Optimal Sequential Selection of a Gambler Assessed by the Prophet

Thesis

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1 Introduction

Regard the following optimal stopping problem: Suppose a finite number n of objects are presented sequentially and are identified by the numerical values of independent random variables $X_1, X_2, \ldots, X_n$. A gambler gets the true values of these offers and pretends to select one according to a nonanticipating stopping time S in order to maximize his payoff. If he rejects an offer it can never be recalled. The payoff function f depends on the gambler's choice X_S and on the overall maximum $Y_n := X_1 \vee \cdots \vee X_n$, which emerges in the very end. Intuitively f should obey certain monotonicity criteria. Evidently the overall maximum Y_n is not known in advance by the gambler, who is uninformed about future values, but it would be chosen by a prophet, who is equivalent to a gambler with complete foresight.
Regarding the abilities of the gambler this means to maximize the functional

$$\mathrm{E}\left(f\left(X_S, Y_n\right)\right)$$

with respect to nonanticipating stopping times S, where the approach is based on the gambler's mean payoff regarding repeated sequential selection. The gambler is informed about n, f and the joint distribution of the offers.

This full information optimal stopping problem arises from subsequent cases: In the discrete time full information best choice problem of Gilbert and Mosteller [18] or Bojdecki [6] the payoff in every realization is either 0 for failure or 1 for a win (i.e. 1 only if $X_S = Y_n$). Suppose for example that as requirement for a win at least 80% of the overall maximum Y_n would be sufficient. More general an offer x is called r–candidate if $x \geq r(Y_n)$, where the socalled relax function r lies below identity. The objective now is to maximize the probability of selection of an r–candidate — a full information good choice problem due to relaxed demands.
Another application of the presented functional is given by the ratio of gambler's choice and prophet's value, i.e. maximization of $\mathrm{E}\left(X_S/Y_n\right)$.
Finally an interesting problem in this setting is to maximize the duration of owning an r–candidate. Here the payoff function additionally depends on the time S of selection. This represents an extension of the full information case of the duration problem given by Ferguson et al. [14].

Resumed in every realization a chosen offer is measured at the end by the overall maximum, the gambler's choice being assessed by the prophet's value.

Abstract

Reviewing first connections to the literature then in chapter 2 for a finite number of stochastically independent offers a payoff function is considered, depending on the chosen value X_S and on the overall maximum Y_n — subject to conditions which ensure the regular case, including dependence on the time S of selection. An optimal stopping time is indicated. As a main subproblem optimal selection of an r–candidate is treated, which includes the asymptotic value in special cases, the inspection of diverse suboptimal stopping times and an extension to a Markov process. Particularly the asymptotic value of sequences of concurrent threshold rules is derived, the myopic stopping time is specified and the access of the gambler is restricted in a sense. Another subproblem is presented by the mean of the ratio of the gambler's choice and the prophet's value, where again threshold rules are studied also.
Subsequent in chapter 3 the environment is extended to a random number of observations: Offers arrive at random times, the periods between arrivals being iid, and the horizon up to which items can be accepted is fixed or random. Concerning fixed horizon for the arrival times stress is layed on the geometric distribution and the exponential distribution, where the problem is verifed to be regular and an optimal stopping time is indicated. For the latter resp. for the Poisson process, referring to selection of an r–candidate, the myopic stopping time is considered and asymptotic equivalence to discrete time is displayed. For a random horizon, referring to selection of an r–candidate, some small cases are worked out and an optimal stopping time is described in the twice exponential as well as in the twice geometric case. In situations where an optimal stopping time seems to be inaccessible due to failure of the regular case the myopic stopping time is specified.
Finally in chapter 4 the concept of an r–candidate is applied to the duration problem, wherefore a distinction between an overall and a temporary r–candidate makes sense. First the duration of owning an r–candidate is investigated for a finite number of offers where with regard to recall the myopic stopping time is verified to be optimal and the asymptotic behaviour is described. Then the duration problem with discounted epochs is resolved. Farther the duration of owning an r–candidate is considered for the Poisson process, where the horizon is taken to be fixed or exponentially distributed. Concerning the former case three kinds of access are distinguished: No recall, permanent recall and event time recall. Optimal stopping times are specified if the problem proves to be regular, otherwise its borders are indicated.

Connections to the Literature

The stochastic optimization problems treated in this thesis are optimal stopping problems based on full information. Particularly for the case of selection of an r–candidate relations to significant problems in the literature are illustrated below, where the order is full information problems first, then some links to related no information cases are given and third topics related to this thesis in the broader sense are mentioned. Finally survey literature is given and some significant directions of variants of related sequential selection problems is listed.

The full information best choice problem for a finite number of offers is given in Gilbert and Mosteller [18] (section 3), being based on heuristic arguments and including several ramifications. An exact solution thereof is published by Bojdecki [6], which also contains the corresponding problem for the Poisson process with finite horizon, while the compact version of the latter case in Gnedin and Sakaguchi [21] includes the specification of the value as a function of the arrival rate. The case of the Poisson process with random horizon is available in Bojdecki [5]. The full information best choice problem with a random number of offers in discrete time is given in Porosinski [24]. These articles correspond with the case $r \equiv id$ of optimal selection of an r–candidate of this thesis.

In the no information best choice problem, the classic secretary problem, items only can be ranked by the gambler. The case with a finite number of secretaries is for example presented in Shiryaev [32] and a random number of offers in discrete time with island solutions is studied Presman and Sonin [25]. The secretary problem in the situation of a Poisson process is given in Cowan and Zabczyk [10].
The kind of assessment investigated in this thesis, particularly the case of finite valued payoff, bears analogy to the no information problem of Yeo and Yeo [38]: A finite number of secretaries, rankable without ties, are associated with nonincreasing weights according to their ranks and presented in order to select, without recall, a secretary for a single position with the aim of maximization of the weighted probability. Explicit expressions are found for main probabilities and numerical methods are required for optimization.

Numerical methods for calculation of the value of maximizing the functional $f(S, X_S)$, where S denotes the stopping time, for a Markov chain without restrictions are presented in Darling [12].
In Chen and Starr [8] selection without replacement from an urn, filled with balls numbered serially, is treated: A gambler intends to select a number with recall in order to maximize functional $\mathrm{E}\left(f(S, Y_S)\right)$, where S is the stopping time and Y_S denotes the present maximum number. The payoff function, nonincreasing resp. nondecreasing in the first resp. second component, obeys conditions which ensure the myopic stopping time to be optimal.
In the field of prophet theory, see Harten et al. [22], the main approach is to compare the mean of the gambler's win and that of the prophet with regard to specific sets of joint distributions of the offers, whereas in this thesis and in the literature it is based on the gambler and the prophet are compared directly for each realization, which may be called maximization regarding (the mean of) repeated selection or maximization of a functional. Particularly the payoff functional $\mathrm{E}\left(X_S/Y_n\right)$ treated in section 2.2 may be an appendage for ratio prophet inequalities.
There is also a relation to the optimal multivariate stopping problem of Assaf and Samuel–Cahn [2], where $h(E(X(S)))$ with d–dimensional random vectors $X(1), \ldots, X(n)$ is the functional (based on d cooperating partners for each component). This value is compared with the socalled classical case $E(h(X(S)))$, the stopping time S referring to gamblers and to prophets. Dependence structures of $X(1), \ldots, X(n)$ are mentioned there in remark 4.4 and in case of $E(h(X(S)))$ this relates to the subject of this thesis.
Two player competitive situations are studied in game theory with zero sum game interpretation, see Sakaguchi [29]. Players are provided with different information referring to the values of the offers (e.g. complete foresight versus nonanticipation), the abilities of access is specified (e.g. recall and no recall) and a dominance or decision rule for joint access is declared.

A survey concerning the secretary problem and its ramifications may be found in chapter 16 of the handbook edited by Ghosh [16], in the reviews of Freeman [15] and a discussion thereof in Ferguson [13]. In Sakaguchi [29] there is a survey concerning game theory with two players. A more general view to optimal stopping referring to choice theory is given in Gnedin [19].

Subsequent significant variants of optimal stopping problems related to this thesis are listed, including references to some exemplary articles:

- The mode of sequential presentation of offers:
  A fixed or a random number of offers in discrete time or continuous time with random interarrival times and with fixed or random horizon.
- Connection resp. joint distribution of the offers:
  iid, independence, Markovian, correlation and dependencies.
- Changes in the grade of information a gambler is provided with:
  No, full and partial information (for the latter for instance imperfect observation referring to excess of a specified level, Sakaguchi [28], or knowledge of a subset of continuous distributions, Petruccelli [23], or information about exchangeability, the game of gogool, Gnedin [19]).
- Enlargement or reduction of access of a gambler to offers:
  Allowance of recall, restricted recall or limited recall (i.e. memory, Tamaki [37]), an object may be unavailable (Ano [1]) and the number of available offers is restricted by random freeze (Samuel–Cahn [31]).
- Comparability of offers:
  Ordinal structure of presented variants (Gnedin [20]) and offers may not possess a total order (Stadje [33]).
- Variation of the functional:
  Relaxations and variants of best choice problems: Maximizing the probability of choosing at least the k–th best (Sakaguchi and Szajowski [30]), the best and second best (Tamaki [36]) or the k–th best offer gets weight w_k (nonincreasing, Yeo and Yeo [38]).
  Maximizing the expected value (for instance Shiryaev [32]), minimizing the expected rank (Assaf and Samuel–Cahn [3]) and the payoff may respect costs of observations (Stadje [35]). The duration of owning a sufficiently good offer (no and full information in Ferguson et. al. [14]).
- Several choices or gamblers:
  Multiple choice (Stadje [34]), collective choice and payment for cooperative gamblers (Assaf and Samuel–Cahn [2]) and game theory with competitive players, specific information structures and selection criteria (Sakaguchi [29]).

2 Optimal Selection in Discrete Time

In this chapter a finite number of objects is presented sequentially whereof a gambler intends to choose one in order to maximize a given payoff function. The gambler watches out for his profit regarding repeated selection, which is equivalent to maximize the mean payoff referring to his strategy.

Mathematical Model

Based on a probability space $(\Omega, \mathcal{A}, \mathrm{P})$ let $X_1, X_2, \ldots$ denote a sequence of iid random variables with continuous distribution function. These are offers presented sequentially to a gambler. Let $Y_k := \max\{X_1, \ldots, X_k\}$, $k \in \mathbb{N}$, denote relative maxima. Set additionally $X_0 := X_\infty := 0$. Let $\mathcal{F}_k := \sigma(X_0, \ldots, X_k)$ contain information unil time $k \in \mathbb{Z}_+$ and let $\mathcal{F} := (\mathcal{F}_k)_{k \in \mathbb{Z}_+}$ represent the filtration. Let $\mathcal{S}$ denote the set of stopping times with respect to $\mathcal{F}$, i.e. random variables S with $\mathrm{P}(S \in \mathbb{Z}_+) = 1$ and $[S_k \leq k] \in \mathcal{F}_k$ for $k \in \mathbb{Z}_+$. Let the horizon $1 < n \in \mathbb{N}$ for selection be fixed. The payoff function f performs the assessment of the gambler's choice by the value of the prophet: Profit $f(X_k, Y_n)$ is payed to the gambler, if he selects the k–th object X_k and if finally the overall maximum Y_n occurs. It is assumed that the joint distribution of the offers, the number n of objects presented and the payoff function f are familiar to the gambler. Furthermore from now on the values $X_1, X_2, \ldots$ of the offers are supposed to be uniformly distributed on $[0, 1]$, where an equivalence to other distributions is mentioned in the paragraph on pages 17f. The payoff function f is assumed to be bounded and monotone according to assumptions (1) below.
The corresponding optimal stopping problem $\mathcal{P}_n = \mathcal{P}_n(f, U([0,1]))$ for $n \in \mathbb{N}$ ($n = 1$ is allowed for convenience) is to find the value of the problem

$$\sup_{S \in \mathcal{S}_n} \mathrm{E}\left(f\left(X_S, Y_n\right)\right)$$

and, if possible, find a stopping time in $\mathcal{S}_n$ attaining this value; here $\mathcal{S}_n \subset \mathcal{S}$ denotes the set of stopping times with respect to $\mathcal{F}$ which don't exceed n. The value chosen by the gambler is $X_S := X_{S(\omega)}(\omega)$ for $\omega \in \Omega$ and $S \in \mathcal{S}$.
For each stopping time $S \in \mathcal{S}_n$ let $v_n(S) := \mathrm{E}(f(X_S, Y_n))$ denote the mean payoff applying S, the value of S. The value of $\mathcal{P}_n$ is abbreviated by $v_n^* := \sup_{S \in \mathcal{S}_n} v_n(S)$. The asymptotic value of $(\mathcal{P}_n)_{n \in \mathbb{N}}$ is $v_\infty^* := \lim_{n \to \infty} v_n^*$, provided this limit exists.

Suppose k offers are presented to a gambler, $1 \leq k \leq n$. Then next to the number k only the values of X_k and Y_{k-1} represent significant information of the gambler. This information is covered by Markov process (time homogenous by the corresponding time space process) and thus problem $\mathcal{P}_n$ corresponds to optimal stopping of $Z := (Z_k)_{k \in \mathbb{Z}_+}$: Initial state $Z_0 := \alpha_0$, $Z_k := (k, X_k, Y_k)$ for $k = 1, \ldots, n$, and final state $Z_k := \alpha_\infty$ for $k > n$. Let $\Delta := \{(x,y) \in [0,1]^2 : x \leq y\}$ and let $E := \{1, \ldots, n\} \times \Delta$. The state space of Z then is $E \cup \{\alpha_0, \alpha_\infty\}$ and transition probabilities are evident. The payoff function is $g(k,x,y) := f(x, y \vee X_{k+1} \vee \cdots \vee X_n)$, where $g(\alpha_0) := 0 =: g(\alpha_\infty)$.

The state space is Δ if the point in time is fixed. In this and in the subsequent chapter the payoff function f is assumed to be bounded on Δ and

$$\begin{aligned} &f(x,y) \text{ is nondecreasing in } x \\ &f(x,y) \text{ is nonincreasing in } y \\ &f(z,z) \text{ is nondecreasing in } z, \end{aligned} \tag{1}$$

where $(x,y) \in \Delta$ and $z \in [0,1]$. Thus f is measurable. Without loss of generality the range of f is $[0,1]$ and $f(0,1) = 0$. Intuitively the payoff $f(x,y)$ should be nondecreasing in the gambler's choice x and nonincreasing in the prophet's value y. To motivate the third assumption suppose payoff function $x(1-y)$, which only violates the monotonicity condition on the diagonal. Then it might be advisible for the gambler to reject a big value X_k (close to 1, the value maximal possible) only because it is a new present maximum, i.e. because $X_k > Y_{k-1}$, which isn't reasonable.
For the proof that $\mathcal{P}_n$ is regular (definition see below) the payoff function will additionally be allowed to depend on the time of selection.

For later reference an abstract of notations (put in italic) and facts concerning optimal stopping of a discrete time stochastic process is given:

General Approach of Optimal Stopping and Notations

Let a stochastic process $Z := (Z_k)_{k \in \mathbb{Z}_+}$ be observed sequentially by a gambler. The state space of Z is assumed to be $E \subset \mathbb{R}^j$, $j \in \mathbb{N}$, equipped with Borel sets and the corresponding filtration is denoted by $\mathcal{F} := (\mathcal{F}_k)_{k \in \mathbb{Z}_+}$, where $\mathcal{F}_k := \sigma(Z_0, \ldots, Z_k)$ for $k \in \mathbb{Z}_+$. Let $h : E \to [0,1]$ denote a payoff function, especially bounded. Set $Z_\infty := \alpha_\infty$ and $h(\alpha_\infty) := 0$. The objective is to find the *value* $v(e) := \sup_{S \in \mathcal{S}} \mathrm{E}\,(\, h(Z_S) \mid Z_0 = e)$ for $e \in E$, where $\mathcal{S}$ denotes the

set of stopping times with respect to $\mathcal{F}$, i.e. event $[S \leq k] \in \mathcal{F}_k$ for $k \in \mathbb{Z}_+$. Particularly the aim is to specify, if possible, a stopping time $S^* \in \mathcal{S}$ such that $v(e) = \mathrm{E}\,(\,h(Z_{S^*}) \mid Z_0 = e)$ for $e \in E$, then S^* is called *optimal*. $S \in \mathcal{S}$ is called *suboptimal* if merely $v(e) \geq \mathrm{E}\,(\,h(Z_S) \mid Z_0 = e)$ is ensured.
This approach is adapted to problem $\mathcal{P}_n$, where a future maximum may affect the payoff by setting $Z_k := (k, X_k, Y_k, \mathrm{E}\,(\,f(X_k, Y_n) \mid X_k, Y_k))$ for $k \in \mathbb{N}$ and $h(Z_k) := \mathrm{E}\,(\mathrm{E}\,(\,f(X_k, Y_n) \mid X_k, Y_k))$, the mean of the fourth component.
The optimal stopping problem is called *monotone* if $\Gamma_0 \subset \Gamma_1 \subset \Gamma_2 \subset \cdots$ where $\Gamma_k := \{e \in E \,:\, h(e) \geq \mathrm{E}\,(\,h(Z_{k+1}) \mid \mathcal{F}_k,\, Z_k = e)$ for $k \in \mathbb{Z}_+$, which is a notation introduced by Chow et al. [9].
The principle of *backward induction* is displayed for the general case of an infinite sequence due to subsequent chapters: For $k \in \mathbb{Z}_+$ let $\mathcal{S}^k$ denote the set of stopping times S with respect to $\mathcal{F}$ with $k \leq S$. For $k \in \mathbb{Z}_+$ the random variables $W_k := \text{ess sup}_{S \in \mathcal{S}^k} \mathrm{E}\,(\,h(Z_S) \mid \mathcal{F}_k)$ are integrable (due to bounded payoff). They satisfy the relation $W_k = \max\{h(Z_k),\, \mathrm{E}\,(\,W_{k+1} \mid \mathcal{F}_k)\}$ for any $k \in \mathbb{Z}_+$. Furthermore the sequence $(W_k)_{k \in \mathbb{Z}_+}$ yields a minimal supermartingale dominating $(h(Z_k))_{k \in \mathbb{Z}_+}$ (due to nonnegative payoff). Then $S^* := \inf\{k \in \mathbb{Z}_+ \,:\, h(Z_k) = W_k\}$, where $\inf_\emptyset := \infty$ yields payoff $h(\alpha_\infty) = 0$, represents an optimal stopping time iff $\mathrm{P}\,(S^* < \infty) = 1$. A proof is given in section 1.5 of Gihman and Skorohod [17]. Randomization of stopping times doesn't increase the value, since in the finite case the initial step and therefore any step of the backward induction would yield the same and in the general case the essential supremum doesn't change (modulo the probability measure) by regarding the specific realizations.
From now on suppose that Z is a discrete time Markov process.
Then W_k represents the value within $\mathcal{S}^k$ given $Z_k = e$ for fixed $e \in E$ (specifying the significant information of $\mathcal{F}_k$), which is denoted by $v_k(e)$ for $k \in \mathbb{Z}_+$. Now stopping sets $\Delta_k := \{e \in E \,:\, h(e) = v_k(e)\}$ for $k \in \mathbb{Z}_+$ are defined whose first hitting time is optimal if it is finite almost surely. The problem is called *regular* or the regular case is valid if $\Delta_0 \subset \Delta_1 \subset \Delta_2 \subset \cdots$.
The *myopic stopping time* or *one step look–ahead rule* compares stopping versus proceeding one step, which formally represents the first hitting time of $\Gamma_k := \{e \in E \,:\, h(e) \geq (\mathbb{P}_k h)(e)\}$ for $k \in \mathbb{Z}_+$ where $(\mathbb{P}_k h)(e) := \mathrm{E}\,(\,h(Z_{k+1}) \mid Z_k = e)$ for $e \in E$ (take infinity if $\Gamma_0, \Gamma_1, \ldots$ are never entered). This myopic stopping time turns out to be optimal in several well–known cases (eventual first suitable transformation). A sufficient criterion for optimality of the myopic stopping time is that its stopping sets prove to be *closed* and *realizable*: $\mathrm{P}\,(\,Z_j \in \Gamma_j\ \forall j > k \mid Z_k = e) = 1$ for $e \in \Gamma_k$ and $k \in \mathbb{Z}_+$ and

$\mathrm{P}(\exists j \in \mathbb{Z}_+ : Z_j \in \Gamma_j) = 1$ — i.e. the stopping sets $\Gamma_0, \Gamma_1, \ldots$ are reached sometime and then never left each with probability 1, in a sense a Markov version of the monotone case. This notation is subject to Porosinski [24] and a proof for sufficiency is given in Cowan and Zabczyk [11], where the situation is an embedded, discrete time Markov process and sufficient for realizability is that the myopic stopping time is finite or that $h(Z_k)$ vanishes as $k \to \infty$, each with probability one.
In case of a homogenous Markov process Z, which is achievable by introducing the corrsponding time space process, the notation is as follows: The first hitting time of the set $\{e \in E : h(e) = v(e)\}$ is optimal if it is finite almost surely. The myopic stopping time is the first hitting time of the set $\{e \in E : h(e) \geq (\mathbb{P}h)(e)\}$, where $\mathbb{P}$ now is independent of time $k \in \mathbb{Z}_+$.

An Optimal Stopping Time

Referring to the preceding paragraph the existence of an optimal stopping time for problem $\mathcal{P}_n$ is ensured via backward induction. For state $(x, y) \in \Delta$ and time $1 \leq k < n$ the mean payoff of stopping and that of proceeding optimally depends on the number $\ell := n - k$ of draws remaining rather on n and k, since ℓ specifies the distribution of $X_{k+1} \vee \cdots \vee X_n$. Thus given $(x, y) \in \Delta$ and $\ell \in \mathbb{Z}_+$ subsequent functions apply for n and k with $n - k = \ell$.

Definition 2.1 *Let state $(x, y) \in \Delta$ be given and suppose $\ell \in \mathbb{Z}_+$ items remain, referring to problem $\mathcal{P}_n$ with $n > \ell$. The mean payoff of stopping in this situation is denoted by*

$$s_\ell(x, y) := \mathrm{E}\,(\, f(x, Y_n) \mid X_{n-\ell} = x,\, Y_{n-\ell} = y\,)\,.$$

The mean payoff of proceeding at least one step and then selecting optimally is $c_0 :\equiv 0$ if $\ell = 0$ and otherwise

$$c_\ell(x, y) := \mathrm{E}\,(\, f(X_i S^*, Y_n) \mid X_{n-\ell} = x,\, Y_{n-\ell} = y,\, S^* > n - \ell \text{ optimal}\,)\,.$$

The mean payoff of an optimal decision or the value of (x, y) is denoted by

$$v_\ell(x, y) := \max\{s_\ell(x, y),\, c_\ell(x, y)\}\,.$$

The subsequent lemma is evident:

Lemma 2.2 *Let $(x, y) \in \Delta$ be given and let $\ell \in \mathbb{Z}_+$. Then*

i) $s_\ell(x, y)$ is nonincreasing in ℓ, nondecreasing in x and nonincreasing in y and

$$s_\ell(x, y) = \int_0^1 f(x, y \vee \zeta)\, d\zeta^\ell = y^\ell f(x, y) + \int_y^1 f(x, \zeta)\, d\zeta^\ell.$$

Particularly $s_\ell(x, y)$ is increasing in x with $x \leq y$ if $f(x, y)$ is.

ii) $c_\ell(x, y) = c_\ell(y)$ is nonincreasing in y and for $\ell \in \mathbb{N}$ the relation $c_\ell(y) = \int_0^y v_{\ell-1}(\xi, y)\, d\xi + \int_y^1 v_{\ell-1}(\xi, \xi)\, d\xi$ is valid.

iii) $v_\ell(x, y) = \max\{s_\ell(x, y),\, c_\ell(y)\}$.

Evidently for fixed $\ell \in \mathbb{N}$ function $c_\ell(y)$ can't increase in y since the demands for future selection grow. On the other hand for fixed y in general the sequence $(c_\ell(y))_{\ell \in \mathbb{N}}$ isn't monotone.

Lemma 2.3 *The optimal stopping problem $\mathcal{P}_n(f)$ is regular.*

The proof is shifted to lemma 2.9, where independent offers with varying continuous distribution function and time dependent payoff are treated.

The following observation proves to be crucial: If it is optimal to stop in state $(x, y) \in \Delta$ where $\ell \in \mathbb{Z}_+$ items remain, then it is also optimal to stop in any state $(\xi, y \vee \xi)$ for $\xi \in (x, 1]$ — due to $f(x, y) \leq f(\xi, y \vee \xi)$, i.e. $s_\ell(x, y) \leq s_\ell(\xi, y \vee \xi)$, and due to $c_\ell(y) \geq c_\ell(y \vee \xi)$. Therefore the following rule specifies an optimal stopping time for problem $\mathcal{P}_n(f)$: "For fixed k and y stop if x exceeds a certain value b^*, otherwise continue" — here b^* depends on y and on $\ell := n-k$ (the number of draws left, which identifies the distribution of $X_{k+1} \vee \cdots \vee X_n$). The following notation is used:

Notation 2.4 A *boundary function* $b_\ell : [0, 1] \to [0, 1]$, combined for $\ell = n-1, \ldots, 0$, is an instruction to specify a stopping time S which considers the present maximum: $S := \inf\{1 \leq k \leq n : X_k \geq b_{n-k}(Y_k)\}$ (set $\inf_\emptyset := \infty$ with resulting payoff 0 almost surely due to $X_\infty = 0$).

Thus for a number $\ell \in \mathbb{Z}_+$ of remaining offers and for present maximum $y \in [0, 1]$ there is a unique critical value $b^*_\ell(y)$ such that it is optimal to select the topical item x if $x \geq b^*_\ell(y)$ — b^*_ℓ is called optimal boundary function.

Definition 2.5 *Sequence of optimal boundary functions* $(b_\ell^*(y))_{\ell\in\mathbb{Z}_+}$*:*
Let $b_0^* :\equiv 0$ *on* $[0,1]$*. For* $\ell \in \mathbb{N}$ *let*

$$b_\ell^*(y) := \inf\left\{x \in [0,1] \,:\, s_\ell(x, y \vee x) \geq c_\ell(y \vee x)\right\}.$$

The corresponding optimal stopping sets Δ_ℓ^* *for* $\ell \in \mathbb{Z}_+$ *are defined by*

$$\Delta_\ell^* := \left\{(x,y) \in \Delta \,:\, x \geq b_\ell^*(y)\right\}.$$

In addition socalled lower boundary points $(\underline{b}_\ell)_{\ell\in\mathbb{Z}_+}$ *are introduced:*

$$\underline{b}_\ell := \inf\left\{x \in [0,1] \,:\, s_\ell(x,x) \geq c_\ell(x)\right\}.$$

Sets for infima in the definition above are nonempty, since $s_\ell(1,1) \geq c_\ell(1)$. The lower boundary point $\underline{b}_\ell$ represents a threshold for selection of a new present maximum $Y_{n-\ell}$ while applying optimal behaviour.
According to lemma 2.3 the regular case guarantees $b_0^* \preceq b_1^* \preceq b_2^* \preceq \cdots$ or

$$\Delta_0^* \supset \Delta_1^* \supset \Delta_2^* \supset \cdots.$$

Therefore $c_\ell(y)$ can be expressed by $b_{\ell-1}^*(y), \ldots, b_0^*(y)$, at least for certain y:

Lemma 2.6 *Let* $(x,y) \in \Delta$ *and let* $\ell \in \mathbb{N}$*. Let* $b_j^*(y)$ *for* $j = \ell-1, \ldots, 0$ *be given. Then for* $y \in [\underline{b}_{\ell-1}, 1]$ *the following holds (set* $\prod_\emptyset := 1$*):*

$$c_\ell(y) \;=\; \sum_{i=0}^{\ell-1}\left(\prod_{j=i+1}^{\ell-1} b_j^*(y)\right)\cdot\int_{b_i^*(y)}^1 s_i(\xi, y\vee\xi)\; d\xi. \tag{2}$$

Proof: Due to monotonicity of the optimal boundary functions $b_\ell^*(y)$

$$\begin{aligned}
c_\ell(y) &= \int_0^{b_{\ell-1}^*(y)} c_{\ell-1}(y)\; d\xi + \int_{b_{\ell-1}^*(y)}^1 s_{\ell-1}(\xi, y\vee\xi)\; d\xi \\
&= c_{\ell-1}(y) b_{\ell-1}^*(y) + \int_{b_{\ell-1}^*(y)}^1 s_{\ell-1}(\xi, y\vee\xi)\; d\xi \\
&= c_{\ell-2}(y) b_{\ell-2}^*(y) b_{\ell-1}^*(y) \;+ \\
&\quad b_{\ell-1}^*(y)\cdot\int_{b_{\ell-2}^*(y)}^1 s_{\ell-2}(\xi, y\vee\xi)\; d\xi \;+\; \int_{b_{\ell-1}^*(y)}^1 s_{\ell-1}(\xi, y\vee\xi)\; d\xi \\
\ldots &= \sum_{i=0}^{\ell-1}\left(\prod_{j=i+1}^{\ell-1} b_j^*(y)\right)\cdot\int_{b_i^*(y)}^1 s_i(\xi, y\vee\xi)\; d\xi
\end{aligned}$$

by iteration, respecting $c_0 \equiv 0$. The smallest y this identity holds is $\underline{b}_{\ell-1}$, since $b^*_{\ell-1}(z) > z$ for $z \in [0, \underline{b}_{\ell-1})$. $\square$

This lemma now permits a recursive representation of the optimal boundary functions:

Theorem 2.7 *Let $\ell \in \mathbb{N}$. For $y \in [0,1]$ suppose $s_\ell(x,y)$ is continuous and increasing in x and let $s_\ell^{-1}(x,y)$ denote its unique inverse with respect to x, both for $x \le y$. Then*

$$b^*_\ell(y) \;=\; s_\ell^{-1}\left(\sum_{i=0}^{\ell-1}\left(\prod_{j=i+1}^{\ell-1} b^*_j(y)\right)\cdot\int_{b^*_i(y)}^{1} s_i(\xi, y\vee\xi)\, d\xi \quad , \, y\right)$$

for $y \in (\underline{b}_\ell, 1]$ and $b^*_\ell(y) = \underline{b}_\ell$ constant for $y \in [0, \underline{b}_\ell]$.

The optimal stopping sets Δ^*_ℓ now are identified according to definition 2.5. If $s_\ell(\cdot, y)$ isn't continuous and increasing, the infimum of those values x such that $s_\ell(x,y) \ge c_\ell(y)$ has to be specified.
Sufficient criteria for increase of $s_\ell(\cdot, y)$: $f(\cdot, y)$ increasing on Δ (section 2.2) or at least inside a subset of Δ (with boundary conditions, see section 2.1). While $b^*_\ell(y)$ equals constant $\underline{b}_\ell$ for $y \le \underline{b}_\ell$, for $y > \underline{b}_\ell$ it may first increase or decrease: As examples based on section 2.1 take $r \in \mathcal{R}$ where $r(\underline{b}_\ell) = \underline{b}_\ell$ resp. $r(\underline{b}_\ell) < \underline{b}_\ell$. The behaviour of function b^*_ℓ concerning monotonicity is exposed regarding finite valued payoff, see the corresponding paragraph on page 34. Besides any stopping time using at each case $b^*_\ell(y)$ as boundary value, wether applying $x > b^*_\ell(y)$ or $x \ge b^*_\ell(y)$, is optimal (more general modifying Δ^*_ℓ on a nullset doesn't affect optimality).

General Distribution Function

Regard problem $\mathcal{P}_n$, where $X_1, \ldots, X_n$ are independent and distributed according to a continuous distribution function F, which is familiar to the gambler. Suppose F is increasing on $R := \{x \in \mathbb{R} \,:\, 0 < F(x) < 1\}$, otherwise adapt ranges of F and the payoff function accordingly (which doesn't affect monotonicity). Set $\Delta_R := \{(x,y) \in R^2 \,:\, x \le y\}$. A payoff function f with respect to F means $f : \Delta_R \to [0,1]$ with monotonicity properties analogue to assumptions (1). Let $\mathcal{P}_n(f,F)$ denote the optimal stopping problem with distribution function F with corresponding payoff function f. Let F^{-1} denote the unique continuous inverse of $F : R \to [0,1]$.

Theorem 2.8 *Let $1 < n \in \mathbb{N}$ and let random variables $\widetilde{X}_1, \ldots, \widetilde{X}_n$ be given, which are iid with distribution function F according to the description above. Let $\widetilde{Y}_k := \max\{\widetilde{X}_1, \ldots, \widetilde{X}_k\}$ for $k = 1, \ldots, n$. Let $\widetilde{f}$ denote a payoff function with respect to F. Regard the optimal stopping problem $\mathcal{P}_n(\widetilde{f}, F)$: Maximize $\mathrm{E}\left(\widetilde{f}(\widetilde{X}_{\widetilde{S}}, \widetilde{Y}_n)\right)$, where $\widetilde{S}$ represents a nonanticipating stopping time with respect to $\widetilde{X}_1, \ldots, \widetilde{X}_n$.*
Let $\widetilde{S}^* := \inf\{1 \le k \le n \,:\, \widetilde{X}_k \ge \widetilde{b}^*_{n-k}(\widetilde{Y}_k)\}$ denote an optimal stopping time. Let $S^* := \inf\{1 \le k \le n \,:\, X_k \ge b^*_{n-k}(Y_k)\}$ denote an optimal stopping time of the optimal stopping problem $\mathcal{P}_n(f, U([0,1])$ with related payoff function $f(x,y) := \widetilde{f}(F^{-1}(x), F^{-1}(y))$ for $(x,y) \in \Delta$.
Then the values of either stopping problem coincide and

$$\widetilde{b}^*_{n-k}(y) \;=\; F^{-1}(b^*_{n-k}(F(y))) \tag{3}$$

where $y \in R$ for $k = 1, \ldots, n$.

Proof: Let $X_k := F(\widetilde{X}_k) \sim U([0,1])$ for $k = 1, \ldots, n$, which are iid. Payoff function f evidently meets the monotonicity properties (1).
Let $\widetilde{S}$ denote a stopping time for $\mathcal{P}_n(\widetilde{f}, F)$ with boundary functions $\widetilde{b}_{n-k}(y)$ where $y \in R$ for $k = 1, \ldots, n$: $\widetilde{S} := \inf\{1 \le k \le n \,:\, \widetilde{X}_k \ge \widetilde{b}_{n-k}(\widetilde{Y}_k)\}$. Define corresponding dual boundary functions $b_{n-k}(y) := F(\widetilde{b}_{n-k}(F^{-1}(y)))$ where $y \in [0,1]$ for $k = 1, \ldots, n$, specifying a stopping time S for $\mathcal{P}_n(f, U([0,1])$.
Then $\widetilde{S} = S$, because $[\widetilde{X}_k \ge \widetilde{b}_{n-k}(\widetilde{Y}_k)]$ iff $[F^{-1}(X_k) \ge F^{-1} b_{n-k} F(F^{-1}(Y_k))]$ iff $[X_k \ge b_{n-k}(Y_k)]$ for $k = 1, \ldots, n$. Thus $\widetilde{X}_{\widetilde{S}} = F(X_S)$.
Let $\mathrm{E}_b(\cdot)$ denote the expectation corresponding to boundary functions $b_{n-k}(y)$, $k = 1, \ldots, n$, for $\mathcal{P}_n(f, U([0,1]))$. $\mathrm{E}_{\widetilde{b}}(\cdot)$ accordingly for $\mathcal{P}_n(\widetilde{f}, F)$. Let $G_k(x,y)$ (resp. $\widetilde{G}_k(x,y)$) denote the joint distribution function of (X_k, Y_n) (resp. of $(\widetilde{X}_k, \widetilde{Y}_n)$) given $S = k$ (resp. $\widetilde{S} = k$). Then the value of S and $\widetilde{S}$ coincide:

$$\begin{aligned}
\mathrm{E}_{\widetilde{b}}\left(\widetilde{f}(\widetilde{X}_{\widetilde{S}}, \widetilde{Y}_n)\right) &= \sum_{k=1}^{n} \mathrm{P}\left(\widetilde{S} = k\right) \int_{\Delta_R} \widetilde{f}(x,y)\, d\widetilde{G}_k(x,y) \\
&= \sum_{k=1}^{n} \mathrm{P}\left(S = k\right) \int_{\Delta_R} \widetilde{f}(x,y)\, dG_k(F(x), F(y)) \\
&= \sum_{k=1}^{n} \mathrm{P}\left(S = k\right) \int_{\Delta} \widetilde{f}(F^{-1}(x), F^{-1}(y))\, dG_k(x,y) \\
&= \mathrm{E}_b\left(f(X_S, Y_n)\right)
\end{aligned}$$

according to the theorem "change of variable" in an integral. Since the values $v_n(S)$ and $v_n(\widetilde{S})$ are equal for particular, related stopping times and since an optimal stopping time is of the stated form, the values of the stopping problems coincide and relation (3) applies. □

Now let $X_1, \ldots, X_n$ be independent and let F_k denote the distribution function of X_k, which is continuous and increasing on $\{x \in \mathbb{R} : 0 < F_k(x) < 1\}$ and which is familiar to the gambler, $k = 1, \ldots, n$. Let R denote the union of these ranges, then a payoff function f is defined on Δ_R as indicated above. The corresponding optimal stopping problem with payoff function f is called $\mathcal{P}_n(f, F_1, \ldots, F_n)$. The formalization of the particular terms is omitted, but the regularity of this problem, extending lemma 2.3, is verified. Moreover in the subsequent proof the payoff function may depend on the time instant a chosen value appeared in order to indicate the range the problem is regular.

Lemma 2.9 *The optimal stopping problem $\mathcal{P}_n(f, F_1, \ldots, F_n)$ is regular.*

Proof: In this proof it is permitted that the payoff function depends on the numbers of remaining draws: Payoff $g_\ell(x, y \vee Y_n)$ is payed in state $(x, y) \in \Delta_R$ if $\ell < n$ items remain. The monotonicity assumption $g_\ell \preceq g_{\ell-1}$ (see definition A.1 in the appendix) for $\ell \in \mathbb{N}$ is made, which seems to be indispensable for validity of the regular case as far as this approach is concerned.
Let state $(x, y) \in \Delta_R$ and let $\ell < n$ denote the number of remaining draws, then $k := n - \ell$ denotes the number of the present draw, $x = X_k$. Further let $F_{k+1}^n :\equiv \prod_{j=k+1}^n F_j$ denote the distribution function of $X_{k+1} \vee \cdots \vee X_n$. Now with notation according to definition 2.1

$$
\begin{aligned}
s_\ell(x, y) &= \int_{-\infty}^{\infty} g_\ell(x, y \vee \zeta)\, dF_{k+1}^n(\zeta) \\
&= F_{k+1}^n(y)\, g_\ell(x, y) + \int_y^{\infty} g_\ell(x, \zeta)\, dF_{k+1}^n(\zeta)
\end{aligned}
$$

and with regard to lemma 2.2

$$
\begin{aligned}
c_\ell(y) &= \int_{-\infty}^{y} v_{\ell-1}(\xi, y)\, dF_{k+1}(\xi) + \int_y^{\infty} v_{\ell-1}(\xi, \xi)\, dF_{k+1}(\xi) \\
&\geq F_{k+1}(y)\, c_{\ell-1}(y) + \int_y^{\infty} s_{\ell-1}(\xi, \xi)\, dF_{k+1}(\xi) \\
F_{k+1}(y) c_{\ell-1}(y) &\leq c_\ell(y) - \int_y^{\infty} s_{\ell-1}(\xi, \xi)\, dF_{k+1}(\xi).
\end{aligned}
$$

The regular case applies if inequality $s_\ell(x,y) \geq c_\ell(y)$ implies validity of $s_{\ell-1}(x,y) \geq c_{\ell-1}(y)$. Sufficient for this implication is

$$F_{k+1}(y)s_{\ell-1}(x,y) \geq c_\ell(y) - \int_y^\infty s_{\ell-1}(\xi,\xi)\ dF_{k+1}(\xi),$$

which proves to be valid since

$$\begin{aligned}
& c_\ell(y) - \int_y^\infty s_{\ell-1}(\xi,\xi)\ dF_{k+1}(\xi) \\
\leq\ & F_{k+1}^n(y)g_\ell(x,y) + \int_y^\infty g_\ell(x,\zeta)\ dF_{k+1}^n(\zeta) \\
& - \int_y^\infty \left(F_{k+2}^n(\xi)g_{\ell-1}(\xi,\xi) + \int_\xi^\infty g_{\ell-1}(\xi,\zeta)\ dF_{k+2}^n(\zeta) \right)\ dF_{k+1}(\xi) \\
=\ & F_{k+1}^n(y)g_\ell(x,y) + \int_y^\infty g_\ell(x,\zeta)\ dF_{k+1}^n(\zeta) - \int_y^\infty F_{k+2}^n(\xi)g_{\ell-1}(\xi,\xi)\ dF_{k+1}(\xi) \\
& - \int_y^\infty \left(\int_y^\zeta g_{\ell-1}(\xi,\zeta)\ dF_{k+1}(\xi) \right)\ dF_{k+2}^n(\zeta) \\
\leq\ & F_{k+1}^n(y)g_\ell(x,y) + \int_y^\infty g_\ell(x,\zeta)\ dF_{k+1}^n(\zeta) - \int_y^\infty F_{k+2}^n(\xi)g_{\ell-1}(x,\xi)\ dF_{k+1}(\xi) \\
& - \int_y^\infty \left(\left[F_{k+1}(\zeta) - F_{k+1}(y)\right] g_{\ell-1}(x,\zeta) \right)\ dF_{k+2}^n(\zeta) \\
\leq\ & F_{k+1}^n(y)g_{\ell-1}(x,y) + \int_y^\infty g_{\ell-1}(x,\zeta)\ dF_{k+1}^n(\zeta) \\
& - \int_y^\infty F_{k+2}^n(\xi)g_{\ell-1}(x,\xi)\ dF_{k+1}(\xi) - \int_y^\infty F_{k+1}(\zeta)g_{\ell-1}(x,\zeta)\ dF_{k+2}^n(\zeta) \\
& + F_{k+1}(y)\int_y^\infty g_{\ell-1}(x,\zeta)\ dF_{k+1}^n(\zeta) \\
=\ & F_{k+1}(y)s_{\ell-1}(x,y),
\end{aligned}$$

where in the last step the medial three terms cancel due to $F_{k+1}^n \equiv F_{k+1} \cdot F_{k+2}^n$ and where in the second last step $g_\ell \preceq g_{\ell-1}$ is applied. □

Evidently theorem 2.7 is valid for problem $\mathcal{P}_n(f, F_1, \ldots, F_n)$ respectively. If offers are iid where for instance $\mathrm{P}(X_1 \in \mathbb{Z}_+) = 1$, then the problem is regular: Spread any mass on an interval of proper length and adapt the payoff function from $\mathbb{Z}_+$ to $\mathbb{R}_+$ accordingly, then the monotonicity assumptions persist.

2.1 Optimal Selection of an r–Candidate

In this section maximization of the probability $\mathrm{P}(X_S \geq r(Y_n))$ is studied, i.e. regard problem $\mathcal{P}_n(f)$ with payoff function $f(x,y) = \mathbf{1}_{[r(y),1]}(x)$, where the gambler either wins one unit or he fails. The maximum Y_n is reduced to $r(Y_n)$, demands for a win are weakened by relax function $r \in \mathcal{R}$, where

$$\mathcal{R} := \Big\{ r : [0,1] \to [0,1] \text{ continuous, increasing, } r(y) \leq y \ \ \forall y \in [0,1] \Big\}.$$

Let $\mathcal{R}_1 := \{r \in \mathcal{R} : r(1) = 1\}$, let $\mathcal{R}^1 := \mathcal{R} \cap C^1([0,1])$ and let $\mathcal{R}_1^1 := \mathcal{R}_1 \cap \mathcal{R}^1$ (one–sided derivatives in 0 and 1; in $\mathcal{R}^1$ derivative ∞ is allowed; in $\mathcal{R}_1^1$ then $r'(1-) \in [1,\infty]$). A relax functino for example is $\vartheta y \in \mathcal{R}^1$ where $\vartheta \in (0,1]$ and $y^d, 1 - \sqrt{1-y} \in \mathcal{R}_1^1$ where $d \in [1,\infty)$.

From now on regard $\mathcal{P}_n(r)$ where $r \in \mathcal{R}$, i.e. maximize $\mathrm{P}(X_S \geq r(Y_n))$. Evidently selection of the k–th offer X_k only makes sense if X_k might lead to a win yet, i.e. if $X_k \geq r(Y_k)$, and finally for win $X_k \geq r(Y_n)$ has to be valid. So the following term is appropriate:

Notation 2.10 For given relax function $r \in \mathcal{R}$ an object $x \in [0,1]$ is called an r*–candidate* with respect to $y \in [0,1]$ if $x \geq r(y)$ holds.

For $\mathcal{P}_n(r)$ the succinct convention is: X_k is an r–candidate if $X_k \geq r(Y_n)$. Retain the stopping sets $\Delta_\ell^* = \{(x,y) \in \Delta : x \geq b_\ell^*(y)\}$ except for $\ell = 0$: For convenience of subsequent formulas and propositions set $b_0^* :\equiv r$ on $[0,1]$, then Δ_0^* is the set of r–candidates (equivalent to payoff according to $b_0^* \equiv 0$ resp. $\Delta_0^* = \Delta$ due to payoff 0 regarding the final state α_∞).

Remark 2.11 Additional motivation for maximization of $\mathrm{P}(X_S \geq r(Y_n))$:

i) This approach means relaxing the demands of selecting the maximum of a sequence, see the full information case of Gilbert and Mosteller [18]. In the following that article is referred to as $r \equiv id$ or $r'(1-) = 1$, the asymptotic value is abbreviated $v_{id}^* \approx 0.5802$ and the asymptotic value within the set of concurrent threshold rules is denoted by $\widetilde{v}_{id}^* \approx 0.5174$.

ii) Suppose $f(x,y) = \mathbf{1}_{\mathbb{R}_+}(h(x,y))$ is monotone according to assumptions (1), where $h : \Delta \to \mathbb{R}$. This fits maximization of $\mathrm{E}(\mathbf{1}_C(X_S, Y_n))$ for a set $C \subset \Delta$. The function $r(y) := \inf\{0 \leq x \leq y : h(x,y) \geq 0\}$ is well defined on $[0,1]$, if $(0,0) \in C$ (consequently the diagonal must be part of C) and r is nondecreasing with regard to the monotonicity of h. Then the curve $\gamma := \{(r(y), y) : y \in [0,1]\}$ describes the lower boundary of C (second coordinate). Maximizing the functional above corresponds to a payoff function, which can be chosen to be equal to 1 above and including γ and 0 below it, i.e. the probability $\mathrm{P}(h(X_S, M_n) \geq 0) = \mathrm{P}(X_S \geq r(Y_n))$ is to be maximized (points $(r(y), y)$ of γ with $r(y) < y$ establish a null set because of the continuous distribution $U([0,1])$).

