(t-x)/\alpha}^{(\alpha)})$. Thus by the remark above,

$$E[l(B_{(t/\alpha)+V_x^{(\alpha)}}^{(\alpha)} \setminus B_{(x/\alpha)+V_x^{(\alpha)}}^{(\alpha)})] \rightarrow \lambda^{-1}(t - x) \quad (2.6.51)$$

uniformly in $x \in [0, t]$. By assumption (b) we have

$$\begin{aligned} E[V_x^{(\alpha)}] &= \int_0^\infty \frac{1 - G((x+u)/\alpha)}{1 - G(x/\alpha)} du \\ &\leq \int_0^\infty (1 - H(u/\alpha)) du = \alpha \int_0^\infty (1 - H(y)) dy \rightarrow 0 \end{aligned} \quad (2.6.52)$$

as $\alpha \rightarrow 0$, uniformly in x . Now (2.6.47) follows from (2.6.50)-(2.6.52).

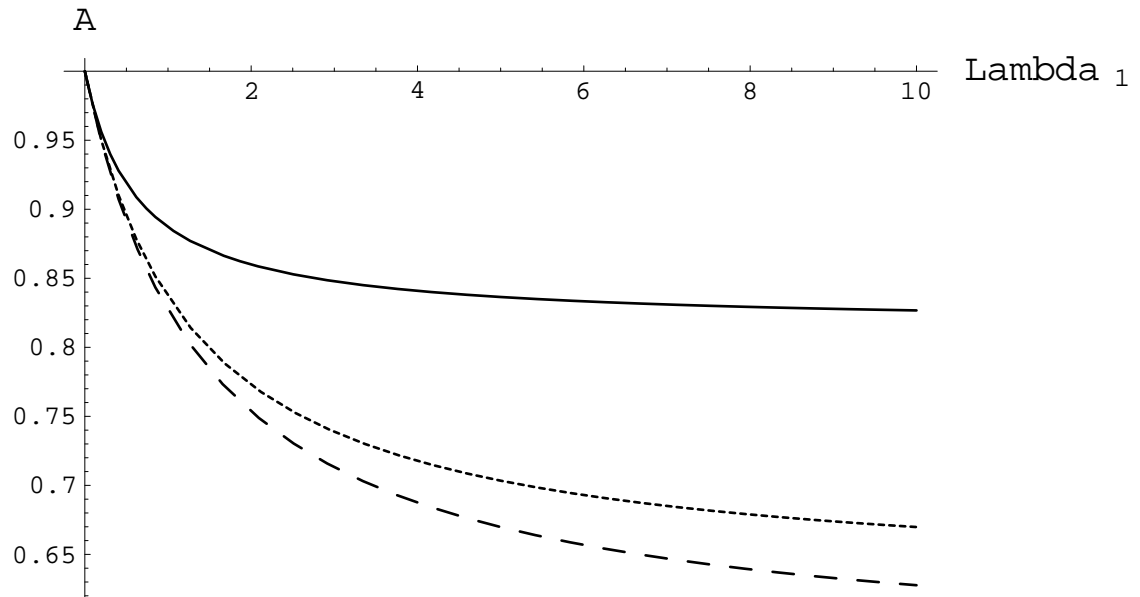


Figure 2.1: The relation between the steady-state availability of the system and λ_1 , where $\lambda_2 = 1.0$ (solid), $\lambda_2 = 5.0$ (dotted), $\lambda_2 = 10.0$ (dashed).

probability of two units operating

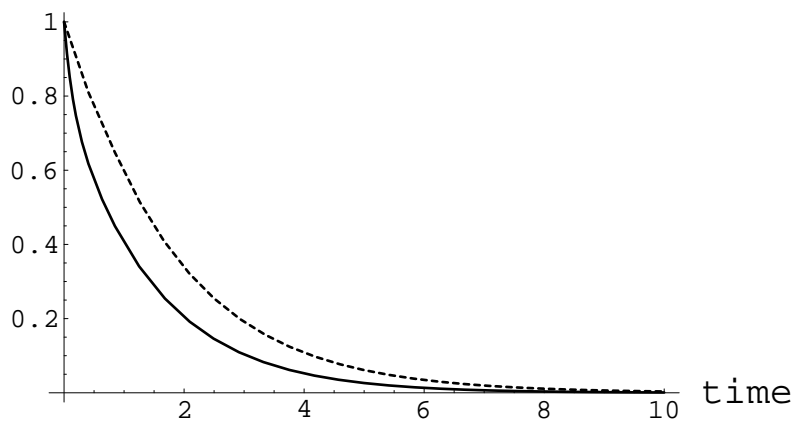


Figure 2.2: Plot of the probability of the two units operating together for Example 1 (solid) and Example 2 (dotted).

probability of two units operating

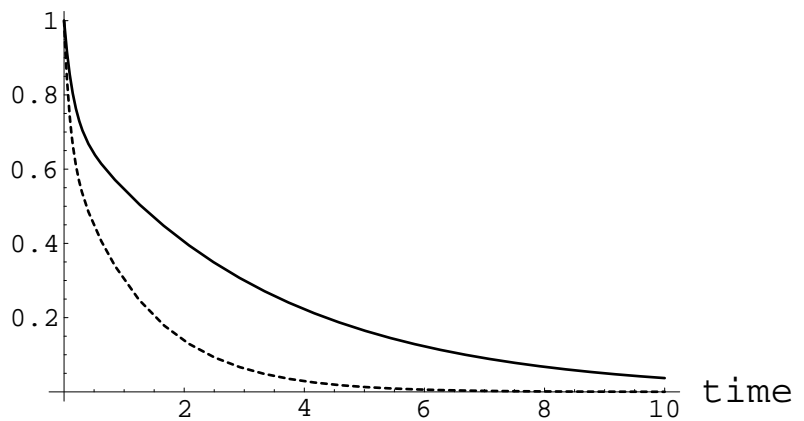


Figure 2.3: Plot of the probability of the two units operating together case (2.3.45), Example 1 (solid) and Example 2 (dotted).

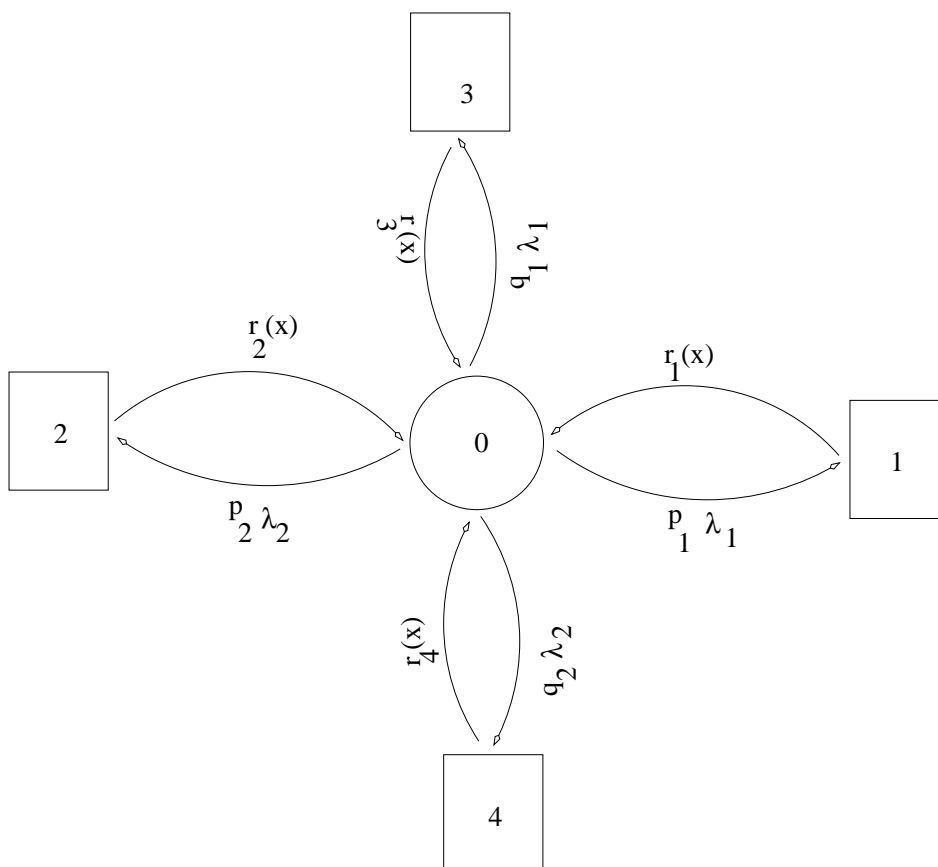


Figure 2.4: Transitions among the states of the series system, where \bigcirc denote the up state and \square denote down states.

availability

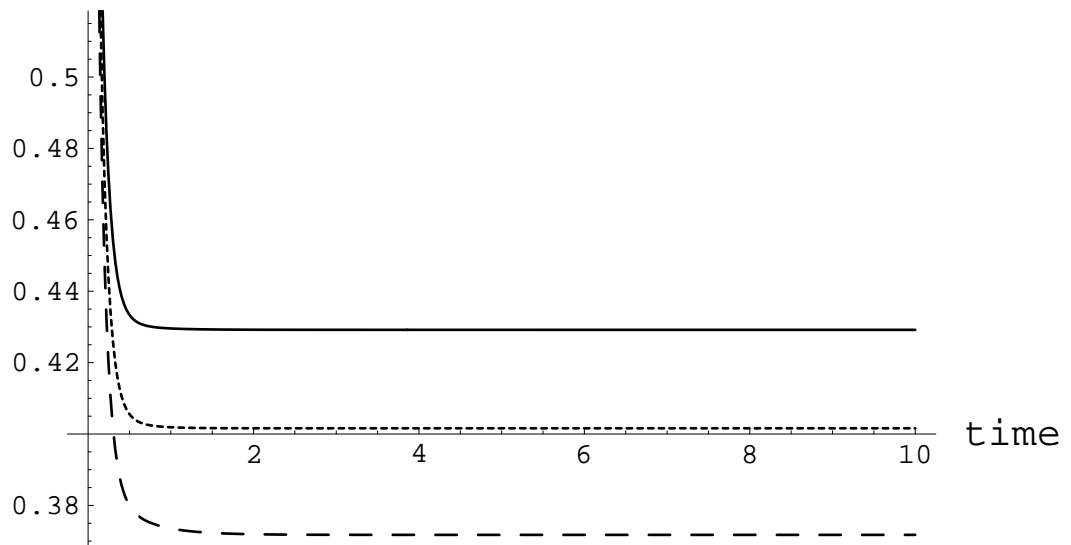


Figure 2.5: Plot of the availability $A(t)$ of the series system as a function of time t , for $\lambda_2 = 0.1$ (solid), $\lambda_2 = 0.5$ (dotted) and $\lambda_2 = 1.0$ (dashed).

failure frequency

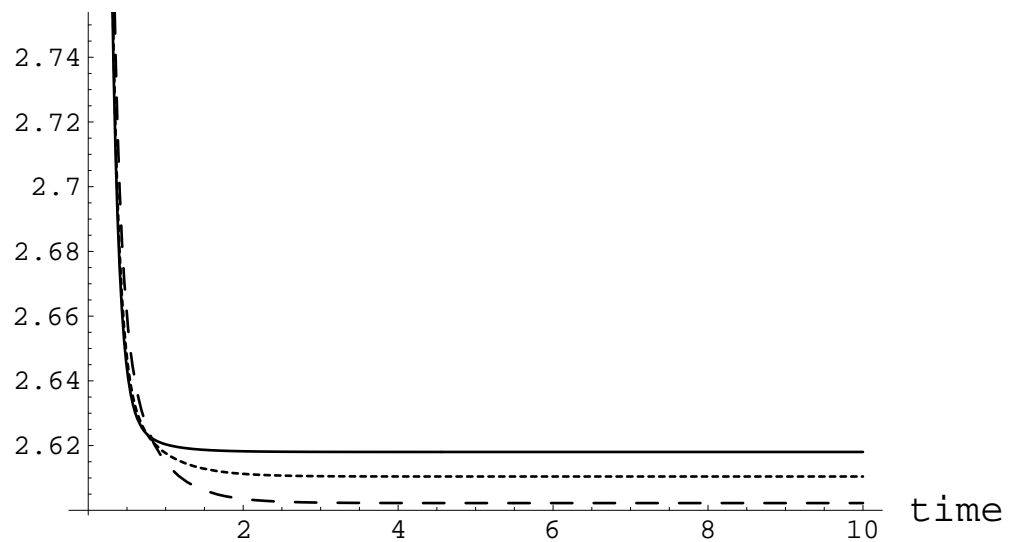


Figure 2.6: Plot of the failure frequency $m_f(t)$ of the series system as a function of time t , for $\lambda_2 = 0.1$ (solid), $\lambda_2 = 0.5$ (dotted) and $\lambda_2 = 1.0$ (dashed).

renewal frequency

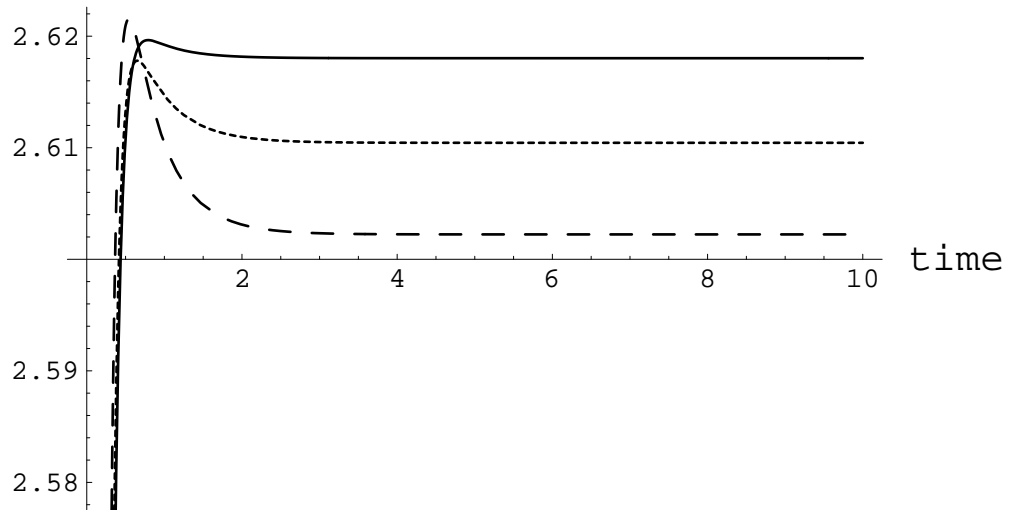


Figure 2.7: Plot of the renewal frequency $m_r(t)$ of the series system as a function of time t , for $\lambda_2 = 0.1$ (solid), $\lambda_2 = 0.5$ (dotted) and $\lambda_2 = 1.0$ (dashed).