An Optimal Stopping Time

Suppose there are $\ell \in \mathbb{N}$ draws left. Heuristically $y = Y_{n-\ell}$ is equal to Y_n with high probability if y is sufficiently close to 1 and then in this time instant $n - \ell$ the only threshold an item should exceed is $r(y)$, i.e. $b_\ell^*(y) = r(y)$ for y sufficiently close to 1. Therefore a socalled upper boundary point $\overline{b}_\ell$ is defined for $\ell \in \mathbb{Z}_+$, in addition to $b_\ell^*(y)$ and $\underline{b}_\ell$ of definition 2.5:

$$\overline{b}_\ell \ := \ \inf\{y \in [0,1] : s_\ell(r(y), y) \geq c_\ell(y)\}.$$

Again this set is nonempty, take $y = 1$, and $\overline{b}_0 = 0$. Below lemma 2.2 will be specified, using the inverse function ϱ of $r \in \mathcal{R}$:

$$\varrho(x) \ := \ \sup\{y \in [0,1] : r(y) \leq x\} \qquad \forall x \in [0,1]. \tag{4}$$

Then $\varrho(x) = 1$ iff $x \in [r(1), 1]$. If $r \in \mathcal{R}^1$ and $y \in [0,1)$ with $r'(y) = 0$, then $\varrho'(r(y)) := \infty$ is declared. In this sense $r \in \mathcal{R}^1$ implies $\varrho \in C^1([0, r(1)])$. Convention $r^k(y) := (r(y))^k$ and $\varrho^k(y) := (\varrho(y))^k$ for $k \in \mathbb{Z}_+$ is used throughout.

Lemma 2.12 *Let $r \in \mathcal{R}$. Then for $\ell \in \mathbb{Z}_+$ the following holds:*

i) For $(x,y) \in \Delta_0^*$, $s_\ell(x,y) = s_\ell(x) = \varrho^\ell(x)$ is continuous. $s_\ell \equiv 1$ on $[r(1), 1]$ and on $(0, r(1))$ it is increasing in x and decreasing in ℓ.

ii) $c_\ell(y)$ is continuous and decreasing in $y \in [0,1]$ for $\ell \in \mathbb{N}$ and

$$c_\ell(y) \ = \ \sum_{i=0}^{\ell-1} r^{\ell-1-i}(y) \left(\int_{r(y)}^{y} v_i(\xi, y)\, d\xi + \int_y^1 v_i(\xi, \xi)\, d\xi \right), \tag{5}$$

while particularly for $y \in [\overline{b}_{\ell-1}, 1]$

$$c_\ell(y) \ = \ \sum_{i=0}^{\ell-1} r^{\ell-1-i}(y) \int_{r(y)}^1 \varrho^i(\xi)\, d\xi. \tag{6}$$

iii) $v_\ell(x,y) = \max\{s_\ell(x), c_\ell(y)\}$ *for* $(x,y) \in \Delta_0^*$ *and it is continuous for* $(x,y) \in \Delta \setminus \{(r(z),z) : z \in (\overline{b}_\ell, 1]\}$.

Proof: For given $\ell \in \mathbb{N}$ regard problem $\mathcal{P}_n$ with $n > \ell$.

i) Since $(x,y) \in \Delta_0^*$, payoff 1 (for selecting x) only can be destroyed by a future value: $s_\ell(x,y) = \mathrm{P}\,(x \geq r(X_j) \text{ for } j = n-\ell+1,\ldots,n) = \varrho^\ell(x)$ due to independence. $s_\ell \equiv 0$ on $\Delta \setminus \Delta_0^*$.

ii) $c_\ell(y) = \mathrm{P}\,(X_S \geq r(Y_n)\,|\,X_{n-\ell} = x, Y_{n-\ell} = y, S > n-\ell \text{ optimal})$ according to definition 2.1, which decreases if y increases (the demands for subsequent objects grow). The representation (5) is obtained by first decomposing values of $X_{n-\ell+1}$ while observing a change of $Y_{n-\ell+1}$ which yields $c_\ell(y) = c_{\ell-1}(y)r(y) + \int_{r(y)}^{y} v_{\ell-1}(\xi,y)\,d\xi + \int_y^1 v_{\ell-1}(\xi,\xi)\,d\xi$ and then by iterating. $c_\ell(y)$ is continuous with regard to (5), since r and integrals concerning v_i are continuous (inductive on ℓ). The representation (6) holds due to lemma 2.6, regularity and due to $b_j^*(y) = r(y)$ if $y \geq \overline{b}_{\ell-1}$ for $j = 0,\ldots,\ell-1$.

iii) $s_\ell(x,y)$ only is discontinuous at points $(r(z),z)$, $z \in [0,1]$, and the definition of $\overline{b}_\ell$ above yields the continuity of $v_\ell(x,y)$ except for the specified curve. □

Suppose there is $\ell = 1$ draw left. Then $s_1(x) = \varrho(x)$ and $c_1(y) = 1 - r(y)$. Thus $\underline{b}_1$ solves $y = r(1-r(y))$ and $\overline{b}_1$ solves $y = 1 - r(y)$ and $b_1^*(y) = r(1-r(y))$ for $y \in [\underline{b}_1, \overline{b}_1]$. A characterization of the optimal boundary functions for $\ell \in \mathbb{N}$ is given in the subsequent theorem, specifying theorem 2.7:

Theorem 2.13 *Let* $r \in \mathcal{R}$. *Then for* $\ell \in \mathbb{N}$ *the following holds:*

i) The upper boundary point $\overline{b}_\ell$ *is unique solution of equation*

$$y^\ell = \sum_{i=0}^{\ell-1} r^{\ell-1-i}(y) \int_{r(y)}^1 \varrho^i(\xi)\,d\xi. \tag{7}$$

ii) The boundary function $b_\ell^* : [0,1] \to [0,1]$ *is given by*

$$b_\ell^*(y) = \begin{cases} \underline{b}_\ell & \text{if } y \in [0, \underline{b}_\ell] \\ r\left(\left[\sum_{i=0}^{\ell-1}\left(\prod_{j=i+1}^{\ell-1} b_j^*(y)\right)\int_{b_i^*(y)}^1 \varrho^i(\xi)\,d\xi\right]^{1/\ell}\right) & \text{if } y \in (\underline{b}_\ell, \overline{b}_\ell) \\ r(y) & \text{if } y \in [\overline{b}_\ell, 1] \end{cases}$$

and $b_\ell^*(y)$ is decreasing in the medial case.

iii) *The lower boundary point $\underline{b}_\ell$ is unique solution of equation $b_\ell^*(y) = y$.*

iv) $\underline{b}_\ell \nearrow r(1)$ *for* $\ell \to \infty$.
$1/2 \leq \overline{b}_\ell \nearrow 1$ *for* $\mathbb{N} \ni \ell \to \infty$.
$r(1/2) \leq b_\ell^*(y) \nearrow r(1)$ *for* $\ell \to \infty$ *for each* $y \in [0,1]$, *or equivalently* $\bigcap_{\ell=0}^{\infty} \Delta_\ell^* = \{(x,y) \in \Delta_0^* : x \geq r(1)\}$.

Proof: Regard theorem 2.7 for $\ell \in \mathbb{N}$ for a problem $\mathcal{P}_n$ with $n > \ell$. Uniqueness in i), ii) and iii) is valid according to lemma 2.12: $c_\ell(y)$ is continuous and decreasing in $y \in [0,1]$ and $s_\ell(x,y)$ is continuous inside Δ_0^* and independent of y for fixed x.

i) $\overline{b}_\ell$ is the unique solution of $s_\ell(r(y),y) = c_\ell(y)$ or $y^\ell = c_\ell(y)$, regard equation (6).

ii) For $y \in [\underline{b}_\ell, \overline{b}_\ell]$ now $b_\ell^*(y) = x$ is the unique solution of $s_\ell(x,y) = c_\ell(y)$ while the remaining range is covered by the definition of the lower resp. upper boundary points. Inside Δ_0^* with respect to y first $s_\ell(x,y)$ is constant and second $c_\ell(y)$ is decreasing, thus $b_\ell^*(y)$ is decreasing in $y \in (\underline{b}_\ell, \overline{b}_\ell)$.

iii) $\underline{b}_\ell$ is the unique solution of $s_\ell(y,y) = c_\ell(y)$ or $y = r([c_\ell(y)]^{1/\ell})$.

iv) Monotone convergence is valid because problem $\mathcal{P}_n$ is regular. $\overline{b}_1$ solves $y = 1 - r(y) \geq 1 - y$, and the solution of $y = 1 - y$ is $1/2$. Selecting an item $x \geq r(1)$ will lead to success resp. to payoff 1 regardless of ℓ and $Y_n \in [0,1]$. On the other hand suppose an item $0 \leq x < r(1)$ is selected. Then $Y_n \in (\varrho(x), 1]$ will prevent success. The probability $(1-\varrho(x))^\ell$ of this event is $o(1)$ for $\ell \to \infty$. In addition $c_\ell(y)$ doesn't vanish for $\ell \to \infty$, not even for the strongest case $r \equiv id$, see Gilbert and Mosteller [18] and remark 2.18 i) below. □

Given $r \in \mathcal{R}$, the boundary function $b_\ell^*(y)$ is constant on $[0, \underline{b}_\ell]$, decreasing on $[\underline{b}_\ell, \overline{b}_\ell]$ and increasing on $[\overline{b}_\ell, 1]$, where the interpretation of these parts is: The present maximum y is inessential, y becomes relevant and finally y is representative with regard to the overall maximum Y_n. Here $r \equiv id$ appears as a marginal case where $\underline{b}_\ell = \overline{b}_\ell$ for $\ell \in \mathbb{Z}_+$.

Example 2.14 Suppose relax function $r(y) = y^4$ for $y \in [0,1]$. Theorem 2.13 yields optimal stopping sets $\Delta_0^* \supset \Delta_1^* \supset \cdots \supset \Delta_5^*$ shown in the figure below, particularly $b_0^* \equiv r$ and $b_1^*(y) = (1-y^4)^4$ for $y \in [\underline{b}_1, \overline{b}_1]$. Proposition 2.15 below will reveal $r(\overline{b}_\ell) \simeq 1 - 4 \cdot 0.3695/\ell$ as $\ell \to \infty$ with $\alpha(4) \approx 0.3695$.

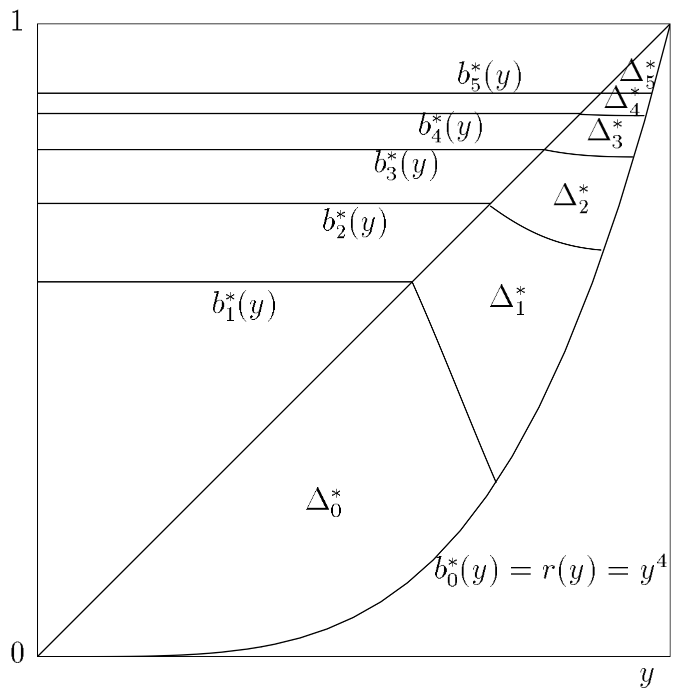

Figure 1: Maximization of $\mathrm{P}\left(X_S \geq Y_n^4\right)$: Optimal boundary functions $b_0^*(y), \ldots, b_5^*(y)$ and the corresponding optimal stopping sets $\Delta_0^* \supset \Delta_1^* \supset \cdots \supset \Delta_5^*$ (with axes x and y reversed). See example 2.14.

The asymptotic behaviour of second order of the upper boundary points $\overline{b}_\ell$ as $\ell \to \infty$ can be specified (next to theorem 2.13 iv)), since they are based directly on relax function r:

Proposition 2.15 *Let $r \in \mathcal{R}_1^1$ with $a := r'(1-) \in [1, \infty)$. The asymptotic behaviour of the upper boundary points $\left(\overline{b}_\ell\right)_{\ell \in \mathbb{N}}$ is $\lim_{\ell\to\infty} \ell(1-\overline{b}_\ell) = \alpha$, where $\alpha = \alpha(a)$ denotes the unique solution inside $(0, \alpha_1]$ of equation*

$$\frac{e^{\alpha(a-1)}}{a} = \int_{\alpha(a-1)}^{\alpha a} \frac{e^\xi - 1}{\xi}\, d\xi, \tag{8}$$

where $\alpha_1 := \alpha(1) \approx 0.8044$ solves $1 = \int_0^\alpha \frac{e^\xi - 1}{\xi}\, d\xi$.

Proof: Assume that $r(y) = ay - a + 1$ for y close to 1 and the idea is that $\overline{b}_\ell = 1 - f(a)/\ell + o(1/\ell)$ as $\ell \to \infty$, where $f(a)$ is nonnegative. Setting $\varepsilon_\ell := \overline{b}_\ell - 1 + f(a)/\ell$ it proves to be justifiable that $\varepsilon_\ell = o(1/\ell)$. For $y = \overline{b}_\ell$ the left side of equation (7) then is $\left(1 - \frac{f(a)-\ell\varepsilon_\ell}{\ell}\right)^\ell$ while the right side is

$$\sum_{i=0}^{\ell-1}\left(1 - \frac{a(f(a)-\ell\varepsilon_\ell)}{\ell}\right)^{\ell-1-i}\frac{a}{i+1}\left[1 - \left(1 - \frac{f(a)-\ell\varepsilon_\ell}{\ell}\right)^{i+1}\right]$$
$$= \left(1 - \frac{a(f(a)-\ell\varepsilon_\ell)}{\ell}\right)^{\ell}\sum_{i=0}^{\ell-1}\frac{a}{i+1}\left[\left(\frac{1}{1-\frac{a(f(a)-\ell\varepsilon_\ell)}{\ell}}\right)^{i+1} - \left(\frac{1-\frac{f(a)-\ell\varepsilon_\ell}{\ell}}{1-\frac{a(f(a)-\ell\varepsilon_\ell)}{\ell}}\right)^{i+1}\right].$$

With regard to lemma A.2 in the appendix the two bases with exponent $i+1$ now are changed to $1 + x_\ell/\ell$, where $x_\ell := a(f(a)-\ell\varepsilon_\ell)\Big/\left(1 - \frac{a(f(a)-\ell\varepsilon_\ell)}{\ell}\right)$ and $x_\ell := (a-1)(f(a)-\ell\varepsilon_\ell)\Big/\left(1 - \frac{a(f(a)-\ell\varepsilon_\ell)}{\ell}\right)$, respectively. Assuming $\ell\varepsilon_\ell = o(1)$ then all terms yield a specific asymptotic behaviour as $\ell \to \infty$ and since $\overline{b}_\ell$ is unique solution of equation (7) it is, aside from ε_ℓ, specified by $f(a)$ — now the assumption $\varepsilon_\ell = o(1/\ell)$ is justified if $f(a)$ is uniquely determined by

$$e^{-f(a)} \simeq ae^{-af(a)}\left[\mathrm{Ei}(af(a)) - \ln\frac{af(a)}{\ell} - \mathrm{Ei}((a-1)f(a)) + \ln\frac{(a-1)f(a)}{\ell}\right]$$

unless $a = 1$, where $1 \simeq \gamma + \mathrm{Ei}(f(1)) - \ln(f(1)/\ell) - \ln(\ell) - \gamma$ results. In both cases $\ln\ell$ cancels. In terms of remark A.4 this leads to equation (8) specifying solution $f(a) = \alpha$, which is existent and unique: Let $g_a(\alpha)$ resp. $h_a(\alpha)$ denote the left resp. the right side of (8). Then $g_a(0) = 1/a$ and $h_a(0) = 0$. Since $g_a'(\alpha) = (a-1)g_a(\alpha)$ and $h_a'(\alpha) = e^{\alpha a}(1-e^{-\alpha})/\alpha$ now $g_a'(0) = 1 - 1/a$ and $h_a'(0) = 1$ and $g_a'(\alpha) < h_a'(\alpha)$, because $(a-1)/a < (e^\alpha - 1)/\alpha$ for $a \ge 1$ and $\alpha \ge 0$. Compare $g_a(\alpha)$ and $h_a(\alpha)$ for $\alpha = 1$: Estimating the integrand $(e^\xi - 1)/\xi \ge (e^{-\xi} - 1)/a$ yields $g_a(1) < h_a(1)$ iff $1 + 2e^{a-1} < e^a$, which is true for $a > 1 - \ln(e-2) \approx 1.3309$, thus at least then α is unique. An upper bound for the solution of equation (8) is specified: For $\alpha = \ln(a)/(a-1)$ the similar estimation by $(e^\xi - 1)(a-1)/(a\ln a)$ yields the corresponding inequality $(a+1)\ln a < a(a-1)(a^{1/(a-1)} - 1)$, which is true at least for $a \ge 4$ (not verified here), i.e. $\ln(a)/(a-1)$ then is an upper bound for the solution $\alpha(a)$ of equation (8). Thus $\alpha \in (0, \alpha_1]$ seems coherent. □

By the boundary function $r(y) \vee r(\overline{b}_\ell)$ an offer isn't rejected the optimal stopping time of theorem 2.13 would select. According to proposition 2.15 above, for $\ell \gg a\alpha$ the boundary function $r(y) \vee r(1 - a\alpha/\ell)$ represents a good approximation. With regard to the behaviour of the curves $b^*_\ell(y)$ inside Δ^*_0 for $r \in \mathcal{R}^1_1$, which seem to become a horizontal line by computations and simulations, the following conjecture is made:

Conjecture 2.16 Regard problem $\mathcal{P}_n(r)$ with $r \in \mathcal{R}^1_1$, $a := r'(1-) \in [1, \infty)$. Let stopping time S_n apply boundary functions $(r(y) \vee r(1 - a\alpha/\ell))_{0<\ell<n}$ (for $\ell = 0$ take $r(y)$). Then the sequence $(S_n)_{n \in \mathbb{N}}$ of stopping times might be asymptotically optimal: $\lim\limits_{n\to\infty} v_n(S_n) = v^*_\infty(a)$. For the value $v_n(S_n)$ see remark 2.32.

For figure 2 below the values $\alpha(a)$ of proposition 2.15 above are computed for some values of $a = r'(1-) \in [1, 20]$ and an approximation from above is plotted, too. The marginal case $\alpha(1) \approx 0.8044$ is in accordance with Gilbert and Mosteller [18] (there in addition the third order is indicated).

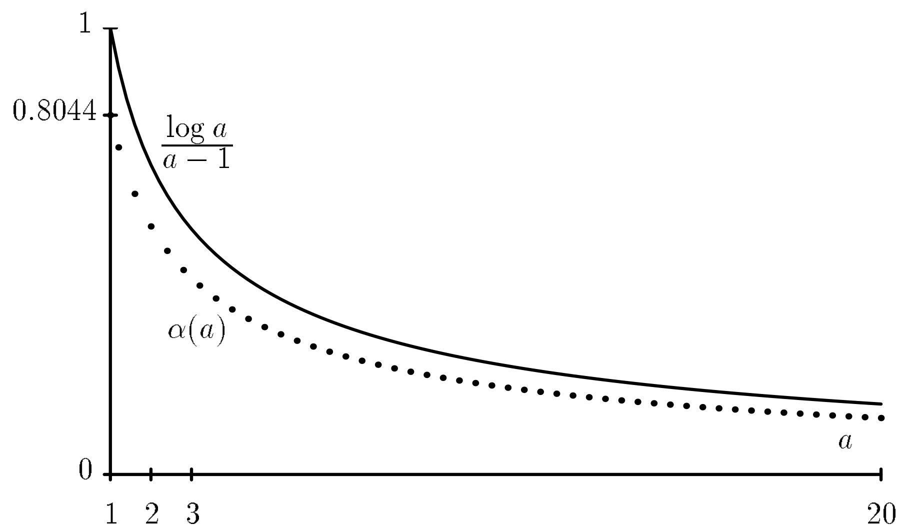

Figure 2: The coefficient $\alpha(a)$ of asymptotic behaviour of second order (plus its approximation from above) of the upper boundary values $\overline{b}_\ell$ as $\ell \to \infty$: $\overline{b}_\ell \simeq 1 - \alpha(a)/\ell$, where $a := r'(1-)$ and $r \in \mathcal{R}^1_1$, see proposition 2.15 and its proof.

Proposition 2.17 *Approximative considerations for stopping problem* $\mathcal{P}_n$*:*

i) *Difference in probability of winning using contiguous stopping times: Let stopping times* S *resp.* $\widetilde{S}$ *according to boundary functions* $b_\ell(y)$ *resp.* $\widetilde{b}_\ell(y)$ *be given. Let* $\beta := \sup_{0\le\ell<n} \sup_{y\in[0,1]} |\, b_\ell(y) - \widetilde{b}_\ell(y)\,| \in [0,1]$*. Then the probability that the payoff (applying* S *resp.* $\widetilde{S}$*) differs is bounded by* $1-(1-\beta)^n$*. This probability vanishes as* $\beta \to 0$*, leaving* n *fixed.*

ii) L^1 *approximation concerning relax function: Let* $r : [0,1] \to [0,1]$ *be nondecreasing with* $0 < r(y) \le y$ *for* $y \in (0,1]$ *and* $r(0) = 0$*. There are sequences* $(\underline{r}_k)_{k\in\mathbb{N}}$*,* $(\overline{r}_k)_{k\in\mathbb{N}} \subset \mathcal{R}$ *with* $\underline{r}_k(1) = \overline{r}_k(1) = r(1)$ *and* $\underline{r}_k \preceq r \preceq \overline{r}_k$ *and* $q_k := \int_0^1 (\overline{r}_k(\zeta) - \underline{r}_k(\zeta))\, d\zeta = o(1)$ *as* $k \to \infty$*.*
Let $b_0 \preceq \cdots \preceq b_{n-1}$ *denote a sequence of provisional boundary functions:* $b_\ell : [0,1] \to [0, r(1)]$ *is nonincreasing, adjust* $b_\ell(y) := y_\ell$ *for* $y \in [0, y_\ell]$*, where* y_ℓ *represents the unique solution of* $b_\ell(y) = y$*,* $0 \le \ell < n$*. With regard to payoff with respect to relax function* r*,* $\underline{r}_k$ *and* $\overline{r}_k$*, boundary functions* $b_\ell \vee r$*,* $b_\ell \vee \underline{r}_k$ *and* $b_\ell \vee \overline{r}_k$ *(for* $0 \le \ell < n$*) yield appropriate stopping times, denoted by* S*,* $\underline{S}_k$ *and* $\overline{S}_k$*, respectively (*$k \in \mathbb{N}$*).*
Let D_k *denote the event that the payoff gained by applying stopping time* $\underline{S}_k$ *resp.* $\overline{S}_k$ *(or* S*) differs,* $k \in \mathbb{N}$*. Then* $\mathrm{P}(D_k) \to 0$ *for* $k \to \infty$*.*

Proof:

i) The probability that the payoff differs is bounded by the probability

$$\mathrm{P}\left(S \ne \widetilde{S}\right) = \sum_{k=1}^{n} \mathrm{P}\left((X_k, Y_k) \in C_k\right) \le \sum_{k=1}^{n} \beta(1-\beta)^{k-1} = 1-(1-\beta)^n,$$

since $\mathrm{P}\left((X_k,Y_k) \in C_k\right) \le \beta(1-\beta)^{k-1}$, where $C_k := (B_k \setminus \widetilde{B}_k) \cup (\widetilde{B}_k \setminus B_k)$, $B_k := \{(x,y) \in \Delta : x \ge b_k(y)\}$ and $\widetilde{B}_k := \{(x,y) \in \Delta : x \ge \widetilde{b}_k(y)\}$ for $k = 1, \ldots, n$.

ii) Let $k \in \mathbb{N}$. Formally $D_k := [\mathbf{1}_{[r(Y_n),1]}(X_{\underline{S}_k}) \ne \mathbf{1}_{[r(Y_n),1]}(X_{\overline{S}_k})]$ and define $\overline{C}_k := \{(x,y) \in \Delta : x \ge \overline{r}_k(y)\}$ and $\underline{C}_k := \{(x,y) \in \Delta : x \ge \underline{r}_k(y)\} \setminus \overline{C}_k$.
The diagonal is not part of $\underline{C}_k$. Now event D_k is partitioned:
On the one hand the payoff may differ if $\underline{S}_k$ is lower than $\overline{S}_k$: Let G_k^j denote the (continuous differentiable) distribution function of Y_{j-1}

given $\underline{S}_k = j$, for $j = 2, \ldots, n$ (case $j = 1$ is void and $G_k^j(r(1)) = 1$). Now $\mathrm{P}\left(D_k \text{ and } \underline{S}_k < \overline{S}_k\right)$ is bounded by the probability (ignoring b_ℓ)

$$\begin{aligned}
\mathrm{P}\left(\underline{S}_k < \overline{S}_k\right) &= \sum_{j=2}^{n} \mathrm{P}\left(\underline{S}_k = j\right) \mathrm{P}\left((X_j, Y_j) \in \underline{C}_k \mid \underline{S}_k = j\right) \\
&\leq \sum_{j=2}^{n} \mathrm{P}\left(\underline{S}_k = j\right) \int_0^{r(1)} \frac{\overline{r}_k(y) - \underline{r}_k(y)}{1 - \underline{r}_k(y)}\, dG_k^j(y),
\end{aligned}$$

which proves to be $o(1)$ for $k \to \infty$: The integrand is bounded by 1, the densities (finitely many) are uniformly bounded and in case of $r(1) = 1$ the integration range is restricted to $[0, \delta_k]$ where $\delta_k \nearrow 1$ for $k \to \infty$ (the integral on $[\delta_k, 1]$ is $o(1)$, which also holds for the corresponding sum). Now use $q_k = o(1)$ for the sum with integral on $[0, \delta_k]$.

On the other hand the payoffs may differ if $\underline{S}_k$ and $\overline{S}_k$ coincide: Let event $E_k^j := [\underline{S}_k = \overline{S}_k = j]$ and let $Y_{j+1}^n := \max\{X_{j+1}, \ldots, X_n\}$ for $j = 1, \ldots, n-1$ (the payoffs don't differ if event $[\underline{S}_k = \overline{S}_k = n]$ occurs, event $[\underline{S}_k = n < \overline{S}_k]$ refers to the first part). Let G_k^j denote the (continuous differentiable) distribution function of Y_{j-1} given E_k^j and let $F_k^{j,y}$ denote the (continuous differentiable) distribution function of X_j given E_k^j and $Y_{j-1} = y$, where $j = 1, \ldots, n-1$ — except for $j = 1$ where $Y_0 := 0$ and $G_k^j \equiv \mathbf{1}_{[0,\infty)}$ but $F_k^{1,0}$ is continuous differentiable. Let $\underline{\varrho}_j$ resp. $\overline{\varrho}_j$ denote the inverse of $\underline{r}_j$ resp. of $\overline{r}_j$ according to equation (4) for $j = 1, \ldots, n-1$. Then (respect D_k is impossible beyond $r(1)$)

$$\begin{aligned}
&\mathrm{P}\left(D_k \text{ and } \underline{S}_k = \overline{S}_k\right) \\
&= \sum_{j=1}^{n-1} \mathrm{P}\left(E_k^j \text{ and } (X_j, Y_j) \in \overline{C}_k \text{ and } (X_j, Y_{j+1}^n) \in \underline{C}_k\right) \\
&= \sum_{j=1}^{n-1} \mathrm{P}\left(E_k^j\right) \int_0^{r(1)} \int_{\overline{r}_k(y)}^{r(1)} \frac{(\underline{\varrho}_k(x))^{n-j} - (\overline{\varrho}_k(x))^{n-j}}{1 - (\overline{\varrho}_k(x))^{n-j}}\, dF_k^{j,y}(x)\, dG_k^j(y),
\end{aligned}$$

which also is $o(1)$ for $k \to \infty$, analogously to the first part: The integrands are bounded by 1 and the densities (finitely many) are uniformly bounded (for $j = 1$ then $y = 0$ and restrict to $F_k^{1,0}$). Furthermore $\underline{\varrho}_k^{n-j} - \overline{\varrho}_k^{n-j} \preceq \underline{\varrho}_k - \overline{\varrho}_k$ on $[0, \delta_k]$ for $j = 1, \ldots, n-1$ for an existing sequence $\delta_k \nearrow r(1)$ as $k \to \infty$. The sum with the innermost integral

restricted to $[\delta_k, r(1)]$ yields $o(1)$ and for the remaining sum referring to the innermost integral on $[0, \delta_k]$ the relation $\int_0^{r(1)}(\underline{\varrho}_k(\zeta) - \overline{\varrho}_k(\zeta))\, d\zeta = q_k = o(1)$ ensures that the entire expression is $o(1)$ for $k \to \infty$.
Since $\underline{S}_k \leq S \leq \overline{S}_k$, this yields $\mathrm{P}\,(D_k) = o(1)$ for $k \to \infty$. $\square$

Remark 2.18

i) Monotonicity of values: $v_n^*(r_1) \geq v_n^*(r_2)$ if $r_1 \preceq r_2$ in $\mathcal{R}$ (also for proper inequalities). Let $r_1, r_2 \in \mathcal{R}_1^1$ with $\infty > r_1'(1-) > r_2'(1-)$. Then $v_n^*(r_1) > v_n^*(r_2)$ holds finally for $n \to \infty$, while asymptotic inequality $v_\infty^*(r_1) > v_\infty^*(r_2)$ only may hold if $v_\infty^*(r_2'(1-)) < 1$ (remark 2.30 vi)).

ii) The domain of any function $r \in \mathcal{R}$ is extended from $[0, 1]$ to $\mathbb{R}$ by

$$r(y) := 0 \text{ for } y < 0 \quad \text{and} \quad r(y) := 1 \text{ for } y > 1.$$

Setting additionally $r(1) := 1$, so r represents a distribution function. Relevant integrals (integrand containing ϱ and integration variable $d\xi$) then can be rewritten according to definition (4) of ϱ by $dr(\xi)$. If integration variable $dr(\xi)$ appears, these extensions are meant implicitely. For example $\int_{r(y)}^1 \varrho^i(\xi)\, d\xi = \int_y^1 \xi^i\, dr(\xi)$, $i \in \mathbb{Z}_+$, where for $y = 0$ this represents the i–th uncentralized moment corresponding to r. If $r \in \mathcal{R}_1^1$ then r' represents a corresponding density (Lebesque almost surely).

General Distribution Function

Suppose $X_1, \ldots, X_n$ are independent. Let F denote the respective distribution function, continuous and increasing on $R := \{x \in \mathbb{R} : 0 < F(x) < 1\}$ with its unique continuous inverse F^{-1} on this set — analogue to the preparations preceding theorem 2.8. Let $r : R \to R$ continuous, increasing and $r \preceq id$ on R. Then $\mathcal{P}_n(r, F)$ can be reduced to $\mathcal{P}_n(\widetilde{r}, U([0, 1])$ by taking $\widetilde{r} :\equiv FrF^{-1} \in \mathcal{R}$, since $\mathbf{1}_{[r(F^{-1}(y)),1]}\,(F^{-1}(x)) = \mathbf{1}_{[\widetilde{r}(y),1]}\,(x)$ for $x \in (0, 1)$.

Example 2.19

i) Let $F(x) = \frac{1}{2} + \frac{1}{\pi}\arctan(x)$ for $x \in \mathbb{R}$ and $r(y) = y - 1$ for $y \in \mathbb{R}$, i.e. select an item at least one unit below the maximum. Then $\widetilde{r}(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(\tan(\pi y - \frac{\pi}{2}) - 1)$ for $y \in [0, 1)$. Thus $\widetilde{r} \in \mathcal{R}_1^1$ and $\widetilde{r}'(1-) = 1$.

ii) Let $F(x) = (1 - e^{-\lambda x})\mathbf{1}_{\mathbb{R}_+}(x)$ for $x \in \mathbb{R}$. If $r(y) = \vartheta y$ for $y \in \mathbb{R}_+$ where $\vartheta \in (0,1]$ then $\widetilde{r}(y) = 1 - (1-y)^\vartheta$ for $y \in [0,1]$ and $\widetilde{r} \in \mathcal{R}_1^1$ with $\widetilde{r}'(1-) = \infty$ unless $\vartheta = 1$ where $\widetilde{r} \equiv id$. If however $r(y) = 0 \wedge (y-1)$ for $y \in \mathbb{R}_+$ (not increasing on $\mathbb{R}_+$), then $\widetilde{r}(y) = 0$ for $y \in [0, 1-e^{-\lambda}]$ and $\widetilde{r}(y) = 1 - e^\lambda + ye^\lambda$ for $y \in (1-e^{-\lambda}, 1]$ and $\widetilde{r} \notin \mathcal{R}$.

Particularly the relax function r remains unchanged if $F(0) = 0$, $F(1) = 1$ and $rF \equiv Fr$, For example $F \equiv id$, $F \equiv r$ and $F \equiv \varrho$ are related. The case where F is continuous but not increasing on R only can be included in this section if r is selected having merged constant parts of F — otherwise r doesn't remain continuous and only general assertions preceeding this section apply. In case of a discrete distribution, for example $\mathrm{P}(X_1 \in \mathbb{Z}_+) = 1$ with corresponding relax function r on $\mathbb{Z}_+$, then any point mass is spread on a proper interval and the relax function r is extended from lattice $\mathbb{Z}_+$ to $\mathbb{R}_+$ by constant continuation. The monotonicity assumptions (1) persist and optimal boundary functions, now consisting of a sequence of values, are specified by an infimum, since concerning functions are not continuous.

Selection with Recall

Suppose problem $\mathcal{P}_n(r)$ where $r \in \mathcal{R}$. If recall is allowed, it is optimal to take the overall maximum Y_n in the last time instant. Suppose however recall is allowed only if the present value turns out to be an r–candidate (which doesn't affect the case $r \equiv id$). In this optimal stopping problem $\mathcal{P}_n$ with restricted recall, one tends to watch the end (to recall at best Y_n) while arrivals of not–r–candidates has to be taken into account.
Now the myopic stopping time proposes to recall $y \in [0, r(1))$ (evidently an offer beyond $r(1)$ is taken anyway) where $\ell \in \mathbb{N}$ items remain provided that the present state $(x, y) \in \Delta_0^*$ if

$$\varrho^\ell(y) \;\ge\; \sum_{i=0}^{\ell-1} r^{\ell-1-i}(y)\left((y - r(y))\varrho^i(y) + \int_y^1 \varrho^i(\xi)\, d\xi\right), \tag{9}$$

by comparing the mean payoff of stopping with that of the one step look–ahead rule concerning epochs of r–candidates (for the latter the maximum of the next r–candidate $\xi \in [r(y), 1]$ and the past maximum y can be taken).

The stopping sets of the myopic stopping time are monotone.

Verification: Fix past maximum $y \in [0, r(1))$ and abbreviate $r = r(y)$, $\varrho = \varrho(y)$ and set $I_i := \int_y^1 \varrho^i(\xi)\, d\xi$ for $i \in \mathbb{Z}_+$. Let $\ell \in \mathbb{N}$ and suppose $\varrho^\ell \geq \sum_{i=0}^{\ell-1} r^{\ell-1-i} \left((y-r)\varrho^i + I_i\right) = (y-r)\varrho^{\ell-1} + I_{\ell-1} + r \sum_{i=0}^{\ell-2} r^{\ell-2-i} \left((y-r)\varrho^i + I_i\right)$ which proves to be not lower than $\varrho \sum_{i=0}^{\ell-2} r^{\ell-2-i} \left((y-r)\varrho^i + I_i\right)$ — then dividing by ϱ the claim is verified — in other words the assertion is the inequality $(y-r)\varrho^{\ell-1} + I_{\ell-1} \geq (\varrho - r) \sum_{i=0}^{\ell-2} r^{\ell-2-i} \left((y-r)\varrho^i + I_i\right)$: On the one hand $\varrho^{\ell-1} \geq (\varrho - r)(r^{\ell-1} - \varrho^{\ell-1})/(r - \varrho) = \varrho^{\ell-1} - r^{\ell-1}$ (cancelling $y - r$ and using the truncated geometric series) and on the other hand inequality $I_{\ell-1} \geq (\varrho - r) \sum_{i=0}^{\ell-2} r^{\ell-2-i} I_i$ holds (by induction on ℓ): Let $\ell = 2$ then $\int_y^1 \varrho(\xi)\, d\xi \geq (\varrho - r)(1-y)$ is obvious. Taking the inequality above as induction assumption then $I_\ell \geq (\varrho - r) \sum_{i=0}^{\ell-1} r^{\ell-1-i} I_i$ holds, since for the left side $I_\ell \geq \varrho I_{\ell-1}$ is valid and the right side is not bigger than $r I_{\ell-1} + (\varrho - r) I_{\ell-1}$ (induction step). Cancelling $I_{\ell-1}$ on both sides $\varrho \geq r + \varrho - r$ results. □

Now the conjecture is that the myopic stopping time can be specified by unique thresholds, i.e. if inequality (9) is valid for $y \in [0,1]$ then for $z \in (y, 1]$, too. The presumed unique solution of inequality (9) as equality is called threshold y_ℓ for $\ell \in \mathbb{Z}_+$. Evidently for $\ell = 0$ threshold $y_0 = 0$ results (provided X_n is an r–candidate) and particularly $\ell = 1$ leads to a threshold $y_1 \in (0, r(1))$, the unique solution of $\varrho(y) + r(y) = 1$. However $h(y) := \varrho^\ell(y) - \sum_{i=0}^{\ell-1} r^{\ell-1-i} \left((y-r)\varrho^i + I_i\right) \in C^1([0, r(1)])$ if $r \in \mathcal{R}^1$ isn't nondecreasing in general, as examples verify, though they don't exclude that there is a unique flaw y_ℓ. Besides $h(0)$ is negative and $h(r(1)) = (r(r(1)))^\ell$ is positive.
Under this hypothesis the stopping sets of the myopic stopping time are closed and realizable and thus they would yield an optimal stopping time. Then evidently $y_\ell \nearrow r(1)$ as $\ell \to \infty$, while the asymptotic behaviour of second order would be as follows: Suppose $r \in \mathcal{R}_1^1$ with $a := r'(1-) \in [1, \infty)$: Then $y_\ell \simeq 1 - \alpha/\ell$ as $\ell \to \infty$, where $\alpha = \alpha(a)$ denotes the unique solution of

$$\frac{1}{a+1} + \frac{1}{a^2+a} e^{\alpha(a-1/a)} \quad = \quad \int_{\alpha(a-1/a)}^{\alpha a} \frac{e^\xi - 1}{\xi}\, d\xi. \tag{10}$$

Verification: Suppose $r(y) = ay - a + 1$ for y close to 1 and assume $y_\ell \simeq 1 - f(a)/\ell + o(1/\ell)$ as $\ell \to \infty$ (a more cautious inspection is omitted in respect of assumption of the hypothesis). Now inequality (9) as an equation yields

$$\left(1 - \frac{f(a)/a}{\ell}\right)^\ell \quad \simeq \quad \sum_{i=0}^{\ell-1} \left(1 - \frac{af(a)}{\ell}\right)^{\ell-1-i} \left(\frac{(a-1)f(a)}{\ell} \left(1 - \frac{f(a)/a}{\ell}\right)^i\right.$$

$$
\begin{aligned}
& \quad + \frac{a}{i+1}\left[1-\left(1-\frac{f(a)/a}{\ell}\right)^{i+1}\right]\Bigg) \\
\left(1-\frac{f(a)/a}{\ell}\right)^{\ell} \simeq & \left(1-\frac{af(a)}{\ell}\right)^{\ell-1} \frac{(a-1)f(a)}{\ell} \sum_{i=0}^{\ell-1} \left(1+\frac{f(a)(a-1/a)}{\ell-af(a)}\right)^{i} \\
& + \left(1-\frac{af(a)}{\ell}\right)^{\ell} \sum_{i=0}^{\ell-1} \frac{a}{i+1} \left(1+\frac{af(a)}{\ell-af(a)}\right)^{i+1} \\
& - \left(1-\frac{af(a)}{\ell}\right)^{\ell} \sum_{i=0}^{\ell-1} \frac{a}{i+1} \left(1+\frac{f(a)(a-1/a)}{\ell-af(a)}\right)^{i+1} .
\end{aligned}
$$

For $a > 1$ the first sum is, due to the truncated geometric series, asymptotically equivalent to $(e^{f(a)(a-1/a)} - 1)(\ell - af(a))/(f(a)(a-1/a))$ and the last two sums are covered by remark A.4 resp. by lemma A.2 in the appendix. Then the following equation holds asymptotically:

$$
e^{-f(a)/a} = e^{-af(a)} \left(\frac{a-1}{a-1/a} \left(e^{f(a)(a-1/a)} - 1\right) + a \int_{f(a)(a-1/a)}^{f(a)a} \frac{e^{\xi}-1}{\xi}\, d\xi \right)
$$

unless $a = 1$, where $1 = \gamma + \mathrm{Ei}(f(1)) - \ln(f(1)/\ell) - \ln(\ell) - \gamma$ results from lemma A.2. Rearrangements produce equation (10) where $f(a) = \alpha$. $\square$

Expression (9) for $r \equiv id$ corresponds to Gilbert and Mosteller [18], since

$$
\sum_{i=0}^{\ell-1} \frac{1}{i+1} y^{\ell-1-i}(1-y^{i+1}) = \sum_{j=0}^{\ell-1} \binom{\ell}{j+1} \frac{1}{j+1} y^{\ell-1-j}(1-y)^{j+1}
$$

for $\ell \in \mathbb{N}$ and $y \in [0,1]$. Here (on the left) a partition with respect to the time instant of the appearance of the next r–candidate is applied, whereas in [18] (on the right) the approach is to use the fact that the present maximum (beyond a certain threshold) emerges as the overall maximum within the number j of future values exceeding this threshold with probability $1/(j+1)$.

The optimal boundary function $b^*_\ell(y)$ has one minimum with respect to y (to specify: at most one strict local minimum in $\overline{b}_\ell$; remind the case $r \equiv id$ where $\underline{b}_\ell = \overline{b}_\ell$). The question answered in the next paragraph is: For general payoff function f, may the optimal boundary function $b^*_\ell(y)$ possess several (local) minima with respect to y, i.e. may fluctuations occur?

Finite Valued Payoff Function

Suppose that the payoff function may attain a finite number of values: Let $f(x,y) := \sum_{i=1}^{i_0} c_i \cdot \mathbf{1}_{C_i}(x,y)$, where $i_0 \in \mathbb{N}$ and $c_1, \dots, c_{i_0} > 0$ and where sets $\Delta \supset C_1 \supset C_2 \supset \cdots \supset C_{i_0}$ are closed and contain $(0,0)$ (related to remark 2.11 ii)). The monotonicity criteria (1) are valid. Set additionally $c_0 := 0$, $C_0 := \Delta$ and $C_{i_0+1} := \emptyset$. f is bounded by 1 by scaling.
Choosing value x while finally the overall maximum y occurs, leads to the mean payoff $\sum_{j=1}^{i} c_j$ if $(x,y) \in C_i \setminus C_{i+1}$, where $i = 0, \dots, i_0$ (with $\sum_\emptyset := 0$). Let r_i define the lower boundary of C_i (confer remark 2.11 ii)) for $i = 0, \dots, i_0$ — obviously $0 \equiv r_0 \preceq r_1 \preceq \cdots \preceq r_{i_0}$, which are assumed to be increasing and continuous, i.e. in $\mathcal{R}$. Set $r_{i_0+1} :\equiv 1$. Let ϱ_i denote the inverse of r_i according to equation (4), $i = 0, \dots, i_0$. Then the mean payoff stopping with $(x,y) \in \Delta$ while $\ell = 0, \dots, n-1$ items remain is given by $s_\ell(x,y) := \sum_{j=1}^{i} c_j \varrho_j^\ell(x)$ if $r_i(y) \le x < r_{i+1}(y)$ where $i = 0, \dots, i_0$. Now $s_\ell(x,y)$ is increasing but not continuous in x inside Δ.

Proposition 2.20 *The optimal boundary function $b_\ell^*(y)$, $\ell \in \mathbb{N}$, for payoff function $\sum_{i=1}^{i_0} c_i \cdot \mathbf{1}_{C_i}(x,y)$ may possess i_0 local minima.*

For a heuristic explanation imagine that the values $c_1, \dots, c_{i_0}$ differ extremely, $0 < c_1 \ll c_2 \ll \cdots \ll c_{i_0}$, and consider the behaviour of function $b_\ell^*(y)$ as y increases from $\underline{b}_\ell$ to 1:
Only c_{i_0} (resp. r_{i_0}) has to be respected first (i.e. for y close to $\underline{b}_\ell$, where either lower boundary values resemble), because values c_i, $i < i_0$, are comparable to 0 (first decreasing, then meeting r_{i_0}). The optimal boundary function $b_\ell^*(y)$ initially fits that of relax function r_{i_0}. As y grows, the magnitude of c_{i_0-1} (compared to 0) takes through, the curve $b_\ell^*(y)$ decreases and then fits the curve r_{i_0-1}, and so on. At the end, if y becomes sufficiently close to 1, the magnitude of c_1 in comparison to 0 steps forward.

Time Dependent Relaxation

Let time dependent relax functions $r_i \in \mathcal{R}$, $i = 1, \dots, n$, be given and suppose the problem of maximizing $\mathrm{P}\left(X_S \ge \max\{r_1(X_1), \dots, r_n(X_n)\}\right)$, denoted by $\mathcal{P}_n(r_1, \dots, r_n)$. Additionally assume $r_{i+1} \preceq r_i$ for $i = 1, \dots, n-1$, i.e. the requirements for the chosen object (with regard to the reduced reference values $r_k(X_k)$ for $k = 1, \dots, n$) weaken as time goes by — in fact they must weaken to ensure the regular case, as the following proof shows:

Lemma 2.21 *The optimal stopping problem $\mathcal{P}_n(r_1, \ldots, r_n)$ is regular.*

Proof: Fix epoch k and let $X_i = x_i$ for $i = 1, \ldots, k$ and let unusually $y_k := \max\{r_1(x_1), \ldots, r_k(x_k)\}$. Assume $x_k \geq y_k$ and let $s_k(x_k)$ denote the mean payoff of stopping at time k with item x_k (now time index for clarity):

$$s_k(x_k) \;=\; \mathrm{P}\left(x_k \geq r_{k+1}(X_{k+1}), \ldots, r_n(X_n)\right) \;=\; \prod_{j=k+1}^{n} \varrho_j(x_k),$$

where ϱ_j represents the inverse of r_j according to equation (4). Analogously $c_k(y_k)$ denotes the mean payoff for proceeding optimally (at least one step, at time k with y_k) and finally $v_k(x_k, y_k)$ denotes the maximum.

$$c_k(y_k) \;=\; \int_0^{y_k} c_{k+1}(y_k)\; dx_{k+1} + \int_{y_k}^1 v_{k+1}\left(x_{k+1}, y_{k+1} \vee r_{k+1}(x_{k+1})\right)\; dx_{k+1}.$$

Here $r_{k+1} \preceq id$ is necessary, since otherwise the first integrand would be $c_{k+1}(y_k \vee r_{k+1}(x_{k+1}))$. No direct generalization like $\mathrm{P}\left(X_S \geq r_i(X_i),\, i \neq S\right)$. For the following separation of the integral implication $r_k(x_k) \leq y_k \Longrightarrow (r_{k+1}(x_{k+1}) \leq y_k$ for $x_{k+1} \leq x_k)$ is applied, which is valid due to $r_{k+1} \preceq r_k$. Without loss of generality $x_k > 0$ and analog to the proof of lemma 2.9

$$\begin{aligned} c_k(y_k) &\geq x_k c_{k+1}(y_k) + \int_{x_k}^1 s_{k+1}(x_{k+1})\; dx_{k+1} \\ x_k c_{k+1}(y_k) &\leq c_k(y_k) - \int_{x_k}^1 s_{k+1}(x_{k+1})\; dx_{k+1}. \end{aligned}$$

Given $s_k(x_k) \geq c_k(y_k)$, it has to be shown $s_{k+1}(x_k) \geq c_{k+1}(y_k)$. Sufficient is

$$\begin{aligned} x_k s_{k+1}(x_k) &\geq c_k(y_k) - \int_{x_k}^1 s_{k+1}(x_{k+1})\; dx_{k+1} \\ \int_{x_k}^1 \prod_{j=k+2}^{n} \varrho_j(x_{k+1})\; dx_{k+1} &\geq \left(\varrho_{k+1}(x_k) - x_k\right) \prod_{j=k+2}^{n} \varrho_j(x_k), \end{aligned}$$

which is obviously valid. Respect that the separation x_k of the integral can't be replaced by y_k, since $x_{k+1} \geq y_k$ doesn't imply $x_{k+1} \geq x_k$. $\Box$

Thus an assertion analogue to theorem 2.7 is valid. Besides this problem with time dependent relax functions can be combined with the case of time indexed distribution functions of lemma 2.9 and the regular case will persist.

The Asymptotic Value for Special Cases

For given relax function $r \in \mathcal{R}$ the limit performance of the sequence $(v_n^*)_{n\in\mathbb{N}}$ of values is studied, which depends on the behaviour of the relax function r in the (lefthand side) neighbourhood of 1. Particularly for $r \in \mathcal{R}_1^1$ and in case of convergence the asymptotic value is denoted by $v_\infty^*(r'(1-))$.
However the pathological case $r(1) < 1$ remains, where the concurrent threshold rule T with threshold $t := r(1)$ (for strict description see notation 2.25 in the next subsection) proves to be asymptotically optimal, since its value $v_n(T) = \mathrm{P}(X_T \geq r(Y_n)) = 1 - r^n(1)$ converges to the maximum value $v_\infty^* = 1$ as $n \to \infty$. Moreover, concerning asymptotic behaviour, no direct connection exists between the optimal stopping problem $\mathcal{P}_n(r)$ for relax function $r \in \mathcal{R}_1$ and problem $\mathcal{P}_n(\vartheta r)$ with decayed relax function ϑr where $\vartheta \in (0,1)$.
The asymptotic value of problem $\mathcal{P}_n$ with $r'(1-) = \infty$ doesn't differ from the asymptotic value if $r(1) < 1$ and an asymptotically optimal sequence of stopping times again is to be found in $(\mathcal{T}_n^c)_{n\in\mathbb{N}}$ (see notation 2.25):

Theorem 2.22 *Let $r \in \mathcal{R}_1^1$ with $r'(1-) = \infty$. Then the asymptotic value $v_\infty^* := \lim_{n\to\infty} v_n^*$ attains its maximum value 1, particularly*

$$v_\infty^* \;=\; \lim_{n\to\infty} \sup_{T\in\mathcal{T}_n^c} v_n(T) \;=\; 1. \tag{11}$$

Proof: Let T denote the concurrent stopping rule with concurrent threshold $0 < t < 1$, i.e. $T := \inf\{1 \leq k \leq n : X_k \in [t,1]\}$ and $X_T \sim U([t,1])$ unless $T = \inf_\emptyset := \infty$ (implies mean payoff 0: $X_\infty = 0$ and $\mathrm{P}(Y_n > 0) = 1$). Then

$$v_n(T) \;:=\; \mathrm{P}(X_T \geq r(Y_n)) \;\geq\; \mathrm{P}(Y_n \in [t, \varrho(t)]) \;=\; \varrho^n(t) - t^n.$$

Assertion (11) is implied by: For each $\varepsilon > 0$ and for each $n_0 \in \mathbb{N}$ there exist $n \geq n_0$ and threshold $t = t(n)$ such that the following last inequality holds:

$$1 \;\geq\; v_n^* \;\geq\; v_n(T) \;\geq\; \varrho^n(t) - t^n \;\geq\; 1 - \varepsilon.$$

Now $r'(1-) = \infty$ implies $1 - \varrho(t) = o(1-t)$ for $t \to 1$, since $0 = \varrho'(1-) = \lim_{x\uparrow 1} \frac{\varrho(1)-\varrho(x)}{1-x} = \lim_{x\uparrow 1} \frac{1-\varrho(x)}{1-x}$.
Expressed in terms of $\delta := 1 - t$ (consider $1 - \varrho(1-\delta) = o(\delta)$ for $\delta \to 0$ and $\delta = \delta(n)$) a sufficient formulation is as follows: For each $\varepsilon > 0$ and for each $n_0 \in \mathbb{N}$ there exist $n \geq n_0$ and $\delta = \delta(n) > 0$ such that

$$h(n,\delta) \;:=\; (1 - o(\delta))^n - (1-\delta)^n \;\geq\; 1 - \varepsilon \tag{12}$$

holds. With regard to n maxima of h only can be attained on the curve

$$n = n(\delta) = \ln\left(\frac{\ln(1-\delta)}{\ln(1-o(\delta))}\right)\Big/\ln\left(\frac{1-o(\delta)}{1-\delta}\right) = -\frac{1}{\delta}\ln\left(\frac{o(\delta)}{\delta}\right)$$

for $\delta \to 0$. These are points where the derivative of h with respect to n equals 0. Though the second derivative might be negative, the subsequent derivations show assertion (12) regardless of values representing maxima or not. Now $n(\delta) \to \infty$ for $\delta \to 0$, thus a sufficient small δ causes $n \geq n_0$. Looking for the maximal values of h, i.e. regarding $h(n(\delta),\delta)$, the assertion sounds as follows: For each $\varepsilon > 0$ there exists a sufficient small δ such that

$$\begin{aligned}(1-o(\delta))^{-\frac{1}{\delta}\ln\left(\frac{o(\delta)}{\delta}\right)} - (1-\delta)^{-\frac{1}{\delta}\ln\left(\frac{o(\delta)}{\delta}\right)} &\geq 1-\varepsilon \\ \left(1-\frac{\frac{o(\delta)}{\delta}}{\frac{1}{\delta}}\right)^{-\frac{1}{\delta}\ln\left(\frac{o(\delta)}{\delta}\right)} - \left(1-\frac{1}{\frac{1}{\delta}}\right)^{-\frac{1}{\delta}\ln\left(\frac{o(\delta)}{\delta}\right)} &\geq 1-\varepsilon.\end{aligned}$$

Now the upper term tends to 1 and the lower term on the lefthand side tends to 0 as $\delta \to 0$. This is valid due to the following equivalent formulation: Given a sequence $(d_k)_{k\in\mathbb{N}}$ with $1 > d_k \searrow 0$ for $k \to \infty$ the following holds:

$$\begin{aligned}\lim_{k\to\infty}\left(1-\frac{d_k}{k}\right)^{-k\ln d_k} &= 1 \\ \lim_{k\to\infty}\left(1-\frac{1}{k}\right)^{-k\ln d_k} &= 0.\end{aligned}$$

The first assertion is valid based on the relations ($o(1) \geq 0$ as $k \to \infty$) $e^{-1-o(1)} \leq \left(1-\frac{d_k}{k}\right)^{\frac{k}{d_k}} \leq e^{-1}$ (for all $k \in \mathbb{N}$). This leads (for $k \to \infty$) to

$$1 \leftarrow \left(d_k^{d_k}\right)^{1+o(1)} = e^{(1+o(1))d_k\ln d_k} \leq \left(\left(1-\frac{d_k}{k}\right)^{\frac{k}{d_k}}\right)^{-d_k\ln d_k} \leq e^{d_k\ln d_k} = d_k^{d_k} \to 1.$$

Analogously for the second assertion the relations ($o(1) \geq 0$ as $k \to \infty$) $e^{-1-o(1)} \leq \left(1-\frac{1}{k}\right)^k \leq e^{-1}$ (for all $k \in \mathbb{N}$) lead (for $k \to \infty$) to

$$0 \leftarrow d_k^{1+o(1)} = e^{(1+o(1))\ln d_k} \leq \left(\left(1-\frac{1}{k}\right)^k\right)^{-\ln d_k} \leq e^{\ln d_k} = d_k \to 0.$$

Thus assertion (12) is verified, which completes the proof. □

Example 2.23 Take $r(y) := 1 - \sqrt{1-y} \in \mathcal{R}_1^1$ with $r'(1-) = \infty$.

Conjecture 2.24 Under the hypothesis, that the asymptotic value $v_\infty^*(d)$ is strictly increasing in $d = r'(1-) \in [1, \infty)$ where $r \in \mathcal{R}_1^1$, the following holds (see remark 2.30 vi)): If the (lefthand sided) difference quotients of function $r \in \mathcal{R}_1$ in 1 vary (finally) in a nontrivial interval $[a, b] \subset [1, \infty]$, say, then a limit of the values $v_n^*(r)$ as $n \to \infty$ doesn't exist.

To specify: Only estimations from below resp. from above (concerning a resp. b) are possible, since the optimal probabilities v_n^* of winning will also vary in a nontrivial interval as n tends to infinity.

Heuristic explanation of conjecture 2.24:
Suppose $r \in \mathcal{R}_1$ where $r'(1-)$ doesn't exist — for simplicity and with regard to the approximation in 2.17 ii) let $\int_0^1 |r(\zeta) - \widetilde{r}(\zeta)| \, d\zeta < \varepsilon$ for $\varepsilon > 0$ where $\widetilde{r}$ is a zigzag curve from $(0,0)$ to $(1,1)$: As an example for $\widetilde{r}$ take the zigzag polygon consisting of horizontal and vertical lines between linear functions $cy - c + 1$ resp. $dy - d + 1$ where $1 \leq c < d$ (adjust $\widetilde{r}(y) > 0$ for $y \in (0,1]$ which for $n \to \infty$ becomes unnecessary). Particularly $\widetilde{r}'(1-)$ doesn't exist.
Let p_j for $j \in \mathbb{N}$ denote the width of the horizontal lines and set $\pi_j := \sum_{i=1}^{j} p_i$ for $j \in \mathbb{N}$ and $\pi_0 := 0$. For each $j \in \mathbb{N}$ items inside interval $[\pi_{j-1}, \pi_j)$ can evidently be identified, since there is no prevention of a win inside these classes (apart from terms depending on ε). Now p_j represents the probability of an item inside the j–th interval, $j \in \mathbb{N}$.
According to Baryshnikov et al. [4] the probability of a tie of the maximal value of offers converges iff $p_j/(1 - \pi_j) \to 0$ as $j \to \infty$ — scores in this situation are called π_j with respective probability p_j, $j \in \mathbb{N}$. Particularly the probability of a tie of the maximal value then converges to 0.
Here $r'(1-)$ resp. $\widetilde{r}'(1-)$ doesn't exist and therefore $p_j/(1 - \pi_j) \not\to 0$ as $j \to \infty$ (for $\widetilde{r}$ the corresponding limit is $(d - c)/c > 0$).
A tie of the maximal value in this situation means that, next to relaxed demands due to function r, there are a couple of items which would lead to a win and additionally this number varies significantly as n grows, which has heavy impact on the value v_n^*.

2.1.1 Suboptimal Stopping Times

Some notations concerning subsets of stopping times are specified:

Notation 2.25 Stopping time T is called *threshold rule* (for $X_1, X_2, \ldots$) if thresholds $t_1, t_2, \ldots \in [0,1]$ exist, such that $T = \inf\{k \in \mathbb{N} : X_k \geq t_k\}$, where $\inf_\emptyset := \infty$ (representing win 0 a.s.) and where for problem $\mathcal{P}_n$ the convention is that the payoff is 0 if $T > n$. If all tresholds are equal to the concurrent threshold t then T is called *concurrent threshold rule*. For $n \in \mathbb{N}$ let $\mathcal{T}_n$ resp. $\mathcal{T}_n^c$ denote the set of these threshold rules resp. concurrent threshold rules.

Now regard optimal selection of an r–candidate, $r \in \mathcal{R}$, in $\mathcal{T}_n$ and in $\mathcal{T}_n^c$.

Optimal Threshold Rule

Let a nonincreasing sequence $1 \geq t_1 \geq \cdots \geq t_n = 0$ of thresholds be given. The value of the corresponding threshold rule $T := \inf\{1 \leq k \leq n \,:\, X_k \geq t_k\}$

$$v_n(T) \;=\; \sum_{k=1}^{n} \int_{t_k}^{1} \left(\prod_{j=1}^{k-1} \min\{t_j, \varrho(x_k)\} \right) \cdot \varrho^{n-k}(x_k)\, dx_k \tag{13}$$

($\prod_\emptyset := 1$), verified by a vivid example: For $n = 4$ event $[T = 3]$ occurs iff $[X_1 < t_1, X_2 < t_2, X_3 \geq \max\{t_3, r(X_1), r(X_2), r(X_4)\}]$ iff, given $X_3 = x_3$, $[X_1 < \min\{t_1, \varrho(x_3)\}, X_2 < \min\{t_2, \varrho(x_3)\}, x_3 \geq t_3, X_4 \leq \varrho(x_3)]$, which leads to expression (13). If T additionally is restricted to acceptance of r–candidates then its value becomes significantly more complicated (in the example above the restriction $X_2 < t_2 \vee r(X_1)$ makes things more intricate) — the value of this stopping time is specified in remark 2.32.

Optimal Concurrent Threshold Rule

In this paragraph the subsets $\mathcal{T}_n^c$ of stopping times is studied for $n \in \mathbb{N}$: Suppose a single threshold $t_n \in [0,1]$ is applied simultaneously to n items, i.e. regard the concurrent threshold rule $T_n := \inf\{1 \leq k \leq n \,:\, X_k \geq t_n\}$ — neglecting the case $X_k < t_n$ for any $1 \leq k \leq n$, since asymptotic behaviour is the primal intention (formally $\inf_\emptyset := \infty$ yields payoff 0 a.s.). The corresponding probability of winning is given by (expression (13) simplifies)

$$v_n(T_n) \;=\; \mathrm{P}\left(X_{T_n} \geq r(Y_n)\right) \;=\; \sum_{k=1}^{n} \left(t_n^{k-1} \int_{t_n}^{1} \varrho^{n-k}(x)\, dx \right). \tag{14}$$

For an optimal concurrent threshold rule T_n^* (by differentiation) the optimal concurrent threshold t_n^* is solution of the following equation:

$$0 = \sum_{k=1}^{n} \left((k-1) t_n^{k-2} \int_{t_n}^{1} \varrho^{n-k}(x)\, dx - t_n^{k-1} \varrho^{n-k}(t_n) \right). \tag{15}$$

Otherwise, setting for example $T_n := \inf\{1 \leq k \leq n : X_k \geq t_n\}$ with $\inf_\emptyset := n$, the upper bound in equation (14) above should be replaced by $n-1$ and the extra term $t_n^{n-1} \left(\int_0^{r(t_n)} \varrho^{n-1}(x)\, dx + 1 - r(t_n) \right)$ occurs.

Example 2.26 Maximize $\mathrm{P}\left(X_{T_n} \geq Y_n^d\right)$ within $\mathcal{T}_n^c$, i.e. let $r(y) = y^d$ for $y \in [0,1]$ where $d \in [1, \infty)$. The optimal concurrent threshold t_n^* uniquely solves

$$0 = \sum_{k=1}^{n} \left(\frac{d(k-1)}{n-k+d} t_n^{k-2} - \frac{n+k(d-1)}{n-k+d} t_n^{(n+k(d-1)-d)/d} \right)$$

and the value $v_n(T_n^*)$ of the optimal concurrent threshold rule T_n^*, whereof some are given numerically in table 1 below for different d,

	$d=1$	$d=2$	$d=3$	$d=10$	$d=20$	$d=30$
$n=$ 2	0.6667	0.7708	0.8223	0.9269	0.9591	0.9715
$n=$ 3	0.6063	0.7196	0.7773	0.9007	0.9418	0.9581
$n=$ 4	0.5808	0.6973	0.7575	0.8886	0.9334	0.9516
$n=$ 5	0.5667	0.6849	0.7463	0.8816	0.9286	0.9478
$n=$ 10	0.5407	0.6617	0.7255	0.8684	0.9193	0.9404
$n=$ 100	0.5196	0.6427	0.7082	0.8574	0.9114	0.9341
$n=$1000	0.5176	0.6409	0.7066	0.8563	0.9107	0.9335

Table 1: The value $\mathrm{P}\left(X_{T_n^*} \geq Y_n^d\right)$ of the concurrent threshold rule T_n^*, which is optimal within $\mathcal{T}_n^c$.

is given by the formula

$$\mathrm{P}\left(X_{T_n^*} \geq Y_n^d\right) = \sum_{k=1}^{n} \frac{d}{n-k+d} (t_n^*)^{k-1} \left(1 - (t_n^*)^{(n-k+d)/d}\right).$$

The first column of table 1 fits searching exactly the overall maximum and its probability converges down to $\widetilde{v}^*_{id} \approx 0.5174$, see remark 2.11 i). However, to what extent does the last row represent the actual asymptotic value with respect to $(\mathcal{T}^c_n)_{n\in\mathbb{N}}$? To specify: Keeping $r'(1-)$ fixed (taking $r \in \mathcal{R}^1_1$ with $r'(1-) < \infty$), how does t^*_n converge to 1 for $n \to \infty$ and what is the corresponding asymptotic value (lower bound for $v^*_\infty(r'(1-))$? This will be answered in the next paragraph.

Asymptotically Optimal Sequences of Concurrent Threshold Rules

Now the sets $\mathcal{T}^c_n$ of concurrent threshold rules are investigated asymptotically. For this purpose let $r \in \mathcal{R}^1_1$ with $d := r'(1-) \in [1, \infty)$, excluding $r'(1-) = \infty$ and dropping the case $r(1) < 1$, see theorem 2.22 and the preceding considerations on page 36. It can be assumed that $r(y) = dy - d + 1$ for y sufficiently close to 1, since the probability for the event $[Y_n < y_0]$ becomes arbitrarily small for any $y_0 < 1$ as n becomes large — other relax functions with derivative d at 1 will coincide in asymptotic behaviour.

The asymptotic probability of winning for the sequence of concurrent threshold rules with concurrent thresholds $(t_n)_{n\in\mathbb{N}}$ is developed (threshold t_n is applied simultaneously to $X_1, \dots, X_n$ for each $n \in \mathbb{N}$): Evidently t_n should tend to 1 as $n \to \infty$. A concurrent threshold rule corresponds to a binomial experiment (each time: exceeding the threshold or not) and therefore $n(1 - t_n)$ represents the mean number of items exceeding t_n. If $n(1 - t_n)$ converges to 0 resp. converges to a positive value resp. diverges to ∞, the number of items exceeding concurrent threshold t_n will (heuristically with regard to the probability of winning) be insufficient resp. is moderate resp. is oversized:

Lemma 2.27 *Let $r \in \mathcal{R}^1_1$ where $d := r'(1-) \in [1, \infty)$. Let thresholds $(t_n)_{n\in\mathbb{N}}$ specify the sequence $(T_n)_{n\in\mathbb{N}}$ of concurrent threshold rules.*

i) If $\lim_{n\to\infty} n(1 - t_n) = 0$, then $\lim_{n\to\infty} v_n(T_n) = 0$.

ii) If $\lim_{n\to\infty} n(1 - t_n) =: \mu \in (0, \infty)$, then

$$\lim_{n\to\infty} v_n(T_n) \;=\; h(\mu, d) \;:=\; de^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^k \left(1 - \left(1 - \frac{1}{d}\right)^k\right)}{k!k}.$$

iii) If $\lim_{n\to\infty} n(1 - t_n) = \infty$ (divergence), then $\lim_{n\to\infty} v_n(T_n) = 0$.

Proof: For $n \in \mathbb{N}$ let K_n denote the number of objects exceeding concurrent threshold t_n, $K_n \sim \mathrm{B}(n, t_n)$. The following inspections are meant for $n \to \infty$, which is equivalent to $t_n \to 1$.

i) It is sufficient to prove that event $K_n = 0$ (which is equivalent to win 0) occurs asymptotically almost surely: $\mathrm{P}(K_n = 0) = t_n^n \to 1$ iff $n \ln(t_n) = n(1 - t_n + o(1 - t_n)) \to 0$, which is true since $t_n \to 1$.

ii) Decomposition with respect to the number $K_n = 0, \ldots, n$ of items exceeding threshold t_n: $\mathrm{P}(K_n = k) = \binom{n}{k}(1-t_n)^k t_n^{n-k}$. Using Poisson approximation this probability asymptotically equals $e^{-\mu}\mu^k/k!$, where $\mu := \lim_{n\to\infty} n(1-t_n) \in (0, \infty)$ (especially exists). The probability of winning given $K_n = k > 0$ items exceed threshold t_n is (for n sufficiently big, arbitrarily accurate) given by

$$\begin{aligned} \mathrm{P}(X_{T_n} \geq r(Y_n) \mid K_n = k) &= \int_0^1 \left(\frac{1}{d}(x + d - 1)\right)^{k-1} dx \\ &= \frac{d}{k} - \frac{d}{k}\left(\frac{d-1}{d}\right)^k \end{aligned}$$

(by rescaling). Consider $\mathrm{P}(X_{T_n} \geq r(Y_n) \text{ and } K_n = 0) = 0$. Thus the limit behaviour of these values is as follows:

$$\begin{aligned} \mathrm{P}(X_{T_n} \geq r(Y_n)) &= \sum_{k=1}^{n} \mathrm{P}(X_{T_n} \geq r(Y_n) \mid K_n = k) \cdot \mathrm{P}(K_n = k) \\ &\overset{n\to\infty}{\to} \sum_{k=1}^{\infty} \left(\frac{d}{k} - \frac{d}{k}\left(\frac{d-1}{d}\right)^k\right) \cdot e^{-\mu}\frac{\mu^k}{k!} \\ &= de^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^k \left(1 - \left(1 - \frac{1}{d}\right)^k\right)}{k!k}. \end{aligned}$$

iii) The probability of failure tends to 1, since, heuristically, $t_n \to 1$ too slowly: Choosing item x, the critical value $\varrho(x)$ will be surpassed, for too much items above threshold t_n will occur. For given $d = r'(1-)$ let $\varepsilon > 0$ and choose $c = c(\varepsilon) \in \mathbb{N}$ such that $\varepsilon \geq \frac{d}{c}\left(1 - \left(1 - \frac{1}{d}\right)^c\right)$. Now fix $\delta > 0$. Due to $n(1 - t_n) \to \infty$ it is possible to choose $n = n(c, \delta)$ such that $\mathrm{P}(K_n \geq c) \geq 1 - \delta$ is valid (a big n will be sufficient, since with regard to the central limit theorem $\mathrm{P}(K_n \geq c) \approx \Phi\left((n(1-t_n)-c)/\sqrt{n(1-t_n)t_n}\right) \approx \Phi\left(\sqrt{n(1-t_n)/t_n}\right) \stackrel{n\to\infty}{\to} 1$). Now X_{T_n} denotes the first (thus choosen) item exceeding concurrent threshold t_n, $X_{T_n} \sim U([t_n,1])$. Subsequent with high probability (at least $1-\delta$) at least $c-1$ objects beyond t_n (possibly preventing a win) will occur (these random variables are iid). The probability of failure is estimated from below (for estimation of ϱ take $d+\zeta$ and $\zeta \to 0$):

$$\begin{aligned}
\mathrm{P}(X_{T_n} < r(Y_n)) &\geq \mathrm{P}(X_{T_n} < r(Y_n) \mid K_n \geq c)\,\mathrm{P}(K_n \geq c) \\
&\geq \left(1 - \int_0^1 \varrho^{c-1}(x)\,dx\right)(1-\delta) \\
&\approx \left(1 - \int_0^1 \left(\frac{x+d-1}{d}\right)^{c-1} dx\right)(1-\delta) \\
&= \left(1 - \frac{d}{c}\left(1 - \left(\frac{d-1}{d}\right)^c\right)\right)(1-\delta) \\
&\geq (1-\varepsilon)(1-\delta).
\end{aligned}$$

Since ε and δ are arbitrary small, $\mathrm{P}(X_{T_n} < r(Y_n))$ tends to 1. □

Theorem 2.28 *Let $r \in \mathcal{R}_1^1$ where $d := r'(1-) \in [1,\infty)$. A sequence $(t_n)_{n\in\mathbb{N}}$ of concurrent thresholds is asymptotically optimal (with respect to $(\mathcal{T}_n^c)_{n\in\mathbb{N}}$) iff $\lim_{n\to\infty} n(1-t_n) = \mu^*$, where $\mu^* = \mu^*(d) \in [\mu_1,\infty)$ (for μ_1 see remark 2.30 ii) below) represents the unique solution of the implicit equation*

$$\int_{\mu\left(1-\frac{1}{d}\right)}^{\mu} \frac{e^\xi - 1}{\xi}\,d\xi = \frac{e^\mu - e^{\mu\left(1-\frac{1}{d}\right)}}{\mu}. \tag{16}$$

In addition the asymptotic value with respect to $(\mathcal{T}_n^c)_{n\in\mathbb{N}}$ is
$\widetilde{v}_\infty^*(d) := \lim_{n\to\infty} \sup_{T\in\mathcal{T}_n^c} v_n(T) = h(\mu^*,d) = de^{-\mu^*}\int_{\mu^*\left(1-\frac{1}{d}\right)}^{\mu^*} \frac{e^\xi-1}{\xi}\,d\xi$ *(lemma 2.27 ii)).*

Proof: With regard to lemma 2.27 sequences where $\lim_{n\to\infty} n(1-t_n)$ is equal to 0 or ∞ can be neglected. Now suppose $\mu := \lim_{n\to\infty} n(1-t_n) \in (0,\infty)$ (if there are several accumulation points, subsequences with different asymptotic probability of winning can be chosen). The asymptotic probability of winning $h(\mu,d)$ of that lemma is to be maximized with respect to μ:

$$\frac{\partial}{\partial\mu}h(\mu,d) = \frac{1-e^{-\mu/d}}{\mu/d} - h(\mu,d).$$

Using the necessary condition $h(\mu^*, d) = \left(1 - e^{-\mu^*/d}\right) d/\mu$, the repeated derivation proves to be negative in its solution $\mu^* = \mu^*(d)$ (if existing):

$$\frac{\partial^2}{\partial \mu^2} h(\mu, d)\bigg|_{\mu=\mu^*} = -\frac{1 - e^{-\mu^*/d}}{(\mu^*)^2/d} + \frac{e^{-\mu^*/d}}{\mu^*} < 0,$$

since this inequality is equivalent to $(d + \mu^*)e^{-\mu^*/d} < d$: For $\mu = 0$ equality would apply and for $\mu > 0$ the derivative of the lefthand side $\frac{\partial}{\partial \mu}\left((d + \mu)e^{-\mu/d}\right)$ $= -\frac{\mu}{d}e^{-\mu/d} < 0$. Furthermore $h(\mu, d)$ is continuous for $\mu \in (0, \infty)$ and $h(\mu, d) \to 0$ as $\mu \to 0$ and as $\mu \to \infty$. Thus a maximum of $h(\mu, d)$ is existing and unique, called $\mu^*(d)$ and represents the limit of $n(1 - t_n)$ for $n \to \infty$ of $(t_n)_{n\in\mathbb{N}}$ in order to be asymptotically optimal with respect to $(\mathcal{T}_n^c)_{n\in\mathbb{N}}$. □

The behaviour of $\mu^*(d)$ and the corresponding value is illustrated in table 2 below for $d = 1\,(1)\,5\,(5)\,20$, including its approximation according to the subsequent theorem 2.29. Heuristically the asymptotic mean number $\mu^*(d)$ of offers beyond the threshold is evidently increasing in d, since due to the decreasing demands it is advisible to consider an increasing number of offers. Equation (16) can (even for $d = 1$) not be solved explicitly, however the following theorem clarifies the asymptotic relation between d and $\mu^*(d)$:

Theorem 2.29 *Let the assumptions and notations of theorem 2.28 be given. Then*

$$\mu^*(d) \simeq \ln(2d + 1) \qquad \text{as } d \to \infty. \tag{17}$$

Proof: Setting $\mu = \mu^*$ the equation $h(\mu, d) = \frac{d}{\mu}(1 - e^{-\mu/d})$ is transformed (rearrangements are valid since d is an upper bound for the absolute series):

$$de^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^k \left(1 - \left(1 - \frac{1}{d}\right)^k\right)}{k!k} = \frac{d}{\mu}(1 - e^{-\mu/d})$$

$$de^{-\mu} \sum_{k=1}^{\infty} \frac{\mu^k}{k!k} \sum_{j=1}^{k} (-1)^{j+1} \binom{k}{j} \frac{1}{d^j} = \frac{d}{\mu}(1 - e^{-\mu/d})$$

$$de^{-\mu} \sum_{1\le j\le k<\infty} (-1)^{j+1} \frac{1}{d^j} \mu^k \frac{1}{j!(k-j)!k} = \frac{d}{\mu}(1 - e^{-\mu/d})$$

$$\begin{aligned} e^{-\mu}\sum_{j=1}^{\infty}\left(\frac{(-1)^{j+1}}{j!}\frac{1}{d^{j-1}}\sum_{i=0}^{\infty}\frac{\mu^{j+i}}{i!(j+i)}\right) &= \frac{d}{\mu}(1-e^{-\mu/d}) \\ \sum_{j=0}^{\infty}(-1)^j\alpha_j\frac{1}{d^j} &= \sum_{j=0}^{\infty}\frac{(-1)^j}{(j+1)!}\left(\frac{\mu}{d}\right)^j \\ \sum_{j=0}^{\infty}(-1)^j\beta_j\frac{1}{d^j} &= 0, \end{aligned} \tag{18}$$

where for $j \in \mathbb{Z}_+$ abbreviations $\alpha_j = \alpha_j(\mu) := \frac{1}{(j+1)!}e^{-\mu}\sum_{i=0}^{\infty}\frac{\mu^{j+1+i}}{i!(j+1+i)}$ and $\beta_j = \beta_j(\mu) := \alpha_j(\mu) - \frac{\mu^j}{(j+1)!}$ are used. In the sequel $\mu > 0$.
Estimation $\alpha_j(\mu) \leq \frac{\mu^{j+1}}{(j+1)!}e^{-\mu}\sum_{i=0}^{\infty}\frac{\mu^i}{(i+1)!} \leq \frac{\mu^{j+1}}{(j+1)!}\frac{1-e^{-\mu}}{\mu} < \frac{\mu^j}{(j+1)!}$ for $j \in \mathbb{N}$. Particularly $\beta_0(\mu) = -e^{-\mu}$ and $\beta_1(\mu) = -\frac{1-e^{-\mu}}{2}$ and estimation $\beta_j(\mu) < 0$ and if $j \geq 2$ and $\mu \geq 2$ then $|\beta_j(\mu)| \leq \frac{\mu^{j+1}}{(j+1)!}$.
Dividing equation (18) by β_0 and grouping with respect to powers of the new variable $x = x(\mu, d) := \frac{1}{d}\frac{\beta_1}{\beta_0} = \frac{1}{2d}(e^{\mu}-1) \in (0,\infty)$, the equivalent expression

$$1 - x + \sum_{j=2}^{\infty}\gamma_j x^j = 0 \tag{19}$$

with new coefficients $\gamma_j = \gamma_j(\mu) := (-1)^j\beta_j\beta_0^{j-1}/\beta_1^j = (-1)^j|\beta_j|\,e^{\mu}\left(\frac{2}{e^{\mu}-1}\right)^j$ for $j \geq 2$ results. Set $F_\mu(x) := \sum_{j=2}^{\infty}\gamma_j x^j$, convergent for $x \in [0,\infty)$ and $\mu \in (0,\infty)$ — $|\gamma_j|(j+1)! \leq \mu e^{\mu}\left(\frac{2\mu}{e^{\mu}-1}\right)^j =: \delta_{\mu,j}$. Now $F_\mu(x) = o(1)$ for $\mu \to \infty$ uniformly in $x \in [a,b]$ with $0 < a < b < \infty$: $\delta_{\mu,j}$ is nonnegative and if $\mu \in [2,\infty)$ then $\delta_{\mu,j}$ is decreasing in $j \geq 2$. On the other hand if $j \geq 2$ then $\delta_{\mu,j} = o(1)$ as $\mu \searrow 0$ or as $\mu \to \infty$ (by the rule of de l'Hospital) and maxima of $\delta_{\mu,j}$ with respect to μ lie on the implicit curve $j = \mu^2(e^{2\mu}-1)/(1+e^{\mu}(\mu-1))$, where the right side is positive, increasing in μ and exceeds 2 for $\mu = 2$. Therefore the maximum of $\delta_{\mu,j}$ for $\mu \geq \mu_0$ and $j \geq 2$ is attained in $\delta_{\mu_0,2} = o(1)$ as $\mu_0 \to \infty$. Thus $\delta_{\mu,j} = o(1)$ for $\mu \to \infty$ independent of $j \geq 2$ and thus $F_\mu(x) \leq o(1)(e^b - b - 1) = o(1)$ as $\mu \to \infty$ uniformly in $x \in [a,b]$.
Since according to theorem 2.28 the equation $1 - x + F_\mu(x) = 0$ has a unique solution x for fixed μ, it is given by $x = (e^{\mu}-1)/(2d) + o(1)$ as $\mu \to \infty$ and its limit is 1. Thus $d \simeq (e^{\mu}-1)/2$ or $\mu \simeq \ln(2d+1)$ as $\mu = \mu^* \to \infty$. □

d	$\mu^*(d)$	$\ln(2d+1)$	$h(\mu^*(d),d)$	$h(\ln(2d+1),d)$
1	1.5029	1.0986	0.5174	0.4974
2	1.9359	1.6094	0.6407	0.6324
3	2.2223	1.9459	0.7064	0.7019
4	2.4387	2.1972	0.7487	0.7460
5	2.6135	2.3979	0.7788	0.7770
10	3.1900	3.0445	0.8562	0.8557
15	3.5468	3.4340	0.8906	0.8904
20	3.8068	3.7136	0.9106	0.9105

Table 2: Optimal $\mu^*(d)$ and optimal asymptotic value $h(\mu^*(d),d)$ within $(\mathcal{T}_n^c)_{n\in\mathbb{N}}$ and approximations thereof, referring to theorems 2.28 and 2.29 above.

Remark 2.30

i) The basic message of theorem 2.28 is the asymptotic behaviour of the optimal concurrent threshold rule t_n^* of $\mathcal{P}_n$: $\lim_{n\to\infty} n(1-t_n^*) = \mu^*(d)$, with corresponding asymptotic value $h(\mu^*(d),d)$.

ii) In Gilbert and Mosteller [18] the case $r \equiv id$ resp. $d=1$ is considered and the corresponding term $\mu_1 := \mu^*(1) \approx 1.5029$ is the solution of equation $g(\mu) := e^{-\mu}\sum_{k=1}^{\infty}\frac{\mu^k}{k!k} = \frac{1}{\mu}(1-e^{-\mu})$. Here $h(\mu,1) \equiv g(\mu)$ and $h(\mu,d) = dg(\mu) - de^{-\mu/d} g\left(\frac{d-1}{d}\mu\right)$.
As $h(\mu^*(1),1) = \widetilde{v}_{id}^* \approx 0.5174 < 0.5802 \approx v_{id}^* = v_\infty^*(1)$ may indicate, the inequality $h(\mu^*(d),d) < v_\infty^*(d)$ may hold for any $d \in [1,\infty)$, see vi).

iii) Function $h(\mu,d)$ solves the following partial differential equations (let $\mu \in (0,\infty)$, $d \in (1,\infty)$):

$$\begin{aligned}
\frac{\partial}{\partial\mu} h(\mu,d) &= \frac{d}{\mu}\left(1-e^{-\mu/d}\right) - h(\mu,d) \\
\frac{\partial}{\partial d} h(\mu,d) &= \frac{1}{d} h(\mu,d) - \frac{1}{d-1}\left(e^{-\mu/d} - e^{-\mu}\right).
\end{aligned}$$

iv) Regard function $\widetilde{h}(\mu) := h(\mu,(e^\mu-1)/2)$, i.e. regard the value referring to the approximation of $\mu^*(d)$. First $\widetilde{h}(\mu)$ is increasing in $\mu \in (0,\infty)$, since it is the solution of the differential equation

$$\frac{d}{d\mu}\widetilde{h}(\mu) \;=\; \frac{1}{e^{\mu}-1}\widetilde{h}(\mu)+\frac{1}{2}\frac{(e^{\mu}-1)\left(e^{\mu}-e^{\mu\frac{e^{\mu}-3}{e^{\mu}-1}}\right)}{\mu e^{\mu}}-\frac{e^{\mu\frac{e^{\mu}-3}{e^{\mu}-1}}-1}{e^{\mu}-3},$$

where the inhomogenous term turns out to be positive for $\mu > 0$. Second $\widetilde{h}(\mu) \to 1$ as $\mu \to \infty$ (in accordance with theorem 2.22 where also only concurrent threshold rules were used), which is verified as follows (set $d := (e^{\mu}-1)/2$): In expression $de^{-\mu}\sum_{j=1}^{\infty}(-1)^{j+1}e^{\mu}\alpha_{j-1}(\mu)/d^{j}$ the leading factor converges to $1/2$, the first term of the series, $j=1$, equals 2, whereas for $j>1$ the j–th term is $O\left(\mu^{j-1}e^{-\mu(j-1)}\right) = o(1)$ for $\mu \to \infty$, since $e^{\mu}\alpha_j(\mu) = O\left(\mu^{j-1}e^{\mu}\right)$.

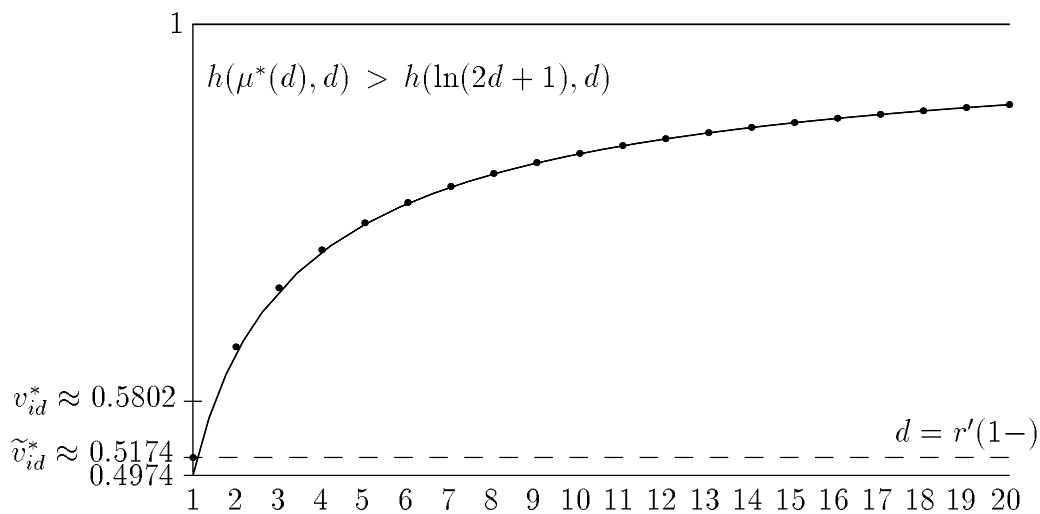

Figure 3: The value $\widetilde{v}^*_{\infty}(d) = h(\mu^*(d),d)$ for $d = 1,\ldots,20$ (dots) and the approximative curve $h(\ln(2d+1),d)$ for $d \in [1,20]$ where $d := r'(1-)$, see remark 2.30 v) below.

v) In figure 3 values $\widetilde{v}^*_{\infty}(d)$, asymptotically optimal within $(\mathcal{T}^c_n)_{n\in\mathbb{N}}$, are plotted for different $d := r'(1-) \in [1,\infty)$: First for fix d exact values $\mu^*(d)$ and probabilities of winning $\widetilde{v}^*_{\infty}(d) = h(\mu^*(d),d)$ are computed (according to theorem 2.28). Second the approximation $h(\ln(2d+1),d)$ from below is plotted, according to theorem 2.29. Due to $r \preceq id$ the relation $v^*_{\infty}(d) \geq \widetilde{v}^*_{\infty}(d) \geq \widetilde{v}^*_{id} \approx 0.5174$ is valid (dashed line in the figure above) and a rough estimation from below is $h(\ln(3),1) \approx 0.4974$

(based on theorem 2.29 and indicated by the baseline in the figure). Relation to the finite case: Fix $d = r'(1-) \in [1,\infty)$ and let $T_n^* = T_n^*(r)$ denote the optimal concurrent threshold rule, $n \in \mathbb{N}$. Varying function $r \in \mathcal{R}_1^1$ while preserving derivative d, then $v_n(T_n^*(r), r) \in [\widetilde{v}_{id}^*, 1)$. In case of $r \equiv id$ then $v_n(T_n^*)$ decreases in n whereas for $r \prec id$ the behaviour of $\widetilde{v}_n^*(d)$ isn't monotone in general. If $d > 1$ particularly $v_n(T_n^*(r))$ may be bigger or lower than $\widetilde{v}_\infty^*(d) = h(\mu^*(d), d)$, since function r might fit (for $y \in [0, 1-\varepsilon]$ where $\varepsilon > 0$ is small) function 0 (maximal probability near 1) or id (minimal probability close to $\widetilde{v}_{id}^*$). In other words $\{v_n(T_n^*(r), r) \ : \ r \in \mathcal{R}_1^1 \text{ with } r'(1-) = d\} = (\widetilde{v}_{id}^*, 1)$, where $d \in (1,\infty)$ and $n \in \mathbb{N}$ are fixed.

vi) Besides $v_\infty^*(1) = v_{id}^* \approx 0.5802$ and $v_\infty^*(\infty) = 1$ the interesting question how $v_\infty^*(d)$ behaves for $d \in (0,\infty)$ must be left open. Particularly it may be possible that $v_\infty^*(d) = 1$ for $d \in (1,\infty]$ and it may be possible that $v_\infty^*(d)$ is continuous on $[1,\infty]$ with $v_\infty^*(d) < 1$ for $d \in [1,\infty)$.

The Myopic Stopping Time

The myopic stopping time referring to selecting an r–candidate means the one step look–ahead rule referring to the subsequence of r–candidates (see the corresponding paragraph on page 81): Compare the mean payoff of stopping with the mean payoff selecting the next r–candidate (if any).
If there is one draw remaining, $\ell = 1$, the optimal and the myopic stopping time are identical for any $r \in \mathcal{R}$ — there is maximal one r–candidate to choose yet. This already implies that the myopic stopping time can be not optimal if $r \prec id$ for $\ell > 1$: The myopic stopping time compares the mean payoff for accepting the present item with the scenario for selecting the next r–candidate — regardless wether this myopic stopping time would really accept the next r–candidate (in the case of proceeding). This difference in anticipation and actual selection is possible as the proper inequalities $r(y) < b_1^*(y) = r(1 - r(y)) < y$ in a nontrivial interval $(\underline{b}_1, \overline{b}_1)$ indicate (see figure 1 on page 25). On the other hand e.g. $\underline{b}_2 = \overline{b}_2$ is possible.
Thus in short the myopic stopping time isn't optimal in general, since its stopping sets may miss closedness, though they may be monotone.