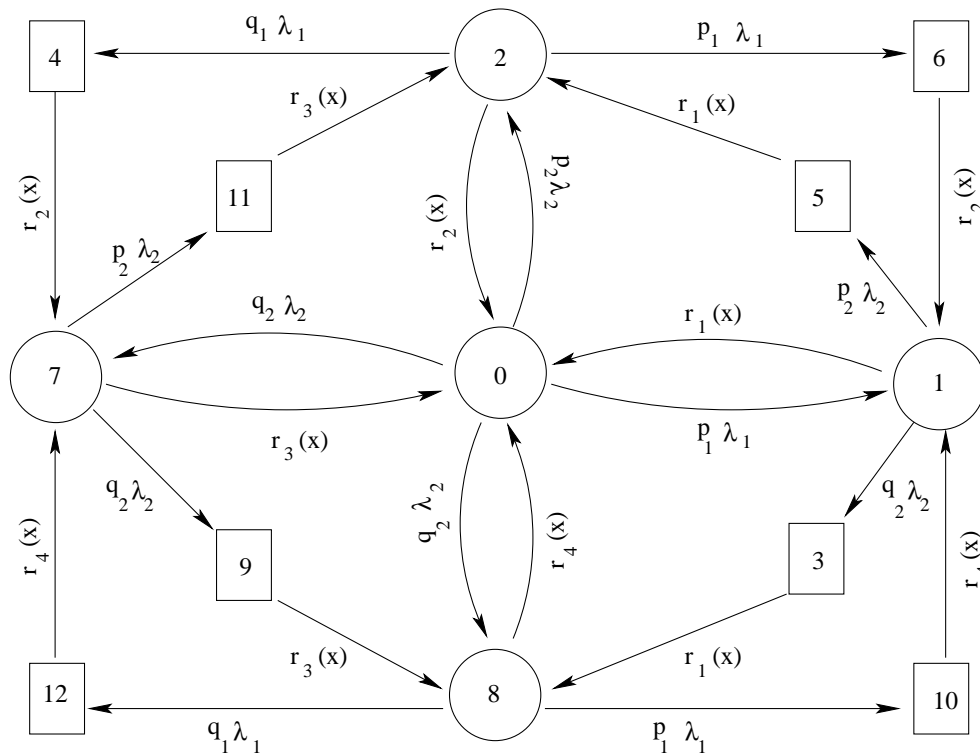


Figure 2.8: Transitions among the states of parallel system, where \bigcirc denote up states and \square denote down states.

steady-state availability

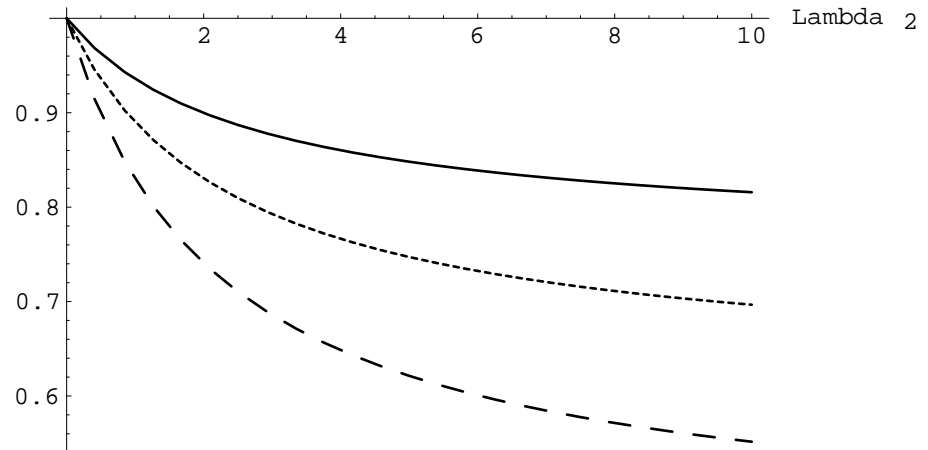


Figure 2.9: The relation between the steady-state availability of the parallel system and λ_2 for $\lambda_1 = 0.5$ (solid), $\lambda_1 = 1.0$ (dotted) and $\lambda_1 = 2.0$ (dashed).

steady-state failure frequency

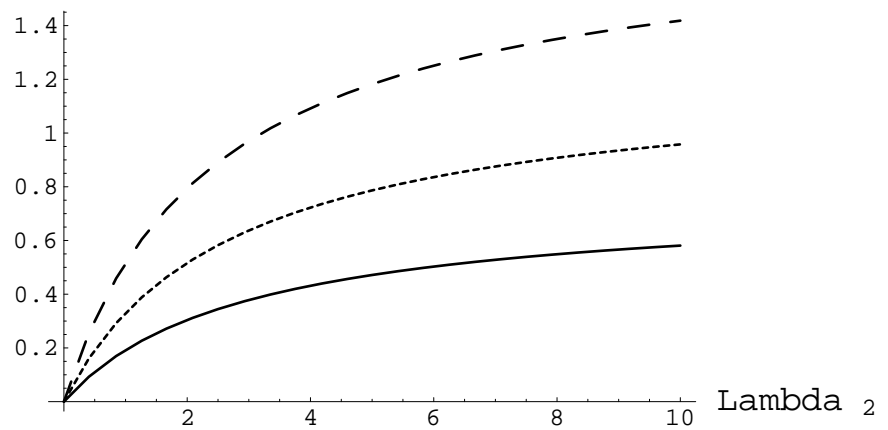


Figure 2.10: The relation between the steady-state failure frequency of the parallel system and λ_2 for $\lambda_1 = 0.5$ (solid), $\lambda_1 = 1.0$ (dotted) and $\lambda_1 = 2.0$ (dashed).

steady-state renewal frequency

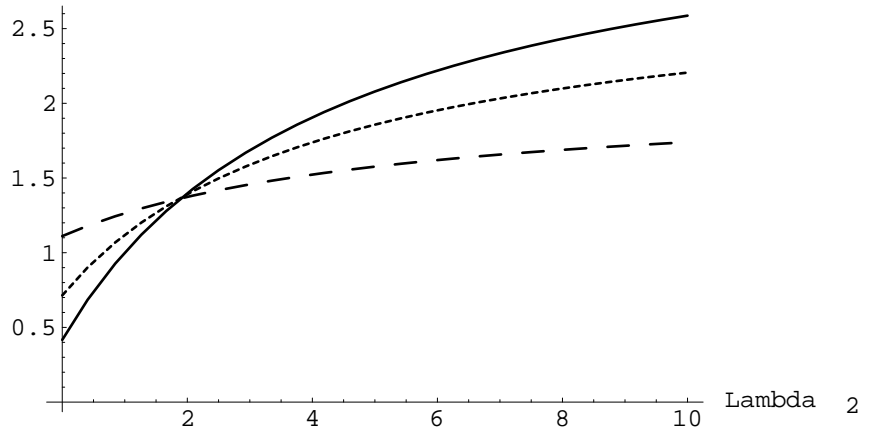


Figure 2.11: The relation between the steady-state renewal frequency of the parallel system and λ_2 for $\lambda_1 = 0.5$ (solid), $\lambda_1 = 1.0$ (dotted) and $\lambda_1 = 2.0$ (dashed).

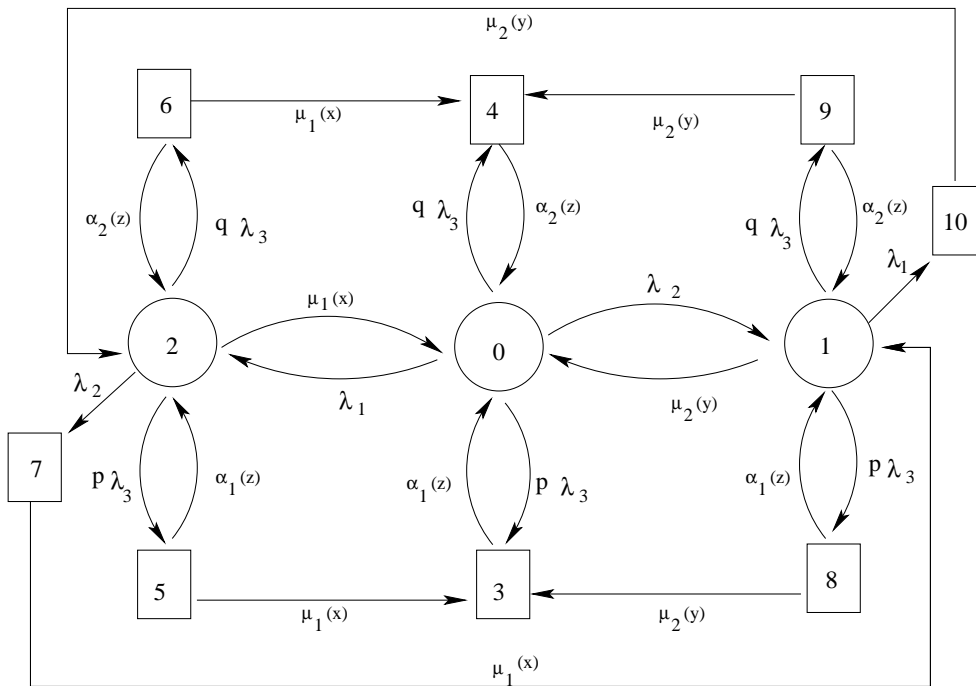


Figure 2.12: Transitions among the states of the system, where \bigcirc denote up states and \square denote down states.

Chapter 3

The cell and the radiation effect

3.1 Introduction

Cell reaction to radiation undoubtedly constitutes an issue of great concern. Cell radiation action comprises all levels of cell organization. It starts with the absorption in essential atoms and molecules and ends with the development of cancer and genetic hazards to future generations and death. To this cause, in Chapter 4, the models for cell survival after irradiation with ionizing radiation are studied.

In section **3.2**, the general structure and functions of the cell are explained; the cell composed of cell membrane, Cytoplasm, Golgi bodies, Mitochondria, Nucleus, etc... .

In section 3.3, The radiation effect on the cell is described; this section consists of three subsections. subsection **3.3.1**, radiation sources are illustrated, the first source is ionizing radiation (*alpha-*, *beta-*, and *gamma-* radiation). Ionizing radiation exposure can occur from a radiation source outside of the body (external radiation) or as a result of taking radioactive material into the body (internal radiation). The second source is optical radiation, and the third source is non-ionizing radiation. In subsection **3.3.2**, the theory of ionizing radiation effect of the cell are illustrated, two theories which prevail in the field of radiobiology (target theory and absorption of radiation) have been explained. In subsection **3.3.3**, the radiation effects on cell constituents are also explained.

3.2 The cell

A cell is the basic unit of life, all living things are made up of cells. Although there is no such thing as a typical cell, all cells have several features in common. The normal human body has about 50 million million cells.

3.2.1 General structure and functions of the cell

Most cells are composed of **protoplasm** : a mixture of carbohydrates, proteins, lipids, nucleic acids, inorganic salts, gases and water (between 70 and 80%).

Living cells are lined by a **plasma membranes or cell membranes**. The plasma membrane in a cell is a very important structure. It has lots of holes called **pores** and **channels** and obviously has an important job to do, keeping the cell together and controlling what substances go in and out. There are also many other membranes which make up several of the cell organelles described below.

The cell can be subdivided into the two main compartments **cytoplasm** and **nucleus** (see [2], [36], [13] and Figure 3.1).

(i) **Cytoplasm**

The cytoplasm surrounds the nucleus and is bounded by the plasma membrane. It consists of a ground substance, **hyaloplasm**, and **organelles**. Hyaloplasm contains enzymes for glycolysis and structural materials. Cytoplasm contains the machinery for carrying out the nuclear instruction. Besides the usual consists of a unit membrane it contains enzymes and energy sources. The organelles of cytoplasm are the following

(1) **Golgi bodies**

The golgi bodies are specialized portions of the endoplasmic reticulum. It is composed of many layers of thin flattened vesicles formed from unit membranes.

The golgi bodies are located on the side of the cell from which substances will be secreted. Some functions of the golgi bodies are as follows:

- a) Proteins and other material made elsewhere in the cell are sometimes transported to the golgi bodies for condensation in membrane bound packets, this allows the transportation of material out of the cell or the storage of material within the cell.
- b) Carbohydrates in the golgi bodies and complexes with proteins coming from the endoplasmic reticulum are sometimes synthesized.
- c) The golgi bodies may also be active in lipoprotein synthesis.

(2) **Mitochondria**

Mitochondria are present in all cells, however the number of mitochondria in a cell varies from a few hundreds to many thousands, depending on the amount of energy required by the cell. Mitochondria are composed of inner and outer unit membranes. They are the major sites of energy production in the cell and contain many important enzymes.

(3) **Endoplasmic Reticulum**

Endoplasmic reticulum is a network of tub-like structures distributed throughout the cytoplasm. Some of these tubes are connected with the nuclear membrane, and some with the cell-membrane. They appear to be associated with enzyme formation, protein synthesis, storage and transport of metabolic products. They may also contribute to the formation of the cellplate in nuclear division, and of the nuclear membrane around the newly-formed nuclei.