Proposition 2.31 *The myopic stopping time S_m for problem $\mathcal{P}_n(r)$ where $r \in \mathcal{R}$ stops in $(x,y) \in \Delta$ with $0 < \ell < n$ remaining offers if $x \geq r(y)$ and*

$$x \;\geq\; r\left(\left[\sum_{i=0}^{\ell-1} r^{\ell-1-i}(y) \int_{r(y)}^{1} \varrho^{i}(\xi)\; d\xi\right]^{1/\ell}\right).$$

Proof: If $(x,y) \in \Delta_0^*$ then $s_\ell(x) = \varrho^\ell(x)$. Set $(\mathbb{P}s)(n,x,y) := 0$ and for $\ell > 0$

$$(\mathbb{P}s)(n-\ell,y) \;=\; \sum_{i=0}^{\ell-1} r^{\ell-1-i}(y) \int_{r(y)}^{1} \varrho^{i}(\xi)\; d\xi,$$

confer page 83: Decompose with respect to the arrival time of the next r–candidate: $\ell-1-i$ non–r–candidates pass and (the first) r–candidate ξ wins with respect to i future values. Similarity to $c_\ell(x,y)$ of (2) and (6). □

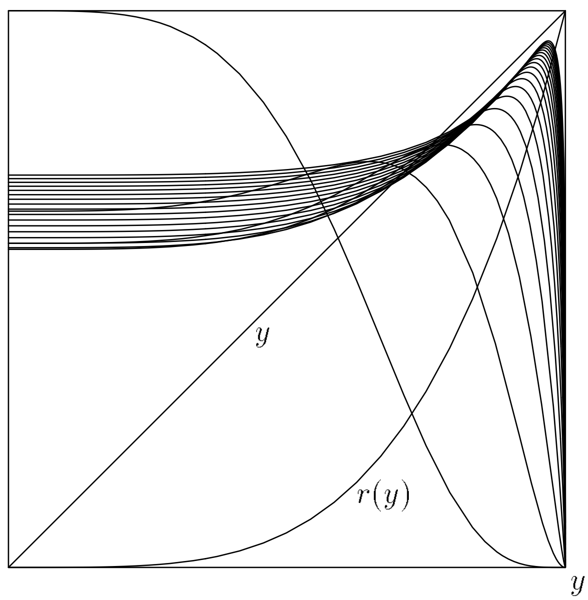

Figure 4: The stopping sets of the myopic stopping time S_m (inside Δ_0^*, axes reversed) aren't monotone: $r(y) = y^4$ on $[0,1]$ and $\ell = 1,\ldots,20$ (for $\ell = 1$ coincidence with figure 1).

The stopping sets of the myopic stopping time seem to be not monotone in general, see figure 4 above. For computations the recursive relation (with reversed time index) $(\mathbb{P}s)(\ell+1,y) = r(y)(\mathbb{P}s)(\ell,y) + \int_{r(y)}^{1} \varrho^\ell(\xi)\; d\xi$ for $\ell \in \mathbb{Z}_+$ with $(\mathbb{P}s)(0,y) := 0$ is useful.

A Random Index to Take Any r–Candidate

In this paragraph the approach of the myopic stopping only to respect, in case of proceeding, selection of the next r–candidate (if any; regardless of its magnitude) is extended to the decision of stopping. Given relax function $\mathcal{R} \ni r \prec id$, regard stopping times based on the following comparison: Either take any present r–candidate or choose the next r–candidate (if any). In the context of stopping problem $\mathcal{P}_n$ such a stopping time is called random index and denoted by I_n. The indicated restriction leads to a rule based on the information: The present maximum Y_i exceeds a threshold t_i (or not) and the present value X_i is an r–candidate (or not), for each time instant $i = 1, \ldots, n$. Thus I_n is specified by a sequence $t_1, \ldots, t_n$, assumed to be nonincreasing, via

$$I_n := \inf \{1 \leq i \leq n \, : \, t_i \leq Y_i \leq \varrho(X_i)\} \tag{20}$$

where $\inf_\emptyset := \infty$. In any case $t_n := 0$ is advisible. The stopping sets of a random index I_n inside Δ_0^* are specified by vertical lines.

In terms of threshold rules this type of restriction means to specify a random time J_n for thresholds $u_1 = \ldots = u_{J_n - 1} = 1$ and $u_{J_n} = \ldots = u_n = r(Y_{J_n - 1})$ with random height $(Y_0 := 0)$. Here $J_n := 1 + \sup\{1 \leq j \leq n \, : \, Y_j < t_j\}$ (where $\sup_\emptyset := 0$) meets random index I_n.

For $n = 2$ the value $\mathrm{P}\left(X_{I_2} \geq r(Y_2)\right) = \int_{t_1}^1 \varrho(x) \; dx + \int_0^{t_1} 1 - r(x) \; dx$ is maximal if t_1 is the unique solution of $r(t) + \varrho(t) = 1$. For $n \in \mathbb{N}$ the value $v_n(I_n) := \mathrm{P}\left(X_{I_n} \geq r(Y_n)\right)$ of I_n according to $t_1 \geq \cdots \geq t_n$, see (20), is

$$\begin{aligned} v_n(I_n) \;\; &= \;\; \sum_{i=1}^{n} \left[t_i^{i-1} \int_{t_i}^{1} \varrho^{n-i}(x) \, dx \right] \\ &\quad + \; \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left[\left(\prod_{k=j+1}^{i-1} t_k \right) \int_{t_{j+1}}^{t_j} \left(\int_{r(y)}^{1} \varrho^{n-i}(x) \, dx \right) \, dy^j \right] \end{aligned} \tag{21}$$

by decomposing with respect to $[I_n = i]$: The first sum covers the case $[Y_{i-1} < t_i \text{ and } X_i \geq t_i]$ for $i = 1, \ldots, n$, whereas the second term covers the events $[Y_j \in [t_{j+1}, t_j) \text{ and } X_{j+1} < t_{j+1}, \ldots, X_{i-1} < t_{i-1} \text{ and } X_i \geq r(Y_j)]$ for $j = 1, \ldots, i-1$ where $i = 2, \ldots, n$ (take $Y_0 := -1$, $\sum_\emptyset := 0$ and $\prod_\emptyset := 1$).

Since maximization of expression (21) seems unaccessible, a more simple approach to a random index (in general suboptimal random index) with an interesting performance is displayed next: Suppose to stop in state $(i,x,y) \in E$ with $(x,y) \in \Delta_0^*$, if the mean payoff of choosing the next r–candidate, $(\mathbb{P}s)(i,y)$, is not bigger than the mean payoff of stopping with any present r–candidate. The smallest value to stop with then is $r(y)$ and this means comparison $(\mathbb{P}s)(i,y) \leq (\varrho(r(y))^{n-i}$. This, with regard to theorem 2.13 i), specifies uniquely the upper boundary value $\overline{b}_{n-i}$ and the corresponding random index is called $\overline{I}_n$ (for fixed n; then $t_i = \overline{b}_{n-i}$ for $i = 1,\ldots,n$ with $\overline{b}_0 := 0$). According to the regular case $\left(\overline{b}_\ell\right)_{\ell\in\mathbb{Z}_+}$ is nondecreasing and proposition 2.15 treats asymptotic behaviour. This random index $\overline{I}_n$ may be a suboptimal random index: Heuristically, supposing a threshold lower than $\overline{b}_{n-i}$, the loss while accepting smaller values (i.e. $x \geq r(y)$ with $s(i,x) < (\mathbb{P}s)(i,y)$) may be prevailed by the other (big) values.
In table 3 below for relax function $r(y) = y^4$ and $n = 2(1)5, 10(10)50, 100, 500$ the threshold $\overline{b}_{n-1}$ (interpretation: $n-1$ items remain) and the value $v_n(\overline{I}_n)$

n	$\overline{b}_{n-1}$	b_{n-1}	$v_n(\overline{I}_n)$	$v_n(I_n)$
2	0.7245	0.6305	0.9498	0.9611
3	0.8427	0.8153	0.9389	0.9742
4	0.8897	0.8768	0.9401	0.9858
5	0.9151	0.9076	0.9449	0.9923
10	0.9605	0.9589	0.9631	0.9904
20	0.9809	0.9805	0.9679	0.9782
30	0.9874	0.9873	0.9680	0.9738
40	0.9906	0.9905	0.9678	0.9716
50	0.9925	0.9925	0.9676	0.9702
100	0.9963	0.9963	0.9670	0.9677
500	0.9993	0.9993	0.9662	0.9662

Table 3: Maximization of $\mathrm{P}\left(X_S \geq Y_n^4\right)$ based on random indexes: Threshold $\overline{b}_{n-1}$ and the value $v_n(\overline{I}_n)$, besides the value $v_n(I_n)$ based on (asymptotic) approximation b_{n-1} of each threshold $\overline{b}_{n-1}$.

are computed. For example if $n = 5$ then $t_1 = \overline{b}_4 \approx 0.9151$ and $t_2 = \overline{b}_3 \approx 0.8897$, and if $n = 4$ then $t_1 = \overline{b}_3 \approx 0.8897$. The approximation $b_{n-1} := 1 - 0.3695/(n-1)$ of $\overline{b}_{n-1}$ for $n > 1$ refers to $\alpha(4) \approx 0.3695$ (see example 2.14 and proposition 2.15), the corresponding random index based on thresholds $b_{n-1} > \cdots > b_0$ being called I_n ($b_0 := \overline{b}_0 := 0$).

In table 3 the values prove to be not monotone in n and $v_n(\overline{I}_n) < v_n(I_n)$ is possible, the approximative thresholds may lead to a better performance. Particularly for $n = 2$ the value of the optimal random index is approximately 0.9620, referring to the thresholds $t_1 \approx 0.5920$ and $t_2 = 0$, where t_1 uniquely solves $t_1^4 + \sqrt[4]{t_1} = 1$.

In figure 5 below the value $v_n(\overline{I}_n)$ is plotted for $n = 1(1)10(5)50$ where $r(y) = y^2, y^3, y^4$. While $\lim_{n\to\infty} n(1 - \overline{b}_n) = \alpha$ is known (proposition 2.15), an analytic description of the asymptotic value $\lim_{n\to\infty} v_n(\overline{I}_n)$ for $r \in \mathcal{R}_1^1$ with $r'(1-) \in (1, \infty)$ via expression (21) seems to be unaccessible. The question wether the values converge for $n \to \infty$ as they suggest in the figure below must be left open — if they do, then they may surpass the value of optimal sequences of concurrent threshold rules and may improve their lower bound for the value $v_\infty^*(r'(1-))$ with $r'(1-) \in (1, \infty)$: See table 2 or figure 3, where for example $h(\mu^*(4), 4) \approx 0.7487$ while here $v_{500}(\overline{I}_{500}) \approx 0.9662$ results for $r'(1-) = 4$.

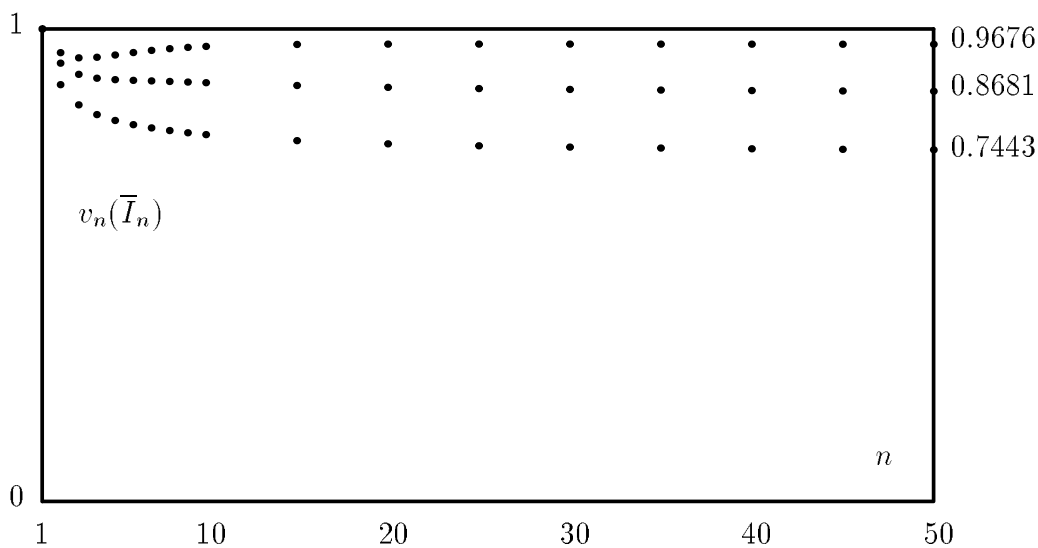

Figure 5: The value $v_n(\overline{I}_n)$ of random index $\overline{I}_n$ (based on $(b_\ell^*)_{0\le\ell<n}$) for $n = 1(1)10(5)50$, where $r(y)$ equals y^2, y^3 and y^4 (corresponding sequences of dots from bottom to top).

Remark 2.32 The approach of expression (21) can be adapted to specify the value of the stopping time S_n for problem $\mathcal{P}_n(r)$ which uses the thresholds $t_1 \geq \cdots \geq t_n$ and only accepts r–candidates, i.e. which applies boundary functions $t_1 \vee r(y), \ldots, t_n \vee r(y)$:

$$\begin{aligned} v_n(S_n) &= \sum_{i=1}^{n} \left[t_i^{i-1} \int_{t_i}^{1} \varrho^{n-i}(x)\, dx \right] \\ &+ \sum_{i=2}^{n} \sum_{j=1}^{i-1} \left[\int_{t_{j+1}}^{t_j} \left(\left(\prod_{k=j+1}^{i-1} t_k \vee r(y) \right) \int_{t_i \vee r(y)}^{1} \varrho^{n-i}(x)\, dx \right) dy^j \right], \end{aligned}$$

since the decomposition for the first sum remains unaffected, $[Y_{i-1} < t_i$ and $X_i \geq t_i]$ for $i = 1, \ldots, n$ with $Y_0 := 0$, while the decomposition for the second sum is different: $[Y_j \in [t_{j+1}, t_j)$ and $X_{j+1} < t_{j+1} \vee r(y), \ldots, X_{i-1} < t_{i-1} \vee r(y)$ and $X_i \geq t_i \vee r(Y_j)]$ for $j = 1, \ldots, i-1$ where $i = 2, \ldots, n$ (again set $\sum_{\emptyset} := 0$ and $\prod_{\emptyset} := 1$).

2.1.2 The Markovian Case

Suppose to maximize the probability of choosing an r–candidate, $r \in \mathcal{R}$, where the sequence of offers $X_1, \ldots, X_n$ represents a Markov process. To permit reasonable results in this Markovian case three minimal properties are claimed:

1. The mean payoff doesn't shrink if the chosen value grows.
2. Unique optimal boundaries exist (see preparation of definition 2.5).
3. The optimal stopping problem is regular.

Let a homogenous Markov process $X = (X_k)_{k \in \mathbb{N}}$ with state space $[0,1]$ be given. Homogeneity is assumed with regard to claim 3., see counterexample 2.35 ii), and $[0,1]$ is taken in order not to restate monotonicity criteria and set $\mathcal{R}$. Let conditional distribution function $F_x(\xi) := \mathrm{P}\,(X_{k+1} \le \xi \mid X_k = x)$ for $x, \xi \in [0,1]$ and $k \in \mathbb{N}$ be continuous and increasing in ξ on the set $\{z \in \mathbb{R} : 0 < F_x(z) < 1\}$ for $x \in [0,1]$. Let $Y_k := \max\{X_1, \ldots, X_k\}$ denote the relative maxima, $k \in \mathbb{N}$.

Let $\mathcal{P}_n^m(r)$, where $r \in \mathcal{R}$ and $1 < n \in \mathbb{N}$, denote the corresponding optimal stopping problem (see specification of the mathematical model in the beginning of this chapter): The objective is optimal stopping of the Markov process $Z = (Z_k)_{k \in \mathbb{Z}_+}$, where $Z_0 := \alpha_0$, $Z_k := (k, X_k, Y_k)$ for $k = 1, \ldots, n$ and $Z_k := \alpha_\infty$ for $k > n$ with payoff function is $\mathrm{P}\,(X_S \ge r(Y_n))$ according to a nonanticipating stopping time $S \in \mathcal{S}_n$.
Suppose $(x, y) \in \Delta$ and let $\ell = n-1, \ldots, 0$ denote the number of remaining draws. Again let $s_\ell(x,y)$ denote the mean payoff of stopping and let $c_\ell(x,y)$ describe the mean payoff for proceeding at least one item and then choosing optimally. Function ϱ is again defined according to expression (4).

Suppose $\ell = 1$: Stopping with the last but one item with $x \ge r(y)$ leads to mean payoff $s_1(x) = \mathrm{P}\,(x \ge r(X_n) \mid X_{n-1} = x) = F_x(\varrho(x))$, which should be nondecreasing in x according to claim 1. A sufficient condition is $F_x(t) \le F_\xi(t)$ for $t \in [0,1]$ and $(x, \xi) \in \Delta$: Then $F_x(\varrho(x)) \le F_x(\varrho(\xi)) \le F_\xi(\varrho(\xi))$. This suggests the following condition for the stochastic behaviour of X: As X_k increases, X_{k+1} should be stochastically nonincreasing, $k \in \mathbb{N}$:

$$\mathrm{P}\,(X_{k+1} \le x \mid X_k = \xi) \quad \ge \quad \mathrm{P}\,(X_{k+1} \le x \mid X_k = \zeta) \tag{22}$$

or $F_\xi(x) \le F_\zeta(x)$ for $x \in [0,1]$ and $(\xi, \zeta) \in \Delta$.

The subsequent lemma specifies the behaviour of function $s_\ell(x,y)$ and $c_\ell(x,y)$.

Lemma 2.33 *Regard problem $\mathcal{P}_n^m(r)$ for a homogenous Markov process X fulfilling condition (22). Let $(x,y)\in\Delta$ be given and let $\ell\in\mathbb{Z}_+$.*

i) If $x\geq r(y)$, then $s_\ell(x)$ is nonincreasing in ℓ and nondecreasing in x.

ii) $c_\ell(x,y)$ is nonincreasing both in x and y.

Proof:

i) Due to homogeneity the mean payoff $s_\ell(x)$ doesn't shrink as ℓ decreases, since the requests for a win become easier to meet (not in a strong sense: Take $\ell=2$ and $X_1=x=\frac{1}{2}$ implying $X_2=X_3=1$, plus homogeneity).

Condition (22) implies $s_\ell(x)$ is nondecreasing in x: The case $\ell=1$ is shown in the introduction above. For $\ell>1$ resp. $k<n-1$ fixed, let $G_\xi(x):=\mathrm{P}\left(Y_{k+2}^n\leq x \;\middle|\; X_{k+1}=\xi\right)$, where $Y_{k+2}^n:=\max\{X_{k+2},\ldots,X_n\}$. Then for $(x,\xi)\in\Delta$

$$\begin{aligned}
s_\ell(x) &= \mathrm{P}\left(x\geq r(X_{k+1}) \text{ and } x\geq r(Y_{k+2}^n) \;\middle|\; X_k=x\right)\\
&= \int_{-\infty}^{\varrho(x)} G_{x_{k+1}}(\varrho(x))\,dF_x(x_{k+1})\\
&\leq \int_{-\infty}^{\varrho(\xi)} G_{x_{k+1}}(\varrho(\xi))\,dF_x(x_{k+1})\\
&\leq \int_{-\infty}^{\varrho(\xi)} G_{x_{k+1}}(\varrho(\xi))\,dF_\xi(x_{k+1})\\
&= s_\ell(\xi),
\end{aligned}$$

where the last inequality is valid since the mass given by $F_\xi-F_x$ is nonnegative.

ii) As the present value x grows, future offers become stochastically smaller, which implies less fortune for the selection in the future meeting the claims demanded by y resp. $x\vee y$. On the other hand $c_\ell(x,y)$ is evidently nonincreasing in y (since the requirements in order to select an r–candidate grow), but it isn't decreasing: Suppose $x\leq y<\zeta\leq 1$ and take $\ell=1$, then possibly $\mathrm{P}\left(X_n\in[r(y),r(\zeta))\mid X_{n-1}=x\right)=0$. □

This lemma provides regularity of problem $\mathcal{P}_n^m(r)$:

Lemma 2.34 *The optimal stopping problem $\mathcal{P}_n^m(r)$ for a homogenous Markov process X obeying condition (22) is regular.*

Proof: Let $(x,y) \in \Delta$ with $x \geq r(y)$ and $\ell \in \mathbb{N}$, Then decomposing with respect to $X_{n-\ell+1} = \xi$ and using monotonicity of lemma 2.33 ii) yields

$$\begin{aligned} c_\ell(x,y) &= \int_{-\infty}^{r(y)} c_{\ell-1}(\xi,y)\, dF_x(\xi) + \int_{r(y)}^{\infty} v_{\ell-1}(\xi, y \vee \xi)\, dF_x(\xi) \\ &\geq \int_{-\infty}^{x} c_{\ell-1}(x,y)\, dF_x(\xi) + \int_{x}^{\infty} s_{\ell-1}(\xi)\, dF_x(\xi) \\ &= F_x(x) c_{\ell-1}(x,y) + \int_{x}^{\infty} s_{\ell-1}(\xi)\, dF_x(\xi), \end{aligned}$$

which leads to the estimation

$$F_x(x) c_{\ell-1}(x,y) \;\leq\; c_\ell(x,y) - \int_x^\infty s_{\ell-1}(\xi)\, dF_x(\xi).$$

Then, supposing $s_\ell(x) \geq c_\ell(x,y)$ with $\ell \in \mathbb{N}$, a sufficient criterion for the desired inequality $s_{\ell-1}(x) \geq c_{\ell-1}(x,y)$ is

$$F_x(x) s_{\ell-1}(x) \;\geq\; c_\ell(x,y) - \int_x^\infty s_{\ell-1}(\xi)\, dF_x(\xi),$$

which proves to be true because the relation $c_\ell(x,y) \leq s_\ell(x) \leq s_{\ell-1}(x)$, see lemma 2.33 i), leads to

$$\int_x^\infty s_{\ell-1}(\xi)\, dF_x(\xi) \;\geq\; (1 - F_x(x))\, s_{\ell-1}(x),$$

verifying the regularity of the problem. □

Thus optimal boundary functions resp. optimal stopping sets for this Markovian setting, in principle, are specified analogously to theorem 2.7.

Homogeneity of the Markov process X and condition (22) are in a sense necessary to enable claims 1., 2. and 3., which is indicated by counterexamples, chosen to be discrete to simplify matters:

Counterexample 2.35 Indication and motivation for conditions posed:

i) Dropping condition (22) may violate claim 1. and claim 2.:
Take $n = 2$. Suppose $X_1 = 1$ implies $X_2 = 0$, thus choosing 1 would be optimal. If now $X_1 = 3$ implies $X_2 = 4$, then choosing 3, seeming more useful than 1, could be not optimal: Take $3 < r(4)$. This effect doesn't even permit an increase of X_2 by the same amount X_1 does (for example $X_1 \sim U([0, 1])$ and $X_2 = 1 + X_1$), since at any rate function r may dominate this increase.

ii) Homogeneity of the Markov process X ensures regularity — claim 3.:
For a nonhomogenous Markov process X the optimal stopping problem isn't regular in general, since $s_\ell(x)$ may be increasing in ℓ (see the last argument in the proof of lemma 2.34, where the case $X_{n-\ell+1} \in (x, \varrho(x)]$ can't be estimated nor circumvented):
Take $n = 3$ and suppose $X_1 = 1$ implies $X_2 = X_3 = 2$ where $1 \geq r(2)$ would ensure a win of 1 unit. Besides suppose $X_2 = 1$ implies $X_3 = 3$ where $1 < r(3)$, which would lead to failure. Thus $s_1(1) < s_2(1)$.

2.2 The Ratio of Gambler's Choice and Prophet's Value

Suppose the payoff function $f(x,y) = x/y$ (unless $y = 0$, where without loss of generality payoff 0 is assumed), for which monotonicity criteria (1) and boundedness in Δ apply. Thus $\mathcal{P}_n(x/y)$ is regular. The mean payoff of stopping in state $(x,y) \in \Delta$ with $y > 0$ where $\ell \in \mathbb{Z}_+$ draws remain is given by

$$s_\ell(x,y) = \int_0^y \frac{x}{y}\, d\zeta^\ell + \int_y^1 \frac{x}{\zeta}\, d\zeta^\ell = \begin{cases} x/y & \text{if } \ell = 0 \\ x(1-\ln y) & \text{if } \ell = 1 \\ \frac{x}{\ell-1}(\ell - y^{\ell-1}) & \text{if } \ell > 1. \end{cases}$$

Thus $s_\ell(x,y)$ is increasing in x inside Δ for any $\ell \in \mathbb{Z}_+$ (since x/y is) and then optimal boundary functions resp. optimal stopping sets are in principle specified according to theorem 2.7.
If one draw remains then it is optimal to stop if the present value x isn't lower than $b_1^*(y) = (1-\frac{1}{2}y)/(1-\ln y)$, since $c_1(y) = \int_0^y \xi/y\; d\xi + \int_y^1 1\; d\xi = 1 - \frac{1}{2}y$.

Regard problem $\mathcal{P}_n$ with respect to the set $\mathcal{T}_n^c$ of concurrent threshold rules: Supposing the concurrent threshold $t \in (0,1]$ the value of the corresponding concurrent threshold rule T is given by (declaring payoff 0 if $Y_n < t$)

$$\begin{aligned} & \mathrm{E}\,(X_T/Y_n) \\ = & \sum_{\ell=0}^{n-1} t^{n-\ell-1} \int_t^1 s_\ell(\xi,\xi)\; d\xi \\ = & t^{n-1}(1-t) + t^{n-2}\int_t^1 \xi(1-\ln\xi)d\xi + \sum_{\ell=2}^{n-1} t^{n-\ell-1}\int_t^1\left(\frac{\ell}{\ell-1}\xi - \frac{1}{\ell-1}\xi^\ell\right)d\xi \\ = & t^{n-1}(1-t) + t^{n-2}\,\frac{3-3t^2+2t^2\ln t}{4} \\ & + \sum_{\ell=2}^{n-1} t^{n-\ell-1}\left(\frac{\ell(1-t^2)}{2(\ell-1)} - \frac{1-t^{\ell+1}}{(\ell-1)(\ell+1)}\right), \end{aligned}$$

to be maximized with respect to t by numerical methods (set $\sum_\emptyset := 0$).
Particularly for $n = 2$ then $t \approx 0.2220$ solving $1 = 3t - t\ln t$ is the optimal concurrent threshold, which yields mean payoff 0.8487. For $n = 2$ the payoff maximal possible, the value v_2^* is approximately 0.9171. This results by regarding $b_1^*(y)$ (setting $x = y$) and since threshold rules will be mentioned

next this value is now verified in terms of threshold rules: Suppose threshold $t_1 \in (0,1]$ and $t_2 = 0$. Then the value of the corresponding threshold rule is given by $\int_0^{t_1} 1 - \frac{1}{2}\xi \; d\xi + \int_{t_1}^1 \xi(1-\ln\xi)\; d\xi = \frac{3}{4} + t_1 - t_1^2 + \frac{1}{2}t_1^2 \ln t_1$, which maximally attains approximately 0.9171 for threshold $t_1 \approx 0.4242$, the unique solution of $1 = \frac{3}{2}t - t\ln t$ in $(0,1]$ (if the first item is taken, i.e. concurrent threshold $t=0$, then the value is 3/4).
Now the value of stopping times referring to the set $\mathcal{T}_n$ of threshold rules is displayed: The value of threshold rule T corresponding to nonincreasing thresholds $t_1 \geq \cdots \geq t_n$ leads to an expression analogue to (21) by applying a similar decomposition:

$$\begin{aligned} \mathrm{E}\,(X_T/Y_n) &= \sum_{i=1}^n \left[t_i^{i-1} \int_{t_i}^1 s_{n-i}(x,x)\; dx \right] \\ &+ \sum_{i=2}^n \sum_{j=1}^{i-1} \left[\left(\prod_{k=j+1}^{i-1} t_k \right) \int_{t_{j+1}}^{t_j} \left(\int_{t_i}^1 s_{n-i}(x, y \vee x)\; dx \right)\; dy^j \right], \end{aligned}$$

where the first sum is computed on the previous page (for addend with index ℓ replace t by $t_{n-\ell}$) and where the innermost integral of the second sum (whose lower limit now is t_i instead of $r(y)$ regarding (21)) should be splitted at value y: This innermost integral from t_i to y is easy to calculate (since $s_\ell(x,y)$ is linear in x) and the integral from y to 1 yields the result of the previous page (replacing t by y and now the upper bound for ℓ is $n-2$).

3 Optimal Selection with Random Arrival Times

In this chapter iid offers arrive at random times, where also the interarrival times are iid. A gambler intends to elect one object in order to maximize the mean of a payoff function, while the time of selection is terminated by a socalled horizon, which is fixed or random.

Mathematical Model

Let $X_1, X_2, \ldots$ be $U([0,1])$ (theorem 2.8 is adaptable). Set $X_0 := X_\infty := 0$.
Let $Y_k := \max\{X_1, \ldots, X_k\}$ denote relative maxima, $k \in \mathbb{N}$. $Y_0 := Y_\infty := 1$.
The relative arrival time of X_k is denoted by A_k, $k \in \mathbb{N}$, where $A_1, A_2, \ldots$ are identically distributed with distribution function G where $G(0) = 0$.
The time horizon is a nonnegative random variable T with distribution function H — then $[0, T]$ is the period the gambler is allowed to select an offer.
Let random variables $X_1, X_2, \ldots$ and $A_1, A_2, \ldots$ and T be independent.
Set $A_0 := 0$ and let B_k denote the absolute arrival time of object X_k: $B_k := A_0 + \cdots + A_k$ for $k \in \mathbb{Z}_+$, its distribution function being denoted by $G^{*(k)}$.
For $t \in \mathbb{R}_+$ let N_t denote the number of items arriving in time interval $[0, t]$: $N_t := \sup\{k \in \mathbb{Z}_+ : B_k \leq t\}$. Then random variable X_{N_t} represents the value with arrival time B_{N_t}. Besides $N := N_T := N_{T(\omega)}(\omega)$ denotes the total number of items arriving in $[0, T]$. Then Y_N represente the overall maximum thereof and the last offer arrives at time instant B_N. Then $\mathrm{P}(N_0 = 0) = 1$ and $\mathrm{P}(N \in \mathbb{Z}_+) = 1$ since $G(0) = 0$ and $H(\infty) = 1$. Set $N_\infty := \infty$.
A decision of a gambler is restricted to epochs of an arrival, i.e. he is confined to the embedded discrete time parameter Markov process $Z := (Z_k)_{k \in \mathbb{Z}_+}$: Let $Z_0 := \alpha_0$ and $Z_k := (B_k, X_k, Y_k)$ for $k \in \mathbb{N}$ with $B_k \leq T$ (i.e. $1 \leq k \leq N$) and $Z_k := \alpha_\infty$ for $k \in \mathbb{N}$ with $B_k > T$ (i.e. $k > N$), where α_0 resp. α_∞ denotes the initial state resp. the final absorbing state. Transition probabilities are evident. Defining $E := \mathbb{R}_+ \times \Delta$ the state space of Z is $E \cup \{\alpha_0, \alpha_\infty\}$.
The profit of the gambler choosing an object x with present maximum y is denoted by $f(x, y \vee Y_N)$, where the bounded payoff function $f : \Delta \to [0,1]$ is assumed to be monotone according to the assumptions (1). Without loss of generality $f(0,1) = 0$. Additionally set $f(\alpha_0) := 0 =: f(\alpha_\infty)$.
Let $\mathcal{F}_t := \sigma(X_0, X_1, \ldots, X_{N_t}; A_0, A_1, \ldots, A_{N_t}; N_t)$ gather the information relevant for the gambler until time $t \in \mathbb{R}_+$ and let $\mathcal{F} := (\mathcal{F}_t)_{t \in \mathbb{R}_+}$contain the

total history. In this notation let $\mathcal{S}$ denote the set of stopping times with respect to $\mathcal{F}$ where $\mathrm{P}\left(S \in \{B_1, B_2, \ldots, B_N, \infty\}\right) = 1$ for $S \in \mathcal{S}$ with the convention that payoff inside event $[S = \infty]$ is $f(X_\infty, Y_N) = 0$ almost surely. Given payoff function f a gambler watches the objective to maximize his mean payoff according to stopping times in $\mathcal{S}$, where it is assumed that the payoff function f, the joint distribution of the draws and the distribution functions G and H are familiar to the gambler, i.e. full information.
Now the optimal stopping problem is to find a stopping time in $\mathcal{S}$ attaining

$$\sup_{S \in \mathcal{S}} \mathrm{E}\left(f(X_{N_S}, Y_N)\right).$$

Let $\mathcal{P} = \mathcal{P}\left(f, U([0,1]), G, H\right)$ denote this optimal stopping problem. Let $v(S) := \mathrm{E}\left(f(X_{N_S}, Y_N)\right)$ denote the corresponding value where stopping time $S \in \mathcal{S}$ is applied. The value of $\mathcal{P}$ is denoted by $v^* := \sup_{S \in \mathcal{S}} v(S)$. In principle existence of an optimal stopping time S^* is ensured: The general approach of optimal stopping on pages 12f is appliable for the embedded,discrete time Markov process Z, since $\mathrm{P}\left(\exists k \in \mathbb{N} : Z_k = \alpha_\infty\right) = 1$ due to $\mathrm{P}\left(N < \infty\right) = 1$. The main subject in this chapter is again given by payoff function $\mathbf{1}_{[r(y),1]}(x)$ where $r \in \mathcal{R}$, see section 2.1: An offer x is called r–candidate (with respect to Y_N) if $x \geq r(Y_N)$. Then the objective is optimal sequential selection of an r–candidate — maximization of the probability $\mathrm{P}\left(X_{N_S} \geq r(Y_N)\right)$. It is advisible to stop only if x is a present r–candidate, i.e. if $x \geq r(y)$ where y denotes the present maximum — in other words if $(x, y) \in \Delta_0^*$.

3.1 Random Arrival Times and Fixed Horizon

Suppose selection is terminated by a fixed point in time $h > 0$. For a state $(b, x, y) \in E$ of the Markov process Z throughout this section let $t := h - b$ denote the remaining time. The mean payoff of stopping in state $(x, y) \in \Delta$ with remaining time $t \in [0, h]$ is

$$s(t, x, y) \;:=\; \mathrm{E}\left(f(x, Y_N) \mid \mathcal{F}_{h-t}, X_{N_{h-t}} = x, Y_{N_{h-t}} = y\right). \tag{23}$$

Let $c(t, y)$ denote the mean payoff of proceeding and then choosing optimally (evidently depending on x only through y): $c(0, y) :\equiv 0$ and for $t \in (0, h]$

$$c(t, y) \;:=\; \mathrm{E}\left(f(X_{N_{S^*}}, Y_N) \mid \mathcal{F}_{h-t},\, Y_{N_{h-t}} = y, S^* > h - t\right). \tag{24}$$

Finally let $v(t, x, y) := \max\{s(t, x, y), c(t, y)\}$ denote the value of this state.

3.1.1 Geometric Arrival Times

Let the arrival times be geometrically distributed: $\mathrm{P}(A_1 = j) = pq^{j-1}$ for $j \in \mathbb{N}$ where $p \in (0,1]$ and $q := 1-p$. Let $n \in \mathbb{N}$ denote the horizon. This optimal stopping problem is denoted by $\mathcal{P}_p$.

Lemma 3.1 *The optimal stopping problem $\mathcal{P}_p$ is regular.*

Proof: Let $(x,y) \in \Delta$ and let $n > \ell \in \mathbb{Z}_+$ denote the the number of draws remaining, which is written as an index for functions s and c and v. As in the last chapter again regularity of this problem is verifed for a payoff function depending on ℓ: Let $g_\ell(x, y \vee Y_n)$ denote the payment in the situation specified above, with the additional assumption $g_\ell \preceq g_{\ell-1}$ on Δ for $\ell \in \mathbb{N}$, which seems to be indispensable to ensure the regular case. Now

$$\begin{aligned}
s_\ell(x,y) &= \sum_{j=0}^{\ell} \binom{\ell}{j} p^j q^{\ell-j} \int_0^1 g_\ell(x, y \vee \zeta)\, d\zeta^j \\
&= (yp+q)^\ell\, g_\ell(x,y) + \int_y^1 \sum_{j=1}^{\ell} \ell \frac{(\ell-1)!}{(\ell-j)!j!} j\zeta^{j-1} p^j q^{\ell-j} g_\ell(x,\zeta)\, d\zeta \\
&= (yp+q)^\ell\, g_\ell(x,y) + p\ell \int_y^1 (\zeta p + q)^{\ell-1}\, g_\ell(x,\zeta)\, d\zeta. \qquad (25)
\end{aligned}$$

Besides $c_0 \equiv 0$ and for $\ell \in \mathbb{N}$ by decomposition

$$\begin{aligned}
c_\ell(y) &= qc_{\ell-1}(y) + p\int_0^1 v_{\ell-1}(\xi, y \vee \xi)\, d\xi \qquad (26) \\
&\geq qc_{\ell-1}(y) + p\int_0^y c_{\ell-1}(y)\, d\xi + p\int_y^1 s_{\ell-1}(\xi,\xi)\, d\xi \\
&= (yp+q)c_{\ell-1}(y) + p\int_y^1 s_{\ell-1}(\xi,\xi)\, d\xi \\
(yp+q)c_{\ell-1}(y) &\leq c_\ell(y) - p\int_y^1 s_{\ell-1}(\xi,\xi)\, d\xi.
\end{aligned}$$

Given $s_\ell(x,y) \geq c_\ell(y)$ for $\ell \in \mathbb{N}$, it has to be shown that $s_{\ell-1}(x,y) \geq c_{\ell-1}(y)$. It is sufficient to show

$$(yp+q)s_{\ell-1}(x,y) \geq c_\ell(y) - p\int_y^1 s_{\ell-1}(\xi,\xi)\, d\xi,$$

which proves to be valid due to the following calculations and estimations:

$$
\begin{aligned}
& c_\ell(y) - p \int_y^1 s_{\ell-1}(\xi,\xi)\; d\xi \\
\le\;& (yp+q)^\ell\, g_\ell(x,y) + p\ell \int_y^1 (\zeta p+q)^{\ell-1}\, g_\ell(x,\zeta)\; d\zeta \\
& -p \int_y^1 \left((\xi p+q)^{\ell-1} g_{\ell-1}(\xi,\xi) + p(\ell-1) \int_\xi^1 (\zeta p+q)^{\ell-2} g_{\ell-1}(\xi,\zeta)\; d\zeta \right)\; d\xi \\
=\;& (yp+q)^\ell\, g_\ell(x,y) + p\ell \int_y^1 (\zeta p+q)^{\ell-1}\, g_\ell(x,\zeta)\; d\zeta \\
& -p \int_y^1 (\xi p+q)^{\ell-1} g_{\ell-1}(\xi,\xi)\; d\xi \\
& -p^2(\ell-1) \int_y^1 \left((\zeta p+q)^{\ell-2} \int_y^\zeta g_{\ell-1}(\xi,\zeta)\; d\xi \right)\; d\zeta \\
\le\;& (yp+q)^\ell\, g_{\ell-1}(x,y) + p\ell \int_y^1 (\zeta p+q)^{\ell-1}\, g_{\ell-1}(x,\zeta)\; d\zeta \\
& -p \int_y^1 (\xi p+q)^{\ell-1} g_{\ell-1}(x,\xi)\; d\xi \\
& -p^2(\ell-1) \int_y^1 \left((\zeta p+q)^{\ell-2} \int_y^\zeta g_{\ell-1}(x,\zeta)\; d\xi \right)\; d\zeta \\
=\;& (yp+q)^\ell\, g_{\ell-1}(x,y) + p(\ell-1) \int_y^1 (\zeta p+q)\,(\zeta p+q)^{\ell-2}\, g_{\ell-1}(x,\zeta)\; d\zeta \\
& -p^2(\ell-1) \int_y^1 (\zeta - y)(\zeta p+q)^{\ell-2} g_{\ell-1}(x,\zeta)\; d\zeta \\
=\;& (yp+q)^\ell\, g_{\ell-1}(x,y) + p(\ell-1)(yp+q) \int_y^1 (\zeta p+q)^{\ell-2}\, g_{\ell-1}(x,\zeta)\; d\zeta \\
=\;& (yp+q) s_{\ell-1}(x,y),
\end{aligned}
$$

since $g_\ell \preceq g_{\ell-1}$ on Δ and since $x \le y \le \xi, \zeta$. $\square$

An Optimal Stopping Time

A recursion formula for $c_\ell(y)$, analogue to lemma 2.6, holds ($y \in [\underline{b}_{\ell-1}, 1]$):

$$\begin{aligned} c_\ell(y) &= qc_{\ell-1}(y) + p\left(\int_0^{b^*_{\ell-1}(y)} c_{\ell-1}(\xi, y)\ d\xi + \int_{b^*_{\ell-1}(y)}^1 s_{\ell-1}(\xi, y \vee \xi)\ d\xi\right) \\ &= (q + pb^*_{\ell-1}(y))c_{\ell-1}(y) + p\int_{b^*_{\ell-1}(y)}^1 s_{\ell-1}(\xi, y \vee \xi)\ d\xi \\ \ldots &= p\sum_{i=0}^{\ell-1}\left(\prod_{j=i+1}^{\ell-1}(q + pb^*_j(y))\right) \cdot \int_{b^*_i(y)}^1 s_i(\xi, y \vee \xi)\ d\xi. \end{aligned} \tag{27}$$

Now analogously to the last chapter optimal stopping sets can be specified, see theorem 2.7: Assuming that $s_\ell(x, y)$, given in (25), is increasing in x, the inverse (with respect to x) together with expression (27) above yield a recursive representation of optimal boundary functions $b^*_\ell(y)$, which describe optimal stopping sets Δ^*_ℓ, $\ell \in \mathbb{Z}_+$.

Selection of an r–Candidate

Now the problem of optimal sequential selection of an r–candidate is considered, i.e. payoff function $\mathbf{1}_{[r(y),1]}(x)$ with $r \in \mathcal{R}$ (confer section 2.1). Then for $(x, y) \in \Delta$ expression (25) simplifies, $s_\ell(x, y) = \mathbf{1}_{[r(y),1]}(x) \cdot (\varrho(x)p + q)^\ell$, which implies slight changes of the representation (27). The myopic stopping time in this context is specified next (it is consistent with the case $p = 1$ of proposition 2.31 and not optimal in general as indicated there):

Proposition 3.2 *The myopic stopping time S_m for $\mathcal{P}_p(r)$ where $r \in \mathcal{R}$ proposes to stop in state $(x, y) \in \Delta$ with $\ell \in \mathbb{N}$ draws remaining if $x \geq r(y)$ and*

$$x \geq r\left(\frac{1}{p}\left(\left[\sum_{i=0}^{\ell-1} r^i(y)\sum_{j=i}^{\ell-1}\binom{j}{i}p^{i+1}q^{j-i}\int_{r(y)}^1 (\varrho(\xi)p + q)^{\ell-1-j}\ d\xi\right]^{1/\ell} - q\right)\right),$$

where relax function r is extended according to remark 2.18 ii).

Proof: $s_\ell(x) = (\varrho(x)p+q)^\ell$ if $x \geq r(y)$ and for $\ell > 0$ (confer page 83)

$$(\mathbb{P}s)(\ell, y) = \sum_{i=0}^{\ell-1} r^i(y) \sum_{j=i}^{\ell-1} \binom{j}{i} p^{i+1} q^{j-i} \int_{r(y)}^1 (\varrho(\xi)p+q)^{\ell-1-j} \, d\xi$$

by decomposition: Waiting $j+1$ time units for the next r–candidate, where i non–r–candidates pass, and respecting the remaining number $\ell - 1 - j$ of epochs resp. the corresponding mean reward choosing r–candidate ξ. □

3.1.2 Exponential Arrival Times — The Poisson Process

Suppose the offers $X_1, X_2, \ldots$ arrive according to a Poisson process with arrival rate $\lambda > 0$, i.e. the interarrival times $A_1, A_2, \ldots$ are iid with distribution function $G(x) := (1 - e^{-\lambda x})\, \mathbf{1}_{\mathbb{R}_+}(x)$. Without loss of generality horizon $h = 1$ is assumed (else rescaling would lead to rate λh), which is equivalent to rate 1 and horizon λ. Then $N = N_1$ is Poisson distributed with parameter λ. This optimal stopping problem is denoted by $\mathcal{P}_\lambda$.

An Optimal Stopping Time

Lemma 3.3 *The optimal stopping problem $\mathcal{P}_\lambda$ is regular.*

Proof: Let state $(x, y) \in \Delta$ be given and let $t \in [0, 1]$ denote the remaining time. Again this problem is verified to be regular for a payoff which depends on t: Let $g(t, x, y \vee Y_N)$ denote the payoff in the situation specified above, with additional assumption that it is continuous differentiable in t and $\frac{\partial}{\partial t} g(t, x, y) \leq 0$. Then

$$\begin{aligned}
s(t, x, y) &:= \mathrm{E}\left(g(t, x, Y_N) \mid \mathcal{F}_{1-t}, X_{N_{1-t}} = x, Y_{N_{1-t}} = y\right) \\
&= \sum_{k=0}^{\infty} \left(\int_0^1 g(t, x, y \vee \zeta)\, d\zeta^k \cdot e^{-\lambda t} \frac{(\lambda t)^k}{k!} \right) \\
&= e^{-\lambda t} g(t, x, y) \sum_{k=0}^{\infty} y^k \frac{(\lambda t)^k}{k!} \\
&\quad + e^{-\lambda t} \int_y^1 \left(g(t, x, \zeta) \sum_{k=1}^{\infty} \frac{(\lambda t)^k}{k!} k \zeta^{k-1} \right) d\zeta \\
&= e^{-\lambda t(1-y)} g(t, x, y) + \lambda t \int_y^1 e^{-\lambda t(1-\zeta)} g(t, x, \zeta)\, d\zeta, \qquad (28)
\end{aligned}$$

and using $\frac{\partial}{\partial t} g(t,x,\zeta) \le 0$ gives the estimation

$$\begin{aligned}\frac{\partial}{\partial t} s(t,x,y) &\le -\lambda(1-y)e^{-\lambda t(1-y)}g(t,x,y) + \lambda \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta \\ &\quad -\lambda^2 t \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta + \lambda^2 t \int_y^1 \zeta e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta.\end{aligned}$$

Functions $c(t,y)$ and $v(t,x,y)$ are defined in the beginning of this section (see equation (24). The optimal stopping problem $\mathcal{P}_\lambda$ is regular if

$$s(t,x,y) \ge c(t,y) \qquad \Longrightarrow \qquad s(u,x,y) \ge c(u,y) \quad \forall u \in [0,t).$$

Now a differential equation for $c(t,y)$ with respect to the (remaining–time) variable t is developed by considering a small time interval of length δ and $\delta \to 0$. Arrivals in $(b-\delta, b]$ are decomposed into the cases of no, one and more than one arrival and relations are given in terms of $o(\delta)$ for $\delta \to 0$ (respect for the last term that the payoff function is bounded):

$$c(t+\delta,y) = [1-\lambda\delta + o(\delta)]\, c(t,y) + [\lambda\delta + o(\delta)] \int_0^1 v(t,\xi,y\vee\xi)\, d\xi + o(\delta).$$

Arranging terms with respect to a difference quotient yields

$$\frac{c(t+\delta,y) - c(t,y)}{\delta} = -\lambda c(t,y) + \lambda \int_0^1 v(t,\xi,y\vee\xi)\, d\xi + o(1).$$

Letting $\delta \to 0$ results in the differential equation resp. in the estimation

$$\frac{\partial}{\partial t} c(t,y) = -\lambda c(t,y) + \lambda \int_0^1 v(t,\xi,y\vee\xi)\, d\xi \tag{29}$$

$$\ge -\lambda(1-y)c(t,y) + \lambda \int_y^1 s(t,\xi,\xi)\, d\xi. \tag{30}$$

Since all functions involved are continuous differentiable with respect to t problem $\mathcal{P}_\lambda$ is regular if the following implication is valid:

$$s(t,x,y) \ge c(t,y) \qquad \Longrightarrow \qquad \frac{\partial}{\partial t} s(t,x,y) \le \frac{\partial}{\partial t} c(t,y) \tag{31}$$

for $t \in (0,1)$. Now assumption $c(t,y) \le s(t,x,y)$ is applied to inequality (30) and the assertion (31) is verified as follows:

$$
\begin{aligned}
&\frac{\partial}{\partial t} c(t,y) \\
\geq\ & -\lambda(1-y)\left(e^{-\lambda t(1-y)} g(t,x,y) + \lambda t \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta\right) \\
& +\lambda \int_y^1 \left(e^{-\lambda t(1-\xi)} g(t,\xi,\xi) + \lambda t \int_\xi^1 e^{-\lambda t(1-\zeta)} g(t,\xi,\zeta)\, d\zeta\right)\, d\xi \\
\geq\ & -\lambda(1-y) e^{-\lambda t(1-y)} g(t,x,y) - \lambda^2 t(1-y) \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta \\
& +\lambda \int_y^1 e^{-\lambda t(1-\xi)} g(t,x,\xi)\, d\xi + \lambda^2 t \int_y^1 \left(\int_y^\zeta e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\xi\right)\, d\zeta \\
\geq\ & -\lambda(1-y) e^{-\lambda t(1-y)} g(t,x,y) - \lambda^2 t(1-y) \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta \\
& +\lambda \int_y^1 e^{-\lambda t(1-\xi)} g(t,x,\xi)\, d\xi + \lambda^2 t \int_y^1 (\zeta - y) e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta \\
=\ & -\lambda(1-y) e^{-\lambda t(1-y)} g(t,x,y) - \lambda^2 t \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta \\
& +\lambda \int_y^1 e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta + \lambda^2 t \int_y^1 \zeta e^{-\lambda t(1-\zeta)} g(t,x,\zeta)\, d\zeta \\
\geq\ & \frac{\partial}{\partial t} s(t,x,y),
\end{aligned}
$$

due to monotonicity of g and the estimation below equation (28). □

Since the regular case applies for problem $\mathcal{P}_\lambda$ now a family of optimal boundary functions $b_t^*(y)$ and optimal stopping sets $\Delta_t^* := \{(x,y) \in \Delta \,:\, x \geq b_t^*(y)\}$ for $t \in [0,1]$ is, in principle, specified. The sets Δ_t^* are nonincreasing in t.

The Inhomogeneous Poisson Process

Suppose items $X_1, X_2, \ldots$ arrive at absolute times $B_1, B_2, \ldots$ (inside time interval $[0,1]$) according to an inhomogeneous Poisson process resp. intensity. Since functions mainly depend on the remaining–time variable t, let $\lambda(t)$ denote the intensity of the process at time $b = 1-t$, where $\lambda(t) : [0,1] \to (0,\infty)$ is continuous, and let $\Lambda(t) := \int_0^t \lambda(u)\, du$.

It is noted that $\mathrm{P}(N_t = k) = e^{-\Lambda(t)}\frac{(\Lambda(t))^k}{k!}$ (by mass–theoretic induction) and $\frac{d}{dt}\mathrm{P}(B_k \le t) = \lambda(t)e^{-\Lambda(t)}\frac{(\Lambda(t))^{k-1}}{(k-1)!}$ (by using equivalence $[N_t \ge k]$ iff $[B_k \le t]$), where $t \ge 0$ and $k \in \mathbb{N}$. Then according to expression (28)

$$s(t,x,y) \;=\; e^{-\Lambda(t)(1-y)}g(t,x,y) + \Lambda(t)\int_y^1 e^{-\Lambda(t)(1-\zeta)}g(t,x,\zeta)\,d\zeta$$

and decomposition similar to lemma 3.3 (in terms of $o(\delta)$ as $\delta \to 0$) yields

$$\frac{\partial}{\partial t}c(t,y) \;=\; -\lambda(t)(1-y)c(t,y) + \lambda(t)\int_y^1 v(t,\xi,y\vee\xi)\,d\xi,$$

since $(\Lambda(t+\delta)-\Lambda(t))/\delta \to \lambda(t)$ as $\delta \to 0$. The monotonicity of the optimal stopping problem is preserved (imitating the arguments used in the proof of lemma 3.3, where the result is that finally the term λ, λt resp. $\lambda^2 t = \lambda \cdot \lambda t$ has to be replaced by $\lambda(t)$, $\Lambda(t)$ resp. $\lambda(t)\Lambda(t)$).

In the remaining part of this subsection the subject is optimal selection of an r–candidate, i.e. payoff function $f(x,y) = \mathbf{1}_{[r(y),1]}(x)$:

r–Candidates: Specification of Main Terms

Take relax function $r \in \mathcal{R}$, section 2.1, and suppose to select an r–candidate, i.e. select an item which exceeds $r(Y_N)$. Then expression (28) simplifies,

$$s(t,x,y) \;=\; \mathbf{1}_{[r(y),1]}(x)\cdot e^{-\lambda t(1-\varrho(x))},$$

and differential equation (29) of the mean payoff of proceeding optimally is

$$\frac{\partial}{\partial t}c(t,y) \;=\; -\lambda(1-r(y))c(t,y) + \lambda\int_{r(y)}^1 v(t,\xi,y\vee\xi)\,d\xi.$$

Comparing $s(t,x,y)$ and $c(t,y)$ leads to a family of optimal boundary functions $b_t^*(y)$ and family of stopping sets $\Delta_t^* := \{(x,y)\in\Delta \,:\, x \ge b_t^*(y)\}$ for $t \in [0,1]$, with lower resp. upper boundary points $\underline{b}_t$ resp. $\overline{b}_t$ where $b_t^*(y) = \underline{b}_t$ for $y \in [0,\underline{b}_t]$ and $b_t^*(y) = r(y)$ for $y \in [\overline{b}_t, 1]$. Alternatively $t \le h$ if $\lambda = 1$ and horizon $h \in \mathbb{R}_+$ is taken. Again Δ_0^* represents the set of r–candidates. Function $b_t^*(y)$ is decreasing on $[\underline{b}_t, \overline{b}_t]$, since $s(t,x,y)$ is constant in y inside Δ_0^* and $c(t,y)$ is decreasing in y.

The difference between the values applying contiguous stopping times, the analogue of proposition 2.17 i), is treated next:

Proposition 3.4 *Let two stopping times S and $\widetilde{S}$ for stopping problem $\mathcal{P}_\lambda(r)$ be given ($r \in \mathcal{R}$, horizon 1). Let their boundary functions be denoted by $b_t(y)$ resp. $\widetilde{b}_t(y)$, $t \in [0,1]$. Let $\beta := \sup_{t\in[0,1]} \sup_{y\in[0,1]} |b_t(y) - \widetilde{b}_t(y)|$. Then $|v(S) - v(\widetilde{S})| \leq 1 - e^{-\lambda\beta}$.*

Proof: Given $N = 0$, $v(S) - v(\widetilde{S}) = 0$. The probability that the payoff differs given $N = n \in \mathbb{N}$ is bounded by

$$\begin{aligned} \mathrm{P}\left(S \neq \widetilde{S} \,\middle|\, N = n\right) &= \sum_{k=1}^{n} \mathrm{P}\left((X_k, Y_k) \in C_{B_k}\right) \\ &\leq \sum_{k=1}^{n} \beta(1-\beta)^{k-1} \\ &= 1 - (1-\beta)^n, \end{aligned}$$

where $C_t := (D_t \setminus \widetilde{D}_t) \cup (\widetilde{D}_t \setminus D_t)$ and $D_t := \{(x,y) \in \Delta : x \geq b_t(y)\}$ and $\widetilde{D}_t := \{(x,y) \in \Delta : x \geq \widetilde{b}_t(y)\}$ where $t \in [0,1]$. Weighted summation yields

$$\begin{aligned} \mathrm{P}\left(S \neq \widetilde{S}\right) &\leq e^{-\lambda} \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left(1 - (1-\beta)^n\right) \\ &= e^{-\lambda}\left(e^{\lambda} - 1 - \left(e^{\lambda(1-\beta)} - 1\right)\right) \\ &= 1 - e^{-\lambda\beta}, \end{aligned}$$

bounding the absolute value of the difference of the payoffs from above. □

r–Candidates: The Myopic Stopping Time

Regarding problem $\mathcal{P}_\lambda(r)$ with $r \in \mathcal{R}$ the myopic stopping time suggests to stop if the mean payoff selecting a present item isn't smaller than the mean payoff choosing the next r–candidate (if any) — the one step look–ahead rule referring to the embedded subsequence of r–candidates, see pages 81f.

Proposition 3.5 *Let problem $\mathcal{P}_\lambda(r)$ with $r \in \mathcal{R}$ be given. The myopic stopping time S_m proposes to stop in state $(x,y) \in \Delta$ with $y < r(1)$ and with remaining time $t > 0$ if $x \geq r(y)$ and (extend r according to remark 2.18 ii))*

$$x \geq r\left(r(y) + \frac{1}{\lambda t} \ln\left(\int_{r(y)}^{1} \frac{e^{\lambda t(\varrho(\xi)-r(y))} - 1}{\varrho(\xi) - r(y)}\, d\xi\right)\right), \tag{32}$$

Proof: For $(b,x,y) \in E$ and $t := 1-b$, $s(t,x) = e^{-\lambda t(1-\varrho(x))}$ for $(x,y) \in \Delta_0^*$, else 0. The mean payoff selecting the next r–candidate (if any): Partition with respect to the arrival time of the next r–candidate, where $k-1$ non–r–candidates pass and his interarrival time is Erlang distributed with parameter k and λ ($k \in \mathbb{N}$; no further r–candidate implies payoff 0), confer term (41):

$$\begin{aligned}
&\int_b^1 \int_{r(y)}^1 \left(e^{-\lambda(1-u)(1-\varrho(\xi))} e^{-\lambda(u-b)} \sum_{k=1}^{\infty} r^{k-1}(y) \frac{\lambda^k}{(k-1)!}(u-b)^{k-1}\right) d\xi\, du \\
= {}& \lambda \int_b^1 e^{-\lambda(u-b)(1-r(y))} \left(\int_{r(y)}^1 e^{-\lambda(1-u)(1-\varrho(\xi))}\, d\xi\right) du,
\end{aligned}$$

i.e. the arrival rate of an r–candidate is $\lambda(u-b)(1-r(y))$ and the term in parentheses divided by $1-r(y)$ represents the corresponding mean payoff choosing the next r–candidate given there is one. Rearranging yields

$$\begin{aligned}
&\lambda \int_{r(y)}^1 \left(e^{-\lambda(1-\varrho(\xi)-b(1-r(y)))} \int_b^1 e^{-\lambda u(\varrho(\xi)-r(y))}\, du\right) d\xi \\
= {}& \lambda e^{\lambda b(1-r(y))} \int_{r(y)}^1 \left(\frac{e^{-\lambda(1-\varrho(\xi))}}{-\lambda(\varrho(\xi)-r(y))} \left[e^{-\lambda u(\varrho(\xi)-r(y))}\right]_b^1\right) d\xi \\
= {}& e^{\lambda b(1-r(y))} \int_{r(y)}^1 \frac{e^{-\lambda(1-\varrho(\xi))}}{\varrho(\xi)-r(y)} \left(-e^{-\lambda(\varrho(\xi)-r(y))} + e^{-\lambda b(\varrho(\xi)-r(y))}\right) d\xi \\
= {}& e^{-\lambda(1-b)(1-r(y))} \int_{r(y)}^1 \frac{e^{\lambda(1-b)(\varrho(\xi)-r(y))} - 1}{\varrho(\xi)-r(y)}.\, d\xi
\end{aligned}$$

The integrand $e^{-\lambda u(\varrho(\xi)-r(y))}$ equals 1 on set $\{z \in [0,1] : r(z) = z\}$ and then subsequent fractions are according to the rule of de l'Hospital, particularly the fraction in the last line there represents $\lambda(1-b)$. These terms agree with section 4 of Bojdecki [6], where integration with respect to ξ can be solved first, because $r \equiv id$ is given. Evidently the myopic stopping time accepts any item $x \geq r(1)$ and any item with remaining time $t = 0$. □

Similar to the discrete case the myopic stopping time for $\mathcal{P}_\lambda$ isn't optimal in general, since the corresponding stopping sets may miss closedness though they may be monotone, see pages 48f. Particularly for $r(y) = y^4$ stopping sets are similar to figure 4 — the diagonal seems disregarded.

Remark 3.6 i) The myopic stopping time for problem $\mathcal{P}_\lambda(r)$, $r \in \mathcal{R}^1$, selects X_1 in any case, if $A_1 \geq 1 - (1 \wedge c/\lambda)$, where $c = c(r)$ denotes the unique solution of equation $I(c) := \int_0^1 \frac{e^{c\zeta}-1}{\zeta}\, dr(\zeta) = 1$ (see remark 2.18 ii); $I(c)$ is increasing in c, $I(0) = 0$ and $I(1) > 1$ for any $r \in \mathcal{R}^1$, since the integrand exceeds 1 on $(0, 1]$). Particularly $c(id) \approx 0.8044$ and $c(r_1) < c(r_2)$ if $r_1 \prec r_2$ in $\mathcal{R}^1$ (changing from r_1 to r_2 masses move to the left and since the integrand of $I(c)$ is increasing in ζ for r_2 a bigger c is necessary for $I(c) = 1$). Thus $c(r) \in (\ln 2, c(id)]$ holds (since $c(\varepsilon_1) = \ln 2$ is not attained inside $\mathcal{R}$).

ii) Let F denote the distribution function of $X_1, X_2, \ldots$, assumed to be continuous on $\mathbb{R}$ and increasing on $R := \{x \in \mathbb{R} : 0 < F(x) < 1\}$. Let $r : R \to R$ be continuous and increasing with $r \preceq id$ on R. Then $s(t, x) = e^{-\lambda t(1-F(\varrho(x)))}$ provided $x \geq r(y)$ (the rate of arrival of an item exceeding $\varrho(x)$ is $\lambda(1 - F(\varrho(x)))$). According to the investigations above $(\mathbb{P}s)(t, y) = e^{-\lambda t(1-F(r(y)))} \int_{r(y)}^1 \frac{e^{\lambda t(F(\varrho(\xi))-F(r(y)))}-1}{F(\varrho(\xi))-F(r(y))}\, dF(\xi)$. For the inhomogenous Poisson process the term λt each time has to be replaced by $\Lambda(t)$, as the direct generalization of the proof of proposition 3.5 shows (notations according to the paragraph on page 67).

The approach of Gnedin and Sakaguchi [21] in order to specify the value $w(\lambda) := v(S_m)$ of the myopic stopping time for $\mathcal{P}_\lambda(r)$ via $w'(\lambda)$ can't be adapted straightforward: If $\delta \in (0, 1)$ and offers below δ are ignored, then boundary functions remain unchanged for offers $U([\delta, 1])$ and rate $\lambda(1 - \delta)$ (except for items below δ). However the corresponding value isn't $w(\lambda(1-\delta))$ but it is the value of $\mathcal{P}_\lambda$ with new relax function $\widetilde{r}(x) := r((x - \delta)/(1 - \delta))$ (rescaled $U([0, 1])$, particularly equal to 0 for $x \in [0, \delta]$). The difference of the values referring to r and $\widetilde{r}$ (with regard to a difference quotient) doesn't seem to be $o(\delta)$ as $\delta \to 0$. Particularly in case of $r(x) = x^a$ for $a \in (1, \infty)$ then $\widetilde{r} \preceq r$ on $[0, 1]$ and regardless taking the difference quotient $(w(\lambda)-w(\lambda(1-\delta)))/\delta$ and letting $\delta \to 0$ then the value $w(\lambda)$ is underestimated, though then the approach can be adapted and specifically this value doesn't vanish as $\lambda \to \infty$.

r–Candidates: Selection with Recall

If permanent recall in the continuous time interval $[0,1]$ is allowed, then choosing Y_N at time 1 is optimal. Suppose however that recall of the present maximum is restricted to time instants where an r–candidate arrives, which signifies a new problem only for a relax function $\mathcal{R} \ni r \prec id$ — analogously to restricted recall in the case of discrete time. This means optimal stopping of the subsequence of r–candidates of the Markov process Z with recall. The relevant states now are contained in $[0,1]^2$ since a new offer only is considered if it is a new maximum. Now regard problem $\mathcal{P}_\lambda(r)$ with this kind of restricted recall:

Theorem 3.7 *Regard $\mathcal{P}_\lambda(r)$ with restricted recall, where $\mathcal{R}^1 \ni r \prec id$. Then stopping time $S^* := \inf\{B_k : (B_k, Y_k) \in \Delta^* \text{ for } 1 \leq k \leq N\}$ is optimal (set $\inf_\emptyset := \infty$), where $\Delta^* := \{(b,y) \in [0,1]^2 : y \geq y^*(1-b)\}$ specifies an optimal stopping set (which has to be restricted to epochs of arrivals). Here $y^*(t) \in [0, r(1))$ depends on the remaining time $t = 1-b$ and denotes the unique solution in $(0, r(1))$ of*

$$\int_y^1 \frac{e^{\lambda t(\varrho(\xi)-r(y))}-1}{\varrho(\xi)-r(y)}\,d\xi \;=\; 1 + (\varrho(y)-y)\frac{e^{\lambda t(\varrho(y)-r(y))}-1}{\varrho(y)-r(y)}$$

if $\int_0^1 \frac{e^{\lambda t \varrho(\xi)}-1}{\varrho(\xi)}\,d\xi > 1$ and $y^*(t) = 0$ otherwise. Regarding remark 3.6 i) then $y^*(t) = 0$ iff $t \leq c(r)/\lambda$.

Proof: The myopic stopping time stops in (t,y) with $y \in [0, r(1))$ if $s(t,y)$ isn't lower than the mean payoff recalling the maximum in the time instant the next r–candidate arrives (if any, else payoff 0), see proposition 3.5:

$$\begin{aligned} e^{-\lambda t(1-\varrho(y))} &\geq e^{-\lambda t(1-r(y))} \int_{r(y)}^1 \frac{e^{\lambda t(\varrho(y\vee\xi)-r(y))}-1}{\varrho(y\vee\xi)-r(y)}\,d\xi \\ e^{\lambda t(\varrho(y)-r(y))} &\geq (y-r(y))\frac{e^{\lambda t(\varrho(y)-r(y))}-1}{\varrho(y)-r(y)} + \int_y^1 \frac{e^{\lambda t(\varrho(\xi)-r(y))}-1}{\varrho(\xi)-r(y)}\,d\xi \end{aligned}$$

and rearranging yields

$$\int_y^1 \frac{e^{\lambda t(\varrho(\xi)-r(y))}-1}{\varrho(\xi)-r(y)}\,d\xi \;\leq\; 1 + (\varrho(y)-y)\frac{e^{\lambda t(\varrho(y)-r(y))}-1}{\varrho(y)-r(y)}.$$

Let $h(t,y) := \int_y^1 g(\xi,y)\ d\xi - 1 - (\varrho(y)-y)g(y,y) \in C^1([0,1]\times[0,r(1)])$, thus stopping in (t,y) is optimal if $h(t,y)\le 0$, where $g(\xi,y) := \frac{e^{\lambda t(\varrho(\xi)-r(y))}-1}{\varrho(\xi)-r(y)}$. $h(0,y) = -1$ and $\frac{\partial}{\partial t}h(t,y) = \lambda\int_y^1 e^{\lambda t(\varrho(\xi)-r(y))}\ d\xi - \lambda(\varrho(y)-y)e^{\lambda t(\varrho(y)-r(y))}$, which evidently is positive for any $t\in[0,1]$ if $y\in[0,r(1))$.
On the other hand there is a unique threshold $y(t)\in[0,r(1))$ such that it is optimal to recall the present maximum if it exceeds $y(t)$: $h(t,r(1)) = -1$ (terms involving g vanish synced) and if $h(t,0)\le 0$, i.e. if $\int_0^1 g(\xi,0)\ d\xi \le 1$, then $y(t)=0$ and else $y(t)\in(0,r(1))$ uniquely, since $h(t,y)$ decreases in y: $\frac{\partial}{\partial y}h(t,y) = -g(y,y)+\int_y^1 f(\xi,y)\ d\xi - (\varrho'(y)-1)g(y,y) - (\varrho(y)-y)\frac{\partial}{\partial y}g(y,y)$ where $f(\xi,y) := r'(y)\frac{(1-\lambda t(\varrho(\xi)-r(y)))e^{\lambda t(\varrho(\xi)-r(y))}-1}{(\varrho(\xi)-r(y))^2}$. Here $g(y,y)$ cancels. Now calculating $\frac{\partial}{\partial y}g(y,y)$ and then rearranging the final term of $\frac{\partial}{\partial y}h(t,y)$ yields $(\varrho(y)-y)\frac{\partial}{\partial y}g(y,y) = -\frac{\varrho(y)-y}{\varrho(y)-r(y)}\varrho'(y)g(y,y) + (\varrho(y)-y)f(y,y)$. Estimating $\frac{\varrho(y)-y}{\varrho(y)-r(y)}\le 1$, the inequality $\frac{\partial}{\partial y}h(t,y)\le 0$ proves to be equivalent to inequality $\int_y^1 f(\xi,y)\ d\xi \le (\varrho(y)-y)f(y,y)$, which in turn is verified as follows: Substitution yields $\int_{\varrho(y)-r(y)}^{1-r(y)} \frac{(1-\lambda tz)e^{\lambda tz}-1}{z^2}\ dr(z+r(y))$ whose integrand is non-positive and nonincreasing in z (deriving leads to $e^{-\lambda tz}\le 1-\lambda tz+(\lambda tz)^2/2 = (1+(1-\lambda tz)^2)/2)$. Thus inserting $\xi = y$ yields the desired inequality: $\int_y^1 f(\xi,y)\ d\xi \le (1-y)f(y,y) \le (\varrho(y)-y)f(y,y)$ (as indicated $f(\xi,y)\le 0$). So $h(t,y)$ is decreasing in $y\in[0,r(1))$ and increasing in $t\in[0,1]$. Thus the stopping sets of the myopic stopping time are closed and they specify an optimal stopping time according to Cowan and Zabczyk [11]. □

r–Candidates: Asymptotic Characterization of the Value Function

In this paragraph the value function $v(t,x,y)$ will be specified for $t\to\infty$, where the horizon is enlarged and the rate is 1 according to the introductory remarks of this subsection. For a function h let h_t denote the partial derivative $\frac{\partial}{\partial t}h$. Let $r\in\mathcal{R}^1$. For $(t,x,y)\in E$ where $x\ne b_t^*(y)$ (according to its definition on page 61)

$$v_t(t,x,y) = \begin{cases} s_t(t,x,y) & \text{if } s(t,x,y) > c(t,y) \\ c_t(t,y) & \text{if } s(t,x,y) < c(t,y). \end{cases}$$

Fixing $(x,y)\in\Delta$ with $y<1$, for t sufficiently big the lower case applies ($e^{-\lambda t(1-\varrho(x))}$ vanishes as $t\to\infty$ and $c(t,y)$ doesn't, since the asymptotic value for $r \equiv id$ is positive, which is also true if $y \in (0,1)$ is given as default, and since demands relax for $r \prec id$). Applying the representation (29) of $c_t(t,y)$ gives

$$v_t(t,x,y) = \lambda\left(-(1-r(y))v(t,x,y) + \int_{r(y)}^{y} v(t,\xi,y)\,d\xi + \int_{y}^{1} v(t,\xi,\xi)\,d\xi\right).$$

Differentiation with respect to y yields

$$\begin{aligned} v_{ty}(t,x,y) &= \lambda\Big(- v_y(t,x,y) + r_y(y)v(t,x,y) + r(y)v_y(t,x,y) \\ &\quad + v(t,y,y)\cdot 1 - v(t,r(y),y)r_y(y) + v(t,1,y)\cdot 0 - v(t,y,y)\cdot 1\Big) \end{aligned}$$

which leads to the retarded partial differential equation

$$v_{yt}(t,x,y) = -\lambda(1-r(y))v_y(t,x,y) + \lambda r_y(y)\Big(v(t,x,y) - v(t,r(y),y)\Big),$$

characterizing the value function $v(t,x,y)$ for t sufficiently big, at least theoretically (boundary conditions: $v(t,x,y) = s(t,x,y)$ for fixed $t > 0$ and $(x,y) \in \Delta_t^*$ (see page 68) and particularly continuity of $v(t,x,y)$ on the curve $\{(y, b_t^*(y)) : y \in [\underline{b}_t, \overline{b}_t]\}$ and discontinuity on $\{(y, r(y)) : y \in (\overline{b}_t, 1]\}$ since $s(t,x,y) = 0$ unless $(x,y) \in \Delta_0^*$).

r–Candidates: Asymptotic Equivalence to the Discrete Case

In this paragraph the values of the optimal stopping problems $\mathcal{P}_n(r)$ and $\mathcal{P}_\lambda(r)$ where $r \in \mathcal{R}_1^1$ are compared asymptotically as n and λ tend to infinity. To ignore small items (which become asymptotically neglectable) let a sequence $(q_n)_{n\in\mathbb{N}} \subset (0,1)$ of thresholds be given, such that $q_n \to 1$ for $n \to \infty$. Let $p_n := 1 - q_n$ for $n \in \mathbb{N}$. Conditions for the remaining items beyond q_n to stay representative will be summarized in assumption (34) below.

Let $X_1, X_2, \ldots \sim U([0,1])$, iid, and regard $\mathcal{P}_n$, $n \in \mathbb{N}$. Extract items beyond threshold q_n: Consider the subsequence $X_{\tau_1(n)}, \ldots, X_{\tau_{K_n}(n)} \sim U([q_n,1])$, iid, where $\tau_0 := 0$ and $\tau_j(n) := \inf\{k > \tau_{j-1}(n) : X_k \geq q_n\}$ for $j \in \mathbb{N}$ (set $\inf_\emptyset := \infty$) and the number of items $K_n := \sup\{k \in \mathbb{Z}_+ : \tau_k(n) \leq n\}$ for $n \in \mathbb{N}$, then $K_n \sim B(n,p_n)$ is Binomial distributed. Besides $\tau_k(n) \leq \tau_k(n+1)$ for $k \in \mathbb{N}$ if $(q_n)_{n\in\mathbb{N}}$ is nondecreasing.

Consider $\mathcal{P}_{\lambda_n}$, resp. the Poisson process on $[0, \lambda_n]$ with rate 1, where $\lambda_n := np_n \to \infty$ as $n \to \infty$. Now analogously items beyond threshold q_n are extracted and an iid subsequence uniformly distributed on $[q_n, 1]$ results. Let L_n denote the corresponding number thereof arriving within time interval $[0, \lambda_n]$, being Poisson distributed with parameter λ_n. For $j \in \mathbb{N}$ let $T_j := B_{\sigma_j}$ denote the arrival time of the j–th item beyond q_n where, setting $\sigma_0 := 0$, $\sigma_j(n) := \inf\{k > \sigma_{j-1} : X_k > q_n\}$ indicates its number within $X_1, X_2, \ldots$ (where $\inf_\emptyset := \infty$). $L_n = \sup\{k \in \mathbb{Z}_+ : B_{\sigma_k(n)} \le \lambda_n\}$.
The sequence of offers presented in $\mathcal{P}_n$ and in $\mathcal{P}_{\lambda_n}$ may be assumed to arise from the same probability space and to be identical. Therefore $\tau_k = \sigma_k$ for $k \in \mathbb{N}$. Then implications concerning probabilities within those optimal stopping problems apply where the random variables are not linked.

Using Poisson approximation now random variables K_n and L_n resp. its binomial distribution P_{K_n} and Poisson distribution P_{L_n} can by maximal coupling be constructed (becoming dependent), such that their original distributions are preserved and simultaneously

$$\mathrm{P}\,(K_n \neq L_n) \;=\; d_{TV}\,(\mathrm{P}_{K_n},\, \mathrm{P}_{L_n}) \;\le\; p_n \quad \text{for } n \in \mathbb{N}, \tag{33}$$

where $d_{TV}\,(\mathrm{P}_{K_n}, \mathrm{P}_{L_n}) := \sup\{|\mathrm{P}_{K_n}(A) - \mathrm{P}_{L_n}(A)| \,:\, A \subset \mathbb{Z}_+\}$ denotes the total variation distance and the estimation is given in Ross [27], page 465 (in this situation the upper bound $(1 \wedge \frac{1}{\lambda_n}) \cdot np_n^2$ simplifies since $\lambda_n = np_n \to \infty$). Once properties of $\mathcal{P}_n$ and $\mathcal{P}_{\lambda_n}$ inside event $[K_n = L_n]$ for any $n \in \mathbb{N}$ are detected then an asymptotic statement whose probability converges to 1 as $n \to \infty$ can be inferred concerning the complete optimal stopping problems.

Assumptions and notations for subsequent considerations:
Let $r \in \mathcal{R}_1^1$ with $d := r'(1-) \in [1, \infty)$, then $r(y) \approx dy - d + 1$ for y sufficiently close to 1. As indicated and illustrated in the introduction above let

$$(p_n)_{n\in\mathbb{N}} \subset (0,1) \quad \text{where} \quad p_n \to 0 \quad \text{and} \quad np_n^2 \to \infty \quad \text{as } n \to \infty. \tag{34}$$

For example take $p_n = \ln(n+8)/\sqrt{n+8}$ or take $p_n = n^{-a}$ where $a \in (0, 1/2)$. Let $q_n := 1 - p_n$ denote the corresponding thresholds. Let $\lambda_n := np_n$ denote the length of the time interval of the Poisson process with rate 1, where the total number of arrivals is N_{λ_n}. The requirements $p_n \to 0$ and $\lambda_n = np_n \to \infty$ as $n \to \infty$ aren't sufficient as the proof of lemma 3.8 ii) shows. Without loss of generality monotone convergence is assumed. Some notations ($n \in \mathbb{N}$):

S_n^* resp. $U_{\lambda_n}^*$ denotes an optimal stopping time of $\mathcal{P}_n$ resp. of $\mathcal{P}_{\lambda_n}$.
$\widetilde{S}_n^*$ resp. $\widetilde{U}_{\lambda_n}^*$ denotes an optimal stopping time of $\mathcal{P}_n$ resp. of $\mathcal{P}_{\lambda_n}$ where offers below q_n are ignored.
v_n^* resp. $w_{\lambda_n}^*$ denotes the value of $\mathcal{P}_n$ resp. of $\mathcal{P}_{\lambda_n}$ (rate 1 on $[0, \lambda_n]$).
$\widetilde{v}_n^*$ resp. $\widetilde{w}_{\lambda_n}^*$ denotes the value of $\mathcal{P}_n$ resp. of $\mathcal{P}_{\lambda_n}$ disregarding offers below q_n — i.e. the value of stopping time $\widetilde{S}_n^*$ resp. of $\widetilde{U}_{\lambda_n}^*$.

Ignoring items below q_n leads to suboptimality but preserves ε_n–optimality:

Lemma 3.8 *There is a nonnegative zero sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ such that:*

i) $v_n^* - \varepsilon_n \le \widetilde{v}_n^* < v_n^*$ *for* $n \in \mathbb{N}$.

ii) $w_{\lambda_n}^* - \varepsilon_n \le \widetilde{w}_{\lambda_n}^* < w_{\lambda_n}^*$ *for* $n \in \mathbb{N}$.

Proof: Both proper inequalities on the right side are evident ($p_n \in (0,1)$).

i) By decomposition

$$\begin{aligned} v_n^* &= \mathrm{P}\left(X_{S_n^*} \ge r(Y_n) \;\middle|\; Y_n < \varrho(q_n)\right) \cdot \mathrm{P}\left(Y_n < \varrho(q_n)\right) \\ &+ \mathrm{P}\left(X_{S_n^*} \ge r(Y_n) \text{ and } Y_n \ge \varrho(q_n)\right), \end{aligned}$$

where the first addend is bounded by a nonnegative zero sequence since $\mathrm{P}\left(Y_n < \varrho(q_n)\right) = \varrho^n(q_n) \le ((q_n + d - 1)/d)^n = (1 - p_n/d)^n$, which is $o(1)$ as $n \to \infty$ (take $d + \delta$ and let $\delta \searrow 0$ for estimation of r).
Now the value $\widetilde{v}_n^*$ represents an upper bound for the second term, since $\mathrm{P}\left(X_{S_n^*} \ge r(Y_n) \text{ and } Y_n \ge \varrho(q_n)\right) \le \mathrm{P}\left(X_{\widetilde{S}_n^*} \ge r(Y_n)\right)$ where the first event implies the second one.

ii) Analogously to i) (with $Y_0 = 0$), based on the relation $\mathrm{P}\left(Y_{N_{\lambda_n}} < \varrho(q_n)\right)$ $\le e^{-\lambda_n} \sum_{k=0}^{\infty} \frac{(\lambda_n(1-p_n/d))^k}{k!} = e^{-\lambda_n} e^{\lambda_n(1-p_n/d)} = e^{-np_n^2/d} \overset{n\to\infty}{\to} 0.$ □

This indicates that the optimal boundary function b_n^*, which lies beyond $r(\overline{b}_n)$, increases faster than q_n but not too fast: $n(1 - r(\overline{b}_n))$ remains finite as $n \to \infty$, where this limit is $a\alpha$ according to proposition 2.15 — restricted to $[\underline{b}_n, \overline{b}_n]$, i.e. separate from the boundary condition induced by r in $[\overline{b}_n, 1]$. The nearly optimal values $\widetilde{v}_n^*$ and $\widetilde{w}_{\lambda_n}^*$ become closely related as n grows:

Lemma 3.9 *There is a nonnegative zero sequence* $(\delta_n)_{n\in\mathbb{N}}$ *such that*

$$\widetilde{v}_n^* - \delta_n \leq \widetilde{w}_{\lambda_n}^* \leq \widetilde{v}_n^* + \delta_n \qquad \text{for } n \in \mathbb{N}. \tag{35}$$

Proof: For $n \in \mathbb{N}$ condition on event $[K_n = L_n]$, which is with regard to the aim of an asymptotic statement justifiable due to relation (33).
Let random data vectors D_n resp. D_{λ_n} contain the essential arrival times of $\mathcal{P}_n$ resp. of $\mathcal{P}_{\lambda_n}$: $D_n := (\tau_1, \ldots, \tau_{K_n})$ resp. $D_{\lambda_n} := (T_1, \ldots, T_{L_n})$. For simplicity it is assumed that $D_n, D_{\lambda_n} \in \mathbb{R}^n$, filled up by zeros.
A discretized version $\mathrm{d}(D_{\lambda_n})$ of the data vector D_{λ_n} is constructed: Subdivide interval $(0, \lambda_n]$ into n subintervals of length p_n each. Every instant $T_1, \ldots, T_{L_n}$ is shifted virtually to the right end point of the corresponding subinterval it is situated. The idea is to identify time point j of $\mathcal{P}_n$ and right end point jp_n of the time interval $((j-1)p_n, jp_n]$ of $\mathcal{P}_{\lambda_n}$ for $j = 1, \ldots, n$.
Assertion: The probability that these arrival times differ becomes negligible:

$$\mathrm{P}\left(D_n \neq \mathrm{d}(D_{\lambda_n})\right) = o(1) \qquad \text{as } n \to \infty. \tag{36}$$

Given $K_n = k$ (where $k = 1, \ldots, n$), the arrival times $\tau_1, \ldots, \tau_k$ of $\mathcal{P}_n$ are distributed just as allocating k balls uniformly into n urns, multiple occupancies excluded. Given $L_n = k$, the right–shifted time points $T_1, \ldots, T_k$ of $\mathcal{P}_{\lambda_n}$ are distributed as allocating k balls uniformly into n urns with multiple occupancies ($(T_1, \ldots, T_k)$ itself is equal in distribution to the order statistic of dimension k). The probability that any two balls arrive at the same urn becomes arbitrarily small:
The restriction to event $[K_n = L_n \leq cnp_n]$ for $n \in \mathbb{N}$ for any $c \in (1, \infty)$ is possible (with regard to an asymptotic inspection) due to the relation $\mathrm{P}\left(K_n \leq cnp_n\right) \approx \Phi\left((c-1)\sqrt{np_n/q_n}\right) \to 1$ as $n \to \infty$.
Suppose allocating $\lceil cnp_n \rceil$ balls uniformly into n urns with multiple occupancies. Then the probability of the event that no urn is met more than once, the left side of (36), is given by $n(n-1)\cdots(n - \lceil cnp_n \rceil + 1)/n^{\lceil cnp_n \rceil}$, which tends to 1 as $n \to \infty$: Let $m := n - \lceil cnp_n \rceil$. Stirling's formula yields:

$$\frac{n!}{m!} \simeq \sqrt{\frac{n}{m}} \frac{n^n}{m^m} \frac{e^{-n}}{e^{-m}} \simeq \frac{n^n}{m^m} e^{-\lceil cnp_n \rceil},$$

where $\sqrt{n/m} \simeq 1/\sqrt{1 - cp_n} \simeq 1$ as $n \to \infty$. Taking logarithms the assertion $n!/m! \simeq n^{\lceil cnp_n \rceil}$ proves to be equivalent to

$$n \ln(n) - m \ln(m) - \lceil cnp_n \rceil \simeq \lceil cnp_n \rceil \ln(n)$$

$$\begin{aligned} \ln(n) - \frac{\lceil cnp_n \rceil}{m} &\simeq \ln(m) \\ n &\simeq m, \end{aligned}$$

which is valid since $\lceil cnp_n \rceil / m \simeq p_n/(c-p_n) = o(1)$ and since $n \simeq n - \lceil cnp_n \rceil$ subject to $p_n = o(1)$ as $n \to \infty$. This proves assertion (36).

Let $\widetilde{v}(S_n)$ resp. $\widetilde{w}(U_{\lambda_n})$ denote the value of stopping time S_n resp. U_{λ_n} for $\mathcal{P}_n$ resp. $\mathcal{P}_{\lambda_n}$ conditioned on $[K_n = L_n]$, i.e. where items below q_n are ignored. For stopping time S_n of problem $\mathcal{P}_n$ let $\mathrm{p}(S_n)$ denote the corresponding continuous time stopping rule for $\mathcal{P}_{\lambda_n}$ which takes state (b, x, y) by the state $(p_n \lceil b/p_n \rceil, x, y)$, i.e. which defers the decision until the right end point $p_n \lceil b/p_n \rceil$ of the time interval containing b. Now relation (36) implies that $\widetilde{w}(\mathrm{p}(S_n)) \geq \widetilde{v}(S_n) - \delta_n$ for $n \in \mathbb{N}$ for a nonnegative zero sequence $(\delta_n)_{n\in\mathbb{N}}$. Referring to optimal stopping times disregarding offers below q_n this yields $\widetilde{w}(\widetilde{U}^*_{\lambda_n}) \geq \widetilde{w}(\mathrm{p}(\widetilde{S}^*_n)) \geq \widetilde{v}(\widetilde{S}^*_n) - \delta_n$, where the outer terms represent the first inequality of assertion (35) and it remains to verify the second inequality.

Construction of a Poissonian version $\mathrm{p}(D_n)$ of discrete arrival times of D_n: For discrete arrival time $\tau_k = j$ define a virtual random arrival time $V_k \sim U((j-1)p_n, jp_n])$ where $k = 1, \ldots, K_n$ and $j \in \{k, \ldots, n\}$. Then $V_k \stackrel{D}{=} B_k$ for $k = 1, \ldots, K_n$ — provided there is an arrival inside $(j-1)p_n, jp_n]$, which is true inside event $[K_n = L_n]$ and $[D_n = \mathrm{d}(D_{\lambda_n})]$ due to the discrete arrival. Resumed for $k = 1, \ldots, K_n$ this means, conditioned on $[K_n = L_n]$:

$$\text{event} \quad [D_n = \mathrm{d}(D_{\lambda_n})] \quad \text{implies} \quad D_{\lambda_n} \stackrel{D}{=} \mathrm{p}(D_n). \tag{37}$$

Let a stopping time U_{λ_n} of problem $\mathcal{P}_{\lambda_n}$ with the family of boundary functions $(b_t(y))_{t\in[0,\lambda_n]}$ be given. This induces a randomized stopping time $\mathrm{d}(U_{\lambda_n})$ for $\mathcal{P}_n$ by applying stopping time U_{λ_n} to the data vector $\mathrm{p}(D_n)$ in the following sense: Suppose item X_j arrives at the time $\tau_k = j$ resp. at the virtual instant V_k ($k = 1, \ldots, K_n$ and $j \in \{k, \ldots, n\}$). Then select X_j iff $X_j \geq b_{\lambda_n - V_k}(Y_j)$ i.e. suppose remaining time $\lambda_n - V_k$ for $\mathcal{P}_{\lambda_n}$. This is equivalent to: $X_j \geq r(Y_j)$ and an additional Bernoulli experiment with success probability ξ succeeds, where $\xi := j - \inf\{\zeta \in (j-1, j] : b_{\lambda_n - \zeta p_n}(y) \leq X_j\}$ with $\inf_\emptyset := j$, particularly accept resp. reject anyway if $X_j \geq b_{\lambda_n - jp_n}(y)$ resp. if $X_j < b_{\lambda_n-(j-1)p_n}(y)$. Now $\widetilde{v}(\mathrm{d}(U_{\lambda_n})) \geq \widetilde{w}(U_{\lambda_n}) - \delta_n$, regarding relation (36) and (37). Referring to optimal stopping times conditioned on event $[K_n = L_n]$ then relation $\widetilde{v}(\widetilde{S}^*_n) \geq \widetilde{v}(\mathrm{d}(\widetilde{U}^*_{\lambda_n}) \geq \widetilde{w}(\widetilde{U}^*_{\lambda_n}) - \delta_n$ applies. This verifies the second inequality of (35) and the proof is complete. □

The preceding lemmas 3.8 and 3.9 result in:

Theorem 3.10 *Let the assumptions and notations on page 76 be given. Then the values of problem $\mathcal{P}_n(r)$ and $\mathcal{P}_{\lambda_n}(r)$ asymptotically coincide:*

$$\lim_{n\to\infty} v_n^* \quad = \quad \lim_{n\to\infty} w_{\lambda_n}^*. \tag{38}$$

An analogue assertion referring to accordance of the asymptotic value of the myopic stopping time of problem $\mathcal{P}_n$ and $\mathcal{P}_{\lambda_n}$ depends on the behaviour of its boundary functions, which isn't evident regarding the corresponding expressions of propositions 2.31 and 3.5 and regarding figure 4.

3.2 Random Arrival Times and Random Horizon

In this section the arrival times of the offers and the horizon T are random, the detailed mathematical model are given on pages 60f. Given relax function $r \in \mathcal{R}$ it is the objective to find a stopping time $S \in \mathcal{S}$ maximizing

$$\mathrm{P}\left(X_{N_S} \geq r(Y_N)\right).$$

Example 3.11 Suppose maximal two objects arrive: $G \equiv \mathbf{1}_{[1,\infty)}$ and $p := \mathrm{P}\left(T = 1\right) = 1 - \mathrm{P}\left(T = 2\right) \in [0, 1)$. The following threshold rule is optimal: $S^* = 1$ if $X_1 \geq t^* := (r + \varrho)^{-1}\left(\frac{1-2p}{1-p} \vee 0\right)$ and $S^* = 2$ otherwise. The value then is given by $v_2^* = p + t^*(1 - 2p) + (1 - p)\left(\int_{t^*}^1 \varrho(\xi)\ d\xi - \int_0^{t^*} r(\xi)\ d\xi\right)$.

Verification: Let $X_1 = x$. $s_1(x) = p+(1-p)\varrho(x)$ and $c_1(x) = (1-p)(1-r(x))$, with usual notation. Stopping is advisible if $p+(1-p)\varrho(x)-(1-p)(1-r(x)) \geq 0$ iff $\varrho(x)+r(x) \geq (1-2p)/(1-p)$. The latter term doesn't exceed 1. Function $(\varrho + r) : [0, 1] \to [0, 2]$ is continuous and increasing and $\varrho(1) + r(1) > 1$. If $p = 0$, then $x \geq r(1 - r(x))$ results. For $p \in [1/2, 1]$ it is optimal to choose X_1 (which is evident if $r \equiv id$ and for $r \prec id$ demands relax). □

Example 3.12 Let the number of items be uniformly distributed on $\{1, 2, 3\}$, $1/3 = \mathrm{P}\left(T = 1\right) = \mathrm{P}\left(T = 2\right) = \mathrm{P}\left(T = 3\right)$, where $G \equiv \mathbf{1}_{[1,\infty)}$. The optimal boundary functions and value functions (number ℓ of remaining draws):

$\ell = 1$: $b_1^* \equiv r$ and $v_1^*(x, y) = \begin{cases} \frac{1}{2}\left(1 + \varrho(x)\right) & \text{if } x \geq r(y) \\ \frac{1}{2}\left(1 - r(y)\right) & \text{if } x < r(y) \end{cases}$ where $(x, y) \in \Delta$.