(4) **Centrosome**

Centrosome is a minute body found in animal cells. It occurs close to the nucleus and usually has two central bodies called *centrioles*. During nuclear division they

pass on to the opposite ends of the cell and organize the nuclear spindle.

(5) **Ribosomes**

Associated with the membrane of the tubes and also occurring free in the cytoplasm are many tiny particles called ribosomes. They are composed of **RNA** (ribonucleic acid) and protein. They synthesize some of the enzymes and are the main seats of protein-synthesis.

(6) **Lysosomes**

They also occur as tiny particles in the cytoplasm. They are spherical in shape, with an outer membrane and dense contents. They are rich in several enzymes and are associated with intra-cellular digestion.

(ii) **Nucleus**

The nucleus is a specialized protoplasmic body much denser than the cytoplasm, and is commonly spherical or oval in shape. It always lies embedded in the cytoplasm.

(1) **Structure**

Each nucleus is surrounded by a thin, transparent membrane known as (1) **the nuclear membrane** which separates the nucleus from the surrounding cytoplasm, within the membrane, completely filling up the space there is a dense but clear mass of protoplasm known as (2) **the nuclear sap or nucleoplasm**. Suspended in the nucleoplasm there are numerous fine crooked threads, loosely connected here and there, forming a sort of network, called (3) **the nuclear reticulum or chromatin network**. The threads are made of a substance known as chromatin or nuclein which is strongly stainable. Chromatin or nuclein is a nucleoprotein. One or more highly refractive, relatively large and usually spherical bodies can be seen in the nucleoplasm; these are known as (4) **the nucleoli**. Two important nucleic acids are **DNA** and **RNA**

(2) **Functions**

The nucleus and the protoplasm are together responsible for the life of a cell and the various vital functions performed by it. If they are separated both of them die. The nucleus, however, is regarded as the controlling centre of the vital activities of the cell in many ways. The specific functions performed by the nucleus are as follows :

- a) The nucleus takes a direct part in reproduction. Two reproductive nuclei called gametes fuse together to give rise to an oospore which grows into an embryo. Thus nuclei are directly concerned in the process of reproduction.
- b) The nucleus takes the initiative in cell division, i.e., it is the nucleus that divides first and this is followed by the division of the cell.
- c) The nucleus is regarded as the *bearer* of hereditary characters. It is to be noted that it is the DNA of the nuclear reticulum that is the sole genetic (hereditary) material of the two reproductive nuclei.

3.3 Radiation effect of the cell

Mathematical models for cell radiation and for cell survival after irradiation are investigated by many groups.

Radiation is the transport of energy without the necessary intervention of a transporting medium. It may be accomplished either by electro-magnetic waves or by particles, e.g. electrons, neutrons or ions (see for example [28] and [43]).

3.3.1 Radiation sources

(1) Ionizing radiation

Ionizing radiation is energy that is carried by several types of particles and rays

emitted by radioactive material, X-ray machines, and fuel elements in nuclear reactors. Ionizing radiation includes α -particles, β -particles (essentially small fast moving pieces of atoms), X-rays and γ -rays (types of electromagnetic radiation).

These radiation particles and rays carry enough energy to knock out electrons from molecules, such as water, protein, and DNA, with which they interact.

The main types of ionizing radiation are called α -, β -, and γ -radiation:-

- (a) β -radiation (particles) are high-energy electrons that some radioactive materials emit when they decay. They exist two types (positive charge or negative charge), depending on the radioactive material that produces them.
- (b) α -radiation is a particle, consisting of two protons and two neutrons, that travels very fast and thus has a lot of kinetic energy.
- (c) γ -radiation is a type of non-visible light, much like radio waves, infrared light, ultraviolet light, and X-rays. When a radioactive atom transforms by emitting an α - or β -particle, it may also give off one or more γ -rays to release excess energy.

Ionizing radiation exposure can occur from a radiation source outside of the body (external radiation) or as a result of taking radioactive material into the body (internal radiation):

- i- external radiation comes from natural and man-made sources of ionizing radiation that are outside the body.
- ii- internal radiation is ionizing radiation that natural and man-made radioactive materials give off while they are inside a body. Radioactive substances enter the body every day since they are in the air, the food and the water.

(2) Optical radiation

The sun is the largest source for optical radiation and the most important source for ultraviolet radiation.

(3) Non-ionizing radiation

Non-ionizing radiation does not carry enough energy to cause ionization. It includes radio-waves, micro-waves, ultrasound and infrared radiation.

3.3.2 Theory of ionizing radiation effect of the cell

A cell is damaged when exposed to ionizing radiation, special when exposed to a burst of ionizing radiation, and the mechanism of damaged has been explained by two theories which prevail in the field of radiobiology.

1. Target theory (Atwood and Norman [5] and Kiefer [28])

This theory postulates that there exist a certain number of sensitive regions which must be hit for damage to result. According to this theory, the amount of damage depends on the probability of an individual ionizing event taking place within a sensitive region of the cell area.

2. Absorption of radiation (Atwood and Norman [5], Gupta [20] and Bansal and Gupta [6])

This theory allows for more general absorption of radiation within a cell to result in the production of chemical changes and cell damage. According to this theory, the damage caused by ionizing radiation to a living cell is likely to be repaired, given enough time and the absence of further hits.

3.3.3 Radiation effects on cell constituents

The following dose rates refer to human cells. Many studies have established that it takes about 3,000 to 5,000 rads of absorbed dose to rupture the **cell membrane**. This major injury to the cell allows extra cellular fluids to enter into the cell. Inversely, it also allows leakage of ions and nutrients from the cell.

Radiation effects on **cytoplasm** are negligible compared to observed effects on structures which are suspended within it. The first involve the **mitochondria**. It requires a few thousand rad to disrupt their function.

Another organelle within the cytoplasm that is effected by radiation is the **lysosome**. The **lysosome** will be ruptured at dose levels between 500 and 1000 rads.

The most radiologically sensitive part of the cell is the **nucleus**. Because there is a wide band of sensitivity for cell nuclei, quantifying a dose range is difficult. The major effect of radiation on the cell nucleus is the inhibition of **DNA** replication.

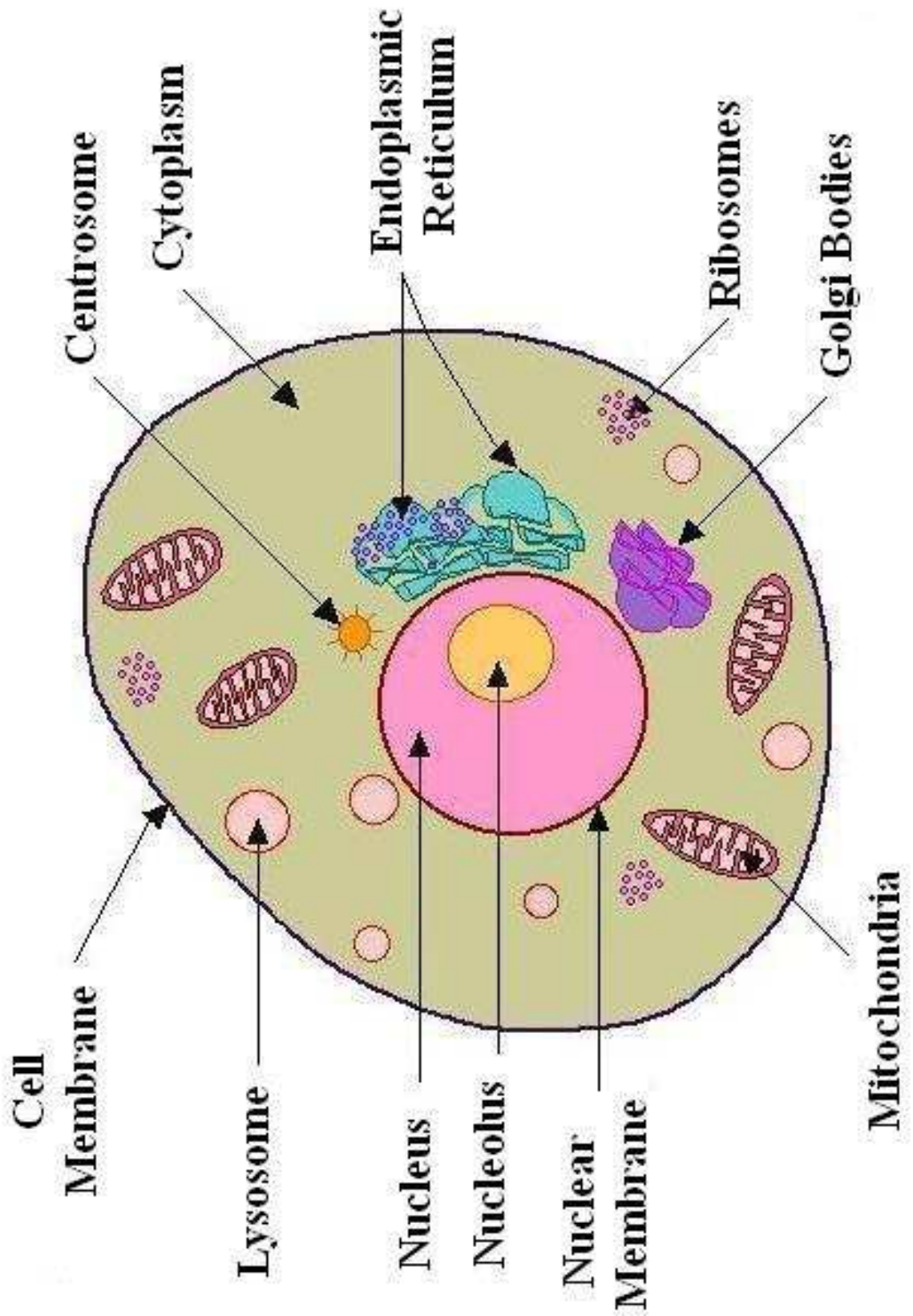


Figure 3.1: The structure of the cell.

Chapter 4

Stochastic models for cell survival after irradiation with ionizing radiation

4.1 Introduction

A cell can be damaged when it is exposed to a burst of ionizing radiation. To explain the damage mechanism, two theories prevail: the target theory and the absorption of radiation theory, which we have explained in Chapter 3.

Within a year after Roentgen's discovery of X -rays in 1895, it was learned that exposure to ionizing radiation could lead to biological damage. Since that time, a tremendous amount of research has been done attempting to interpret the reactions which take place from the moment that radiation enters a living cell until some permanent damage is produced.

Cell radiobiology has proven to be a fertile field for the application of mathematical, especially stochastic models. Probabilistic methods of data analysis have been inseparably linked with experimental research in this field [25].

The first mathematical model of the effects of radiation on microorganisms was proposed by Atwood and Norman in 1949 [5], they studied multi-hit survival curves and introduced

the target theory. The other hypothesis of the biological effect of radiation has been introduced by Kiga [29] in 1952. In 1967 Gupta studied the probability of survival of a cell with ionizing radiation [19], and in 1969 he studied a two compartment model for cell survival after ionizing radiation [20].

In 1971 Gupta and Bansal studied a stochastic model for cell survival under irradiation with exponential repair and general damage time distribution [21], In 1975 Gupta and Bansal studied a two compartment model for cell survival under irradiation with general repair time and exponential damage time distribution [22].

Neyman and Puri [39] in 1976 studied a structural model of radiation effects in living cells, a stochastic model for cell survival after irradiation was studied by Bansal and Gupta [6] in 1978, and by Agrafiotis [1] in 1982.

In 1982 Jaiswal, Karmeshu and Rangaswamy [26] studied a semi-Markovian model for cell survival after irradiation.

In this chapter, we consider several stochastic models for cell survival after irradiation.

- (a) a stochastic two compartment model for cell survival after irradiation generalizing from Bansal and Gupta [6].
- (b) a n compartment model for cell survival after ionizing irradiation generalizing Agrafiotis [1].
- (c) a semi-Markovain model for the behavior of a living cell exposed to radiations generalizing Jaiswal, Karmeshu and Rangaswamy [26].

4.2 A stochastic two compartment model for cell survival after irradiation

4.2.1 The stochastic model

Let the cell consist of two compartments (nucleus and cytoplasm), where cytoplasm consists of a ground substance, hyaloplasm, and organelles (golgi bodies, mitochondria, ... etc, see Chapter 3).

We make the following assumptions:

- (1) With respect to the effects of an irradiation process, the cell consists of two compartments C_1 (nucleus) and C_2 (cytoplasm); each compartment is in a different state of radio-sensitivity which remains constant throughout the considered period of irradiation.
- (2) As soon as a compartment is damaged a repair process begins.
- (3) If a compartment has been repaired after being damaged, the cell behaves like a normal one.
- (4) Four alternative states are possible for the condition of a cell after irradiation:
 - (i) the normal state, S_0 , in which there is no damage in the cell;
 - (ii) the reduced efficiency state S_1 , in which only the compartment C_1 is damaged and will be eventually repaired;
 - (iii) the reduced efficiency state S_2 , in which only the compartment C_2 is damaged and eventually repaired;
 - (iv) the damage state, S_3 , in which both compartments are damaged, i.e., the cell is in state S_1 and compartment C_2 is damaged before repair completion, or in

state S_2 and compartment C_1 is damaged before repair completion. This state is absorbing. Transitions among the states are shown in Figure 4.1.