$\ell = 2$: On the one hand $b_2^*(y) = -\frac{1}{2} + \sqrt{\frac{1}{4} - r^2(y) + \int_{r(y)}^1 \varrho(\xi)\ d\xi}$ for $y \in [\underline{b}_2, \overline{b}_2]$ (which is extended on $[0, \underline{b}_2)$ by $\underline{b}_2$ and on $(\overline{b}_2, 1]$ by r) and on the other hand

$$v_2^*(x, y) = \begin{cases} \frac{1}{3}\left(1 + \varrho(x) + \varrho^2(x)\right) & \text{if } x \geq b_2^*(y) \\ \frac{1}{3}\left(1 - r^2(y) + \int_{r(y)}^1 \varrho(\xi)\ d\xi\right) & \text{if } x < b_2^*(y) \end{cases} \quad \text{where } (x, y) \in \Delta.$$

Verification: For $\ell = 1$ take example 3.11 with $p = 1/2$. For $\ell = 2$ evidently $s_2(x) = \frac{1}{3}\left(1 + \varrho(x) + \varrho^2(x)\right)$, recursion, straightforward since $b_1^* \equiv r$, gives $c_2(y) = \frac{1}{3}\left(1 - r^2(y) + \int_{r(y)}^1 \varrho(\xi)\ d\xi\right)$, which yield $b_2^*(y)$ and $v_2^*(x, y)$. □

For the case $r \equiv id$ where maximal three objects arrive see Porosinski [24]. A generalization of the uniform distribution of example 3.12 fails, see page 84.

Since the remaining time is not known in advance, in this section $s(b,x,y)$ and $c(b,x,y)$ (resp. $s_k(x,y)$ and $c_k(x,y)$) represent the mean payoff of stopping and that of skipping the present item and then proceeding optimally in state $(b,x,y)\in E$ (resp. $(k,x,y)\in E$).
It is convenient to introduce the moment generating function of N_t,

$$m(t,z) \;:=\; \mathrm{E}\left(z^{N_t}\right) \;=\; \sum_{n=0}^{\infty} z^n \mathrm{P}\left(N_t=n\right), \tag{39}$$

denoting the mean payoff in the moment of an arrival if the remaining time is supposed to be $t\geq 0$ and where value $z\in[0,1]$ represents a boundary subsequent values shouldn't exceed (referring to $U([0,1])$).
The mean payoff of stopping in state $(b,x,y)\in E$ where $x\geq r(y)$ is

$$\begin{aligned} s(b,x) &= \mathrm{E}\left(m\left(T-b,\varrho(x)\right) \mid T\geq b\right) \\ &= \frac{1}{1-H(b-)}\int_b^\infty m(t-b,\varrho(x))\,dH(t). \end{aligned} \tag{40}$$

The Subsequence of r–Candidates

It is advisible only to select an item, which is an r–candidate referring to previous offers. A transformation of the Markov process $Z=(Z_k)_{k\in\mathbb{Z}_+}$ to the subsequence concerning r–candidates is performed: Set $\tau_0:=0$ and for $k\in\mathbb{N}$ let, using $\inf_\emptyset:=\infty$ and $Y_0=0$,

$$\tau_k \;:=\; \inf\left\{n\in\mathbb{N}\,:\,\tau_{k-1}<n\leq N \text{ and } X_n\geq r(Y_{\tau_{k-1}})\right\},$$

representing numbers of r–candidates. Now define for $k\in\mathbb{Z}_+$

$$R_k := \begin{cases} \alpha_0 & \text{if } k=0 \\ (\tau_k, B_{\tau_k}, X_{\tau_k}, Y_{\tau_k}) & \text{if } k>0 \text{ and } \tau_k<\infty \\ \alpha_\infty & \text{if } k>0 \text{ and } \tau_k=\infty. \end{cases}$$

Particularly $\tau_1=1$ only if $B_1\leq T$. Setting $E:=\mathbb{N}\times\mathbb{R}_+\times\Delta$ the state space of the stochastic process $R:=(R_k)_{k\in\mathbb{Z}_+}$ is $E\cup\{\alpha_0,\alpha_\infty\}$, where α_0 resp. α_∞ denotes the initial resp. final state with $f(\alpha_0):=0=:f(\alpha_\infty)$. Now R is a homogenous Markov process (due to $Y_{\tau_{k+1}}=Y_{\tau_k}\vee X_{\tau_{k+1}}$, the definition of the τ_k and due to independence of $X_1,X_2,\ldots,A_1,A_2,\ldots,T$). A situation for a decision consists of a quadrupel $(i,b,x,y)\in E$. The transition probabilities result from the following expressions: $\mathrm{P}(R_0 = \alpha_0) = 1$ and $\mathrm{P}(R_{k+1} = \alpha_\infty \mid R_k = \alpha_\infty) = 1$ for $k \in \mathbb{N}$ (homogeneity) and

$$\mathrm{P}(R_1 = \alpha_\infty \mid R_0 = \alpha_0) = \mathrm{P}(B_1 > T) = \int_0^\infty (1 - G(t))\, dH(t),$$

and for $\beta \in (0,\infty)$, $\xi \in [0,1]$

$$\mathrm{P}(\tau_1 = 1, B_1 \le \beta, X_1 = Y_1 \le \xi \mid R_0 = \alpha_0) = \xi \int_0^\infty G(\beta \wedge t)\, dH(t),$$

and for $i > k \in \mathbb{N}$

$$\begin{aligned} &\mathrm{P}(R_{k+1} = \alpha_\infty \mid \tau_k = i, B_i = b, X_i = x, Y_i = y) \\ = \;& \frac{1}{1 - H(b-)} \int_b^\infty \sum_{j=0}^\infty r^j(y) \mathrm{P}(N_u = i + j)\; dH(u) \end{aligned}$$

and for $i > k \in \mathbb{N}$, $\beta \in (0,\infty)$, $(\xi, \zeta) \in \Delta$

$$\begin{aligned} &\mathrm{P}(\tau_{k+1} = j, B_j \le \beta, X_j \le \xi, Y_j \le \zeta \mid \tau_k = i, B_i = b, X_i = x, Y_i = y) \\ = \;& r^{j-i-1}(y)(\xi - r(y)) \mathrm{P}(B_j \le (\beta \wedge (T - b)) \mid T \ge b) \\ = \;& r^{j-i-1}(y)(\xi - r(y)) \frac{1}{1 - H(b-)} \int_b^\infty G^{*(j-i)}(\beta \wedge (t - b))\, dH(t) \end{aligned}$$

if $j > i$, $\beta \ge b$, $\xi \ge r(y)$ and $\zeta \ge y$ and this probability equals 0 otherwise. Here the fact is exploitet, that the joint distribution of X_j and Y_j given $X_j \ge r(Y_i)$ possesses virtually a one–dimensional density.

The natural filtration $\mathcal{F} := (\mathcal{F}_k)_{k \in \mathbb{Z}_+}$ with $\mathcal{F}_k := \sigma(X_0, \ldots, X_k; A_0, \ldots, A_k)$. Set $R_\infty := \alpha_\infty$. The set $\mathcal{S}$ of stopping times with respect to $\mathcal{F}$ is, without loss, reduced

$$\mathcal{S}_0 := \{S \in \mathcal{S} : (S = k \implies X_k \ge r(Y_k)) \;\forall k \in \mathbb{N}\}$$

for any stopping time $S \in \mathcal{S}_0$ define a related stopping time

$$S_0 := \begin{cases} k & \text{if } S = \tau_k < \infty, \text{ for } k \in \mathbb{N} \\ \infty & \text{if } S = \infty. \end{cases}$$

$S_0 \in \mathcal{S}_0$ and S_0 is a stopping time with respect to the filtration $(\mathcal{F}_{\tau_k})_{k \in \mathbb{Z}_+}$.

The transformed problem now consists of optimal stopping of the homogenous Markov process $R := (R_k)_{k\in\mathbb{Z}_+}$ with reward function f_0, representing the probability that the value x of state $(i,b,x,y) \in E$ remains an r–candidate: $f_0(i,b,x,y) = f_0(b,x) := \frac{1}{1-H(b-)}\int_b^\infty m(t,\varrho(x))\ dH(t)$ if $H(b-) < 1$ and $f_0(b,x) := 0$ otherwise, with additional convention $f_0(\alpha_0) := 0 =: f_0(\alpha_\infty)$. Then $\mathrm{P}\left(X_S \geq r(Y_N)\right) = f_0(R_{S_0})$ for $S \in \mathcal{S}_0$.

The Myopic Stopping Time

The mean payoff of stopping in $(b,x,y) \in E$ in the general case is given by expression (40). The mean payoff of proceeding until the next r–candidate arrives is generally denoted formally by $\mathbb{P}s$ and equals 0 if $H(b-) = 1$ and otherwise, applying the transition probabilities of the preceeding paragraph,

$$\begin{aligned}&(\mathbb{P}s)(b,y) \qquad (41)\\ &= \frac{1}{1-H(b-)}\int_b^\infty\left(\sum_{k=1}^\infty r^{k-1}(y)\int_0^{t-b}\left(\int_{r(y)}^1 s(b+u,\xi)\ d\xi\right)dG^{*(k)}(u)\right)dH(t).\end{aligned}$$

Here $k-1$ non–r–candidates pass ($k \in \mathbb{N}$, payoff 0 if no r–candidate arrives). Now $s \geq \mathbb{P}s$ specifies the myopic stopping time or one step look–ahead rule of epochs of arrivals, which in general proves to be not optimal, see page 48.

Subsequent the myopic stopping time is illustrated in situations where an optimal stopping time seems intricate. Suppose deterministic arrival times: $G \equiv \mathbf{1}_{[1,\infty)}$. Then without loss of generality the horizon T is assumed to be discrete: $\mathrm{P}\left(T \in \mathbb{Z}_+\right) = 1$. Let $\pi_k := \mathrm{P}\left(T = k\right)$ and let $\overline{\pi}_k := \sum_{j=k}^\infty \pi_j$ denote the tail probability where $k \in \mathbb{Z}_+$. Let state $(k,x,y) \in E$ with $x \geq r(y)$ be given. Then the mean payoff of stopping is

$$s(k,x) \;=\; \frac{1}{\overline{\pi}_k}\sum_{j=k}^\infty \pi_j \varrho^{j-k}(x)$$

while choosing the next r–candidate (if any) yields the mean payoff

$$\begin{aligned}(\mathbb{P}s)(k,y) &= \frac{1}{\overline{\pi}_k}\sum_{j=k+1}^\infty \pi_j \sum_{i=1}^{j-k} r^{i-1}(y)\int_{r(y)}^1 s(k+i,\xi)\ d\xi\\ &= \sum_{j=k+1}^\infty \frac{\overline{\pi}_j}{\overline{\pi}_k} r^{j-k-1}(y)\int_{r(y)}^1 s(j,\xi)\ d\xi.\end{aligned}$$

In this situation for $r \equiv id$ and unrestricted distribution of T socalled stopping islands may occur, see Porosinski [24] whose approach can't be adapted since the myopic stopping time isn't optimal in general if $r \prec id$.
Particularly the problem of selection of an r–candidate where the number of the draws is uniformly distributed on $\{1, \ldots, n\}$ seems to be not regular. In case of $r \equiv id$ the myopic stopping time proves to be optimal, see the cited article, and in the general case $\mathcal{R} \ni r \prec id$ already the proof that the problem is regular (see section 2.1 or 3.1) fails. In addition there is a connection to the problem of maximizing the duration of owning a temporary r–candidate without recall, see the corresponding paragraph on page 97 of section 4.1.2.

Geometric Horizon

Let the arrival times and let the horizon be geometrically distributed: Let $\mathrm{P}\left(A_1 = k\right) = p(1-p)^{k-1}$ for $k \in \mathbb{N}$ where $p \in (0, 1]$, including the case $p = 1$ of deterministic arrivals $G \equiv \mathbf{1}_{[1,\infty)}$, and let $\mathrm{P}\left(T = k\right) = \pi(1-\pi)^k$ for $k \in \mathbb{Z}_+$ where $\pi \in (0, 1)$ (the alternative geometric distribution is mentioned below). Let this optimal stopping problem be denoted by $\mathcal{P}_{p,\pi}$. Let $q := p(1-\pi)/\pi$.

Theorem 3.13 *For optimal stopping problem $\mathcal{P}_{p,\pi}(r)$ where $r \in \mathcal{R}^1$ the stopping time $S^* := \inf\{1 \le k \le T \,:\, X_{N_k} \ge x^*\}$ (with $\inf_\emptyset := \infty$) is optimal — take the first value above x^*. Here $x^* \in [0, r(1))$ denotes the unique solution in $(0, r(1))$ of*

$$\frac{1 + q(1-x)}{1 + q(1-\varrho(x))} \;=\; \int_x^1 \frac{q}{1 + q(1-\varrho(\xi))}\, d\xi$$

if $\int_0^1 \frac{1}{1+q(1-\varrho(\xi))}\, d\xi > \frac{1}{q}$ and $x^* = 0$ otherwise ($q := p(1-\pi)/\pi$).
The value of $\mathcal{P}_{p,\pi}(r)$ — the probability of winning applying S^* — then is

$$\mathrm{P}\left(X_{N_{S^*}} \ge r(Y_N)\right) \;=\; \frac{q}{1 + q(1-x^*)} \int_{x^*}^1 \frac{1}{1 + q(1-\varrho(\xi))}\, d\xi.$$

Proof: Suppose first that recall is allowed. The mean payoff of recalling the value $y \in [0, r(1))$ is independent of the time of recall since G and H is memoryless (given the horizon hasn't terminated the choosing yet; an offer beyond $r(1)$ ensures maximal payoff 1 anytime):

$$s(y) \;=\; \pi \sum_{k=0}^{\infty} (1-\pi)^k \sum_{j=0}^{k} \binom{k}{j} (1-p)^{k-j} p^j \varrho^j(y)$$

$$
\begin{aligned}
&= \pi \sum_{k=0}^{\infty} (1-\pi)^k \left(1-p+p\varrho(y)\right)^k \\
&= \frac{\pi}{\pi + (1-\pi)p(1-\varrho(y))} \\
&= \frac{1}{1+q(1-\varrho(y))}.
\end{aligned}
$$

The one step look–ahead rule watches for the mean payoff of choosing the maximum of y and the new offer in the moment of the next arrival (in case of no further arrival payoff 0). The probability for another arrival is $\mathrm{P}(A_1 \le T) = q/(1+q)$ and thus the myopic stopping time stops if

$$
\begin{aligned}
\frac{1}{1+q(1-\varrho(y))} &\ge \frac{q}{1+q}\int_0^1 \frac{1}{1+q(1-\varrho(y\vee\xi))}\, d\xi \\
\frac{1+q(1-y)}{1+q(1-\varrho(y))} &\ge \int_y^1 \frac{q}{1+q(1-\varrho(\xi))}\, d\xi.
\end{aligned}
$$

Let $h(y) := \frac{1+q(1-y)}{1+q(1-\varrho(y))} - \int_y^1 \frac{q}{1+q(1-\varrho(\xi))}\, d\xi \in C^1([0,r(1)])$. $h(y)$ is increasing, since $h'(y) \ge 0$ iff $\varrho'(y)(1-\pi)p(\pi+(1-\pi)p(1-y)) \ge 0$. $h(r(1)) = 1$. $h(0) \ge 0$ is equivalent to $\int_0^1 \frac{q}{1+q(1-\varrho(\xi))}\, d\xi \le 1$, i.e. take the first item if this is true and else there is a unique solution inside $(0,r(1))$ solving $h(x)=0$. Thus the stopping sets of the myopic stopping time are closed and realizable (the mean payoff vanishes as time grows to infinity) and thus they specify an optimal rule. If $S^* = \infty$ then $\mathrm{P}(X_\infty \ge r(Y_N)) = 0$, including the case $N=0$ with resulting payoff 0 due to $Y_0 = 1$.

The probability of winning applying S^*: Since $\mathrm{P}(A_1 \le T) = q/(1+q)$ decomposition with respect to the number $k \in \mathbb{N}$ of arrivals and then with respect to the number $j = 1,\ldots,k$ of the first value beyond x^* yields

$$
\begin{aligned}
\mathrm{P}(X_{S^*} \ge r(Y_N)) &= \frac{1}{1+q}\sum_{k=1}^{\infty}\left(\frac{q}{1+q}\right)^k \sum_{j=1}^{k} (x^*)^{j-1} \int_{x^*}^1 \varrho^{k-j}(\xi)\, d\xi \\
&= \frac{1}{1+q}\sum_{j=1}^{\infty}\left(\frac{q}{1+q}\right)^j \int_{x^*}^1 \varrho^{j-1}(\xi)\, d\xi \sum_{j=k}^{\infty}\left(\frac{qx^*}{1+q}\right)^{j-k} \\
&= \frac{1}{1+q}\,\frac{q}{1+q(1-x^*)}\sum_{j=1}^{\infty}\left(\frac{q}{1+q}\right)^{j-1} \int_{x^*}^1 \varrho^{j-1}(\xi)\, d\xi \\
&= \frac{q}{1+q(1-x^*)}\int_{x^*}^1 \frac{1}{1+q(1-\varrho(\xi))}\, d\xi,
\end{aligned}
$$

which in other words means: An offer beyond x^*, called ξ, occurs with probability $q(1-x^*)/(1+q(1-x^*))$ and the integral divided by $1-x^*$ represents the mean payoff choosing ξ given there is one beyond x^*. □

Evidently the optimal threshold is $x^* = 0$ if $q \leq 1$, i.e. if $p \leq \pi/(1-\pi)$. If in theorem 3.13 alternatively $\mathrm{P}(T=k) = \pi(1-\pi)^{k-1}$, $k \in \mathbb{N}$, is taken, the mean payoff is $s(y) = \frac{\pi - \pi p(1-\varrho(y))}{\pi + (1-\pi)p(1-\varrho(y))}$.

Example 3.14 Suppose $r(x) = \vartheta x$ for $x \in [0,1]$ where $\vartheta \in (0,1]$. Then $\int_x^1 \frac{q}{1+q(1-\varrho(\xi))}\, d\xi = \vartheta \ln(1 + q(1-x/\vartheta)) + q(1-\vartheta)$. Therefore $x^* = 0$ is the optimal threshold iff $\vartheta \ln(1+q) + q(1-\vartheta) \leq 1$ iff $\vartheta \geq \frac{1-q}{\ln(1+q)-q}$. Otherwise $x^* \in (0,\vartheta)$ uniquely solves equation $\frac{1+q(1-x)}{1+q(1-\varrho(x))} = \vartheta \ln(1+q(1-x/\vartheta)) + q(1-\vartheta)$. Particularly $x^* = 0$ if $q \leq 1$ and $x^* > 0$ if $q > e-1$, whereas $x^* \in [0,\vartheta)$ in case of $q \in (1, e-1]$. The value of $\mathcal{P}_{p,\pi}(\vartheta x)$ is $\frac{\ln(1+q(1-x^*/\vartheta)) + q(1-\vartheta)}{1+q(1-x^*)}$.
Taking $p = 1$ and $\vartheta = 1$ this meets the geometric case of Porosinski [24]: $q = (1-\pi)/\pi$ yields $x^* = 0$ if $\pi \geq 1/e$ and otherwise $x^* = \frac{1-\pi e}{1-\pi}$. The optimal probability of winning is $-\pi \ln \pi$ and $1/e$, respectively.

Now for geometric horizon and general arrival times the myopic stopping time is displayed: Let $\mathrm{P}(T=k) = \pi(1-\pi)^{k-1}$ for $k \in \mathbb{N}$ and distribution function G. Let $(b,x,y) \in E$, where $x \geq r(y)$. Since $[T \geq b]$ implies $[T \geq \lceil b \rceil]$ now $\mathrm{P}(T = i + \lceil b \rceil \mid T \geq \lceil b \rceil) = \pi(1-\pi)^{i+\lceil b \rceil - 1}/(1-\pi)^{\lceil b \rceil - 1}$ for $i \in \mathbb{Z}_+$:

$$s(b,x) \;=\; \sum_{j=1}^{\infty} m(j + \lfloor b \rfloor - b, \varrho(x))\; \pi(1-\pi)^{j-1}.$$

Due to (41) choosing the next r–candidate (if any) yields the mean payoff

$$\begin{aligned}
(\mathbb{P}s)(b,y) &= \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} r^{i-1}(y) \int_0^{j+\lfloor b \rfloor - b} \left(\int_{r(y)}^1 s(\xi)\; d\xi \right) dG^{*(i)}(u) \pi(1-\pi)^{j-1} \\
&= \int_{r(y)}^1 s(\xi)\; d\xi \cdot \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} r^{i-1}(y) G^{*(i)}(j + \lfloor b \rfloor - b) \pi(1-\pi)^{j-1} \\
&= \int_{r(y)}^1 s(\xi)\; d\xi \cdot \sum_{j=1}^{\infty} \frac{1 - m(j + \lfloor b \rfloor - b, r(y))}{1 - r(y)} \pi(1-\pi)^{j-1} \\
&= \frac{1 - s(r(r(y)))}{1 - r(y)} \int_{r(y)}^1 s(\xi)\; d\xi, \qquad (42)
\end{aligned}$$

where particularly the third equality sign is based on the following identities (set $z := r(y) \in [0,1)$ and let $u \geq 0$):

$$\begin{aligned}
\frac{1-m(u,z)}{1-z} &= \frac{1}{1-z} - \left(\sum_{i=0}^{\infty} z^i\right)\left(\sum_{j=0}^{\infty} z^j \mathrm{P}\left(N_u = j\right)\right) \\
&= \frac{1}{1-z} - \sum_{j=0}^{\infty} z^j \mathrm{P}\left(N_u \leq j\right) \\
&= \frac{1}{1-z} - \sum_{j=0}^{\infty} z^j \left(1 - G^{*(j+1)}(u)\right) \\
&= \sum_{j=0}^{\infty} z^j G^{*(j+1)}(u),
\end{aligned}$$

because events $[N_u \leq j]$ and $[A_1 + \cdots + A_{j+1} > u]$ are equivalent.

Exponential Horizon

Let the horizon be exponentially distributed, $T \sim \exp(\mu)$ with $\mu > 0$, while the distribution function of A_1 is G. Let $(b, x, y) \in E$ where $x \geq r(y)$. Then

$$s(b,x,y) \;=\; s(x) \;=\; \int_0^{\infty} m(t, \varrho(x))\mu e^{-\mu t}\, dt, \tag{43}$$

independent of b due to memorylessness of T. The myopic stopping time is illustrated: Let $y < r(1)$ and referring to expression (41) rearrange the order of integrals (respect $\frac{1}{1-H(b)} \int_b^{\infty} \int_0^{t-b} 1\; dG^{*(k)}(u) dH(t) = \int_0^{\infty} G^{*(k)}(t)\mu e^{-\mu t}\, dt$):

$$\begin{aligned}
(\mathbb{P}s)(y) &= \int_0^{\infty} \left(\sum_{k=1}^{\infty} r^{k-1}(y) \int_{r(y)}^1 s(\xi)\, d\xi \;\, G^{*(k)}(t)\mu e^{-\mu t}\right) dt \\
&= \int_{r(y)}^1 s(\xi)\, d\xi \cdot \int_0^{\infty} \frac{1-m(u,r(y))}{1-r(y)} \mu e^{-\mu u}\, du \\
&= \frac{1-s(r(r(y)))}{1-r(y)} \int_{r(y)}^1 s(\xi)\, d\xi,
\end{aligned}$$

where the second equality sign is verified similar to the discrete case, see below expression (42).

Now the interarrival times and the horizon are assumed to be exponentially distributed: Let $A_1 \sim \exp(\lambda)$ and $T \sim \exp(\mu)$, where $\lambda, \mu > 0$ and $\nu := \mu/\lambda$. Let this optimal stopping problem be denoted by $\mathcal{P}_{\lambda,\mu}(r)$ for $r \in \mathcal{R}$.
By an appropiate limiting procedure problem $\mathcal{P}_{\lambda,\mu}$ represents the limit of the twice geometric case $\mathcal{P}_{p,\pi}$ of theorem 3.13, in this sense the latter is more general. However this problem $\mathcal{P}_{\lambda,\mu}$ proves to be equivalent to the duration problem $\mathcal{D}^t_{\lambda,\mu}$, see page 119, and therefore its solution is given in detail:

Theorem 3.15 *For problem $\mathcal{P}_{\lambda,\mu}(r)$ where $r \in \mathcal{R}^1$ with $\nu := \mu/\lambda$ the stopping time $S^* := \inf\{b \in [0,T] : X_{N_b} \geq x^*\}$ (with $\inf_\emptyset := \infty$) is optimal — take the first value above x^*. Here $x^* \in [0, r(1))$ denotes the unique solution in $(0, r(1))$ of*

$$\frac{\nu+1-x}{\nu+1-\varrho(x)} = \int_x^1 \frac{1}{\nu+1-\varrho(\xi)}\, d\xi,$$

if $\int_0^1 \frac{1}{\nu+1-\varrho(\xi)}\, d\xi > 1$ and $x^* = 0$ otherwise.
The value of $\mathcal{P}_{\lambda,\mu}(r)$ — the probability of winning applying S^* — is

$$\mathrm{P}\left(X_{N_{S^*}} \geq r(Y_N)\right) = \frac{\nu}{\nu+1-x^*} \int_{x^*}^1 \frac{1}{\nu+1-\varrho(\xi)}\, d\xi.$$

Proof: Suppose first that recall is allowed. Due to memoryless of G and H the mean payoff of recalling $y \in [0,1]$ is independent of the moment of recall:

$$s(y) = \mu \int_0^\infty e^{-(\lambda(1-\varrho(y))+\mu)u}\, du = \frac{\mu}{\mu+\lambda(1-\varrho(y))} = \frac{\nu}{\nu+1-\varrho(y)}.$$

Regarding now event time recall, i.e. recall only in moments of an arrival, this expresseion is compared with the mean payoff recalling the topical maximum in the moment of the next arrival, which also is time independent. Since any further arrival occurs with probability $\mathrm{P}(A_1 \leq T) = \lambda/(\mu+\lambda) = 1/(\nu+1)$ this means

$$\frac{\nu}{\nu+1-\varrho(y)} \geq \frac{1}{\nu+1}\int_0^1 \frac{\nu}{\nu+1-\varrho(y\vee\xi)}\, d\xi \tag{44}$$

$$\frac{\nu+1-y}{\nu+1-\varrho(y)} \geq \int_y^1 \frac{1}{\nu+1-\varrho(\xi)}\, d\xi,$$

a condition verified to be consistent with selection in moments of an arrival but without recall and with the infinitesimal look–ahead rule based on permanent recall in continuous time. This condition specifies when to choose an offer independent of time and thus recall is redundant.
Let $h(x) := \frac{\nu+1-x}{\nu+1-\varrho(x)} - \int_x^1 \frac{1}{\nu+1-\varrho(\xi)}\, d\xi \in C^1([0, r(1)])$. Now h is increasing, since $h'(x)$ proves to be nonnegative iff $\varrho'(x)(\nu+1-x)/(\nu+1-\varrho(x))^2 \geq 0$. If $h(0) = 1 - \int_0^1 \frac{1}{\nu+1-\varrho(\xi)}\, d\xi \geq 0$ then take the first item, if any. Otherwise the solution of $h(x) = 0$ yields an unique threshold inside $(0, r(1))$, since $h(r(1)) = 1$. Since the stopping sets of this myopic stopping time are closed and realizable an optimal stopping time is specified. If $S^* = \infty$ then the mean payoff is $\mathrm{P}\,(X_\infty \geq r(Y_N)) = 0$ ($X_\infty = 0$, including the case $N = 0$ due to $Y_0 = 1$).
The optimal probability of winning applying S^*: Decomposition with respect to the total number of offers and watching for the first one beyond x^* yields (due to $\mathrm{P}\,(A_1 \leq T) = 1/(\nu+1)$)

$$\begin{aligned}
\mathrm{P}\,(X_{S^*} \geq r(Y_N)) &= \frac{\nu}{\nu+1} \sum_{k=1}^{\infty} \left(\left(\frac{1}{\nu+1} \right)^k \sum_{j=1}^{k} (x^*)^{j-1} \int_{x^*}^1 \varrho^{k-j}(\xi)\, d\xi \right) \\
&= \frac{\nu}{\nu+1} \sum_{j=1}^{\infty} \left(\left(\frac{1}{\nu+1} \right)^j \int_{x^*}^1 \varrho^{j-1}(\xi)\, d\xi \sum_{k=j}^{\infty} \left(\frac{x^*}{1+\nu} \right)^{k-j} \right) \\
&= \frac{\nu}{\nu+1} \frac{1}{\nu+1-x^*} \sum_{j=1}^{\infty} \left(\left(\frac{1}{\nu+1} \right)^{j-1} \int_{x^*}^1 \varrho^{j-1}(\xi)\, d\xi \right) \\
&= \frac{1}{\nu+1-x^*} \int_{x^*}^1 \frac{\nu}{\nu+1-\varrho(\xi)}\, d\xi,
\end{aligned}$$

in other words: Since the rate λ of an arrival beyond x^* is $\lambda(1-x^*)$ (for each arrival an additional and independent Bernoulli experiment), the probability of an arrival beyond x^* is $\lambda(1-x^*)/(\mu + \lambda(1-x^*)) = (1-x^*)/(\nu+1-x^*)$. Besides the integral divided by $1-x^*$ represents the mean payoff choosing an offer beyond x^* given there is an offer beyond x^*. $\square$

Particularly $x^* = 0$ is the optimal threshold if $\mu \geq \lambda$, since in mean the process will end before an item occurs (in the proof the integrand then is lower than 1 on $[0, 1)$).

Example 3.16 Suppose $r(x) = \vartheta x$ for $x \in [0,1]$ where $\vartheta \in (0,1]$. The condition for $x^* = 0$ is $\int_0^\vartheta \frac{1}{\nu+1-\xi/\vartheta}\,d\xi + \frac{1-\vartheta}{\nu} \leq 1$ or $\vartheta \geq \frac{\nu-1}{\nu\ln(1+1/\nu)-1}$. If this is not the case $x^* \in (0,\vartheta)$ is specified by $\frac{\nu+1-x}{\nu+1-x/\vartheta} = \vartheta \ln\left(\frac{\nu+1-x/\vartheta}{\nu}\right) + \frac{1-\vartheta}{\nu}$.
Particularly $x^* = 0$ if $\nu \geq 1$ and $x^* \in (0,\vartheta)$ if $\nu < \frac{1}{e-1} \approx 0.5820$, while $x^* \in [0,\vartheta)$ in case of $\nu \in [\frac{1}{e-1}, 1)$. The value of $\mathcal{P}_{\lambda,\mu}(\vartheta x)$, i.e. the optimal probability of winning is $\frac{1-\vartheta}{\nu+1-x^*} + \frac{\nu\vartheta}{\nu+1-x^*}\ln\left(\frac{\nu+1-x^*/\vartheta}{\nu}\right)$.
This is in accordance with the case $\vartheta = 1$ resp. $r \equiv id$ of Bojdecki [6]: Here $x^* = 0$ if $\nu \geq 1/(e-1)$ and otherwise $x^* = 1 - \nu(e-1)$. The optimal probability of winning is $\frac{\nu}{\nu+1}\ln\left(\frac{\nu+1}{\nu}\right)$ and $1/e$, respectively.
Particularly take $\vartheta = 0.8$. Then $x^* = 0$ if $\nu \geq 0.6959$, approximately. For example the value is 0.3773 if $\nu = 1$ and if $\nu = 1/2$, computing $x^* \approx 0.2847$, the value is 0.4370.

4 The Duration Problem Based on r–Candidates

In this chapter the full information case of the duration problem of Ferguson et al. [14] (sections 3 and 4.2) is applied to r–candidates, where offers are compared via relax function $r \in \mathcal{R}$ (see section 2.1): On the one hand the subject in this article is to select an item in order to maximize the upcoming period until the end provided it proves to be best overall — here the item finally has to emerge as an r–candidate (an overall r–candidate). On the other hand it is the aim to select an item while proposing to own it as long as possible until the moment where it is surpassed — here the criterion will be the duration the selected item remains an r–candidate (a temporary r–candidate). The concept of an r–candidate, notation 2.10, will be specified in the corresponding subsections.
In this chapter the sequence $X_1, X_2, \ldots$ of offers is assumed to be independent and $U([0,1])$ (see the final remark 4.20), relative maxima are denoted by $Y_k := \max\{X_1, \ldots, X_k\}$ for $k \in \mathbb{N}$ and stopping times refer to this sequence. The corresponding optimal stopping problems are called duration of owning an overall resp. temporary r–candidate and they are denoted by $\mathcal{D}^o(r)$ resp. $\mathcal{D}^t(r)$ for given relax function $r \in \mathcal{R}$, where no recall as well as allowance of recall is investigated.
Anytime the case $r \equiv id$ or $r \in \mathcal{R}_1^1$ with $r'(1-) = 1$ refers to the article [14].

4.1 The Duration Problem in Discrete Time

In this section let $1 < n \in \mathbb{N}$ be fixed and regard the specification of an optimal, nonanticipating stopping time referring to the sequence $X_1, \ldots, X_n$. This discrete optimal stopping problem, which is denoted by $\mathcal{D}_n^o$ resp. $\mathcal{D}_n^t$ corresponding to an overall and a temporary r–candidate, is investigated in two subsections below, where the definition of an r–candidate will be specified formally.

4.1.1 The Duration of Owning an Overall r–Candidate

For $k = 1, \ldots, n$ the value X_k is called an *overall* r–candidate if $X_k \geq r(Y_n)$, the corresponding duration then being $n - k + 1$. Selecting according to a nonanticipating stopping time S, the duration of owning X_S concerning an overall r–candidate is $D := (n + 1 - S) \cdot \mathbf{1}_{[r(Y_n),1]}(X_S)$.

The prophet isn't restricted to nonanticipating stopping times, but has complete foresight. A gambler's duration of owning an overall r–candidate is bounded by that of the prophet, in every realization and thus also in mean:

The Prophet's Choice

For any realization the prophet elects the maximum of the durations of any overall r–candidate of $X_1, \ldots, X_n$, which in mean leads to the payoff

$$\int_0^1 \sum_{k=1}^{n} \left((y - r(y))^{k-1} r^{n-k}(y)\, k \left[n \binom{n}{k} - \binom{n}{k+1} \right] \right) dy. \tag{45}$$

Verification: The prophet would choose the first overall r–candidate, i.e. the first item x for which $x \geq r(y)$ holds, where $Y_n = y$. Decomposing then with respect to the number k of items fulfilling this property, the maximal duration is $n - i$, if there are i not–r–candidates preceding the first of them (where $i = 0, \ldots, n-k$); the remaining $k-1$ r–candidates and $n-k-i$ not–r–candidates may be in any order: $\sum_{i=0}^{n-k} (n-i) \binom{n-1-i}{k-1} = n\binom{n}{k} - \binom{n}{k+1}$ (using $\sum_{i=k-1}^{m} \binom{i}{k-1} = \binom{m+1}{k}$ repeatedly while telescoping $\sum_{i=k-1}^{j} \binom{i}{k-1}$ for $j = k-1, \ldots, n-2$, then $\binom{n}{k}$ occurs n times in place of k times). The factor k reflects the number of positions the maximum y can take. $\square$

In the case $r \equiv id$ expression (45) corresponds with $\frac{1}{n} \sum_{i=1}^{n} i = \frac{n+1}{2}$ (there is a single overall r–candidate, $k = 1$).

Example 4.1 For $n \in \mathbb{N}$ the mean of the maximal duration of any overall r–candidate of $X_1, \ldots, X_n$ with $r(x) = \vartheta x$ for $x \in [0,1]$ where $\vartheta \in [0,1)$ is given by the concave function

$$d_n(\vartheta) \;:=\; n - \frac{\vartheta}{1-\vartheta} + \frac{\vartheta\,(1-\vartheta^n)}{n(1-\vartheta)^2}.$$

Verification: Reversing sum and integration in expression (45), the integral yields $1/n$ (extracting factors), which gives

$$\sum_{k=1}^{n} k \binom{n}{k} \vartheta^{n-k} (1-\vartheta)^{k-1} - \sum_{k=1}^{n-1} \frac{k}{n} \binom{n}{k+1} \vartheta^{n-k} (1-\vartheta)^{k-1}$$

$$\begin{aligned}
&= n - \sum_{k=1}^{n-1} \frac{k}{k+1} \binom{n-1}{k} \vartheta^{n-k} (1-\vartheta)^{k-1} \\
&= n - \frac{\vartheta}{1-\vartheta} \sum_{k=1}^{n-1} \binom{n-1}{k} \vartheta^{n-1-k} (1-\vartheta)^{k} + \sum_{k=1}^{n-1} \frac{1}{k+1} \binom{n-1}{k} \vartheta^{n-k} (1-\vartheta)^{k-1} \\
&= n - \frac{\vartheta}{1-\vartheta} \left(1-\vartheta^{n-1}\right) + \frac{1}{n} \frac{\vartheta}{(1-\vartheta)^2} \left(1-\vartheta^{n} - n(1-\vartheta)\vartheta^{n-1}\right) \\
&= n - \frac{\vartheta}{1-\vartheta} + \frac{\vartheta\,(1-\vartheta^{n})}{n(1-\vartheta)^2},
\end{aligned}$$

where the second sum in the third last line needs extra computation: Write $(1-\vartheta)^{-2}(1-\vartheta)^{k+1}/(k+1)$ as an integral, and add and substract two terms to get an expression according to the binomial theorem.
The case $\vartheta = 0$ gives $d_n(0) = n$ (take the first item) and also the marginal case $\vartheta \to 1$ or $r \equiv id$ is preserved: $d_n(\vartheta) \to (n+1)/2$ as $\vartheta \to 1$ (applying two times the rule of de l'Hospital to the unified two last terms of $d_n(\vartheta)$ altogether yields $n + (2n - n(n+1)\vartheta^{n-1})/(2n)$ as $\vartheta \to 1$).
The concavity of $d_n(\vartheta)$, i.e. $d_n''(\vartheta) \le 0$, is equivalent to the nonpositivity of

$$4 - 2n + 2(n+1)\vartheta - n(n+1)\vartheta^{n-1} + 2(n-2)(n+1)\vartheta^{n} - (n-2)(n-1)\vartheta^{n+1}.$$

First $d_2'' \equiv 0$ and now let $n > 2$ and $\vartheta \in (0,1)$: $d_n''(\vartheta) \le 0$, since $d_n''(1) = 0$ and $d_n^{(3)}(\vartheta) \le 0$, where the latter in turn is true since $d_n^{(3)}(1) = 0$ and $d_n^{(4)}(\vartheta) \le 0$ — in this last inequality only consecutive exponents of ϑ appear, with factors which are integer multiples of the positive product $(n-2)(n-1)n(n+1)$. Thus inequality $d_n^{(4)}(\vartheta) \le 0$ holds iff $-1 + 2\vartheta - \vartheta^2 = -(1-\vartheta)^2 \le 0$. □

Selection without Recall

The optimal stopping problem $\mathcal{D}_n^o$ without recall seems not to be regular in general. Let $(x,y) \in \Delta$. The mean payoff choosing x with $x \ge r(y)$ is 1 if $\ell = 0$ and if $\ell \in \mathbb{N}$ items remain it is given by

$$s_\ell(x) \;=\; (\ell+1)\varrho^{\ell}(x).$$

Alternatively proceeding at least one step and then choosing optimally yields

$$c_\ell(y) \;=\; r(y)c_{\ell-1}(y) + \int_{r(y)}^{1} v_{\ell-1}(\xi, y \vee \xi)\; d\xi,$$

which leads to the estimation $yc_{\ell-1}(y) \leq c_\ell(y) - \ell \int_y^1 \varrho^{\ell-1}(\xi)\, d\xi$. Assuming $s_\ell(x) \geq c_\ell(y)$ this optimal stopping problem would be regular if

$$\begin{aligned} y\ell\varrho^{\ell-1}(x) &\geq (\ell+1)\varrho^\ell(x) - \ell \int_y^1 \varrho^{\ell-1}(\xi)\, d\xi \\ \int_y^1 \varrho^{\ell-1}(\xi)\, d\xi &\geq \left(\frac{\ell+1}{\ell}\varrho(x) - y\right)\varrho^{\ell-1}(x), \end{aligned}$$

which in turn is valid if $(\ell+1)/\ell\varrho(x) \leq 1$ iff $x \leq r(\ell/(\ell+1))$ (for $r \in \mathcal{R}_1^1$ with $r'(1-) = a$ this would require roughly $x \leq (\ell+1-a)/(\ell+1)$). Only if the reverse inequality would imply that stopping is optimal, which doesn't seem to be evident, then problem $\mathcal{D}_n^o$ without recall would be regular.

Selection with Recall

For problem $\mathcal{D}_n^o$ with recall the duration is $D := (n+1-S) \cdot \mathbf{1}_{[r(Y_n),1]}(Y_S)$ according to stopping time S. Heuristically the aim is to wait sufficiently long in order to recall an item which will be an overall r–candidate and to respect the simultaneous decrease of the (possibly) outstanding duration.

Theorem 4.2 *An optimal stopping time for $\mathcal{D}_n^o(r)$ with recall and $r \in \mathcal{R}^1$ — maximizing the duration of owning an overall r–candidate with recall — is given by $S^* := \inf\{1 \leq k \leq n : Y_k \geq y^*_{n-k}\}$, where $y_0^* = 0$ and y_ℓ^* for $\ell \in \mathbb{N}$ is the unique solution inside $[r(1/2), r(1))$ of equation*

$$\left(\frac{\ell+1}{\ell}\varrho(y) - y\right)\varrho^{\ell-1}(y) = \int_y^1 \varrho^{\ell-1}(\xi)\, d\xi. \tag{46}$$

The sequence $(y_\ell^*)_{\ell\in\mathbb{Z}_+}$ of optimal thresholds is strictly increasing.

Proof: The mean of the duration choosing $y \in [0,1]$ where $\ell = 0, \ldots, n-1$ items remain is $s_\ell(y) = (\ell+1)\varrho^\ell(y)$. The one step look–ahead rule yields

$$c_\ell(y) = ys_{\ell-1}(y) + \int_y^1 s_{\ell-1}(\xi)\, d\xi.$$

The myopic stopping time suggests to stop, if $s_\ell(y) \geq c_\ell(y)$, i.e. if

$$\begin{aligned} (\ell+1)\varrho^\ell(y) &\geq \ell y\varrho^{\ell-1}(y) + \ell\int_y^1 \varrho^{\ell-1}(\xi)\, d\xi \\ \left(\frac{\ell+1}{\ell}\varrho(y) - y\right)\varrho^{\ell-1}(y) &\geq \int_y^1 \varrho^{\ell-1}(\xi)\, d\xi. \end{aligned} \tag{47}$$

The corresponding stopping sets are monotone (let $y \in [0, r(1))$: Supposing $s_\ell(y) \geq c_\ell(y)$, i.e. inequality (47), and using $\left(\frac{\ell+1}{\ell}\varrho(y) - y\right)\varrho^{\ell-1}(y) < \left(\frac{\ell}{\ell-1}\varrho(y) - y\right)\varrho^{\ell-2}(y)\varrho(y)$ and $\int_y^1 \varrho^{\ell-1}(\xi)\, d\xi \geq \varrho(y)\int_y^1 \varrho^{\ell-2}(\xi)\, d\xi$, the desired inequality $s_{\ell-1}(y) > c_{\ell-1}(y)$ follows (dividing by $\varrho(y) > 0$), where $1 < \ell \in \mathbb{N}$. Let $\ell \in \mathbb{N}$ and let $h(y) := \int_y^1 \varrho^{\ell-1}(\xi)\, d\xi - \left(\frac{\ell+1}{\ell}\varrho(y) - y\right)\varrho^{\ell-1}(y) \in C^1([0, r(1)])$. Now $h(y)$ is decreasing, because $[(\ell-1)y - (\ell+1)\varrho(y)]\,\varrho^{\ell-2}(y)\varrho'(y) \leq 0$ iff $h'(y)$ is nonpositive. Besides $h(0) = \int_0^1 \varrho^{\ell-1}(\xi)\, d\xi > 0$ and $h(r(1)) = -1/\ell$. Particularly for $\ell = 1$ an optimal decision is given via threshold $y_1^* := r(1/2)$. Therefore a unique threshold $y_\ell^* \in [r(1/2), r(1))$ is specified by equation (46). Thus the stopping sets of the myopic stopping time are closed and realizable (since Y_k is nondecreasing). Besides monotonicity $y_1^* < y_2^* < \cdots$ holds. □

While evidently y_ℓ^* tends to $r(1)$ as $\ell \to \infty$ the asymptotic behaviour of second order is as follows:

Proposition 4.3 *Let $r \in \mathcal{R}_1^1$ and $a := r'(1-) \in [1, \infty)$. The asymptotic behaviour of optimal thresholds of theorem 4.2 is $\lim_{n\to\infty} \ell(1 - y_\ell^*) = a\alpha$, where $\alpha = \alpha(a)$ represents the unique solution inside $(0, \ln(2)]$ of equation*

$$a \;=\; \frac{\alpha - 1}{\alpha + 1 - e^\alpha}.$$

Proof: Assume $r(y) = a(y-1) + 1$ or $\varrho(y) = (y + a - 1)/a$ in the neighbourhood of 1. Then equation (46) yields an asymptotic relation as $\ell \to \infty$:

$$\left(\frac{\ell+1}{\ell}\frac{y_\ell^* + a - 1}{a} - y_\ell^*\right)\left(\frac{y_\ell^* + a - 1}{a}\right)^{\ell-1} \simeq \frac{a}{\ell}\left[\left(\frac{\xi + a - 1}{a}\right)^{\ell}\right]_{y_\ell^*}^1$$

$$\frac{y_\ell^*(a + \ell + 1 - a\ell) + (a + \ell + 1)(a - 1)}{a\ell}\left(\frac{y_\ell^* + a - 1}{a}\right)^{\ell-1} \simeq \frac{a}{\ell}.$$

In order to get a second order specification of y_ℓ^* set $\varepsilon_\ell := y_\ell^* - 1 + f(a)/\ell$ for $\ell \in \mathbb{N}$, where $f : [1, \infty) \to [0, \infty)$ with $f(1) = \ln 2$ (heuristically and finally the optimal threshold y_ℓ^* should decrease as a increases, i.e. as the demands relax, thus $f(a)$ should be increasing and its range being contained in $[\ln 2, \infty)$, see below). This produces, applying $(y_\ell^* + a - 1)/a \simeq 1 - f(a)/(a\ell) + \varepsilon_\ell/a$, the following relation:

$$\frac{(f(a) - \ell\varepsilon_\ell)\left(a - 1 - \frac{a+1}{\ell}\right) + a^2 + a}{a\ell}\left(1 - \frac{f(a) + \ell\varepsilon_\ell}{a\ell}\right)^{\ell-1} \;\simeq\; \frac{a}{\ell}.$$

Cancel $1/\ell$, drop term $f(a)(a+1)/\ell = o(1)$ and suppose $\varepsilon_\ell = o(1/\ell)$ as $\ell \to \infty$, then this leads to a transcendental equation, which specifies $f(a)$ (its uniqueness, i.e. that of y_ℓ^*, will justify the assumption concerning ε_ℓ):

$$\frac{f(a)}{a}(a-1) + a + 1 \quad = \quad a e^{f(a)/a},$$

preserving the case $f(1) = \ln(2)$. Define $g(a) := f(a)/a$ for $a \in [1, \infty)$. Now g is positive and g is differentiable if f is supposed to be. Then

$$g(a)(a-1) + a + 1 \quad = \quad a e^{g(a)}$$

or $G(a, g(a)) := g(a)(a-1) + a + 1 - a e^{g(a)} \equiv 0$, which by derivation reveals $g'(a) = (1 + g(a) - e^{g(a)})/(1 - a + a e^{g(a)})$. Since this term is negative, g is decreasing with range contained in $(0, \ln(2)]$. Substituting its inverse — take $g(a) = \alpha$ and $a = h(\alpha)$ — yields the expression

$$h(\alpha) \quad = \quad \frac{\alpha - 1}{\alpha + 1 - e^\alpha},$$

where $h : (0, \ln(2)] \to [1, \infty)$ is decreasing with $h(\ln(2)) = 1$ and $h(0+) = \infty$ (see figure 6 on page 100). Thus the assumption $\varepsilon_\ell = o(1/\ell)$ is justified.
Resumed this means: Given a, specify α solving equation $h(\alpha) = a$. Then $h(\alpha)\alpha = a g(a) = f(a)$ is the desired value to build $y_\ell^* \simeq 1 - a\alpha/\ell$.
Besides f proves to be increasing: $f'(a) = g(a) + a g'(a)$, which is positive iff $(1 + g(a)/a)/(1 - g(a)) > e^{g(a)}$ (valid by estimating $g(a)/a > 0$). Particularly $f'(1) = g(1) + g'(1) = \ln 2 + 1/h'(\ln 2) = (3\ln(2) - 1)/2 \approx 0.5397$. □

Example 4.4 Let $r(y) = y^5$ for $y \in [0,1]$, i.e. let $a = r'(1-) = 5$. The following optimal thresholds y_ℓ^* (based on theorem 4.2) and approximating thresholds $y_\ell := 1 - a\alpha/\ell$ (based on $\alpha(5) \approx 0.4391$, proposition 4.3) result:

ℓ	10	20	30	40	50	100
y_ℓ^*	0.7952	0.8939	0.9285	0.9460	0.9567	0.9782
y_ℓ	0.7805	0.8902	0.9268	0.9451	0.9561	0.9781

4.1.2 The Duration of Owning a Temporary r–Candidate

The duration of owning a *temporary* r–candidate for a $k = 1, \ldots, n$ means the period object X_k with $X_k \geq r(Y_k)$ stays an r–candidate with regard to the remaining sequence $Y_{k+1}, \ldots, Y_n$. Selecting according to a nonanticipating stopping time S the duration D of owning a temporary r–candidate is given by the number of time units the present r–candidate X_S remains an r–candidate: $D := \sum_{j=0}^{n-S} \mathbf{1}_{\left[r\left(Y_{S+j}\right),1\right]}(X_S)$.

The prophet's choice in this context and the corresponding mean of the maximal duration of a temporary r–candidate seems awkward and inaccessible.

Selection without Recall

The optimal stopping problem $\mathcal{D}_n^t(r)$ without recall doesn't seem to be regular in general. It is related to the problem $\mathcal{P}(r)$ of selecting an r–candidate where the number of values presented is uniformly distributed on $\{1, \ldots, n\}$: The mean payoff of stopping in state $(x, y) \in \Delta$ with $\ell \in \mathbb{Z}_+$ remaining items is evidently $\frac{1}{\ell+1} \sum_{j=0}^{\ell} \varrho^j(x)$ provided $x \geq r(y)$. Apart from factor $\frac{1}{\ell+1}$ this coincides with the mean payoff of $\mathcal{D}_n^t$ given in expression (49) below (replace y by x and presume $x \geq r(y)$).

Selection with Recall

For $\mathcal{D}_n^t(r)$ with recall the number of time units Y_S stays an r–candidate matters, formally the mean of $D := \sum_{j=0}^{n-S} \mathbf{1}_{\left[r\left(Y_{S+j}\right),1\right]}(Y_S)$ is the relevant functional.

Theorem 4.5 *An optimal stopping time for $\mathcal{D}_n^t(r)$ with recall where $r \in \mathcal{R}^1$ — maximizing the duration of owning a temporary r–candidate with recall — is given by $S^* := \inf\{1 \leq k \leq n \,:\, Y_k \geq y_{n-k}^*\}$, where $y_0^* = y_1^* = 0$ and y_ℓ^* for $1 < \ell \in \mathbb{N}$ is the unique solution inside $(0, r(1))$ of equation*

$$\varrho^\ell(y) + (1-y) \sum_{j=0}^{\ell-1} \varrho^j(y) \;\; = \;\; \sum_{j=0}^{\ell-1} \int_y^1 \varrho^j(\xi)\, d\xi. \qquad (48)$$

The sequence $(y_\ell^*)_{\ell \in \mathbb{N}}$ of optimal thresholds is strictly increasing.

Proof: Suppose present maximum $y \in [0,1]$ where $\ell \in \mathbb{Z}_+$ offers remain, then the mean duration of owning y as an overall r–candidate is

$$s_\ell(y) \;=\; (\ell+1)\varrho^\ell(y) + (1-\varrho(y))\sum_{j=0}^{\ell-1}(j+1)\varrho^j(y) \;=\; \sum_{j=0}^{\ell}\varrho^j(y), \quad (49)$$

by decomposing with respect to the number $j+1$ of the item finishing the beginning duration. For $\ell \in \mathbb{N}$ now let $c_\ell(y)$ represent the mean payoff of the one step look–ahead rule, i.e. the mean duration choosing $Y_{n-\ell+1}$:

$$c_\ell(y) \;=\; y s_{\ell-1}(y) + \int_y^1 s_{\ell-1}(\xi)\, d\xi.$$

The myopic stopping time suggests to stop if $s_\ell(y) \geq c_\ell(y)$, i.e. if

$$\sum_{j=0}^{\ell}\varrho^j(y) \;\geq\; y\sum_{j=0}^{\ell-1}\varrho^j(y) + \sum_{j=0}^{\ell-1}\int_y^1 \varrho^j(\xi)\, d\xi$$

$$\varrho^\ell(y) + (1-y)\sum_{j=0}^{\ell-1}\varrho^j(y) \;\geq\; \sum_{j=0}^{\ell-1}\int_y^1 \varrho^j(\xi)\, d\xi. \quad (50)$$

The corresponding stopping sets are monotone: Let $1 < \ell \in \mathbb{N}$, $y \in [0, r(1))$ and suppose $s_\ell(y) \geq c_\ell(y)$ (evidently for $y \geq r(1)$ stopping anytime). Using both $\varrho^\ell(y) < \varrho^{\ell-1}(y)$ and $\varrho(\xi) \geq \varrho(y)$ then inequality (50) leads to

$$\varrho^{\ell-1}(y) + (1-y)\sum_{j=0}^{\ell-1}\varrho^j(y) \;>\; \sum_{j=0}^{\ell-2}\int_y^1 \varrho^j(\xi)\, d\xi + (1-y)\varrho^{\ell-1}(y),$$

which is equivalent to $s_{\ell-1}(y) > c_{\ell-1}(y)$ (substracting $(1-y)\varrho^{\ell-1}(y)$).
Let $h(y) := \sum_{j=0}^{\ell-1}\int_y^1 \varrho^j(\xi)\ d\xi - \varrho^\ell(y) + (1-y)\sum_{j=0}^{\ell-1}\varrho^j(y) \in C^1([0,r(1)])$ where $\ell \in \mathbb{N}$. Now $h(y)$ is decreasing, since $h'(y)$ proves to be nonpositive iff $-\left[\ell\varrho^{\ell-1}(y) + (1-y)\sum_{j=1}^{\ell-1} j\varrho^{j-1}(y)\right]\varrho'(y) \leq 0$. Moreover $h(r(1)) = -1$ and $h(0) = \sum_{j=1}^{\ell-1}\varrho^j(\xi)\ d\xi > 0$ if $\ell > 1$ and $h(0) = 0$ if $\ell = 1$ (in the latter case even $y = 0$ could be chosen to stay optimal). Therefore a unique threshold $y_\ell^* \in (0, r(1))$ is specified by equation (48).
Thus the myopic stopping sets are closed and realizable (Y_k is nondecreasing) and monotonicity $y_0^* = y_1^* < y_2^* < \cdots$ holds. □

Obviously $y_\ell^* \nearrow r(1)$ as $\ell \to \infty$ while the second order behaviour is as follows:

Proposition 4.6 *Let $r \in \mathcal{R}_1^1$ and $a := r'(1-) \in [1,\infty)$. The asymptotic behaviour of optimal thresholds of theorem 4.5 is $\lim_{\ell\to\infty} \ell(1-y_\ell^*) = a\alpha$, where $\alpha = \alpha(a)$ represents the unique solution of equation*

$$1 - \frac{a-1}{ae^\alpha} \quad = \quad \int_{-\alpha}^0 \frac{e^\xi - 1}{\xi}\, d\xi \tag{51}$$

inside $(0,\alpha_1]$, where $\alpha_1 := \alpha(1) \approx 1.3450$ is the solution of $1 = \int_{-\alpha}^0 \frac{e^\xi-1}{\xi}\, d\xi$.

Proof: Regard equation (48) and suppose $\varrho(y) = (y-1+a)/a$ for y close to 1. Then the asymptotic behaviour of y_ℓ^* as $\ell \to \infty$ is given by

$$\left(\frac{y_\ell^* + a - 1}{a}\right)^\ell + (1-y_\ell^*)\sum_{j=0}^{\ell-1}\left(\frac{y_\ell^*+a-1}{a}\right)^j \simeq \sum_{j=0}^{\ell-1}\frac{a}{j+1}\left(1-\left(\frac{y_\ell^*+a-1}{a}\right)^{j+1}\right).$$

Assuming $y_\ell^* \simeq 1 - f(a)/\ell$ this yields (setting $y_\ell^* = 1 - f(a)/\ell + \varepsilon_\ell$ as in proposition 4.3 again justifies $\varepsilon_\ell = o(1/\ell)$ by exactly the same arguments)

$$\left(1-\frac{f(a)}{a\ell}\right)^\ell + \frac{f(a)}{\ell}\sum_{j=0}^{\ell-1}\left(1-\frac{f(a)}{a\ell}\right)^j \;\simeq\; \sum_{j=0}^{\ell-1}\frac{a}{j+1}\left(1-\left(1-\frac{f(a)}{a\ell}\right)^{j+1}\right).$$

The asymptotic behaviour as $\ell \to \infty$ of either leftmost term is evident, $\sum_{j=0}^{\ell-1}(1-f(a)/(a\ell))^j \simeq \left(1-e^{-f(a)/a}\right) a\ell/f(a)$ according to the truncated geometric series and $\sum_{j=0}^{\ell-1}\frac{1}{j+1}\left(1-\frac{f(a)}{a\ell}\right)^{j+1} \simeq \mathrm{Ei}(-f(a)/a) - \ln\left(f(a)/(a\ell)\right)$ according to lemma A.2 in the appendix (set $x_\ell := -f(a)/a$). This yields

$$e^{-f(a)/a} + a\left(1-e^{-f(a)/a}\right) \quad\simeq\quad a\left(\ln(\ell) + \gamma + \ln\left(\frac{f(a)}{a\ell}\right) - \mathrm{Ei}\left(-\frac{f(a)}{a}\right)\right).$$

Cancel $\ln(\ell)$, introduce $g(a) := f(a)/a$ (for $a \in [1,\infty)$ with range contained in $(0,\infty)$) and respect remark A.4, then equation (51) results from

$$a + (a-1)e^{-g(a)} \quad = \quad a\int_0^{g(a)} \frac{1-e^{-\xi}}{\xi}\, d\xi,$$

where $f(a) = ag(a)$. Now analogously to the case of an overall r–candidate the inverse of g is displayed (g is differentiable if f is supposed to be): $G(a, g(a)) \;:=\; 1 - \frac{a-1}{a}e^{-g(a)} - \int_0^{g(a)} \frac{1-e^{-\xi}}{\xi}\, d\xi \;\equiv\; 0$, which by derivation reveals

$-\frac{e^{-g(a)}}{a^2} + g'(a)\left(\frac{a-1}{a}e^{-g(a)} - \frac{1-e^{-g(a)}}{g(a)}\right) \equiv 0$ or $g'(a) = 1 \Big/ \left(a^2 - a - a^2\frac{e^{g(a)}-1}{g(a)}\right)$, which is negative since g is positive. Thus g is decreasing and the inverse of $g(a) = \alpha$ denoted by $h(\alpha) = a$ exists (also decreasing, with $h(0+) = \infty$):

$$h(\alpha) \quad = \quad 1 \Big/ \left(1 - e^{\alpha}\left(1 - \int_0^{\alpha} \frac{1-e^{-\xi}}{\xi}\, d\xi\right)\right).$$

Thus, given a, specify α solving $h(\alpha) = a$. Then $h(\alpha)\alpha = ag(a) = f(a)$ yields the desired coefficient of the term $y_\ell^* \simeq 1 - a\alpha/\ell$ as $\ell \to \infty$.
Besides $f(a)$ may be increasing: $f'(a) = g(a) + ag'(a)$ proves to be positive iff $1 + g(a) + (1 - g(a))/a < e^{g(a)}$, which is evident only if $g(a) \geq 1$. Particularly $f'(1) = g(1) + g'(1) = \alpha_1 + 1/h'(\alpha_1) = \alpha_1(e^{\alpha_1} - 2)/(e^{\alpha_1} - 1) \approx 0.8711$.
Finally $\int_{-\alpha_1}^{0}(e^{\xi} - 1)/\xi\, d\xi = \int_{-1}^{0}(e^{\alpha_1\xi} - 1)/\xi\, d\xi = \int_0^1(1 - e^{-\alpha_1\xi})/\xi\, d\xi$. $\square$

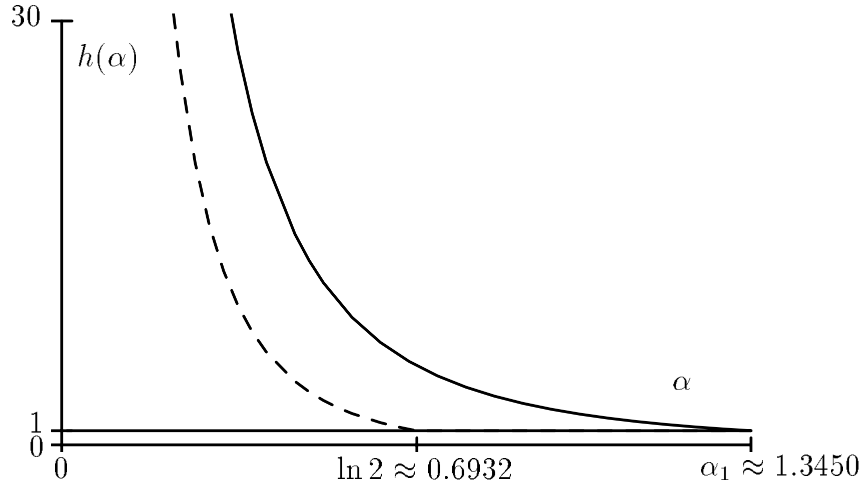

Figure 6: Function $h(\alpha)$ for $\mathcal{D}_n^o$ resp. $\mathcal{D}_n^t$ with recall (dashed resp. solid line) reveals $\lim_{\ell\to\infty} \ell(1 - y_\ell^*) = f(a)$ via $f(a) = \alpha h(\alpha)$ and $a = h(\alpha)$. Particularly $f(1) \approx 0.6932$ resp. 1.3450 and besides $f'(1) \approx 0.5397$ resp. 0.8711. See the proof of proposition 4.3 resp. 4.6.

Example 4.7 Take $r(y) = y^5$ for $y \in [0, 1]$, i.e. take $a = r'(1-) = 5$. For some ℓ the optimal thresholds y_ℓ^* according to theorem 4.5 and approximations $y_\ell := 1 - a\alpha/\ell$ where $\alpha(5) \approx 0.7261$, see proposition 4.6, are given:

ℓ	10	20	30	40	50	100
y_ℓ^*	0.6731	0.8275	0.8830	0.9115	0.9288	0.9641
y_ℓ	0.6370	0.8185	0.8790	0.9092	0.9274	0.9640

4.2 The Discounted Duration of Owning an r–Candidate

Suppose to maximize the duration of owning an r–candidate where future epochs are discounted by an amount of $\delta \in (0,1)$ per period. For relax function $r \in \mathcal{R}^1$ this socalled discounted duration problem is denoted by $\mathcal{D}_\delta(r)$. Heuristically it is advisible to watch for a big item whose quality as an r–candidate will last but on the other hand not to hesitate too long, since the valence of periods decrease.
Suppose first that recall is allowed such that formally it is the objective to maximize random variable $D := (1-\delta)\sum_{j=1}^{\infty}\delta^j \mathbf{1}_{\left[r\left(Y_{S+j}\right),1\right]}(Y_S)$, where selection according to stopping time S is applied (factor $1-\delta$ for standardization; set payoff 0 if $S=\infty$). The mean duration selecting resp. recalling object $Y_k = y$ in epoch $k \in \mathbb{N}$ then is given by

$$\begin{aligned} s_k(y) &= (1-\delta)\delta^k \left(1+\sum_{j=1}^{\infty}\delta^j \mathrm{P}\left(y \geq r(X_{k+1}),\ldots,r(X_{k+j})\right)\right) \\ &= \frac{(1-\delta)\delta^k}{1-\delta\varrho(y)} \end{aligned} \tag{52}$$

(which simplifies if $y \in [r(1),1]$). According to the statements on page 13 (and boundedness of the payoff by 1) the following stopping time is optimal:

$$S^* \;:=\; \inf_{k\in\mathbb{N}}\left\{ s_k(Y_k) \geq \operatorname*{ess\,sup}_{S\in\mathcal{S}^k} \mathrm{E}\left(s_S(Y_S) \mid X_1,\ldots,X_k\right)\right\}. \tag{53}$$

Let $w(y)$ denote the value of the duration with discounting, if stopping below $y \in [0,1]$ is avoided. $w(y)$ is nonincreasing in y since the set of stopping times allowed shrinks. Now time invariance holds in the following sense: $\operatorname{ess\,sup}_{S\in\mathcal{S}^k}\mathrm{E}\left(s_S(Y_S) \mid X_1,\ldots,X_k\right) = \delta^k w(Y_k)$. Applying this to the inequality given in (53), using (52) and setting $Y_k = y$ this leads to the relation

$$\frac{1-\delta}{1-\delta\varrho(y)} \;\geq\; w(y).$$

Since the left side is increasing in y and the right side is nonincreasing in y and since this relation is independent of the epoch y occurs, a concurrent threshold rule proves to be optimal — this in turn is based on the X_k and therefore this concurrent threshold rule will also be optimal for the discounted duration problem without recall.

Theorem 4.8 *For the discounted duration problem $\mathcal{D}_\delta(r)$ with or without recall, where $r \in \mathcal{R}^1$ and $\delta \in (0,1)$ is the discount factor, the stopping time $S^* := \inf\{k \in \mathbb{N} : X_k \geq x^*\}$ (with $\inf_\emptyset := \infty$) is optimal — take the first value above x^*. Here $x^* \in [0, r(1))$ denotes the unique solution inside $(0, r(1))$ of*

$$\frac{\delta}{1-\delta x} \;=\; -\frac{\partial}{\partial x}\ln\left(\int_x^1 \frac{1}{1-\delta\varrho(\xi)}\,d\xi\right) \tag{54}$$

(or of equation (56) below) if $\mu(\delta) := \int_0^1 1/\left(1/\delta - \varrho(\xi)\right)\,d\xi > 1$ and $x^* = 0$ (choose the first item) if $\mu(\delta) \leq 1$.
The value of $\mathcal{D}_\delta(r)$ — the mean $\mathrm{E}\,(D^*)$ of the optimal discounted duration D^* applying S^* — is given by $(1-\delta)/(1-\delta\varrho(x^*))$ and $(1-\delta)\mu(\delta)$, respectively.

Proof: The optimality of a concurrent threshold rule S is verified in the introduction. Supposing concurrent threshold $x \in [0, r(1))$ first $X_S \sim U([x,1])$ holds and second $\mathrm{E}\left(\delta^S\right) = (1-x)\sum_{j=1}^\infty \delta^j x^{j-1} = (1-x)\delta/(1-\delta x)$, leading to the mean duration $\mathrm{E}\,(D)$ according to S (using expression (52)):

$$\begin{aligned}
\mathrm{E}\,(D) &= \mathrm{E}\left(\frac{(1-\delta)\delta^S}{1-\delta\varrho(X_S)}\right)\\
&= \frac{1-\delta}{1-x}\mathrm{E}\left(\delta^S\right)\int_x^1 \frac{1}{1-\delta\varrho(\xi)}\,d\xi\\
&= \frac{(1-\delta)\delta}{1-\delta x}\int_x^1 \frac{1}{1-\delta\varrho(\xi)}\,d\xi.
\end{aligned} \tag{55}$$

Supposing $x > 0$, a necessary condition for maximal payoff is that its derivative

$$\frac{(1-\delta)\delta^2}{(1-\delta x)^2}\int_x^1 \frac{1}{1-\delta\varrho(\xi)}\,d\xi - \frac{(1-\delta)\delta}{1-\delta x}\frac{1}{1-\delta\varrho(x)}$$

with respect to x vanishes, which proves to be equivalent to

$$\int_x^1 \frac{1}{1-\delta\varrho(\xi)}\,d\xi \;=\; \frac{1}{\delta}\,\frac{1-\delta x}{1-\delta\varrho(x)}. \tag{56}$$

Let $h(x) := \int_x^1 \frac{1}{1-\delta\varrho(\xi)}\,d\xi - \frac{1}{\delta}\frac{1-\delta x}{1-\delta\varrho(x)} \in C^1([0, r(1)])$. Now h is decreasing, since $h'(x) = -\varrho'(x)(1-\delta x)/(1-\delta\varrho(x))^2$. If $h(0) \leq 0$, then $x = 0$ (selecting the first item) is an optimal threshold, since positive thresholds lead to minor payoff.

If $h(0) > 0$, then $h(r(1)) = -1/\delta$ combined with continuity of h ensures uniqueness of an optimal threshold in $(0, r(1))$ and $h'(x) \leq 0$ is sufficient for the existence of a maximum. Altogether this shows, that a unique optimal threshold inside $[0, r(1))$ exists.
The mean $\mathrm{E}(D^*)$ for $x^* > 0$ is computed by applying identity (56) to expression (55) and for $x^* = 0$ it is given directly by expression (55). □

Remark 4.9

i) For fixed δ an optimal threshold x^* is invariant inside the set of functions $r \in \mathcal{R}^1$, where ϱ varies on $(0, x^*)$ and remains unchanged on $[x^*, 1]$.

ii) Given relax function $r \in \mathcal{R}^1$, there is a crucial $\delta_0 = \delta_0(r) \in (1/2, 1-1/e]$ such that $x^*(\delta) = 0$ for $\delta \in (0, \delta_0]$: Since function $\frac{1}{1/\delta - x}$ is increasing in $x \in [0,1]$, the relation $\varepsilon_1 \prec r \preceq id$ yields

$$-\ln(1-\delta) = \int_0^1 \frac{1}{\frac{1}{\delta} - \xi}\, d\xi \leq \int_0^1 \frac{1}{\frac{1}{\delta} - \xi}\, dr(\xi) < \int_0^1 \frac{1}{\frac{1}{\delta} - \xi}\, d\varepsilon_1(\xi) = \frac{\delta}{1-\delta}$$

where the leftmost term doesn't exceed 1 for $\delta \leq \delta_0(id) = 1 - 1/e \approx 0.6321$ and the rightmost term remains lower or equal to 1 if $\delta \in (0, 1/2]$. Besides $\delta_0(r_1) < \delta_0(r_2)$ holds generally if $r_1 \prec r_2$ in $\mathcal{R}^1$.

iii) Given relax function $r \in \mathcal{R}^1$, the optimal threshold $x^*(\delta)$ is nondecreasing in δ (with regard to ii) above increasing only on $[\delta_0, 1]$) — heuristically: Weaker discounting (growing δ) allows upgrading the requirements, i.e. x^*. Furthermore it is heuristically evident, that $x^*(\delta) \to r(1)$ as $\delta \to 1$. Besides $r_1 \preceq r_2$ in $\mathcal{R}^1$ doesn't imply $x^*(r_1) \leq x^*(r_2)$, as figure 7 indicates.

iv) The mean $\mathrm{E}(D^*)$ of the optimal discounted duration D^* particularly for $\delta = \delta_0$ simplifies to $1 - \delta_0$ (since $\mu(\delta_0) = 1$). For $r \equiv id$ in the case $0 < x^* = (1 - e(1-\delta))/\delta$ the mean reduces to $1/e$ independent of δ.

Example 4.10 Some special cases treated here correspond to figure 7 below.

i) Let $r(x) = \vartheta x$ for $x \in [0,1]$ where $\vartheta \in (0,1]$. Then $\delta_0 = f^{-1}(\vartheta)$ where f^{-1} is the unique inverse function of $f : (1/2, 1-1/e] \to (0,1]$ setting $f(\delta) = (2\delta - 1)/(\delta + (1-\delta)\ln(1-\delta))$ — here $f(1/2) = 0$, $f(1-1/e) = 1$

and $f \in C^1([1/2, 1-1/e])$ and $f'(\delta) > 0$ for $\delta \in (1/2, 1-1/e]$.
Particularly for $\vartheta = 1/2$, where $\delta_0(x/2) \approx 0.5471$ results, the optimal threshold $x^*(\delta)$ for $\delta \in (\delta_0, 1)$ is determined uniquely by equation
$\int_x^\vartheta \frac{1}{1-\delta\xi/\vartheta}\, d\xi + \frac{1-\vartheta}{1-\delta} = \frac{\vartheta}{\delta}\left[\ln\left(1-\frac{\delta\xi}{\vartheta}\right)\right]_x^\vartheta + \frac{1-\vartheta}{1-\delta} = \frac{1}{\delta}\frac{1-\delta x}{1-\delta x/\vartheta}$.
The case $\vartheta = 1$: $\delta_0(id) = 1-1/e$ and $x^*(\delta) = 0 \vee \frac{1-e(1-\delta)}{\delta}$ (dashed line).

ii) Let $r(x) = 1-\sqrt{1-x}$, $x \in [0,1]$. Then $\delta_0(1-\sqrt{1-x}) \approx 0.5747$ and the optimal threshold $x^*(\delta)$ for $\delta \in (\delta_0, 1)$ is determined by equation $\int_0^{1-x} \frac{1}{1-\delta+\delta\xi^2}\, d\xi = \arctan\left((1-x)\sqrt{\frac{\delta}{1-\delta}}\right)\Big/\sqrt{\delta(1-\delta)} = \frac{1}{\delta}\frac{1-\delta x}{1-\delta+\delta(1-x)^2}$.

iii) Suppose relax function $r(x) = x^3$, $x \in [0,1]$, which leads to $\delta_0(x^3) \approx 0.5610$ and the optimal threshold $x^*(\delta)$ for $\delta \in (\delta_0, 1)$ is determined by $\int_x^1 \frac{1}{1-\delta\sqrt[3]{\xi}}\, d\xi = \frac{1}{2\delta^3}\left[-3\delta\xi(2+\delta\xi) - 6\ln(1-\delta\xi)\right]_{\sqrt{x}}^1 = \frac{1}{\delta}\frac{1-\delta x}{1-\delta\sqrt[3]{x}}$.

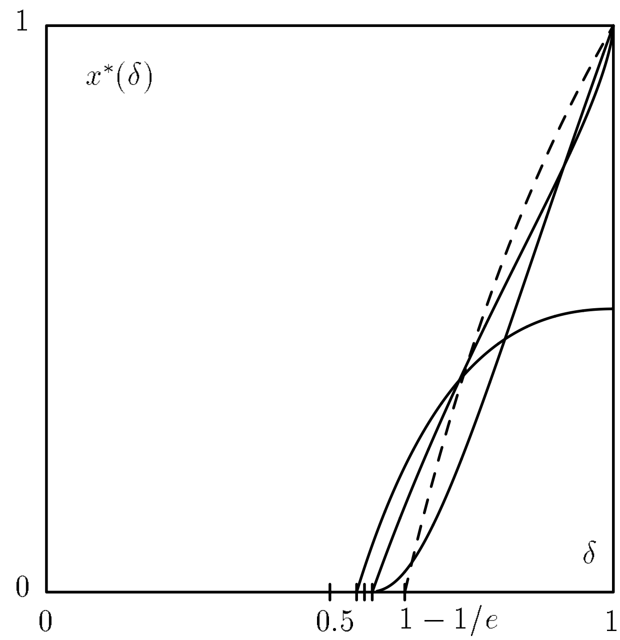

Figure 7: The optimal threshold $x^*(\delta)$ as a function of the discount factor δ, where relax function $r(x)$ equals x (dashed line) or $x/2$ resp. $1-\sqrt{1-x}$ resp. x^3 (solid lines, see example 4.10 i), ii) and iii), in the lower part of the figure from left to right, respectively).

4.3 The Duration Problem Referring to the Poisson Process

In this section offers arrive according to a Poisson process and the time horizon up to which a gambler may select an offer is constant or exponentially distributed. For relax function $r \in \mathcal{R}$, analogously to the discrete case in section 4.1, the objective is to maximize the duration of owning an overall r–candidate resp. a temporary r–candidate, specified and treated in two subsections below — see figure 8 for an illustration of the difference.
An abstract of notations, analogue to chapter 3: A random number N of offers $X_1, \dots, X_N$ arrive at times $B_1, \dots, B_N$ and T denotes the time horizon. $X_1, X_2, \dots$ are assumed to be $U([0,1])$, $Y_k := X_1 \vee \cdots \vee X_k$ and $B_k := A_1 + \cdots + A_k$ for $k \in \mathbb{N}$, where $A_1, A_2, \dots$ are iid and exponentially distributed with parameter $\lambda > 0$. The time horizon T is assumed to be constant (without loss of generality equal to 1) or exponentially distributed with rate $\mu > 0$. $X_1, X_2, \dots, A_1, A_2, \dots, T$ are assumed to be independent. N_t denotes the number of items arriving during time interval $[0,t]$ for $t \in \mathbb{R}_+$ while $N :\equiv N_T$ denotes the total number of items offered, $\mathrm{P}\left(N \in \mathbb{Z}_+\right) = 1$. The history is $\mathcal{F} := (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, where $\mathcal{F}_t := \sigma\left(X_0, \dots, X_{N_t}; A_0, \dots, A_{N_t}; N_t\right)$ for $t \in \mathbb{R}_+$. Let $\mathcal{S}$ denote the set of stopping times with respect to $\mathcal{F}$ with the possible restriction to times of arrivals, as specified below. For marginal cases set $X_0 := X_\infty := 0$, $Y_0 := 2$, $Y_\infty := 1$, $B_0 := 0$, $N_\infty := \infty$ and set $r(2) := 1$ for any $r \in \mathcal{R}$.

In this situation for $r \in \mathcal{R}$ the problem of maximizing the duration of owning an overall resp. temporary r–candidate is denoted by $\mathcal{D}^o_\lambda$ resp. $\mathcal{D}^t_\lambda$ if the horizon is equal to 1. If the horizon is exponentially distributed with parameter $\mu > 0$ it is called $\mathcal{D}^o_{\lambda,\mu}$ resp. $\mathcal{D}^t_{\lambda,\mu}$.
Three kinds of access are considered for problem $\mathcal{D}^o_\lambda$ and $\mathcal{D}^t_\lambda$: No recall (concerning the discrete time Markov process $(B_k, X_k, Y_k)_{1 \le k \le N}$), event time recall (concerning the discrete time Markov process $(B_k, Y_k)_{1 \le k \le N}$) and permanent recall (concerning the continuous time Markov process $(t, Y_{N_t})_{t \in [0,T]}$). Recall proves to be unessential for problem $\mathcal{D}^o_{\lambda,\mu}$ and $\mathcal{D}^t_{\lambda,\mu}$. Any process is equipped with an initial state α_0 and an absorbing final state α_∞ and payoff of stopping in α_0 or α_∞ is defined to be 0. Stopping times for discrete time processes are restricted to the times of arrivals, while for continuous time they aren't restricted.

According to the sketch of optimal stopping on pages 12f the myopic stopping time which refers to a sequence of offers is optimal if its stopping sets prove to be closed and realizable. The latter is valid because the final state is reached with probability 1, due to the final state and $\mathrm{P}(N < \infty) = 1$. For the continuous time process the infinitesimal look–ahead rule is considered. Now for later reference some main terms for fixed horizon are resumed:

Lemma 4.11 *Let $r \in \mathcal{R}$ and let problem $\mathcal{D}^o_\lambda(r)$ or $\mathcal{D}^t_\lambda(r)$ be given. Let $s(t,x,y)$ resp. $s(t,y)$ represent, dependent on no recall or recall, the corresponding mean duration of stopping with $(x,y) \in \Delta$ resp. $y \in [0,1]$ with remaining time $t \in [0,1]$. Three kinds of access of a gambler are considered:*

i) If recall is not allowed then the problem is regular if

$$\frac{\partial}{\partial t} s(t,x,y) \;\; \leq \;\; -\lambda(1-y)s(t,x,y) + \lambda \int_y^1 s(t,\xi,\xi)\, d\xi. \qquad (57)$$

ii) If event time recall is allowed (concerning the discrete time Markov process) then the one step look–ahead rule suggests to stop if

$$s(t,y) \;\; \geq \;\; \lambda \int_0^t e^{-\lambda(t-u)} \left(y s(u,y) + \int_y^1 s(u,\xi)\, d\xi \right) du \qquad (58)$$

and is optimal if the corresponding stopping sets prove to be closed.

iii) If permanent recall is allowed (concerning the continuous time Markov process) then the infinitesimal look–ahead rule proposes to stop if

$$\frac{\partial}{\partial t} s(t,y) \;\; \geq \;\; -\lambda(1-y)s(t,y) + \lambda \int_y^1 s(t,\xi)\, d\xi \qquad (59)$$

and it is optimal provided the corresponding stopping sets are closed.

Proof: In this section function $s(t,x,y)$ resp. $s(t,y)$ proves to be differentiable in t, bounded and continuous. For problem $\mathcal{D}^o_\lambda$ and $\mathcal{D}^t_\lambda$ in case of event $[S \geq 1]$ the declaration $S := 1$ can be made with resulting duration 0.

i) The development is presented in the proof of lemma 3.3 of section 3.1.2.

ii) Compare the mean duration recalling $y \in [0,1]$ and the mean payoff of proceeding one item (if any) and then recalling the topical maximum:

$$s(t,y) \;\geq\; \lambda \int_{1-t}^{1} \left(\int_0^1 s(1-w, y \vee \xi) e^{-\lambda(w-(1-t))} \, d\xi \right) \, dw,$$

where the arrival time of the next offer ξ is called w, the rate of an arrival being $\lambda(w-(1-t))$, and the maximum of y and the present offer ξ can be selected. This results in inequality (58). Through this an optimal stopping time is specified according to the general approach of optimal stopping on pages 12f (based on Cowan and Zabczyk [11]), if the corresponding stopping sets prove to be closed.

iii) Suppose present maximum $y \in [0,1]$. Then the mean payoff of the δ–look–ahead rule is, supposing remaining time $t+\delta$ and waiting a period of length $\delta \in (0, 1-t)$,

$$\begin{aligned} & (1-\lambda\delta+o(\delta))s(t,y) + (\lambda\delta+o(\delta)) \left(y s(t,y) + \int_y^1 s(t,\xi)\, d\xi \right) + o(\delta) \\ = \;& s(t,y) - \lambda\delta(1-y)s(t,y) + \lambda\delta \int_y^1 s(t,\xi)\, d\xi + o(\delta). \end{aligned}$$

Hereafter it seems advisible to choose y with time $t+\delta$ to go if this expression doesn't exceed $s(t+\delta, y)$, i.e. if

$$\frac{s(t+\delta,y)-s(t,y)}{\delta} \;\geq\; -\lambda(1-y)s(t,y) + \lambda \int_y^1 s(t,\xi)\, d\xi + o(\delta).$$

Letting $\delta \to 0$ this yields the condition (59). If the corresponding stopping sets additionally prove to be closed then this infinitesimal look–ahead rule is optimal according to Ross [26] — in other words stop as early as the infinitesimal operator (which is equal to the right minus the left side) becomes nonpositive. In the cited article take $\lambda = 0$ and take $c \equiv 0$ then the set B_0 is closed and specifies an optimal stopping time (stopping times here are bounded by 1). $\square$

In figure 8 below the difference between the subjects of the next two subsections, i.e. between problem $\mathcal{D}_\lambda^o$ and $\mathcal{D}_\lambda^t$ (referring to no recall), is illustrated.

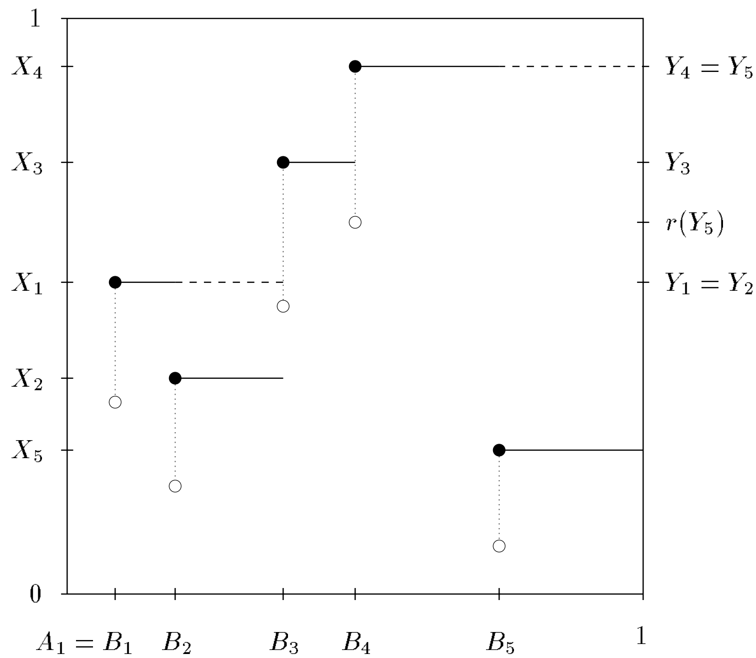

Figure 8: A sample path of $(X_{N_t})_{t\in[0,1]}$ with $N = 5$ where X_k resp. $r(X_k)$ is indicated by filled resp. empty dots, $k = 1, \ldots, 5$. For example X_3 is an overall r–candidate (with duration $1 - B_3$, chosen in B_3) and X_1 is not (duration as temporary r–candidate is $B_4 - B_1$, chosen in B_1).

4.3.1 The Duration of Owning an Overall r–Candidate

Maximizing the duration of owning an overall r–candidate without recall means maximizing the mean of random variable

$$D := (T - S) \cdot \mathbf{1}_{[r(Y_N),1]}(X_{N_S}),$$

which represents the remaining time choosing value X_{N_S} provided he finally proves to be an r–candidate with respect to Y_N. The horizon T is equal to 1 or exponentially distributed and selection takes place according to a nonanticipating stopping time S with convention $S = T$ in case of event $[S \geq T]$ with resulting duration 0. If recall is permitted X_{N_S} is replaced by Y_{N_S}.

Duration of Owning an Overall r–Candidate with Permanent Recall

Theorem 4.12 *Let the duration problem $\mathcal{D}^o_\lambda$ with $r \in \mathcal{R}^1$ be given. Then an optimal stopping time is given by $S^* := \inf\{b \in [0,1] : (b, Y_{N_b}) \in \Delta^*\}$ (set $\inf_\emptyset := 1$) with optimal stopping set $\Delta^* := \{(b,y) \in [0,1]^2 : y \geq y^*(1-b)\}$. Here $y^*(t)$, where $t := 1-b$ denotes the remaining time, is the unique solution in $(0, r(1))$ of*

$$\int_y^1 e^{\lambda t \varrho(\xi)}\, d\xi \;\; = \;\; \left(\frac{1}{\lambda t} + \varrho(y) - y\right) e^{\lambda t \varrho(y)}$$

if $\lambda t \int_0^1 e^{\lambda t \varrho(\xi)}\, d\xi > 1$ and $y^*(t) = 0$ otherwise.

Proof: Recalling the value y with time $t > 0$ to go yields the mean duration

$$s(t,y) \;\; = \;\; t e^{-\lambda t(1-\varrho(y))}, \tag{60}$$

where the rate of an arrival of an item beyond $\varrho(y)$ is $\lambda(1-\varrho(y))$. Now $\frac{\partial}{\partial t} s(t,y) = \Big(1 - \lambda t(1-\varrho(y))\Big) e^{-\lambda t(1-\varrho(y))}$, i.e. $s(t,y)$ is increasing in t iff $\lambda t(1-\varrho(y)) < 1$ and therefore an optimal stopping doesn't stop inside the set $\{(1-t, y) : \lambda t(1-\varrho(y)) > 1\}$ since waiting is profitable, see remark 4.14. The infinitesimal look–ahead rule of inequality (59) states the condition

$$(1-\lambda t(1-\varrho(y)))e^{-\lambda t(1-\varrho(y))} \geq -\lambda t(1-y)e^{-\lambda t(1-\varrho(y))} + \lambda \int_y^1 t e^{-\lambda t(1-\varrho(\xi))}\, d\xi$$

$$\int_y^1 e^{\lambda t \varrho(\xi)}\, d\xi \;\; \leq \;\; \left(\frac{1}{\lambda t} + \varrho(y) - y\right) e^{\lambda t \varrho(y)}.$$

Let $h(t,y) := \int_y^1 e^{\lambda t \varrho(\xi)}\, d\xi - \Big(\frac{1}{\lambda t} + \varrho(y) - y\Big) e^{\lambda t \varrho(y)} \in C^1((0,1] \times [0, r(1)])$. Now $\frac{\partial}{\partial t} h(t,y)$ proves to be positive iff $\int_y^1 \varrho(\xi) e^{\lambda t \varrho(\xi)}\, d\xi > (\varrho(y) - y)\varrho(y) e^{\lambda t \varrho(y)}$, which is true if $y \in [0, r(1))$, and on the other hand
$\frac{\partial}{\partial y} h(t,y) = -\varrho'(y) e^{\lambda t \varrho(y)} (2 + \lambda t(\varrho(y) - y))$. Thus $h(t,y)$ is increasing in t and decreasing in y. If $h(t,0) = \int_0^1 e^{\lambda t \varrho(\xi)}\, d\xi - 1/(\lambda t)$ is nonpositive, then $y^*(t) = 0$ is optimal (for $t = 0$, too), else there is a unique solution $y^*(t)$ inside $(0, r(1))$, because $h(t, r(1)) = -1/(\lambda t)$. Since $y^*(0) = 0$ the set for the infimum of S^* isn't empty unless $N = 0$. □

Example 4.13 Let $r(y) = \vartheta y$ for $y \in [0,1]$ where $\vartheta \in (0,1]$. Then regard theorem 4.12: $\int_0^1 e^{\lambda t \varrho(\xi)}\, d\xi = \frac{\vartheta}{\lambda t}\left(e^{\lambda t} - 1\right) + (1-\vartheta)e^{\lambda t} \leq \frac{1}{\lambda t}$ is equivalent to $\vartheta \geq h(\lambda t)$ where $h(u) := \frac{1-ue^u}{e^u - 1 - ue^u}$ for $u > 0$. Now $h(u) < 0$ for $u \in (0, u_0)$, $h(u) \in (0,1)$ for $u \in (u_0, \ln(2))$ and $h(u) > 1$ for $u > \ln(2)$ where $u_0 \approx 0.5671$ solves $1 = ue^u$ ($h(u_0) = 0$, $h(\ln(2)) = 1$ and h is increasing at least on $(0,1)$). Thus $y^*(t) = 0$ if $t \leq u_0/\lambda$ and $y^*(t) \in (0,\vartheta)$ if $t > \ln(2)/\lambda$, regardless of ϑ. If however $t \in (u_0/\lambda, \ln(2)/\lambda]$, then $y^*(t) = 0$ if $\vartheta \geq h(\lambda t)$ and otherwise $y^*(t) \in (0,\vartheta)$ is the unique solution of $(\lambda t(1-\vartheta)+\vartheta)e^{\lambda t} = (1+\vartheta+\lambda t y(1/\vartheta - 1))e^{\lambda t y/\vartheta}$. An instance (where $\vartheta = 0.8$ and rate $\lambda = 9$) is plotted in figure 9 on page 116, along with the corresponding problem $\mathcal{D}_\lambda^t$ with permanent recall.