The damages due to irradiation are assumed to arrive according to Poisson processes with intensities λ and μ for the two compartments, respectively (see Kiefer [28], chapters 7 and 16). Thus, for a small interval of time $(t, t + h)$, the probability of a damage of compartment C_1 is $\lambda h + o(h)$. Let $\eta_1(x)h$ be the first-order probability that a cell whose compartment C_1 has been damaged recovers to the normal state S_0 during the time interval $(x, x + h)$, given that the repair has not been completed before time x . The relation between $\eta_1(x)$ and the repair time density function of compartment C_1 , say $D_1(x)$, is

$$D_1(x) = \eta_1(x) \exp\left\{-\int_0^x \eta_1(u) du\right\}. \quad (4.2.1)$$

Similarly, for a small time interval $(t, t + h)$, the probability of a damage of compartment C_2 is $\mu h + o(h)$. We define $\eta_2(y)h$ as the first-order probability that a cell with damaged compartment C_2 recovers to the normal state S_0 in a time interval $(y, y + h)$ after the damage, given that it has not been repaired before. Again the relation between $\eta_2(y)$ and the repair time density function of compartment C_2 , say $D_2(y)$, is given by

$$D_2(y) = \eta_2(y) \exp\left\{-\int_0^y \eta_2(u) du\right\}. \quad (4.2.2)$$

When a cell is in the reduced efficiency state S_1 , the time until the next damage of compartment C_1 follows an exponential distribution with parameter λ (as a consequence of the Poisson assumption), whereas the repair time follows a general distribution determined by (4.2.1), and when a cell is in the reduced efficiency state S_2 , the time until the next damage of compartment C_2 follows an exponential distribution with parameter μ , while the repair time has the distribution determined by (4.2.2). If compartment C_1 is damaged while the cell is in state S_0 , a transition from S_0 to S_1 takes place. If a repair is completed while the cell is in state S_1 , there is a transition from S_1 to S_0 . If compartment C_2 is damaged while

the cell is in state S_1 , the cell is considered to pass into the irreparably damaged state, i.e., the transition will be $S_1 \rightarrow S_3$. Similarly, if compartment C_2 is damaged while the cell is in state S_0 , we have a transition $S_0 \rightarrow S_2$; if repair is completed in state S_2 , the transition leads from S_2 to S_0 , and if compartment C_1 is damaged while the cell is in state S_2 , we have a transition from S_2 to the absorbing state S_3 .

The probabilistic quantities of interest to us are:

- 1- $P_0(t)$, the probability that at time t a cell is in the normal state S_0 ;
- 2- $P_1(x, t) dx$, the probability that at time t a cell which is in the reduced efficiency state S_1 is under repair and the elapsed time of repair is in the infinitesimal interval $(x, x + dx)$;
- 3- $P_2(y, t) dy$, the probability that at time t , a cell which is in the reduced efficiency state S_2 is under repair and the elapsed time of repair is in the interval $(y, y + dy)$;
- 4- $P_3(t)$, the probability that at time t a cell is in the state S_3 of irreparable damage.

Our model assumptions lead to a system of differential equations for these probability functions. Consider the two real intervals $(0, t]$ and $(t, t + h]$ where h is very small. The forward equations for the process can be written as

$$P_1(x + h, t + h) = P_1(x, t)\{(1 - \eta_1(x)h)(1 - \mu h)\} + o(h), \quad (4.2.3)$$

$$P_2(y + h, t + h) = P_2(y, t)\{(1 - \eta_2(y)h)(1 - \lambda h)\} + o(h), \quad (4.2.4)$$

$$P_0(t + h) = P_0(t)\{(1 - \lambda h)(1 - \mu h)\} + \int_0^\infty P_1(x, t)\eta_1(x)h dx + \int_0^\infty P_2(y, t)\eta_2(y)h dy + o(h). \quad (4.2.5)$$

As $h \rightarrow 0$ in equations (4.2.3), (4.2.4) and (4.2.5), we obtain the differential equations

$$\frac{\partial P_1(x, t)}{\partial x} + \frac{\partial P_1(x, t)}{\partial t} + \{\eta_1(x) + \mu\}P_1(x, t) = 0, \quad (4.2.6)$$

$$\frac{\partial P_2(y, t)}{\partial y} + \frac{\partial P_2(y, t)}{\partial t} + \{\eta_2(y) + \lambda\}P_2(y, t) = 0, \quad (4.2.7)$$

and

$$\begin{aligned} \frac{dP_0(t)}{dt} = & -\{\lambda + \mu\}P_0(t) \\ & + \int_0^\infty P_1(x, t)\eta_1(x)dx + \int_0^\infty P_2(y, t)\eta_2(y)dy. \end{aligned} \quad (4.2.8)$$

Equations (4.2.6), (4.2.7) and (4.2.8) have to be solved subject to the following boundary conditions:

- (i) $P_1(0, t) = \lambda P_0(t)$ and $P_2(0, t) = \mu P_0(t)$; these equations specify that as soon as the cell enters one of the reduced efficiency states the repair process is started.
- (ii) $P_0(0) = 1$; this is the assumption that initially the cell is in the normal state.

Taking Laplace transforms in equations (4.2.6), (4.2.7) and (4.2.8), we find that

$$\frac{\partial P_1^*(x, s)}{\partial x} + \{s + \eta_1(x) + \mu\}P_1^*(x, s) = 0, \quad (4.2.9)$$

$$\frac{\partial P_2^*(y, s)}{\partial y} + \{s + \eta_2(y) + \lambda\}P_2^*(y, s) = 0, \quad (4.2.10)$$

$$\begin{aligned} \{s + \lambda + \mu\}P_0^*(s) = 1 & + \int_0^\infty P_1^*(x, s)\eta_1(x)dx \\ & + \int_0^\infty P_2^*(y, s)\eta_2(y)dy, \end{aligned} \quad (4.2.11)$$

and the boundary conditions (i) transform into

$$P_1^*(0, s) = \lambda P_0^*(s), \quad (4.2.12)$$

$$P_2^*(0, s) = \mu P_0^*(s). \quad (4.2.13)$$

Equations (4.2.9) and (4.2.10) yield

$$P_1^*(x, s) = P_1^*(0, s) \exp \left\{ - (s + \mu)x - \int_0^x \eta_1(u) du \right\}, \quad (4.2.14)$$

$$P_2^*(y, s) = P_2^*(0, s) \exp \left\{ - (s + \lambda)y - \int_0^y \eta_2(u) du \right\}. \quad (4.2.15)$$

Inserting (4.2.14) and (4.2.15) in (4.2.11), we obtain

$$\begin{aligned} \{s + \lambda + \mu\}P_0^*(s) &= 1 + P_1^*(0, s) \int_0^\infty \eta_1(x) e^{\{-(s+\mu)x - \int_0^x \eta_1(u) du\}} dx \\ &+ P_2^*(0, s) \int_0^\infty \eta_2(y) e^{\{-(s+\lambda)y - \int_0^y \eta_2(u) du\}} dy, \end{aligned} \quad (4.2.16)$$

and we have

$$\{s + \lambda + \mu\}P_0^*(s) = 1 + P_1^*(0, s)D_1^*(s + \mu) + P_2^*(0, s)D_2^*(s + \lambda), \quad (4.2.17)$$

where $D_1^*(s)$ and $D_2^*(s)$ are the Laplace transform of the density functions $D_1(x)$ and $D_2(y)$ defined by equations (4.2.1) and (4.2.2). Now using (4.2.12) and (4.2.13) in (4.2.17), we get

$$P_0^*(s) = \frac{1}{\{s + \lambda[1 - D_1^*(s + \mu)] + \mu[1 - D_2^*(s + \lambda)]\}}. \quad (4.2.18)$$

Finally, formula (4.2.18) for $P_0^*(s)$ can be substituted in equations (4.2.12) and (4.2.13), yielding

$$P_1^*(0, s) = \frac{\lambda}{\{s + \lambda[1 - D_1^*(s + \mu)] + \mu[1 - D_2^*(s + \lambda)]\}}, \quad (4.2.19)$$

$$P_2^*(0, s) = \frac{\mu}{\{s + \lambda[1 - D_1^*(s + \mu)] + \mu[1 - D_2^*(s + \lambda)]\}}. \quad (4.2.20)$$

Hence, for any repair-time probability densities we can determine the Laplace transforms of the probability functions $P_0(t)$, $P_1(x, t)$ and $P_2(y, t)$ in terms of the Laplace transforms $D_1^*(s)$ and $D_2^*(s)$.

4.2.2 Availability analysis of the model:

a) Availability of the model for cell survival after irradiation, denoted by $AC(t)$, is the probability that the cell is a live at time t , by using definition 1.1.3. Then

$$AC(t) = P_0(t) + \int_0^\infty P_1(x, t)dx + \int_0^\infty P_2(y, t)dy, \quad (4.2.21)$$

by the Laplace transform, we have

$$AC^*(s) = P_0^*(s) + \int_0^\infty P_1^*(x, s)dx + \int_0^\infty P_2^*(y, s)dy, \quad (4.2.22)$$

and from equations (4.2.14) and (4.2.15), we have

$$\begin{aligned} AC^*(s) &= P_0^*(s) + P_1^*(0, s) \int_0^\infty e^{\{-(s+\mu)x - \int_0^x \eta_1(u)du\}} dx \\ &+ P_2^*(0, s) \int_0^\infty e^{\{-(s+\lambda)y - \int_0^y \eta_2(u)du\}} dy, \end{aligned} \quad (4.2.23)$$

let

$$\begin{aligned} r_1(x) &= \frac{D_1(x)}{\eta_1(x)}, \\ r_2(y) &= \frac{D_2(y)}{\eta_2(y)}, \end{aligned}$$

we have

$$AC^*(s) = P_0^*(s) + P_1^*(0, s)r_1^*(s + \mu) + P_2^*(0, s)r_2^*(s + \lambda), \quad (4.2.24)$$

and from equations (4.2.18)-(4.2.20), we obtain

$$AC^*(s) = \frac{1 + \lambda r_1^*(s + \mu) + \mu r_2^*(s + \lambda)}{\{s + \lambda[1 - D_1^*(s + \mu)] + \mu[1 - D_2^*(s + \lambda)]\}}. \quad (4.2.25)$$

The steady-state availability of the model is given by

$$AC(\infty) = \lim_{t \rightarrow \infty} AC(t) = \lim_{s \rightarrow 0} s(AC^*(s)). \quad (4.2.26)$$

b) Renewal frequency of the model for cell survival after irradiation, let $m_r(t)$, denote the renewal frequency, i.e., the derivative of the expected number of renewal of the model having occurred up time t . A renewal of the model means return the cell to the normal state S_0 . By using definition 1.1.5, we have

$$m_r(t) = \int_0^\infty P_1(x, t) \eta_1(x) dx + \int_0^\infty P_2(y, t) \eta_2(y) dy, \quad (4.2.27)$$

by the Laplace transform, we have

$$m_r^*(s) = \int_0^\infty P_1^*(x, s) \eta_1(x) dx + \int_0^\infty P_2^*(y, s) \eta_2(y) dy, \quad (4.2.28)$$

and from equations (4.2.14) and (4.2.15), we have

$$\begin{aligned} m_r^*(s) &= P_1^*(0, s) \int_0^\infty \eta_1(x) e^{\{-(s+\mu)x - \int_0^x \eta_1(u) du\}} dx \\ &+ P_2^*(0, s) \int_0^\infty \eta_2(y) e^{\{-(s+\lambda)y - \int_0^y \eta_2(u) du\}} dy, \end{aligned} \quad (4.2.29)$$

we have

$$m_r^*(s) = P_1^*(0, s) D_1^*(s + \mu) + P_2^*(0, s) D_2^*(s + \lambda), \quad (4.2.30)$$

and from equations (4.2.18)-(4.2.20), we obtain

$$m_r^*(s) = \frac{\lambda D_1^*(s + \mu) + \mu D_2^*(s + \lambda)}{\{s + \lambda[1 - D_1^*(s + \mu)] + \mu[1 - D_2^*(s + \lambda)]\}}. \quad (4.2.31)$$

The steady-state renewal frequency of the model is given by

$$m_r(\infty) = \lim_{t \rightarrow \infty} m_r(t) = \lim_{s \rightarrow 0} s m_r^*(s). \quad (4.2.32)$$

4.2.3 Special cases

Let $\eta_1(x)$ and $\eta_2(y)$ be constant, say $\eta_1(x) \equiv \theta_1$ and $\eta_2(y) \equiv \theta_2$. In this exponential case we have

$$\begin{aligned} D_1^*(s + \mu) &= \frac{\theta_1}{s + \mu + \theta_1}, \\ D_2^*(s + \lambda) &= \frac{\theta_2}{s + \lambda + \theta_2}, \end{aligned}$$

and

$$\begin{aligned} r_1^*(s + \mu) &= \frac{1}{s + \mu + \theta_1}, \\ r_2^*(s + \lambda) &= \frac{1}{s + \lambda + \theta_2}. \end{aligned}$$

Equations (4.2.18)-(4.2.20), (4.2.25) and (4.2.31) become

$$P_0^*(s) = \frac{(s + \mu + \theta_1)(s + \lambda + \theta_2)}{\mathfrak{R}}, \quad (4.2.33)$$

$$P_1^*(0, s) = \frac{\lambda(s + \mu + \theta_1)(s + \lambda + \theta_2)}{\mathfrak{R}}, \quad (4.2.34)$$

$$P_2^*(0, s) = \frac{\mu(s + \mu + \theta_1)(s + \lambda + \theta_2)}{\mathfrak{R}}, \quad (4.2.35)$$

$$AC^*(s) = \frac{(s + \mu + \theta_1)(s + \lambda + \theta_2) + \lambda(s + \lambda + \theta_2) + \mu(s + \mu + \theta_1)}{\mathfrak{R}}, \quad (4.2.36)$$

and

$$m_r^*(s) = \frac{\lambda\theta_1(s + \lambda + \theta_2) + \mu\theta_2(s + \mu + \theta_1)}{\mathfrak{R}}, \quad (4.2.37)$$

where

$$\mathfrak{R} = s(s + \lambda + \theta_2)(s + \mu + \theta_1) + \lambda(s + \mu)(s + \lambda + \theta_2) + \mu(s + \lambda)(s + \mu + \theta_1), \quad (4.2.38)$$

and the steady state availability and renewal frequency of the model are given by

$$AC(\infty) = m_r(\infty) = 0. \quad (4.2.39)$$

We have derived formulas for the probabilities of survival of the cell in the different states after irradiation in terms of Laplace transforms.