Duration of Owning an Overall r–Candidate, Event Time Recall

Regard problem $\mathcal{D}_\lambda^o$ where recall is restricted to time instants of arrivals. The myopic stopping time suggests to stop if the mean payoff recalling $y \in (0, r(1)]$, expression (60), isn't lower than the payoff recalling the maximum in the time instant of the next arrival of an item (if any), inequality (58):

$$te^{-\lambda t(1-\varrho(y))} \;\geq\; \int_b^1 \left(\lambda e^{-\lambda(u-b)} \int_0^1 (1-u)e^{-\lambda(1-u)(1-\varrho(y\vee\xi))}\, d\xi\right) du,$$

and since $\int_b^1 (1-u)e^{-cu}\, du = [-(1-u)e^{-cu}/c + e^{-cu}/c^2]_b^1$ this gives

$$\begin{aligned}
te^{-\lambda t(1-\varrho(y))} &\geq \lambda e^{-\lambda(1-b)} \int_0^1 \left(e^{\lambda\varrho(y\vee\xi)} \int_b^1 (1-u)e^{-\lambda u\varrho(y\vee\xi)}\, du\right) d\xi \\
te^{\lambda t\varrho(y)} &\geq \lambda \int_0^1 \left(e^{\lambda\varrho(y\vee\xi)} \left(t\frac{e^{-\lambda b\varrho(y\vee\xi)}}{\lambda\varrho(y\vee\xi)} + \frac{e^{-\lambda\varrho(y\vee\xi)} - e^{-\lambda b\varrho(y\vee\xi)}}{(\lambda\varrho(y\vee\xi))^2}\right)\right) d\xi \\
te^{\lambda t\varrho(y)} &\geq \int_0^1 \left(t\frac{e^{\lambda(1-b)\varrho(y\vee\xi)}}{\varrho(y\vee\xi)} + \frac{1}{\lambda}\frac{1-e^{\lambda(1-b)\varrho(y\vee\xi)}}{\varrho^2(y\vee\xi)}\right) d\xi \\
te^{\lambda t\varrho(y)} &\geq yt\frac{e^{\lambda t\varrho(y)}}{\varrho(y)} + \frac{1}{\lambda}\frac{1-e^{\lambda t\varrho(y)}}{\varrho^2(y)} + \int_y^1 \left(t\frac{e^{\lambda t\varrho(\xi)}}{\varrho(\xi)} + \frac{1}{\lambda}\frac{1-e^{\lambda t\varrho(\xi)}}{\varrho^2(\xi)}\right) d\xi
\end{aligned}$$

and thus the myopic stopping time proposes to stop if (respect $y \in (0, r(1)]$)

$$\left(1 - \frac{y}{\varrho(y)}\right)\lambda te^{\lambda t\varrho(y)} - y\frac{1-e^{\lambda t\varrho(y)}}{\varrho^2(y)} \;\geq\; \int_y^1 \left(\lambda t\frac{e^{\lambda t\varrho(\xi)}}{\varrho(\xi)} + \frac{1-e^{\lambda t\varrho(\xi)}}{\varrho^2(\xi)}\right) d\xi.$$

Let $h(t,y) := \left(1 - \frac{y}{\varrho(y)}\right) \lambda t e^{\lambda t \varrho(y)} - y \frac{1-e^{\lambda t \varrho(y)}}{\varrho^2(y)} - \int_y^1 \left(\lambda t \frac{e^{\lambda t \varrho(\xi)}}{\varrho(\xi)} + \frac{1-e^{\lambda t \varrho(\xi)}}{\varrho^2(\xi)}\right) \, d\xi$ which is in $C^1([0,1] \times (0, r(1)])$, so it is suggested to stop if $h(t,y) \geq 0$.
Now $\frac{\partial}{\partial y} h(t,y) = \varrho'(y) \left(\lambda^2 t^2 \frac{\varrho(y)-y}{\varrho(y)} e^{\lambda t \varrho(y)} + 2y\lambda t \frac{1}{\varrho^2(y)} e^{\lambda t \varrho(y)} + 2y \frac{1-e^{\lambda t \varrho(y)}}{\varrho^3(y)}\right)$, where the first term in parantheses is nonnegative and the sum of the last two addends, too, since rearrangement yields $e^{-\lambda t \varrho(y)} \geq 1 - \lambda t \varrho(y)$. This representation implies that $h(t,y)$ is increasing in $y \in (0, r(1)]$ for any $t \in (0,1]$. Besides $h(t, r(1)) = r(1)(e^{\lambda t} - 1)$ is positive if $t \in (0,1]$. Thus, given t, the myopic stopping time accepts the present maximum y if $y \geq y^*(t)$ with a unique threshold $y^*(t) \in [0, r(1))$.
But the stopping sets of the myopic stopping time don't seem to be monotone in general: $\frac{\partial}{\partial t} h(t,y) = \lambda \left((1 + \lambda t(\varrho(y) - y)) e^{\lambda t \varrho(y)} - \lambda t \int_y^1 e^{\lambda t \varrho(\xi)} \, d\xi\right)$, proven to be negative only if $\lambda t(1 - \varrho(y)) > 1$: While $\lambda t(\varrho(y) - y) e^{\lambda t \varrho(y)} \leq \lambda t \int_y^{\varrho(y)} e^{\lambda t \varrho(\xi)} \, d\xi$ holds generally, for a universal conclusion the first inequality of $e^{\lambda t \varrho(y)} \leq \lambda t(1 - \varrho(y)) e^{\lambda t \varrho(y)} \leq \lambda t \int_{\varrho(y)}^1 e^{\lambda t \varrho(\xi)} \, d\xi$ seems to be necessary.
Thus the stopping sets of the myopic stopping time for $\mathcal{D}_\lambda^o$ with event time recall can be verified to be closed in general only if $\lambda t(1 - \varrho(y)) > 1$, see remark 4.14 below.

Duration of Owning an Overall r–Candidate — No Recall

Problem $\mathcal{D}_\lambda^o$ without recall doesn't prove to be regular in general: The mean duration of stopping in $(x,y) \in \Delta$ with $x \geq r(y)$ and time t to go is, according to expression (60), $s(t,x) = t e^{-\lambda t(1-\varrho(x))}$. This problem is regular if inequality (57) holds, which in this case is

$$\Big(1 - \lambda t(1 - \varrho(x))\Big) e^{-\lambda t(1-\varrho(x))} \;\leq\; -\lambda t(1-y) e^{-\lambda t(1-\varrho(x))} + \lambda t \int_y^1 e^{-\lambda t(1-\varrho(\xi))} \, d\xi$$

where rearrangement yields

$$\Big(1 + \lambda t(\varrho(x) - y)\Big) e^{\lambda t \varrho(x)} \;\leq\; \lambda t \int_y^1 e^{\lambda t \varrho(\xi)} \, d\xi.$$

This is verified to be true by partition of the integral in $\varrho(x)$ if $\lambda t(1-\varrho(x)) \geq 1$ holds. Thus problem $\mathcal{D}_\lambda^o$ without recall is verified to be regular at most in the beginning of the process or in C_λ, see the subsequent remark 4.14.

Remark 4.14 For problem $\mathcal{D}^o_\lambda$, regardless of the kind of access, the set $C_\lambda := \{(b,y) \in [0,1]^2 \,:\, \lambda(1-b)(1-\varrho(y)) \geq 1\}$ plays an essential role for regularity. C_λ represents the beginning of the process, where b and Y_{N_b} is small. $C_\lambda = \emptyset$ for $\lambda \in (0,1)$, $C_1 = \{(0,0)\}$ and $C_\lambda \nearrow [0,1) \times [0, r(1))$ as $\lambda \to \infty$.

Duration of Owning an Overall r–Candidate, Exponential Horizon

Inspect problem $\mathcal{D}^o_{\lambda,\mu}$, i.e. the Poisson process with arrival rate $\lambda > 0$ and exponentially distributed horizon T with parameter $\mu > 0$.

Theorem 4.15 *For problem $\mathcal{D}^o_{\lambda,\mu}(r)$ where $r \in \mathcal{R}^1$ with $\nu := \mu/\lambda$ the stopping time $S^* := T \wedge \inf\{b \geq 0 \,:\, X_{N_b} \geq x^*\}$ is optimal (set $\inf_\emptyset := \infty$) — take the first value above x^*. Here $x^* \in [0, r(1))$ denotes the unique solution in $(0, r(1))$ of*

$$\frac{\nu + 1 - x}{(\nu + 1 - \varrho(x))^2} \;=\; \int_x^1 \frac{1}{(\nu + 1 - \varrho(\xi))^2}\, d\xi$$

if $\int_0^1 1/(\nu + 1 - \varrho(\xi))^2 \, d\xi > 1/(\nu+1)$ and $x^* = 0$ otherwise.
The value of $\mathcal{D}^o_{\lambda,\mu}(r)$ — the mean duration applying S^* — then is

$$\mathrm{E}\left((T - S^*) \cdot \mathbf{1}_{[r(Y_N),1]}(X_{N_{S^*}})\right) \;=\; \frac{1}{\lambda}\frac{1}{\nu + 1 - x^*}\int_{x^*}^1 \frac{1}{(\nu + 1 - \varrho(\xi))^2}\, d\xi.$$

Proof: Supposing first that permanent recall is allowed, the mean payoff recalling $y \in [0, r(1)]$ is

$$s(y) \;=\; \int_0^\infty u e^{-\lambda(1-\varrho(y))u} \mu e^{-\mu u}\, du \;=\; \frac{\mu}{(\mu + \lambda(1 - \varrho(y)))^2} \;=\; \frac{1}{\lambda}\frac{1}{(\nu + 1 - \varrho(y))^2},$$

independent of the elapsed time and being proportional to $1/\lambda$ while keeping the rescaled rate $\nu = \mu/\lambda$ fixed. Now compare $s(y)$ and the mean payoff of proceeding one item (if any), similarly to inequality (44) in the proof of theorem 3.15 (any further arrival occurs with probability $\lambda/(\mu + \lambda)$):

$$\begin{aligned} \frac{\mu}{(\mu + \lambda(1 - \varrho(y)))^2} \;&\geq\; \frac{\lambda}{\mu + \lambda}\int_0^1 \frac{\mu}{(\mu + \lambda(1 - \varrho(y \vee \xi)))^2}\, d\xi \\ \frac{\nu + 1 - y}{(\nu + 1 - \varrho(y))^2} \;&\geq\; \int_y^1 \frac{1}{(\nu + 1 - \varrho(\xi))^2}\, d\xi. \end{aligned}$$

Let $h(y) := \frac{\nu+1-y}{(\nu+1-\varrho(y))^2} - \int_y^1 \frac{1}{(\nu+1-\varrho(\xi))^2}\, d\xi \in C^1([0, r(1)])$. Then h is increasing, since $h'(y) = 2\varrho'(y)/(\nu+1-\varrho(y))$. If $h(0) = \frac{1}{\nu+1} - \int_0^1 \frac{1}{(\nu+1-\varrho(\xi))^2}\, d\xi \geq 0$, then taking the first item is optimal, otherwise there is a unique solution of $h(0) = 0$ in $(0, r(1))$ representing an optimal threshold, since $h(r(1)) = 1/\nu$ — thus recall is superfluous/redundant.
The optimal mean duration applying S^* is evident analogously to the interpretation indicated at the end of the proof of theorem 3.15: An item exceeding x^* arrives with probability $(1-x^*)/(\nu+1-x^*)$ and then take the mean payoff choosing him given there is one ($1-x^*$ cancels).
In case of an empty set of the infimum then $S^* = T$ with corresponding payoff 0 (event $[T = \infty]$ is irrelevant as a nullset). □

Example 4.16 Suppose $r(y) = \vartheta y$ for $y \in [0,1]$ where $\vartheta \in (0,1]$. Then the equation $\frac{\nu+1-x}{(\nu+1-x/\vartheta)^2} = \frac{\vartheta}{\nu} - \frac{\vartheta}{\nu+1-x/\vartheta} + \frac{1-\vartheta}{\nu^2}$ uniquely determines an optimal threshold $x^* \in (0, \vartheta)$ unless $\vartheta \geq 1 + \nu - \nu^2$, where $x^* = 0$ is optimal. Particularly $x^* \in (0, \vartheta)$ if $\nu \leq 1$, on the opposite side the optimal threshold is $x^* = 0$ if $\nu \geq \frac{1+\sqrt{5}}{2} \approx 1.6180$ and in remaining cases $x^* \in [0, r(1))$ may occur. The value of $\mathcal{D}^o_{\lambda,\mu}(\vartheta x)$ is $\frac{1}{\lambda}\frac{1}{\nu+1-x^*}\left(\frac{\vartheta}{\nu} - \frac{\vartheta}{\nu+1-x^*/\vartheta} + \frac{1-\vartheta}{\nu^2}\right)$.

4.3.2 The Duration of Owning a Temporary r–Candidate

Maximizing the duration of owning a temporary r–candidate without recall means maximizing the mean of

$$D := (T - B_N) \cdot \mathbf{1}_{[r(Y_N),1]}(X_{N_S}) + \sum_{j=0}^{N-N_S-1} \left(A_{N_S+j+1} \cdot \mathbf{1}_{[r(Y_{N_S+j}),1]}(X_{N_S}) \right),$$

representing the epoch a value X_{N_S}, which is an r–candidate with respect to Y_{N_S}, stays an r–candidate with respect to $Y_{N_S+1}, \ldots, Y_N$ (set $\sum_\emptyset := 0$). Here A_k denotes the relative arrival time of object X_k, $k = 1, \ldots, N$. If recall is permitted X_{N_S} has to be replaced by Y_{N_S}. Here S represents a nonanticipating stopping time and for simplicity, to cover all cases, set $D := 0$ in case of event $[S < A_1]$ or event $[S > T]$.

Duration of Owning a Temporary r–Candidate, Permanent Recall

Regard problem $\mathcal{D}_\lambda^t$ with permanent recall, i.e. take horizon $T = 1$.

Theorem 4.17 *Let the duration problem $\mathcal{D}_\lambda^t$ with $r \in \mathcal{R}^1$ be given. Then an optimal stopping time is given by $S^* := \inf\{b \in [0,1] : (b, Y_{N_b}) \in \Delta^*\}$ (set $\inf_\emptyset := 1$) with optimal stopping set $\Delta^* := \{(b,y) \in [0,1]^2 : y \geq y^*(1-b)\}$. Here $y^*(t)$, where $t := 1-b$ denotes the remaining time, is the unique solution in $(0, r(1))$ of*

$$\int_y^1 \frac{1 - e^{-\lambda t(1-\varrho(\xi))}}{1 - \varrho(\xi)} \, d\xi \;=\; 1 + (\varrho(y) - y) \, \frac{1 - e^{-\lambda t(1-\varrho(y))}}{1 - \varrho(y)}$$

if $t > c/\lambda$ and $y^*(t) = 0$ if $t \leq c/\lambda$, where constant $c = c(r)$ is specified in lemma 4.18 below.

Proof: If the present state is (t,y) where $y \in [0, r(1))$, then the distribution of the arrival time of the next y–beating r–candidate is $\exp(\lambda(1 - \varrho(y)))$ and the probability that no such arrives is $e^{-\lambda t(1-\varrho(y))}$. Therefore the corresponding mean duration of stopping is given by

$$\begin{aligned} s(t,y) &= t e^{-\lambda t(1-\varrho(y))} + \int_0^t u \lambda (1 - \varrho(y)) e^{-\lambda u(1-\varrho(y))} \, du \\ &= \int_0^t e^{-\lambda u(1-\varrho(y))} \, du \\ &= \frac{1 - e^{-\lambda t(1-\varrho(y))}}{\lambda(1-\varrho(y))}, \end{aligned} \tag{61}$$

which consistently for $y \in [r(1), 1]$ yields t by the rule of de l'Hospital. It is suggested to stop by the infinitesimal look–ahead rule, inequality (59), if

$$\begin{aligned} e^{-\lambda t(1-\varrho(y))} &\geq -\lambda(1-y)s(t,y) + \lambda \int_y^1 s(t,\xi)\, d\xi \\ \int_y^1 \frac{1-e^{-\lambda t(1-\varrho(\xi))}}{1-\varrho(\xi)}\, d\xi &\leq 1 + (\varrho(y)-y)\frac{1-e^{-\lambda t(1-\varrho(y))}}{1-\varrho(y)}. \end{aligned}$$

Let $h(t,y) := \int_y^1 \frac{1-e^{-\lambda t(1-\varrho(\xi))}}{1-\varrho(\xi)} - 1 - (\varrho(y)-y)\frac{1-e^{-\lambda t(1-\varrho(y))}}{1-\varrho(y)} \in C^1([0,1]\times[0,r(1)])$ and it is advisible to stop if $h(t,y) \leq 0$. $\frac{\partial}{\partial t}h(t,y)$ is positive iff inequality $\int_y^1 e^{\lambda t \varrho(\xi)}\, d\xi > (\varrho(y)-y)e^{\lambda t \varrho(y))}$ holds, which is true if $y \in [0, r(1))$. Thus this stopping problem is monotone. On the other hand $\frac{\partial}{\partial y}h(t,y) = \frac{\varrho'(y)}{(1-\varrho(y))^2}\left[(\varrho(y)-y)(1-\varrho(y))\lambda t e^{-\lambda t(1-\varrho(y))} - (1-y)\left(1-e^{-\lambda t(1-\varrho(y))}\right)\right]$ where the term in brackets is negative for $y \in [0, r(1))$, because this is based on $1+\lambda t(1-\varrho(y)) < e^{\lambda t(1-\varrho(y))}$ (estimating $\frac{\varrho(y)-y}{1-y} \leq 1$). Thus $h(t,y)$ decreases in $y \in [0, r(1))$ for $t \in [0,1]$ and the following holds: If $h(t,0)$ is nonpositive, or, with regard to lemma 4.18, if $t \leq c/\lambda$, then $y^*(t) = 0$ is an optimal threshold. Otherwise, if $t > c/\lambda$, there is a unique solution $y^*(t)$ inside $(0, r(1))$, such that stopping is optimal if $y \geq y^*(t)$, since $h(t, r(1)) = -1$. The set the infimum is specified by isn't empty unless $N = 0$ (payoff 0). □

Lemma 4.18 *For $r \in \mathcal{R}$ let $c = c(r) \in (1, c_1]$ denote the unique solution of*

$$\int_0^1 \frac{1-e^{-c(1-\varrho(\xi))}}{1-\varrho(\xi)}\, d\xi \;=\; 1, \tag{62}$$

where $c_1 \approx 1.3450$ solves $\int_0^1 \frac{1-e^{-c\zeta}}{\zeta}\, d\zeta$. Besides $c(r_1) < c(r_2)$ if $r_1 \prec r_2$ in $\mathcal{R}$.

Proof: Let $I(c)$ denote the left side of the following equivalent equation:

$$\int_0^1 \frac{1-e^{-c\xi}}{\xi}\, dq(\xi) \;=\; 1$$

with distribution function $q(\xi) := 1 - r(1-\xi)$ for $\xi \in \mathbb{R}$ (using remark 2.18 ii); mass $1 - r(1)$ in 0) and where the integrand will be called $f_c(\xi)$. Since $I(c)$ is continuous and increasing in c (because the integrand is), $I(0) = 0$

and $I(2) > 1$, a value $c(r)$ exists and is unique — $I(0) = 0$ holds because the integrand vanishes due to the rule of de l'Hospital and also $I(2) > 1$ holds for any $r \in \mathcal{R}$: For $c > 0$ a proper lower bound of the integral $I(c)$ is given by the area of the triangle with the edges $(0,0)$, $(0,c)$ and $(1 \wedge (c^3/2), 0)$, whose area is, for $c \geq \sqrt[3]{2}$, given by $c/2$: First $f_c(0) = c$ and $f_c'(0) = -c^2/2$. Second $f_c'(\xi) \leq 0$ and $f_c''(\xi) \geq 0$ (for $\xi \in (0,1]$ leading to the series expansion of $e^{c\xi}$ to first and second order, respectively). $f_c'(\xi) \leq 0$ and $id \preceq q$ now imply that the Lebesgue–measure on $[0,1]$ minimizes the area of the triangle for $r \in \mathcal{R}$. For the supplement let $r_1, r_2 \in \mathcal{R}$. Now $r_1 \prec r_2$ implies $q(r_2) \prec q(r_1)$. Thus $c(r_1) < c(r_2)$, since $f_c(\xi)$ is decreasing in ξ ($f_c'(\xi) \leq 0$, see above). Particularly the maximal value is $c_1 := c(id) \approx 1.3450$, and the minimal value (not attained inside $\mathcal{R}$) is $c(\varepsilon_1) = 1$. □

Example 4.19 Let $r(y) = \vartheta y$ for $y \in [0,1]$ where $\vartheta \in (0,1]$. The constant $c(\vartheta y)$ is determined by $\int_0^1 \frac{1-e^{-c(1-\varrho(\xi))}}{1-\varrho(\xi)}\, d\xi = \vartheta \int_0^1 \frac{1-e^{-c\zeta}}{\zeta}\, d\zeta + (1-\vartheta)c = 1$. Then $y^*(t) = 0$ if $t \leq c/\lambda$ and otherwise $y^*(t) \in (0, \vartheta)$ is the unique solution of equation $\vartheta \int_0^{1-y/\vartheta} \frac{1-e^{-\lambda t\zeta}}{\zeta}\, d\zeta + (1-\vartheta)\lambda t = 1 + (y/\vartheta - y)\, \frac{1-e^{-\lambda t(1-y/\vartheta)}}{1-y/\vartheta}$.

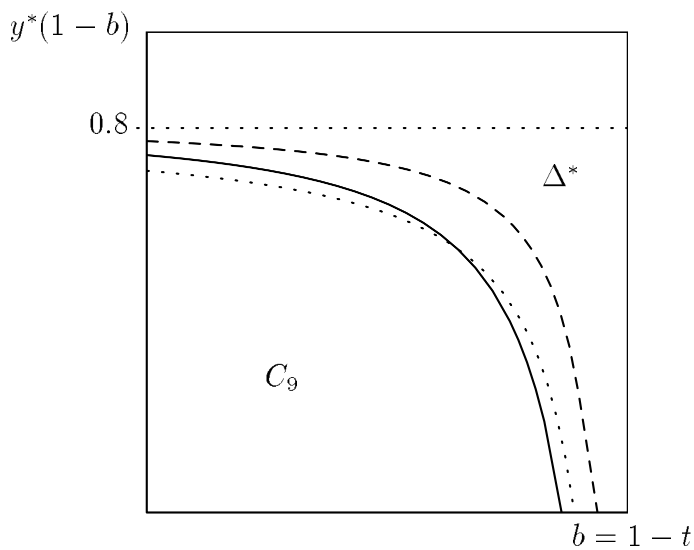

Figure 9: The optimal stopping set Δ^* (the area beyond curve $y^*(t)$) of problem $\mathcal{D}_\lambda^o$ resp. $\mathcal{D}_\lambda^t$ (dashed resp. solid line) each with permanent recall where $r(y) = 0.8y$ and $\lambda = 9$, see example 4.13 resp. 4.19. The area beyond 0.8 evidently is contained in Δ^*. For $\mathcal{D}_\lambda^o$ it is advisible not to stop inside set C_9 indicated by the dotted line $r(1 - 1/(9t))$, since waiting is profitable (remark 4.14 and proof of theorem 4.12).

Duration of Owning a Temporary r–Candidate, Event Time Recall

Regard problem $\mathcal{D}_\lambda^t$ where recall is restricted to time instants of arrivals. The myopic stopping time recalls $y \in (0, r(1))$ according to inequality (58) if

$$\begin{aligned}
\frac{1-e^{-\lambda(1-b)(1-\varrho(y))}}{\lambda(1-\varrho(y))} &\geq \int_b^1 \left(\lambda e^{-\lambda(u-b)} \int_0^1 \frac{1-e^{-\lambda(1-u)(1-\varrho(y\vee\xi))}}{\lambda(1-\varrho(y\vee\xi))}\, d\xi\right) du \\
\frac{1-e^{-\lambda(1-b)(1-\varrho(y))}}{\lambda(1-\varrho(y))} &\geq e^{-\lambda t} \int_0^1 \left(\int_b^1 \frac{e^{\lambda(1-u)} - e^{\lambda(1-u)\varrho(y\vee\xi)}}{1-\varrho(y\vee\xi)}\, du\right) d\xi, \quad (63)
\end{aligned}$$

respecting that the integrand is bounded. Now the innermost integral is (using abbreviation $\varrho = \varrho(y \vee \xi)$)

$$\begin{aligned}
&\int_b^1 \sum_{k=1}^\infty \frac{(\lambda(1-u))^k(1-\varrho^k)}{k!(1-\varrho)}\, du \\
=\;& \frac{1}{\lambda}\left[-\sum_{k=1}^\infty \frac{(\lambda(1-u))^{k+1}(1-\varrho^k)}{(k+1)!(1-\varrho)}\right]_b^1 \\
=\;& \frac{1}{\lambda(1-\varrho)} \sum_{k=1}^\infty \frac{(\lambda t)^{k+1}(1-\varrho^k)}{(k+1)!} \\
=\;& \frac{1}{\lambda(1-\varrho)}\left(-1-\lambda t + e^{\lambda t} - \frac{1}{\varrho}\left(-1-\lambda t\varrho + e^{\lambda t\varrho}\right)\right) \\
=\;& \frac{1-\varrho+\varrho e^{\lambda t} - e^{\lambda t\varrho}}{\lambda\varrho(1-\varrho)},
\end{aligned}$$

which is identical to the formal integration of the corresponding fraction. Resumed inequality (63) is equivalent to (resolve $\varrho(y \vee \xi)$ by partition of the integration range with respect to ξ, apply the rule of de l'Hospital for $\xi \in [r(1), 1]$ resp. for $\varrho = 1$)

$$\begin{aligned}
\frac{1-e^{-\lambda t(1-\varrho(y))}}{1-\varrho(y)} \geq\;& \frac{y}{1-\varrho(y)}\left(1-e^{-\lambda t} + \frac{e^{-\lambda t} - e^{-\lambda t(1-\varrho(y))}}{\varrho(y)}\right) \\
&+ \int_y^{r(1)} \frac{1-e^{-\lambda t}}{1-\varrho(\xi)} + \frac{e^{-\lambda t} - e^{-\lambda t(1-\varrho(\xi))}}{\varrho(\xi)(1-\varrho(\xi))}\, d\xi \\
&+ (1-r(1))\left(t - \frac{1-e^{-\lambda t}}{\lambda}\right),
\end{aligned}$$

which by rearrangement yields

$$\frac{(1-y)\varrho(y)-y(1-\varrho(y))e^{-\lambda t}-(\varrho(y)-y)e^{-\lambda t(1-\varrho(y))}}{\varrho(y)(1-\varrho(y))}$$
$$\geq \int_y^{r(1)} \frac{\varrho(\xi)+(1-\varrho(\xi))e^{-\lambda t}-e^{-\lambda t(1-\varrho(\xi))}}{\varrho(\xi)(1-\varrho(\xi))}\,d\xi+(1-r(1))\left(t-\frac{1-e^{-\lambda t}}{\lambda}\right).$$

Now let $h(t,y)$ denote the difference of the left side minus the right one, $h\in C([0,1]\times(0,r(1))$. Thus the myopic stopping time proposes to stop if $h(t,y)\geq 0$. Now on the one hand $\frac{\partial}{\partial t}h(t,y)=$
$\lambda\frac{ye^{-\lambda t}+(\varrho(y)-y)e^{-\lambda t(1-\varrho(y))}}{\varrho(y)}+\lambda\int_y^{r(1)}\frac{e^{-\lambda t}-e^{-\lambda t(1-\varrho(\xi))}}{\varrho(\xi)}\,d\xi-(1-r(1))(1-e^{-\lambda t})$,
not nonpositive in general, as examples suggest. On the other hand $\frac{\partial}{\partial y}h(t,y)=$
$\varrho'(y)\Big[\left(1-y+ye^{-\lambda t}-(1+\lambda t(\varrho(y)-y))e^{-\lambda t(1-\varrho(y))}\right)\varrho(y)(1-\varrho(y))\ -$
$\left((1-y)\varrho(y)-y(1-\varrho(y))e^{-\lambda t}-(\varrho(y)-y)e^{-\lambda t(1-\varrho(y))}\right)(1-2\varrho(y))\Big]\Big/$
$[\varrho^2(y)(1-\varrho(y))^2]$, where the nominator seems to be nonnegative, but seems to be not easy to verify.

Thus, analogously to problem $\mathcal{D}^o_\lambda$ with event time recall, here the stopping sets of the myopic stopping time don't seem to be closed in general and an optimal stopping time isn't specified through this.

Duration of Owning a Temporary r–Candidate — No Recall

Problem $\mathcal{D}^t_\lambda$ without recall isn't resolved, but the main terms are given, regarding the regular case: According to expression (61) the mean payoff of stopping in $(x,y)\in\Delta$ with $x\geq r(y)$ and time t to go is

$$s(t,x) \;=\; \frac{1-e^{-\lambda t(1-\varrho(x))}}{\lambda(1-\varrho(x))}.$$

Now inequality (57) in this setting yields

$$e^{-\lambda t(1-\varrho(x))}\;\leq\;-\lambda(1-y)\frac{1-e^{-\lambda t(1-\varrho(x))}}{\lambda(1-\varrho(x))}+\lambda\int_y^1\frac{1-e^{-\lambda t(1-\varrho(\xi))}}{\lambda(1-\varrho(\xi))}\,d\xi,$$

where rearrangements yield

$$1+(\varrho(x)-y)\frac{1-e^{-\lambda t(1-\varrho(x))}}{1-\varrho(x)}\;\leq\;\int_y^1\frac{1-e^{-\lambda t(1-\varrho(\xi))}}{1-\varrho(\xi)}\,d\xi.$$

This inequality proves to be true if

$$1 \quad \leq \quad \int_{\varrho(x)}^{1} \frac{1 - e^{-\lambda t(1-\varrho(\xi))}}{1 - \varrho(\xi)} \, d\xi$$

by partitioning the integral in $\varrho(x)$ and respecting that the integrand is non-decreasing in ξ and $\xi \geq y \geq x$.
If however the reverse inequality would be true, then at least the following can be concluded: The mean duration choosing x exceeds the mean duration for choosing the next r–candidate beyond $\varrho(x)$ (if there is none then payoff is 0, if there is one then pretend he is as early as possible, i.e. time t to go):

$$\frac{1 - e^{-\lambda t(1-\varrho(x))}}{\lambda(1 - \varrho(x))} \quad > \quad \frac{1 - e^{-\lambda t(1-\varrho(x))}}{1 - \varrho(x)} \int_{\varrho(x)}^{1} \frac{1 - e^{-\lambda t(1-\varrho(\xi))}}{\lambda(1 - \varrho(\xi))} \, d\xi,$$

where however arrivals with value inside $[r(y), \varrho(x))$ aren't considered.

Duration of Owning a Temporary r–Candidate, Exponential Horizon

The optimal stopping problem $\mathcal{D}^t_{\lambda,\mu}(r)$ is equivalent to problem $\mathcal{P}_{\lambda,\mu}(r)$ of theorem 3.15, since the mean payoff recalling $y \in [0, r(1))$ is given by

$$\begin{aligned} s(y) &= \int_0^\infty \frac{1 - e^{-\lambda u(1-\varrho(y))}}{\lambda(1 - \varrho(y))} \mu e^{-\mu u} \, du \\ &= \frac{\mu}{\lambda(1 - \varrho(y))} \left(\left[-\frac{1}{\mu} e^{-\mu u} \right]_0^\infty - \left[\frac{1}{\lambda(1 - \varrho(y)) + \mu} e^{-(\lambda(1-\varrho(y))+\mu)u} \right]_0^\infty \right) \\ &= \frac{1}{\lambda(1 - \varrho(y))} \left(1 - \frac{\mu}{\lambda(1 - \varrho(y)) + \mu} \right) \\ &= \frac{1}{\lambda(1 - \varrho(y)) + \mu} \\ &= \frac{1}{\lambda} \frac{1}{1 - \varrho(y) + \nu}, \end{aligned}$$

which differs from the mean payoff of selecting an r–candidate by factor $\lambda\nu$, see theorem 3.15. The factor $1/\lambda$ again covers proportionality to time keeping $\nu := \mu/\lambda$ fixed. For an instance see example 3.16, where, next to ν, λ has to be specified and then the values must be divided by $\lambda\nu$.

Lastly the duration problems of this whole chapter are considered for a more general distribution function of the offers:

Remark 4.20 Suppose $X_1, X_2, \ldots$ are iid with distribution function F, where F is increasing on $R := \{x \in \mathbb{R} : 0 < F(x) < 1\}$ and absolute continuous with a density nonvanishing almost everywhere on R. As relax function take $r : R \to R$ continuous and increasing with $r \preceq id$ on R. The duration problems of theorems in this chapter where the myopic stopping time proves to be optimal are resolved in the same manner, where now the density of F appears and the argumentation is the same. The results are as follows:
Regard the main equations in theorems of this chapter, which specify a solution y depending on the remaining time ℓ or t: A term $\varrho(y)$ resp. y has to be replaced by $F(\varrho(y))$ resp. $F(y)$ and integration is with respect to $dF(\xi)$. Exemplary for the discrete setting take equation (48), which changes to $(F(\varrho(y)))^\ell + (1 - F(y)) \sum_{j=0}^{\ell-1} (F(\varrho(y)))^j = \sum_{j=0}^{\ell-1} \int_y^1 (F(\varrho(\xi)))^j \, dF(\xi)$, with a unique solution y_ℓ inside $(0, r(\sup\{x \in \mathbb{R} : F(x) < 1\}))$ for $1 < \ell \in \mathbb{N}$.

Regarding problem $\mathcal{D}_n^o$ and $\mathcal{D}_n^t$ with recall the optimal thresholds of the former seem to be bigger than those of the latter. Particularly this is true asymptotically as figure 6 suggests. This may reflect the phenomenon that the chosen offer in one case must represent an overall r–candidate, while in the other case an intermediate r–candidate is worthwhile. Consistently this relation seems to persist for problem $\mathcal{D}_\lambda^o$ compared with $\mathcal{D}_\lambda^t$ each with permanent recall, see figure 9.

Concluding Remarks

In this thesis an optimal stopping problem with full information and mainly with iid offers has been investigated in discrete and continuous time. The dedicated functional is given by the mean of a payoff function, which depends in its most general form on the stopping time S, the value of the chosen offer X_S and the overall maximum Y_N, where N denotes the total number of presented offers. In chapter 2 and 3 this payoff function obeys monotonicity assumptions which seem to be indispensable in order to ensure in general that the problem is regular. In this case optimal stopping sets are, in principle, specified. Optimal selection of an r–candidate has been the main task, a generalization of the full information best choice problem. Threshold rules have been considered and the myopic stopping time has been specified and verified to be optimal or indicated to be not optimal. In the last chapter the socalled duration problem has been investigated based on r–candidates.

In the case of optimal selection of an r–candidate it would be interesting to specify the value of the myopic stopping time, particularly in the case of the Poisson process, though the performance seems to be not promising regarding its stopping sets.
A main question is the behaviour of the asymptotic value referring to optimal selection of an r–candidate as a function of $r'(1-)$. For this purpose the indicated threshold rules restricted to r–candidates may be helpful.
As an extension for optimal selection of an r–candidate observation costs can be taken into account, which seems to be accessible only for selection with recall due to the regular case. Further offers may supposed to be unavailable with a certain probability or the period of accepting offers may be terminated by a freezing random variable, while the total number of offers remains unaffected.
Regarding the duration problem based on r–candidates investigation of threshold rules would be interesting in the case of no recall in discrete and in continuous time in order to get access to the problem.

A Appendix

Definition A.1 *Let two real functions g_1, g_2 with identic domain G be given. The relation $g_1 \preceq g_2$ is defined to be valid, if $g_1(x) \leq g_2(x)$ for $x \in G$. The relation $g_1 \prec g_2$ holds if $g_1 \preceq g_2$ and if there is an $x \in G$ such that $g_1(x) < g_2(x)$ is valid.*

An application of this notation: Let two distribution functions F and G with $F \prec G$ be given and let $h : [0,1] \to (0,\infty)$ be nonincreasing. Then $\int_0^1 h(x)\, dF(x) < \int_0^1 h(x)\, dG(x)$ or $\mathrm{E}\,(h(V)) < \mathrm{E}\,(h(W))$, where F resp. G is the distribution function of random variable V resp. W, i.e. V is stochastically lower than W. This can be verified by mass theoretical induction based on nonincreasing functions.

Lemma A.2 *Let $(x_\ell)_{\ell\in\mathbb{N}} \subset \mathbb{R}$ denote a convergent sequence with $x := \lim_{\ell\to\infty} x_\ell$.*

$$\sum_{k=1}^{\ell} \frac{1}{k}\left(1+\frac{x_\ell}{\ell}\right)^k \simeq \begin{cases} \gamma + \ln \ell & \text{if } x = 0 \\ \mathrm{Ei}(x) - \ln|x| + \ln \ell & \text{if } x \neq 0 \end{cases} \quad \text{as } \ell \to \infty, \quad (64)$$

where the exponential integral function Ei *is defined below.*

Proof: Let $\varepsilon_\ell := x_\ell - x = o(1)$ as $\ell \to \infty$. Taking $x = 0$ the relation $\sum_{k=1}^{\ell} \frac{1}{k}\left(1+\frac{\varepsilon_\ell}{\ell}\right)^k \simeq \gamma + \ln \ell$, known if $\varepsilon_\ell = 0\ \ \forall \ell \in \mathbb{N}$, is valid for $\varepsilon_\ell = o(1)$: The left side is equal to $\sum_{k=1}^{\ell} \frac{1}{k}\left(1 + o\left(\frac{1}{\ell}\right)\right) = o(1) + \sum_{k=1}^{\ell} \frac{1}{k}$, using the binomial theorem ($o(\frac{1}{\ell})$ depends on k). The assertion is, as $\ell \to \infty$,

$$\sum_{k=1}^{\ell} \frac{1}{k}\left(1+\frac{x_\ell}{\ell}\right)^k = \sum_{k=1}^{\ell} \left[\frac{1}{\ell}\left(\left(1+\frac{x_\ell}{\ell}\right)^\ell\right)^{k/\ell} \Big/ \frac{k}{\ell}\right] \simeq \gamma + \int_{1/\ell}^{1} \frac{e^{x\xi}}{\xi}\, d\xi. \quad (65)$$

Now fix $\varepsilon_\ell = o(1)$ as $\ell \to \infty$ and restrict x on interval $[a,b] \subset \mathbb{R}$. Then define $g_\ell(x) := \sum_{k=1}^{\ell} \frac{1}{k}\left(1+\frac{x+\varepsilon_\ell}{\ell}\right)^k - \int_{1/\ell}^{1} \frac{e^{x\xi}}{\xi}\, d\xi$. Now g_ℓ' is uniformly convergent to 0 on $[a,b]$: $g_\ell'(x) = \sum_{k=1}^{\ell} \frac{1}{\ell}\left(1+\frac{x+\varepsilon_\ell}{\ell}\right)^{k-1} - \left[\frac{e^{x\xi}}{x}\right]_{1/\ell}^{1} = \frac{\left(1+\frac{x+\varepsilon_\ell}{\ell}\right)^\ell - 1}{x+\varepsilon_\ell} - \frac{e^x - e^{x/\ell}}{x}$, according to lemma A.3 below (pathological terms are understood according to the rule of de l'Hospital). Due to $\lim_{\ell\to\infty} g_\ell(0) = \gamma$ the following holds on $[a,b]$ (with fixed $\varepsilon_\ell = o(1)$): g_ℓ is uniformly convergent, the limit $\lim_{\ell\to\infty} g_\ell(x) =: g(x)$ exists, g is differentiable and $g'(x) = \lim_{\ell\to\infty} g_\ell'(x)$.

Then $g' \equiv 0$ or $g \equiv \gamma$ on this interval, because $g(0) = \gamma$. Thus relation (65) is verified for any $x \in \mathbb{R}$ and $\varepsilon_\ell = o(1)$ as $\ell \to \infty$. Besides the lower integration limit $1/\ell$ in (65) can't be replaced by $o(1)$ due to the case $x = 0$. The right side of (65) equals $\gamma + \mathrm{Ei}(x) - \mathrm{Ei}\,(x/\ell)$, where $\mathrm{Ei}(x) := \int_{-\infty}^{x} \frac{e^\xi}{\xi}\,d\xi$ denotes the exponential integral function (for $x \neq 0$; for $x > 0$ the principal value is taken). The relation $\mathrm{Ei}(x/\ell) - \ln|x/\ell| \to \gamma$ as $\ell \to \infty$ results for example from the identity $\gamma = -\int_0^\infty e^{-\xi} \ln\xi\, d\xi$ via integration by parts. □

Lemma A.3 *Function* $\frac{\left(1+\frac{x+\varepsilon_\ell}{\ell}\right)^\ell - 1}{x+\varepsilon_\ell} - \frac{e^x - e^{x/\ell}}{x}$ *is uniformly convergent to 0 on interval* $[a,b] \subset \mathbb{R}$ *as* $\ell \to \infty$ *(see the proof of lemma A.2).*

Proof: Set $c := \max\{|a|,|b|\} + \sup_{\ell\in\mathbb{N}} |\varepsilon_\ell| < \infty$, then $|x|, |x+\varepsilon_\ell| \le c$ for $\ell \in \mathbb{N}$. Telescope $\frac{\left(1+\frac{x+\varepsilon_\ell}{\ell}\right)^\ell - 1}{x+\varepsilon_\ell} - \frac{e^{x+\varepsilon_\ell}-1}{x+\varepsilon_\ell} + \frac{e^{x+\varepsilon_\ell}-1}{x+\varepsilon_\ell} - \frac{e^x-1}{x} + \frac{e^x-1}{x} - \frac{e^x-e^{x/\ell}}{x}$, then all three differences are uniformly convergent to 0 on $[a,b]$ as $\ell \to \infty$:
The last difference is $\left|\frac{e^x-1}{x} - \frac{e^x-e^{x/\ell}}{x}\right| = \frac{1}{\ell}\sum_{k=0}^\infty \frac{|x/\ell|^k}{(k+1)!} \le e^c/\ell = o(1)$.
The medial difference yields $\left|\frac{e^{x+\varepsilon_\ell}-1}{x+\varepsilon_\ell} - \frac{e^x-1}{x}\right| = \left|\sum_{k=2}^\infty \frac{(x+\varepsilon_\ell)^{k-1} - x^{k-1}}{k!}\right|$
$= |\varepsilon_\ell| \sum_{k=2}^\infty \frac{1}{k!} \sum_{j=0}^{k-2} \binom{k-2}{j} |x+\varepsilon_\ell|^{k-2-j}|x|^j \le |\varepsilon_\ell| \sum_{k=2}^\infty \frac{1}{k!} c^{k-2} 2^{k-2} \le |\varepsilon_\ell| e^{2c}$, which is again $o(1)$.
The first difference: Function $h_\ell(y) := \frac{\left(1+\frac{y}{\ell}\right)^\ell - e^y}{y}$ is nonincreasing resp. nondecreasing in ℓ for $\ell > c$ on $[-c,0]$ resp. on $[0,c]$ and its limit is 0 for each $y \in [-c,c]$ ($h_\ell(0) = 0$ for $\ell \in \mathbb{N}$, applying the rule of de l'Hospital). According to the theorem of Dini the convergence then is uniform on $[-c,0]$ resp. on $[0,c]$ and thus on $[-c,c]$. Regarding $h_\ell(x+\varepsilon_\ell)$ the uniform convergence is preserved: If $x < 0$ then $x + \varepsilon_\ell < 0$ finally, if $x > 0$ then $x + \varepsilon_\ell > 0$ finally and if $x = 0$ then the sign of $x + \varepsilon_\ell$ may alternate but negative and positive values are covered concerning a uniform bound on $[-c,c]$ and $h_\ell(0) = 0$.
All three differences prove to be uniformly convergent to 0 on $[a,b]$ and thus their sum is, too. □

Remark A.4 The entire exponential integral function is defined for $x \in \mathbb{R}$: $\mathrm{Ein}(x) := \int_0^x \frac{1-e^{-\xi}}{\xi}\,d\xi = -\sum_{k=1}^\infty \frac{(-x)^k}{k!k} = \gamma + \ln|x| - \mathrm{Ei}(-x)$ (with $\mathrm{Ein}(0) = 0$). Let $\lim\limits_{\ell\to\infty} x_\ell = x \neq 0$ and $\lim\limits_{\ell\to\infty} y_\ell = y \neq 0$. Then, regarding lemma A.2:

$$\lim_{\ell\to\infty} \sum_{k=1}^{\ell} \frac{1}{k}\left(\left(1+\frac{x_\ell}{\ell}\right)^k - \left(1+\frac{y_\ell}{\ell}\right)^k\right) = \mathrm{Ei}(x) - \ln|x| - \mathrm{Ei}(y) + \ln|y|$$

$$= -\mathrm{Ein}(-x) + \mathrm{Ein}(-y) = \int_{-x}^{-y} \frac{1-e^{-\xi}}{\xi}\,d\xi = \int_y^x \frac{e^\xi - 1}{\xi}\,d\xi.$$

Notations

$b_\ell^*(y)$	optimal boundary function, see definition 2.5 and its preliminaries
$C^1(A)$	functions on A, continuous differentiable, derivative $\pm\infty$ allowed
Δ, Δ_ℓ^*	$\{(x,y) \in [0,1]^2 : x \le y\}$ resp. optimal stopping sets (definition 2.5)
ε_x	point mass in $x \in \mathbb{R}$, with corresponding distribution function $\mathbf{1}_{[x,\infty)}$
f	represents the payoff function of chapter 2 and 3
γ	Euler constant, approximately 0.5772
id	the identity function $id(x) = x$ for $x \in \mathbb{R}$
$\mathbb{N}$	$\{1, 2, \ldots\}$
N_t, N	number of objects arriving in $[0,t]$ resp. $[0,T]$, see page 60
$o(\cdot)$, $O(\cdot)$	little and big o–notation, Landau symbols
Φ	distribution function of the standard normal distribution
$\mathbb{P}$	denotes the transition function applied to s, see equation (41)
r, ϱ	generally a function inside $\mathcal{R}$ resp. its inverse see pages 21f
$\mathcal{R}$, $\mathcal{R}_1^1$	sets of specific functions $r : [0,1] \to [0,1]$, see page 21
$\sigma(\cdots)$	smallest σ–algebra containing $\cdots$
S^*	notation for an optimal stopping time
S_m	notation for the myopic stopping time, referring to r–candidates
$\mathcal{T}_n$, $\mathcal{T}_n^c$	set of threshold resp. concurrent threshold rules, see page 39
$U([a,b])$	the uniform distribution on interval $[a,b] \subset \mathbb{R}$
$v(\cdot)$	the value of a state, stopping time or of the problem itself
v_n^*, v_λ^*, v_∞^*	the value of problem $\mathcal{P}_n$ and of problem $\mathcal{P}_\lambda$, asymptotic value
Y_k	the maximum of $X_1, \ldots, X_k$
$\mathbb{Z}_+$	$\{0\} \cup \mathbb{N}$
$\sim$	indicates the distribution of a random variable
$\simeq$	asymptotic equivalence ($x_k \simeq c + y_k$, $k \to \infty$ means $\lim_{k\to\infty}(x_k - y_k) = c$)
$\approx$	approximative specification of a number
$\stackrel{D}{=}$	sign for equality in distribution
$\ll$, $\gg$	significantly or sufficiently lower resp. bigger than
$\vee$, $\wedge$	maximum and minimum sign
$\nearrow$, $\searrow$	monotone convergence (nondecreasing resp. nonincreasing)
$\prec$, $\preceq$	relations for real functions, see definition A.1, page 122
$\mathbf{1}_A(x)$	indicator function of set $A \subset \mathbb{R}$
$\lceil x \rceil$, $\lfloor x \rfloor$	ceil resp. floor: $\inf\{n \in \mathbb{Z} : n \ge x\}$ resp. $\sup\{n \in \mathbb{Z} : n \le x\}$
$[X \le x]$	brackets for an event

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