In the case of exponential repair time distributions, (4.2.33)-(4.2.37) show how these survival probabilities depend on the parameters λ , μ , θ_1 and θ_2 . The most important function is the probability $P_0(t)$ that the cell is in the normal state S_0 at time t .

As a numerical example, we have estimated the parameters λ , μ , θ_1 and θ_2 in equation (4.2.33) from the experimental data on survival of human lymphocytes following exposure

to ionizing radiation reported by Madhavanath [37]. The estimates are given in Table 1. Laplace inversion of (4.2.33) then leads to the graphs of $P_0(t)$ for various doses of radiation displayed in Figure 4.2. The graphs show an exponential-type decrease of the probability of survival of a cell in the normal state over time; for higher doses of radiation the decrease is much faster, as expected. The renewal frequencies in figures 4.3 and 4.4 are seen to increase strongly towards a maximum and then to decrease to zero exponentially.

Dose	λ	μ	θ_1	θ_2
20 rads	0.21	0.2	0.01	0.009
80 rads	0.35	0.3	0.01	0.009
200 rads	0.57	0.4	0.01	0.009

Table 4.1: Parameters used in cell survival probability

4.3 A n compartments model for cell survival after ionizing irradiation

4.3.1 The model

Let the cell consist of n compartments (nucleus, cytoplasm, plasma membrane, golgi material, ...)

We consider the following for irradiation effects:

- (1) With respect to the effects of an irradiation process, the cell consists of n regions C_1, C_2, \dots, C_n ;
- (2) Each region is in a different state of radio-sensitivity which remains constant throughout the period of irradiation;
- (3) As soon as a region is damaged a repair process starts immediately;
- (4) If a region has been repaired after being damaged, the cell behaves like a normal one;
- (5) $n + 2$ alternative states are possible for the subsequent condition of a cell which has been irradiated:
 - (i) the normal state, say S_0 , where there is no damage in the cell,
 - (ii) the reduced efficiency states $S_i, i = 1, \dots, n$, in which the region C_i is damaged and eventually repaired,
 - (iii) the damaged state (the cell is not viable), say S_{n+1} .

When a cell is in the normal state S_0 and subject to continuous irradiation, the time until region $C_i, (i = 1; 2, \dots, n)$ is damaged while the other regions $C_1, C_2, \dots, C_{i-1}, C_{i+1}, \dots, C_n$ are functioning with normal efficiency follows an arbitrary distribution function $F_i(t), i =$

1; 2; ...; n , respectively, the cell is transferred to the state S_i , $i = 1; 2; \dots; n$, the transition is denoted $S_0 \rightarrow S_i$, ($i = 1; 2; \dots; n$).

When a cell is in the state S_i , ($i = 1; \dots; n$), the repair time of the damaged region C_i , ($i = 1; \dots; n$) obeys an arbitrary distribution function $G_i(t)$, ($i = 1; \dots; n$), if repair is completed the cell transfer to the state S_0 , the transition is denoted $S_i \rightarrow S_0$, ($i = 1; \dots; n$).

When a cell is in the reduced efficiency state S_i , ($i = 1; \dots; n$), and at the time which all another regions have damage, while C_i , ($i = 1; \dots; n$) is still under repair, the entire cell passes into the damaged state S_{n+1} , the transition is denoted $S_i \rightarrow S_{n+1}$, ($i = 1; \dots; n$).

Let $P(t)dt$ be the probability that the cell goes to the damaged state S_{n+1} at time t under the above conditions, when it is subject to continuous irradiation starting at time 0.

It is given by

$$P(t) = \int_0^t a_1(x)P_{1,n+1}(t-x)dx + \int_0^t a_2(x)P_{2,n+1}(t-x)dx + \dots + \int_0^t a_n(x)P_{n,n+1}(t-x)dx, \quad (4.3.1)$$

where

$$a_i(x) = f_i(x) \prod_{j=1, j \neq i}^n \bar{F}_j; \quad i = 1; \dots; n, \quad (4.3.2)$$

$$\bar{F}_i(x) = \int_x^\infty f_i(u)du; \quad i = 1; \dots; n, \quad (4.3.3)$$

$$P_{i,n+1}(t) = \int_0^t b_i(x)P(t-x)dx + w_i(t); \quad i = 1; \dots; n, \quad (4.3.4)$$

$$b_i(x) = g_i(x) \prod_{j=1, j \neq i}^n \bar{F}_j; \quad i = 1; \dots; n, \quad (4.3.5)$$

$$w_i(x) = \sum_{j=1, j \neq i}^n [f_j(x) \prod_{r=1, r \neq j, r \neq i}^n \bar{F}_r(x)] \int_x^\infty g_i(u)du; \quad i = 1; \dots; n. \quad (4.3.6)$$

Taking the Laplace transforms from (4.3.1) and (4.3.4), we have

$$P^*(s) = a_1^*(s)P_{1,n+1}^*(s) + a_2^*(s)P_{2,n+1}^*(s) + \dots + a_n^*(s)P_{n,n+1}^*(s), \quad (4.3.7)$$

$$P_{i,n+1}^*(s) = b_i^*(s)P^*(s) + w_i^*(s); \quad i = 1; 2; \dots; n. \quad (4.3.8)$$

Then from equations (4.3.7) and (4.3.8), we have

$$P^*(s) = \frac{\sum_{i=1}^n a_i^*(s)w_i^*(s)}{1 - \sum_{i=1}^n a_i^*(s)b_i^*(s)}. \quad (4.3.9)$$

From the above results, the probability that the cell is viable at time t is

$$R(t) = \int_t^\infty P(x) dx, \quad (4.3.10)$$

so that

$$R^*(s) = \frac{1 - P^*(s)}{s} \quad (4.3.11)$$

$$= \frac{1 - \sum_{i=1}^n a_i^*(s)[b_i^*(s) + w_i^*(s)]}{s[1 - \sum_{i=1}^n a_i^*(s)b_i^*(s)]}. \quad (4.3.12)$$

Then the mean life-time of the cell is given by

$$T = R^*(s)|_{s=0} \quad (4.3.13)$$

$$= \lim_{s \rightarrow 0} \frac{1 - \sum_{i=1}^n a_i^*(s)[b_i^*(s) + w_i^*(s)]}{s[1 - \sum_{i=1}^n a_i^*(s)b_i^*(s)]}. \quad (4.3.14)$$

4.3.2 Special cases and conclusion

Let $f_i(t) = \lambda_i e^{-\lambda_i t}$; $i = 1; 2; \dots; n$, and $k = \sum_{i=1}^n \lambda_i$, we have

$$a_i^*(s) = \frac{\lambda_i}{s + k},$$

$$b_i^*(s) = g_i(s + k - \lambda_i),$$

$$w_i^*(s) = \frac{k - \lambda_i}{s + k - \lambda_i} (1 - g_i(s + k - \lambda_i)); \quad i = 1; 2; \dots; n,$$

in equations (4.3.9), (4.3.12) and (4.3.14), we have

$$P^*(s) = \frac{\sum_{i=1}^n \frac{\lambda_i (k - \lambda_i)}{(k + s - \lambda_i)} (1 - g_i^*(s + k - \lambda_i))}{k + s - \sum_{i=1}^n \lambda_i g_i^*(s + k - \lambda_i)}, \quad (4.3.15)$$

$$\begin{aligned} R^*(s) &= \frac{s + k - \sum_{i=1}^n \frac{\lambda_i}{k + s - \lambda_i} [k - \lambda_i + s g_i^*(s + k - \lambda_i)]}{s [k + s - \sum_{i=1}^n \lambda_i g_i^*(s + k - \lambda_i)]} \\ &= \frac{1 + \sum_{i=1}^n \frac{\lambda_i [1 - g_i^*(s + k - \lambda_i)]}{k + s - \lambda_i}}{k + s - \sum_{i=1}^n \lambda_i g_i^*(s + k - \lambda_i)}, \end{aligned} \quad (4.3.16)$$

mean life-time of the cell is given by:

$$\begin{aligned}
 T &= \lim_{s \rightarrow 0} \frac{s + k - \sum_{i=1}^n \frac{\lambda_i}{k + s - \lambda_i} [k - \lambda_i + s g_i^*(s + k - \lambda_i)]}{s [k + s - \sum_{i=1}^n \lambda_i g_i^*(s + k - \lambda_i)]} \\
 &= \frac{1 + \sum_{i=1}^n \frac{\lambda_i [1 - g_i(k - \lambda_i)]}{k - \lambda_i}}{k - \sum_{i=1}^n \lambda_i g_i(k - \lambda_i)}, \tag{4.3.17}
 \end{aligned}$$

at $n=2$, we have

$$P^*(s) = \frac{\lambda_1 \lambda_2 [(s + \lambda_1)(1 - g_1^*(s + \lambda_2)) + (s + \lambda_2)(1 - g_2^*(s + \lambda_1))]}{(s + \lambda_1)(s + \lambda_2)[s + \lambda_1(1 - g_1^*(s + \lambda_2)) + \lambda_2(1 - g_2^*(s + \lambda_1))]}, \tag{4.3.18}$$

and the mean life-time of the cell is given by:

$$T = \frac{\lambda_1 \lambda_2 + \lambda_1^2 (1 - g_1(\lambda_2)) + \lambda_2^2 (1 - g_2(\lambda_1))}{\lambda_1 \lambda_2 [\lambda_1 (1 - g_1(\lambda_2)) + \lambda_2 (1 - g_2(\lambda_1))]}, \tag{4.3.19}$$

where

$$g_i(\lambda_j) = \int_0^{\infty} e^{-\lambda_j t} g_i(t) dt, \tag{4.3.20}$$

this result is obtained by Agrafiotis [1].

Let $g_i(t) = \mu_i e^{-\mu_i t}$; $i = 1; 2; \dots; n$, we have

$$a_i^*(s) = \frac{\lambda_i}{s + k},$$

$$b_i^*(s) = \frac{\mu_i}{s + \mu_i - \lambda_i + k},$$

$$w_i^*(s) = \frac{k - \lambda_i}{s + \mu_i - \lambda_i + k}; \quad i = 1; 2; \dots; n,$$

in equations (4.3.15)-(4.3.17) we have

$$P^*(s) = \frac{\sum_{i=1}^n \frac{\lambda_i(k - \lambda_i)}{k + s + \mu_i - \lambda_i}}{k + s - \sum_{i=1}^n \frac{\lambda_i \mu_i}{k + s + \mu_i - \lambda_i}}, \quad (4.3.21)$$

$$\begin{aligned} R^*(s) &= \frac{s + k - \sum_{i=1}^n \frac{\lambda_i(k + \mu_i - \lambda_i)}{k + s + \mu_i - \lambda_i}}{s[k + s - \sum_{i=1}^n \frac{\lambda_i \mu_i}{k + s + \mu_i - \lambda_i}]} \\ &= \frac{1 + \sum_{i=1}^n \frac{\lambda_i}{k + s + \mu_i - \lambda_i}}{k + s - \sum_{i=1}^n \frac{\lambda_i \mu_i}{k + s + \mu_i - \lambda_i}}, \end{aligned} \quad (4.3.22)$$

the mean life-time of the cell is given by:

$$\begin{aligned} T &= \lim_{s \rightarrow 0} \frac{s + k - \sum_{i=1}^n \frac{\lambda_i(k + \mu_i - \lambda_i)}{k + s + \mu_i - \lambda_i}}{s[k + s - \sum_{i=1}^n \frac{\lambda_i \mu_i}{k + s + \mu_i - \lambda_i}]} \\ &= \frac{1 + \sum_{i=1}^n \frac{\lambda_i}{k + \mu_i - \lambda_i}}{k - \sum_{i=1}^n \frac{\lambda_i \mu_i}{k + \mu_i - \lambda_i}}. \end{aligned} \quad (4.3.23)$$

The relation between the probability that the cell is viable and, time the cell exposed to a burst of ionizing radiation shown in figure 4.5 at $n = 6$ and figure 4.6 at $n = 2$.

These results are agreement with Madhavanath [37], Bansal and Gupta [6] and Section 4.2.

- (1) Let $\lambda_i = \lambda$ and $\mu_i = \mu, i = 1; 2; \dots; n$, we have $k = n\lambda$ and

$$P^*(s) = \frac{n(n-1)\lambda^2}{s^2 + s[\mu + (2n-1)\lambda] + n(n-1)\lambda^2}, \quad (4.3.24)$$

$$R^*(s) = \frac{s + \mu + (2n-1)\lambda}{s^2 + s[\mu + (2n-1)\lambda] + n(n-1)\lambda^2}, \quad (4.3.25)$$

The mean life-time of the cell is given by:

$$T = \frac{\mu + (2n - 1)\lambda}{n(n - 1)\lambda^2}; (n > 1), \quad (4.3.26)$$

at $n = 2$, we have

$$P^*(s) = \frac{2\lambda^2}{s^2 + s[\mu + 3\lambda] + 2\lambda^2}, \quad (4.3.27)$$

and

$$T = \frac{\mu + 3\lambda}{2\lambda^2}. \quad (4.3.28)$$

(2) Let $n = 2$, $\lambda_1 \neq \lambda_2$ and $\mu_1 \neq \mu_2$ we have

$$P^*(s) = \frac{\lambda_1\lambda_2(\lambda_1 + \lambda_2 + \mu_1 + \mu_2 + 2s)}{C}, \quad (4.3.29)$$

$$R^*(s) = \frac{C_1}{C}, \quad (4.3.30)$$

where

$$C = (\lambda_1 + \lambda_2 + s)(\lambda_1 + \mu_2 + s)(\lambda_2 + \mu_1 + s) - \lambda_1\mu_1(\lambda_1 + \mu_2 + s) - \lambda_2\mu_2(\lambda_2 + \mu_1 + s),$$

$$C_1 = (\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \mu_1\mu_2 - \lambda_1\lambda_2 + s(2(\lambda_1 + \lambda_2) + \mu_1 + \mu_2) + s^2,$$

$$T = \frac{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \mu_1 + \mu_2) + \mu_1\mu_2 - \lambda_1\lambda_2}{(\lambda_1 + \lambda_2)(\lambda_1 + \mu_2)(\lambda_2 + \mu_1) - \lambda_1\mu_1(\lambda_1 + \mu_2) - \lambda_2\mu_2(\lambda_2 + \mu_1)}, \quad (4.3.31)$$

at $\lambda_1 = \lambda_2 = \lambda$ and $\mu_1 = \mu_2 = \mu$ we have the result in equation (4.3.27) and (4.3.28).

4.4 A semi-Markovian model for cell survival after irradiation with ionizing radiation

4.4.1 The model

In this model for cell survival after irradiation one assumes the existence of be sensitive targets. We number the states from $0, \dots, k$ where 0 denotes the normal state, k the state of a damage in the nucleus and $1, \dots, k - 1$ other damage states. An ionizing event or hit of a target initiates changes leading to an observable effect like genetic change or death of the cell. When a target i is hit, the cell is brought into damage state i . It is likely that the cell sooner or later recovers from damage state i and returns to the normal state (state 0) by virtue of a self-repair mechanism. It may also happen that due to further irradiation, the cell is unable to recover and becomes functionally impaired or undergoes alteration. In damage states $1, 2, \dots, k - 1$, the probability of recuperation is high since the nucleus is unaffected.

For example, water can be converted by radiation into highly reactive compounds of short life span such as hydrogen peroxide and these highly reactive compounds may in turn act upon the genetic material of the cell inflicting further damage on the cell, bringing it from state i where $i = 1, \dots, k - 1$ to the damaged state k .

In state k the probability of recovery is obviously smaller than that in either of the states $1, 2, \dots, k - 1$. The cell in state k therefore either undergoes mutation with a high probability to become an altered cell or returns to normal state with a low probability. The cell damage due to irradiation and subsequent recovery the cell may moves stochastically among the states $0, 1, \dots, k$ to become finally an altered cell state $k + 1$.

The assumptions underlying the model are as follows:

- (i) A living cell is a single entity with respect to the effects of an irradiation process.

- (ii) The cell may be in one of $k + 2$ states: 0 denotes the normal state; $i = 1, 2, \dots, k$, denote the damage states; $k + 1$ denotes the cell in altered state. Irradiation may bring the cell from state 0 to one of the damage states i , where $i = 1, 2, \dots, k$. A repair process immediately begins and if accomplished, the cell returns to state 0; otherwise if in state k the cell goes into the altered state $k + 1$ and if in state $i \in \{1, \dots, k - 1\}$ the cell can go to k or $k + 1$. Transitions among the states are shown in Figure 4.7.
- (iii) The cell moves from one state to another with random sojourn time.
- (iv) The successive states visited form a Markov chain and the sojourn time has a distribution which depends on the state being visited and the next state to be entered.

Under the above assumptions, the stochastic behavior of the cell can be described by an "absorbing" semi-Markov process resulting from the Markov renewal process $\{X_n, T_n, n \geq 0\}$, where T_n represents the n -th transition epoch and X_n is $\{0, 1, \dots, k, k + 1\}$ -valued and represents the state entered by the system at T_n . Let $T_0 = 0$ and

$$Q(t) = [Q_{ij}(t)], \tag{4.4.1}$$

where

$$Q_{ij}(t) = Pr\{X_{n+1} = j, T_{n+1} - T_n \leq t | X_n = i\}, \tag{4.4.2}$$

is the semi-Markov kernel of the process. Then

$$Q(t) = \begin{pmatrix} 0 & Q_{01}(t) & Q_{02}(t) & Q_{03}(t) & \dots & Q_{0k-1}(t) & Q_{0k}(t) & 0 \\ Q_{10}(t) & 0 & 0 & 0 & \dots & 0 & Q_{1k}(t) & Q_{1k+1}(t) \\ Q_{20}(t) & 0 & 0 & 0 & \dots & 0 & Q_{2k}(t) & Q_{2k+1}(t) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ Q_{k-10}(t) & 0 & 0 & 0 & \dots & 0 & Q_{k-1k}(t) & Q_{k-1k+1}(t) \\ Q_{k0}(t) & 0 & 0 & 0 & \dots & 0 & 0 & Q_{kk+1}(t) \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & Q_{k+1k+1}(t) \end{pmatrix}. \tag{4.4.3}$$

The associated Markov chain has the transition probability matrix

$$\mathbb{P} = [Q_{ij}(\infty)] = \begin{pmatrix} 0 & p_{01} & p_{02} & p_{03} & \dots & p_{0k-1} & p_{0k} & 0 \\ p_{10} & 0 & 0 & 0 & \dots & 0 & p_{1k} & p_{1k+1} \\ p_{20} & 0 & 0 & 0 & \dots & 0 & p_{2k} & p_{2k+1} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ p_{k-10} & 0 & 0 & 0 & \dots & 0 & p_{k-1k} & p_{k-1k+1} \\ p_{k0} & 0 & 0 & 0 & \dots & 0 & 0 & p_{kk+1} \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix} \quad (4.4.4)$$

Let $B = \{0, 1, \dots, k\}$ be the set of transient states. At any time t the state of the system is denoted by the variable

$$Y_t = \begin{cases} X_n \in B; & \text{if } T_n \leq t < T_{n+1} < L, n \geq 0, \\ k + 1; & \text{if } t \geq L, \end{cases} \quad (4.4.5)$$

where

$$L = \sup_n \{T_n\},$$

is the life span of the cell, i.e., the time taken by the cell to be transformed from the normal state into the "genetic change or death of the cell" altered state. We have that $Y = \{Y_t, t \geq 0\}$ is a semi-Markov process.

We study the following probabilities:

- (1) Distribution of L , the life time of the cell,
- (2) $\phi_{ij}(t) = Pr\{Y_t = j | Y_0 = i\}$; $i, j \in B$,
- (3) Expected time spent in each state during the cell life.

4.4.2 The life span of the cell

Let L_i represent the life span of the cell starting from state $i \in B$ at $t = 0$ and let

$$\alpha_i(t) = Pr\{L_i \leq t\}; \quad i = 0, 1, \dots, k. \quad (4.4.6)$$

Then

$$\begin{aligned}
\alpha_0(t) &= \alpha_1(t) \odot Q_{01}(t) + \alpha_2(t) \odot Q_{02}(t) + \dots + \alpha_i(t) \odot Q_{0i}(t) \\
&\quad + \dots + \alpha_k(t) \odot Q_{0k}(t); \\
\alpha_1(t) &= Q_{1k+1}(t) + \alpha_0(t) \odot Q_{10}(t) + \alpha_k(t) \odot Q_{1k}(t); \\
\alpha_2(t) &= Q_{2k+1}(t) + \alpha_0(t) \odot Q_{20}(t) + \alpha_k(t) \odot Q_{2k}(t); \\
\dots &= \dots; \\
\alpha_i(t) &= Q_{ik+1}(t) + \alpha_0(t) \odot Q_{i0}(t) + \alpha_k(t) \odot Q_{ik}(t); \\
\dots &= \dots; \\
\alpha_{k-1}(t) &= Q_{k-1k+1}(t) + \alpha_0(t) \odot Q_{k-10}(t) + \alpha_k(t) \odot Q_{k-1k}(t); \\
\alpha_k(t) &= Q_{kk+1}(t) + \alpha_0(t) \odot Q_{k0}(t).
\end{aligned} \tag{4.4.7}$$

By using Laplace transforms in the system of equations (4.4.7), we have

$$\begin{aligned}
\alpha_0^*(s) &= \alpha_1^*(s)Q_{01}^*(s) + \alpha_2^*(s)Q_{02}^*(s) + \dots + \alpha_i^*(s)Q_{0i}^*(s) \\
&\quad + \dots + \alpha_k^*(s)Q_{0k}^*(s); \\
\alpha_1^*(s) &= Q_{1k+1}^*(s) + \alpha_0^*(s)Q_{10}^*(s) + \alpha_k^*(s)Q_{1k}^*(s); \\
\alpha_2^*(s) &= Q_{2k+1}^*(s) + \alpha_0^*(s)Q_{20}^*(s) + \alpha_k^*(s)Q_{2k}^*(s); \\
\dots &= \dots; \\
\alpha_i^*(s) &= Q_{ik+1}^*(s) + \alpha_0^*(s)Q_{i0}^*(s) + \alpha_k^*(s)Q_{ik}^*(s); \\
\dots &= \dots; \\
\alpha_{k-1}^*(s) &= Q_{k-1k+1}^*(s) + \alpha_0^*(s)Q_{k-10}^*(s) + \alpha_k^*(s)Q_{k-1k}^*(s); \\
\alpha_k^*(s) &= Q_{kk+1}^*(s) + \alpha_0^*(s)Q_{k0}^*(s).
\end{aligned} \tag{4.4.8}$$

From (4.4.8), we have

$$\alpha_0^*(s) = \frac{\sum_{i=1}^k Q_{0i}^*(s)Q_{ik+1}^*(s) + \sum_{i=1}^{k-1} Q_{0i}^*(s)Q_{ik}^*(s)Q_{kk+1}^*(s)}{1 - \sum_{i=1}^k Q_{0i}^*(s)Q_{i0}^*(s) - \sum_{i=1}^{k-1} Q_{0i}^*(s)Q_{ik}^*(s)Q_{k0}^*(s)}. \quad (4.4.9)$$

Let $k = 2$. We have

$$\alpha_0^*(s) = \frac{Q_{01}^*(s)Q_{13}^*(s) + Q_{02}^*(s)Q_{23}^*(s) + Q_{01}^*(s)Q_{12}^*(s)Q_{23}^*(s)}{1 - Q_{01}^*(s)Q_{10}^*(s) - Q_{02}^*(s)Q_{20}^*(s) - Q_{01}^*(s)Q_{12}^*(s)Q_{20}^*(s)}. \quad (4.4.10)$$

This result was obtained by Jaiswal et al. [26].

4.4.3 Probability of the cell being in different states

Let $\phi_{ij} = Pr\{Y_t = j | Y_0 = i\}$ denote the conditional probability of the cell being in state j where starting from state i . Let $R_{ij}(t)$ denote the mean number of visits to the state j during $[0, t]$ if the system started in state i and

$$H_j(t) = 1 - \sum_{i \in B} Q_{ji}(t), j \in B, t \geq 0. \quad (4.4.11)$$

Lemma 4.4.1. *Çinlar [9], p.337*

For any $i, j \in B$ and $t \geq 0$ we have

$$\phi_{ij}(t) = \int_0^t dR_{ij}(u)H_j(t-u). \quad (4.4.12)$$

Then by standard arguments, where $R(t) = [R_{ij}(t)]$ is the Markov renewal kernel corresponding to $Q(t)$ defined in equation (4.4.3), the LST of $R(t)$ is given by

$$R^*(s) = [1 - Q^*(s)]^{-1}$$

$$= \frac{1}{\delta} \begin{pmatrix} r_{00} & r_{01} & r_{02} & \dots & r_{0k-1} & r_{0k} \\ r_{10} & r_{11} & r_{12} & \dots & r_{1k-1} & r_{1k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ r_{k0} & r_{k1} & r_{k2} & \dots & r_{kk-1} & r_{kk} \end{pmatrix}. \quad (4.4.13)$$

where

$$\begin{aligned} \delta &= 1 - \sum_{i=1}^k Q_{0i}^*(s)Q_{i0}^*(s) - \sum_{i=1}^{k-1} Q_{0i}^*(s)Q_{ik}^*(s)Q_{k0}^*(s), \\ r_{00} &= 1, \\ r_{0i} &= Q_{0i}^*(s); \quad i = 1, 2, \dots, k-1, \\ r_{0k} &= Q_{0k}^*(s) + \sum_{i=1}^{k-1} Q_{0i}^*(s)Q_{ik}^*(s). \end{aligned}$$

We assume the existence of density functions and denote them by lower case letters.

From equations (4.4.12), we have

$$\begin{aligned} \phi_{00}^*(s) &= \frac{R_{00}^*(s)H_0^*(s)}{s\delta} \\ &= \frac{1}{s} \left(1 - \sum_{i=1}^k q_{0i}^*(s) \right) \\ &\quad \left\{ 1 - \sum_{i=1}^k q_{0i}^*(s)q_{i0}^*(s) - \sum_{i=1}^{k-1} q_{0i}^*(s)q_{ik}^*(s)q_{k0}^*(s) \right\}^{-1}. \end{aligned} \quad (4.4.14)$$

For $k = 2$ we see that

$$\begin{aligned} \phi_{00}^*(s) &= \frac{1}{s} (1 - q_{01}^*(s) - q_{02}^*(s)) \\ &\quad \left\{ 1 - q_{01}^*(s)q_{10}^*(s) - q_{02}^*(s)q_{20}^*(s) - q_{01}^*(s)q_{12}^*(s)q_{20}^*(s) \right\}^{-1}. \end{aligned} \quad (4.4.15)$$

This result was obtained by Jaiswal et al. [26].

4.4.4 Expected time spent in each state

Let u_{ij} denote the expected time in state j during $[0, t]$, the initial state being i . Then (see Çinlar [9], p.339),

$$u_{ij}(t) = \int_0^t dR_{ij}(u) \int_0^{t-u} H_j(u) du, \quad (4.4.16)$$

and the expectation of the total time spent in state j is

$$u_{ij} = R_{ij}(\infty)m_j = ([1 - \mathbb{P}]^{-1})_{ij}m_j, \quad (4.4.17)$$

where

$$m_j = \int_0^\infty H_j(u) du, \quad (4.4.18)$$

is the mean sojourn time in state j and \mathbb{P} defined by equation (4.4.4).

Therefore

$$\begin{aligned} u_{00} &= \frac{m_0}{\delta_1}, \\ u_{0i} &= \frac{m_i p_{0i}}{\delta_1}; \quad i = 1, 2, \dots, k-1, \\ u_{0k} &= \frac{m_k}{\delta_1} \left(\sum_{i=1}^{k-1} p_{0i} p_{ik} + p_{0k} \right), \end{aligned}$$

where

$$\delta_1 = 1 - \sum_{i=1}^k p_{0i} p_{i0} - \sum_{i=1}^{k-1} p_{0i} p_{ik} p_{k0}.$$

Then the expected life span of the cell is

$$E(L_0) = \frac{1}{\delta_1} \left(m_0 + \sum_{i=1}^k p_{0i} m_i + m_k \sum_{i=1}^{k-1} p_{0i} p_{ik} \right). \quad (4.4.19)$$

Equation (4.4.19) can be obtained directly from equation (4.4.9) or from Barlow and Proschan [7]. The variance of L_0 can also be obtained from equation (4.4.9).

For $k = 2$ we have

$$E(L_0) = \frac{1}{\delta_2}(m_0 + m_1 p_{01} + m_2 p_{02} + m_2 p_{01} p_{02}), \quad (4.4.20)$$

where

$$\delta_2 = 1 - p_{01} p_{10} - p_{02} p_{20} - p_{01} p_{12} p_{20},$$

a result obtained by Jaiswal et al. [26].

4.4.5 Special cases

Consider the cell irradiation problem with Poisson damage and arbitrary repair process.

Let λ_{ij} , $i \neq j$, be the damage rate from state i to state j , ($i = 0, 1, \dots, k, j = 1, 2, \dots, k+1$)

and $d_{i0}(t)$ be the repair time density from the state i , $i = 1, 2, \dots, k$ to the normal state 0.

We have

$$q_{0i}(t) = \lambda_{0i} \exp\left[-\sum_{j=1}^k \lambda_{0j} t\right]; \quad i = 1; 2; \dots; k, \quad (4.4.21)$$

$$q_{k0}(t) = d_{k0}(t) \exp[-\lambda_{kk+1} t], \quad (4.4.22)$$

and where $i = 1; \dots; k-1$,

$$q_{i0}(t) = d_{i0}(t) \exp[-(\lambda_{ik+1} + \lambda_{ik}) t], \quad (4.4.23)$$

$$q_{ik}(t) = \lambda_{ik} \exp[-(\lambda_{ik} + \lambda_{ik+1}) t]. \quad (4.4.24)$$

For the Laplace transforms we have

$$q_{0i}^*(s) = \frac{\lambda_{0i}}{(\sum_{j=1}^k \lambda_{0j} + s)}; \quad i = 1; 2; \dots; k, \quad (4.4.25)$$

$$q_{i0}^*(s) = d_{i0}^*(\lambda_{ik+1} + \lambda_{ik} + s); \quad i = 1; 2; \dots; k-1, \quad (4.4.26)$$

$$q_{k0}^*(s) = d_{k0}^*(\lambda_{kk+1} + s), \quad (4.4.27)$$

$$q_{ik}^*(s) = \frac{\lambda_{ik}}{(s + \lambda_{ik} + \lambda_{ik+1})}; \quad i = 1; 2; \dots; k-1. \quad (4.4.28)$$

Let

$$\lambda = \sum_{j=1}^k \lambda_{0j}.$$

From equation (4.4.14), we have

$$\begin{aligned} \phi_{00}^*(s) = & \left\{ (\lambda + s) - \lambda_{0k} d_{k0}^*(\lambda_{kk+1} + s) \right. \\ & \left. - \sum_{i=1}^{k-1} \lambda_{0i} \left(d_{i0}^*(\lambda_{ik} + \lambda_{ik+1} + s) + \frac{\lambda_{ik} d_{i0}^*(\lambda_{kk+1} + s)}{(\lambda_{ik} + \lambda_{ik+1} + s)} \right) \right\}^{-1}. \end{aligned} \quad (4.4.29)$$

For $k = 2$, it follows that

$$q_{0i}^*(s) = \frac{\lambda_{0i}}{(\sum_{j=1}^2 \lambda_{0j} + s)}; \quad i = 1, 2, \quad (4.4.30)$$

$$q_{10}^*(s) = d_{10}^*(\lambda_{13} + \lambda_{12} + s), \quad (4.4.31)$$

$$q_{20}^*(s) = d_{20}^*(\lambda_{23} + s), \quad (4.4.32)$$

$$q_{12}^*(s) = \frac{\lambda_{12}}{(s + \lambda_{12} + \lambda_{13})}, \quad (4.4.33)$$

and

$$\begin{aligned} \phi_{00}^*(s) = & \left\{ (\lambda_{01} + \lambda_{02} + s) - \lambda_{02}d_{20}^*(\lambda_{23} + s) \right. \\ & \left. - \lambda_{01} \left(d_{10}^*(\lambda_{12} + \lambda_{13} + s) + \frac{\lambda_{12}d_{20}^*(\lambda_{23} + s)}{(\lambda_{12} + \lambda_{13} + s)} \right) \right\}^{-1}. \end{aligned} \quad (4.4.34)$$

A similar result for $k = 2$ was obtained by Jaiswal et al. [26], however his equations corresponding to (4.4.30)-(4.4.34) were not correct.

Let $d_{10}(t) = \theta_1 e^{-\theta_1 t}$ and $d_{20}(t) = \theta_2 e^{-\theta_2 t}$ we have

$$\phi_{00}^*(s) = \frac{\mathfrak{R}_1}{\mathfrak{R}_2}, \quad (4.4.35)$$

where

$$\begin{aligned} \mathfrak{R}_1 &= (\lambda_{12} + \lambda_{13} + \theta_1 + s)(s + \lambda_{23} + \theta_2)(\lambda_{12} + \lambda_{13} + s) \\ \mathfrak{R}_2 &= (\lambda_{01} + \lambda_{02} + s)(\lambda_{12} + \lambda_{13} + \theta_1 + s)(s + \lambda_{23} + \theta_2)(\lambda_{12} + \lambda_{13} + s) \\ &\quad - \theta_2(\lambda_{12} + \lambda_{13} + \theta_1 + s) \{ \lambda_{02}(\lambda_{12} + \lambda_{13} + s) + \lambda_{01}\lambda_{12} \} \\ &\quad - \theta_1\lambda_{01}(\lambda_{23} + \theta_2 + s)(\lambda_{12} + \lambda_{13} + s). \end{aligned}$$

The relation between the conditional probability of the cell being in state 0 where starting from state 0 and the time the cell was exposed to a burst of ionizing radiation is shown in Figure 4.8.

Set

$$\begin{aligned} \lambda_{12} &= \lambda_{02} = \lambda_{23} = \theta_2 = 0, \\ \lambda_{01} &= \lambda_{13} = \lambda, \\ \theta_1 &= \mu, \end{aligned}$$

in equation (4.4.35). We find that

$$\phi_{00}^*(s) = \frac{s + \lambda + \mu}{(s + \lambda + \mu)^2 - \mu(s + \lambda + \mu) - \lambda\mu}, \quad (4.4.36)$$

a result obtained by Bansal and Gupta [6]. Putting

$$\begin{aligned} \lambda_{12} &= 0, \\ \lambda_{02} &= \lambda_{13} = \mu, \\ \lambda_{01} &= \lambda_{23} = \lambda, \end{aligned}$$

in equation (4.4.35), we get

$$\phi_{00}^*(s) = \frac{(s + \mu + \theta_1)(s + \lambda + \theta_2)}{(s + \mu + \theta_1)(s + \lambda + \theta_2) + \lambda(s + \mu)(s + \lambda + \theta_2) + \mu(s + \mu + \theta_1)(s + \lambda)}. \quad (4.4.37)$$

This result was already obtained in Section 4.2 (equation (4.2.33)).

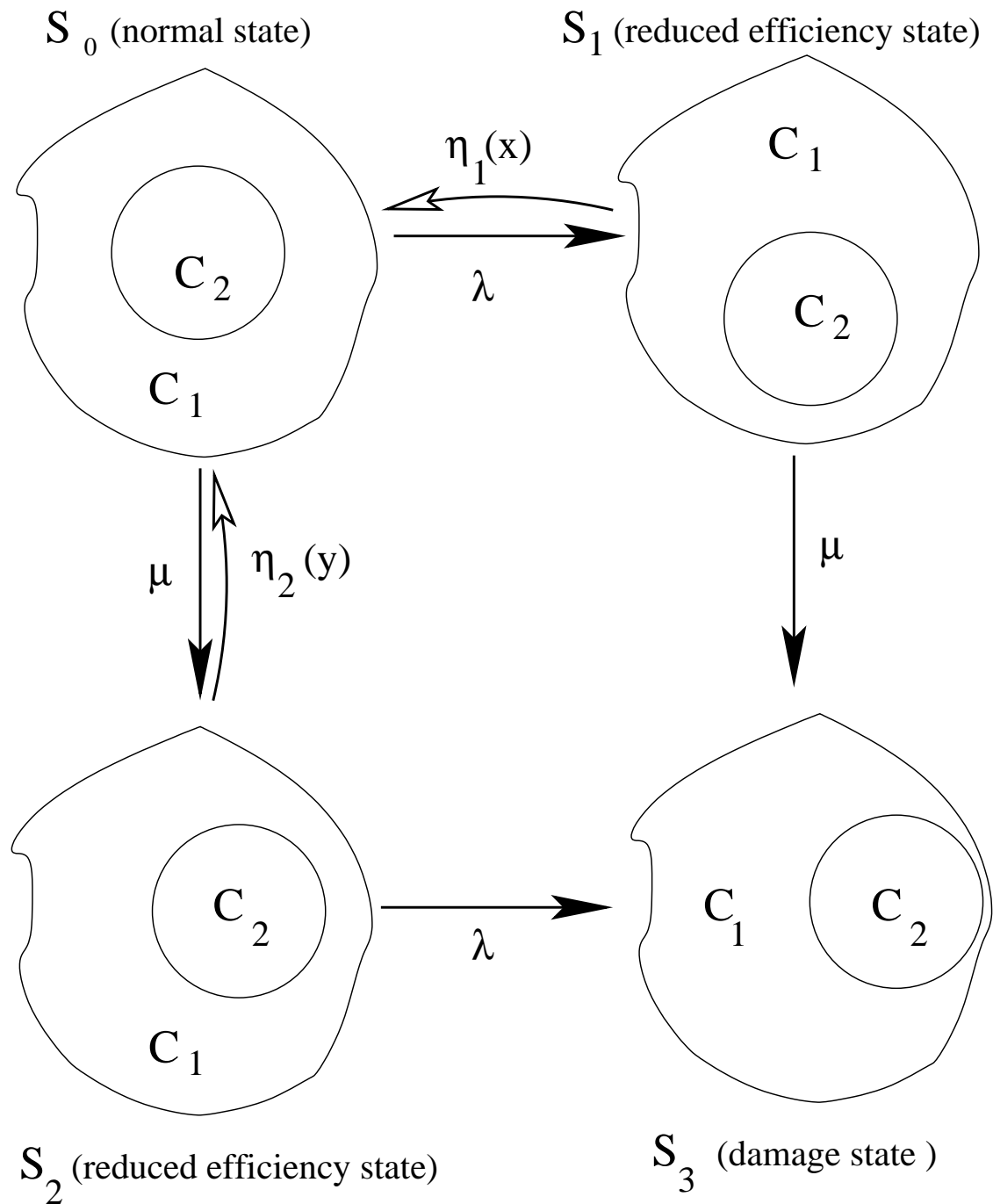


Figure 4.1: Transitions among the states of the model.

survival probability

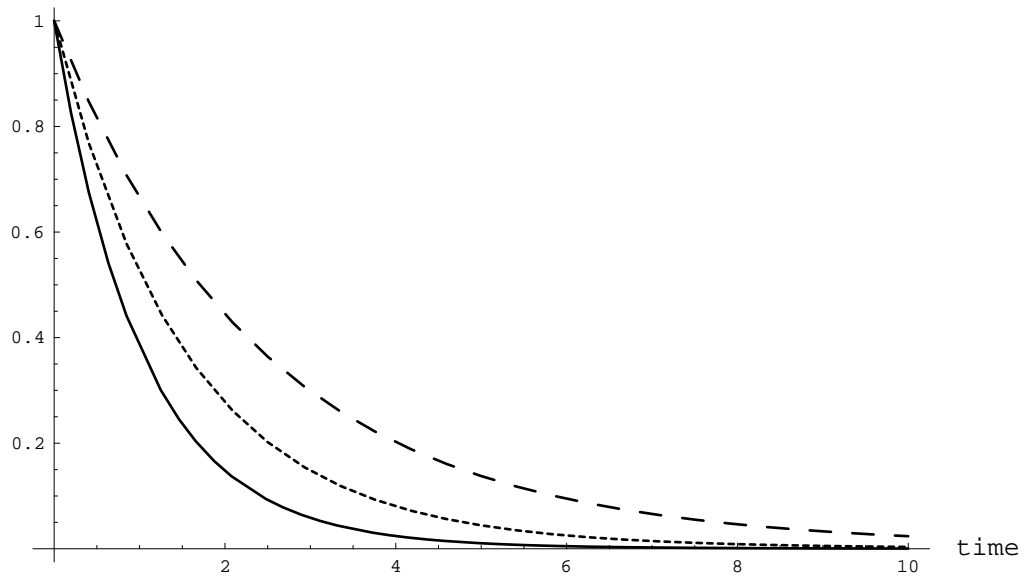


Figure 4.2: The survival probability of the cell after ionizing radiation, under 200 rads (solid), 80 rads (dotted) and 20 rads (dashed).

availability

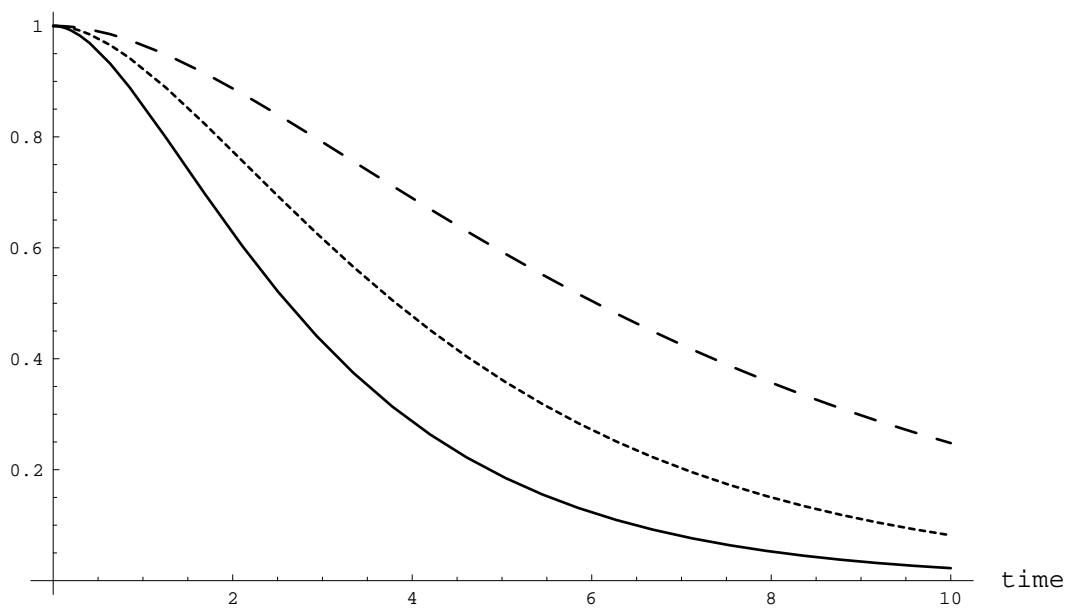


Figure 4.3: The availability of cell after ionizing radiation, under 200 rads (solid), 80 rads (dotted) and 20 rads (dashed).

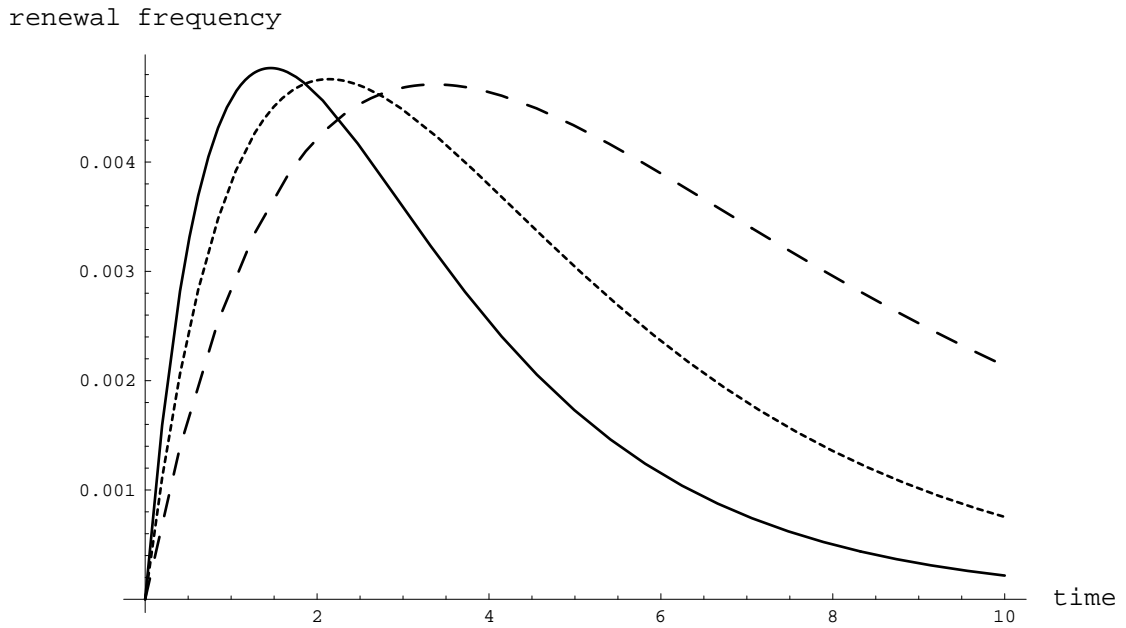


Figure 4.4: The renewal frequency of cell after ionizing radiation, under 200 rads (solid), 80 rads (dotted) and 20 rads (dashed).

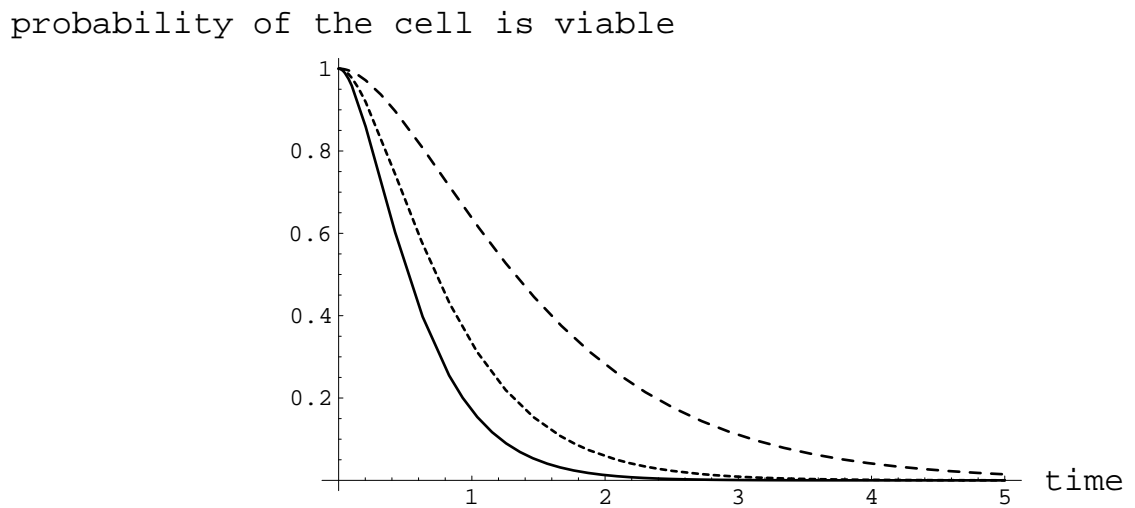


Figure 4.5: The relation between the probability that the cell is viable and, time at $n = 6$, under 200 rads (solid), 80 rads (dotted) and 20 rads (dashed).

probability of the cell is viable

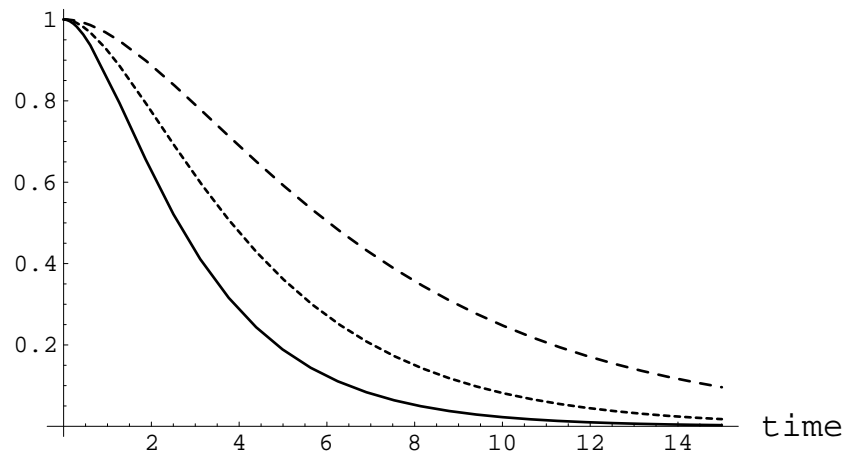


Figure 4.6: The relation between the probability that the cell is viable and time t at $n = 2$, under 200 rads (solid), 80 rads (dotted) and 20 rads (dashed).

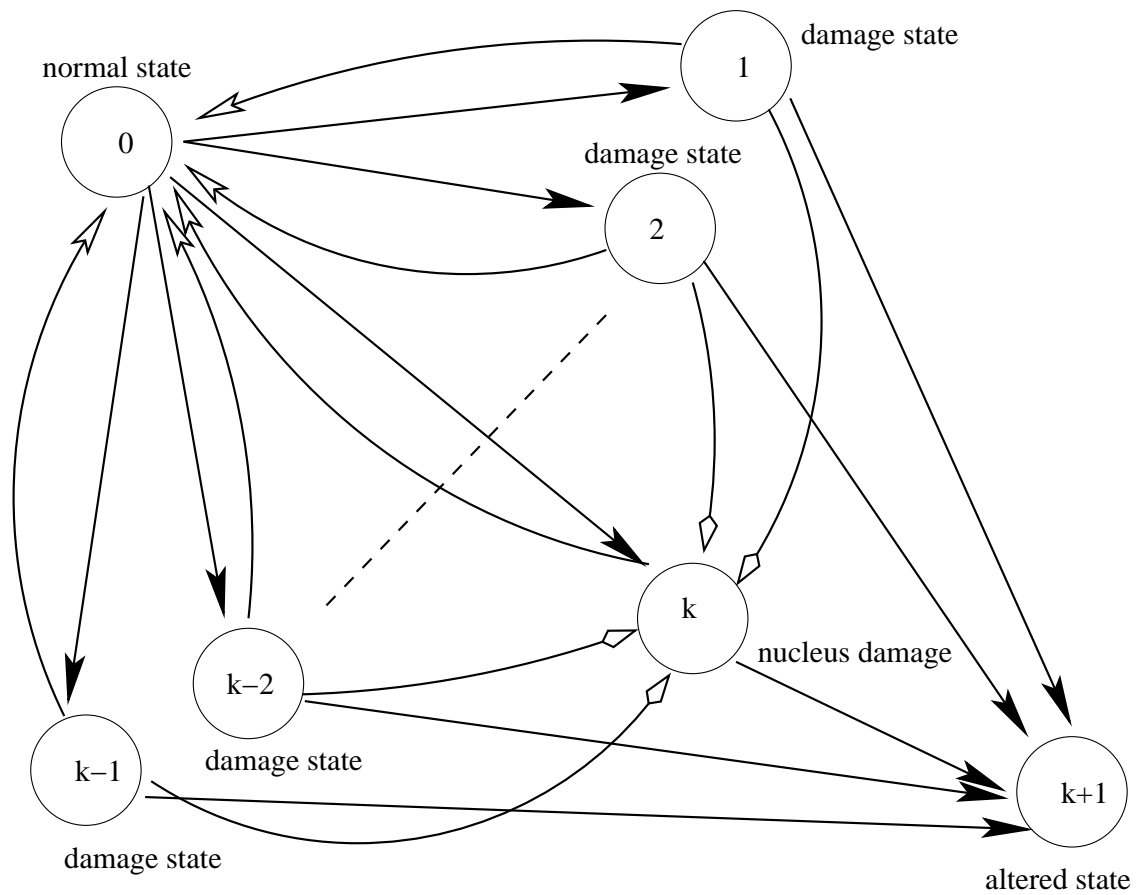


Figure 4.7: Transitions among the states of the model.

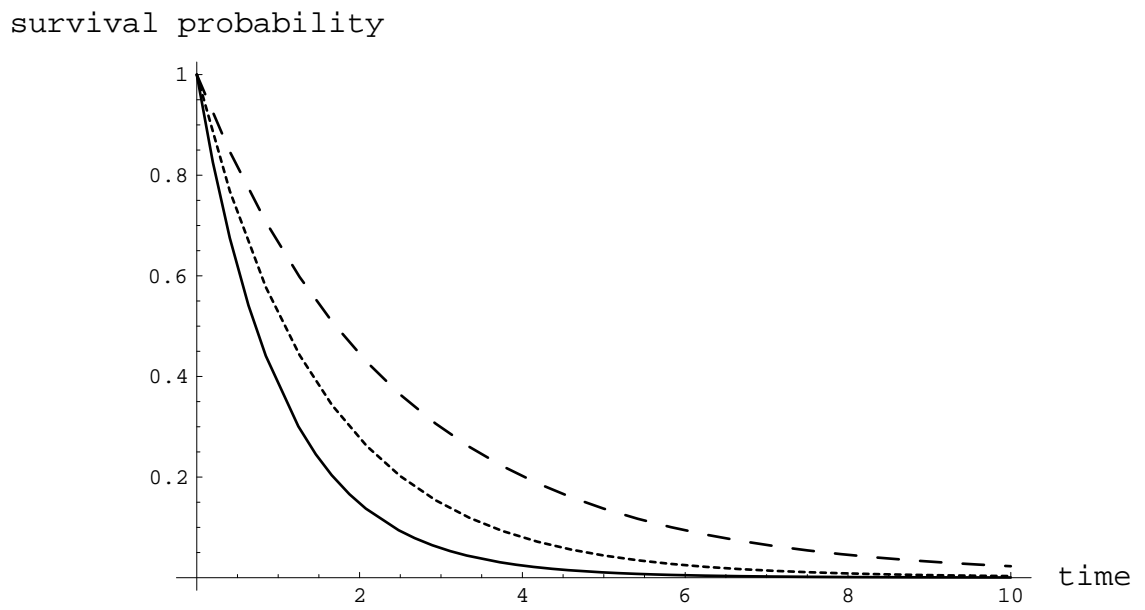


Figure 4.8: The survival probability of the cell after ionizing radiation, under 200 rads (solid), 80 rads (dotted) and 20 rads (dashed).

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